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# FIRST COHOMOLOGY GROUPS FOR FINITE GROUPS OF LIE TYPE IN DEFINING CHARACTERISTIC

ALISON E. PARKER AND DAVID I. STEWART

ABSTRACT. Let  $G$  be a finite group of Lie type, defined over a field  $k$  of characteristic  $p > 0$ . We find explicit bounds for the dimensions of the first cohomology groups for  $G$  with coefficients in simple  $kG$ -modules. We proceed by bounding the number of composition factors of Weyl modules for simple algebraic groups independently of  $p$  and using this to deduce bounds for the 1-cohomology of simple algebraic groups. If  $\gamma_l$  denotes the (finite) maximum of the dimensions of the 1-cohomology groups over all Lie groups of rank  $l$  we find bounds for the growth rate of the sequence  $\{\gamma_l\}$ . We show that  $\log \gamma_l$  is  $O(l^3 \log l)$ .

## 1. INTRODUCTION

In [Gur86] R. Guralnick made a conjecture that there should be a universal bound on the dimension of the first cohomology groups  $H^1(G, V)$  where  $G$  is a finite group and  $V$  is an absolutely irreducible faithful representation for  $G$ . The conjecture reduces to the case where  $G$  is a finite simple group, thence, using the classification of finite simple groups, readily to the case where  $G$  is a finite group of Lie type or alternating group, since the number of possibilities for  $G$  and  $V$  when  $G$  is sporadic and  $H^1(G, V) \neq 0$  is finite.

Very recently, computer calculations of Frank Lübeck, complemented by those of Leonard Scott and Tim Sprowl have provided strong evidence that the Guralnick conjecture may unfortunately be false.<sup>1</sup> Finding an infinite sequence of increasing dimensions of  $H^1(G, V)$  has proven to be very challenging, but assuming such a sequence is eventually produced, the natural question, as originally proposed in [Sco03] is to find upper and lower bounds for the growth rate of  $\dim H^1(G, V)$  as the Lie rank of  $G$  grows.

On the question of finding upper bounds, there are two distinct cases, admitting very different techniques. In defining characteristic—that is where  $V$  is a representation for a finite group of Lie type  $G(p^r)$  over a field of characteristic  $p$ —the result [CPS09, Theorem 7.10] uses algebraic group methods to assert the implicit existence of a bound on 1-cohomology.<sup>2</sup> In cross-characteristic, [GT11] is more specific, giving  $|W| + e$  as a bound on  $\dim H^1(G, V)$ , where  $e$  is the twisted rank

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<sup>1</sup>Due to Scott–Sprowl (as yet unpublished), one knows there are simple  $SL_n$ -modules  $V_n$  such that  $\dim H^1(SL_n, V_n)$  takes the values 1, 1, 1, 2, 3, 16, 469, 36672 for  $n = 2, \dots, 9$  respectively. All bar the last of these values is confirmed by Lübeck. It follows there are values of  $q = p^r$  such that  $\dim H^1(PSL_n(q), V_n)$  takes these same values. We thank L. Scott for the provision of these numbers.

<sup>2</sup>In [PS11] this was generalised in several directions, notably to show that the same bound works for the dimension of  $\text{Ext}^1$  between simples.

of  $G$  and  $W$  is the Weyl group  $W(\Phi)$  of the root system  $\Phi$  of  $G$ .<sup>3</sup> If the alternating groups are thought of as Lie type groups over the field of one element, then they are also covered by this case.

The Cline–Parshall–Scott result for defining characteristic suffers in comparison to the cross-characteristic case by not furnishing any explicit bounds. The reason is that it relies on a large piece of machinery due to Andersen–Jantzen–Soergel [AJS94]: if  $G$  is a semisimple algebraic group defined over a field of characteristic  $p$ , then there is a prime  $p_0$  such that for all  $p \geq p_0$  a significant amount of the representation theory of  $G$  is independent of  $p$ , including a character formula for all restricted simple modules. Using this theorem one can reduce to the case of dealing with each  $p$  separately: [CPS09] shows that there are only finitely many values of  $p$  for which the maximum value of  $\dim H^1(G, V)$  may be greater than the generic case. Unfortunately, in the original result of [AJS94],  $p_0$  was implicit. Nowadays one does know bounds for  $p_0$ , courtesy of Peter Fiebig [Fie], but these are simply too big: in combination with the [CPS09] result, the Fiebig result leads only to a bound for  $\dim H^1(G, V)$  which grows super-exponentially with the rank.

The main purpose of this paper is to find a new proof in the defining characteristic case. Our proof has certain advantages over the previous one. First of all, we get, as in [GT11], explicit bounds; secondly, our proof is quite direct, and uniform over all  $p$ . In particular we make no use of the Lusztig character formula, nor of the representation theory of quantum groups.

Certain aspects of our proof are similar to those in [CPS09]. We start by finding a uniform bound for  $\dim H^1(G, V)$  with  $G$  a simple algebraic group of fixed root system and  $V$  a simple  $G$ -module; for this our innovation is to make use of the sum formula (Theorem 3.1)—one of the few tools available in the theory which is uniform with  $p$ . We use the sum formula to bound the length (i.e. number of composition factors) of a Weyl module  $V = V(\lambda)$  with restricted high weight  $\lambda \in X_1$ . If  $L(\lambda)$  is the corresponding simple head of  $V$ , it is a well known fact that  $H^1(G, L(\lambda)) \cong \text{Hom}_G(\text{rad } V, k)$ , showing that a bound on the number of composition factors of  $V$  also bounds the dimension of  $H^1(G, L(\lambda))$ . After that, a Frobenius kernel argument (Proposition 5.1) gives us a bound for all simples (not just the restricted ones). From here we follow the same route as [CPS09] by using results of Bendel–Nakano–Pillen to relate algebraic group cohomology to finite group cohomology.

Our main result is

**Theorem A.** *Let  $G$  be a finite simple group of Lie type with associated Coxeter number  $h$ , defined over an algebraically closed field  $k$ . Let  $V$  be an irreducible  $kG$ -module. Then*

$$\dim H^1(G, V) \leq \max \left\{ \frac{z_p^{\lfloor h^3/6 \rfloor} - 1}{z_p - 1}, \frac{1}{2} (h^2(3h - 3)^3)^{\frac{h^2}{2}} \right\}$$

where  $z_p = \lfloor h^3/6(1 + \log_p(h-1)) \rfloor \leq \lfloor h^3/6(1 + \log_2(h-1)) \rfloor$ ; if  $p \geq h$  then we may take  $z_p = \lfloor h^3/6 \rfloor$ .<sup>†</sup>

We have been fairly imprecise in order to get a uniformly expressed bound (in terms of  $h$  here) but chasing through our proof having fixed a specific root system would yield slightly finer results. However, our main interest is in applying this result together with that of Guralnick–Tiep, to get a growth rate result. Let  $\{\gamma_l\}$  be the sequence  $\{\max \dim H^1(G, V)\}$ , where the maximum is over all finite simple groups of Lie type of Lie rank  $l$  and irreducible representations  $V$  for  $G$ .

**Theorem B.** *We have  $\log \gamma_l = O(l^3 \log l)$ .*

<sup>3</sup>A very recent calculation of Frank Lübeck calculating Kazhdan–Lusztig polynomials for a root system of type  $E_6$  implies that the Guralnick–Tiep bound will not be sufficient for defining characteristic.

<sup>†</sup>The fraction in the displayed inequality needs to be expanded first as a polynomial when  $z_p = 1$ .

Let  $p_0$  be an integer such that whenever  $p \geq p_0$  the Lusztig character formula holds for an ideal of weights containing all the restricted ones. At the end of the paper we get a bound for the dimensions of cohomology groups in terms of  $p_0$ . For this, we use our approach from bounding the length of Weyl modules to improve the Cline–Parshall–Scott bounds for specific  $p$ , removing a dependence on the Kostant partition function. We get

**Theorem C.** *Whenever  $p \geq p_0$ , suppose the Lusztig Character Formula holds for all  $p$ -restricted weights. Let  $G$  be a finite group of Lie type defined over an algebraically closed field  $k$ . Then if  $V$  is an irreducible  $kG$ -module, we have*

$$\dim H^1(G, V) \leq \max \left\{ p_0^{h^2/2}, \frac{1}{2} (h^2(3(h-3)^3)^{\frac{h^2}{2}} \right\}.$$

*In particular,  $\log \gamma_l = O(l^2 \log p_0)$ .*

It was recently announced by G. Williamson in [Wil13] that the minimum value of  $p_0$  used in the hypotheses of the theorem should be much bigger than  $h$ . Specifically  $p_0 = p_0(h)$  must grow strictly faster than any linear function on  $h$ , and is probably exponential in  $h$ . If the latter speculation is correct, Theorem C gives us no better bound on  $\log \gamma_l$ , making it conceivable that the bound in Theorem B is in fact sharp.

Finding bounds for cohomology groups is currently an active area of research. Papers such as [BNP<sup>+</sup>12] and [PSS] use methods to compare algebraic group cohomology with finite group cohomology in order to assert the existence of implicit bounds for the higher cohomology groups  $H^n(G, V)$  ( $n \geq 2$ ) but they both rely on [PS11, Theorem 7.1]. Since this uses [AJS94] the bounds are consequently implicit. Hence it is an open question to generalise the bounds in this paper to higher cohomology degrees. One outcome is that in defining characteristic, the only explicit upper bound known for (non-trivial) cohomology groups which depends only on the Lie rank of  $G$  is that provided in our Theorem A. Thus the only known corresponding growth rate is that provided by our Theorem B.

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## 2. NOTATION

Most of our notation will be that used in Jantzen [Jan03] and the reader is referred there for the proper definitions.

Throughout this paper  $k$  will be an algebraically closed field of characteristic  $p$  and  $G$  will be a group, finite or algebraic.

Suppose  $G$  be a connected, simple, simply connected algebraic group. We fix a maximal torus  $T$  of  $G$  of dimension  $l$ , the rank of  $G$ . We also fix  $B$ , a Borel subgroup of  $G$  with  $B \supseteq T$  and let  $W$  be the Weyl group of  $G$ .

Let  $X(T) = X$  be the weight lattice for  $G$  and  $Y(T) = Y$  the dual weights. The natural pairing  $\langle -, - \rangle : X \times Y \rightarrow \mathbb{Z}$  is bilinear and induces an isomorphism  $Y \cong \text{Hom}_{\mathbb{Z}}(X, \mathbb{Z})$ . We take  $\Phi$  to be the roots of  $G$ . For each  $\alpha \in \Phi$  we take  $\alpha^\vee \in Y$  to be the coroot of  $\alpha$ . Let  $\Phi^+$  be the

positive roots, chosen so that  $B$  is the negative Borel and let  $\Pi$  be the set of simple roots. Set  $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha \in X(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ .

We have a partial order on  $X(T)$  defined by  $\mu \leq \lambda \iff \lambda - \mu \in \text{NS}$ . A weight  $\lambda$  is *dominant* if  $\langle \lambda, \alpha^\vee \rangle \geq 0$  for all  $\alpha \in \Pi$  and we let  $X^+(T) = X^+$  be the set of dominant weights.

Take  $\lambda \in X^+$  and let  $k_\lambda$  be the one-dimensional module for  $B$  which has weight  $\lambda$ . We define the induced module,  $H^0(\lambda) = \text{Ind}_B^G(k_\lambda)$ . This module has simple socle  $L(\lambda)$ , the irreducible  $G$ -module of highest weight  $\lambda$ . One also has the Weyl module  $V(\lambda) = H^0(-w_0(\lambda))^*$  with  $L(\lambda)$  as its simple head. Both modules have formal character given by Weyl's character formula,  $\chi(\lambda)$ . Any finite dimensional, rational irreducible  $G$ -module is isomorphic to  $L(\lambda)$  for a unique  $\lambda \in X^+$ .

We return to considering the weight lattice  $X(T)$  for  $G$ . There are also the affine reflections  $s_{\alpha, mp}$  for  $\alpha$  a positive root and  $m \in \mathbb{Z}$  which act on  $X(T)$  as  $s_{\alpha, mp}(\lambda) = \lambda - (\langle \lambda, \alpha^\vee \rangle - mp)\alpha$ . These generate the affine Weyl group  $W_p$ . We mostly use the dot action of  $W_p$  on  $X(T)$  which is the usual action of  $W_p$ , with the origin shifted to  $-\rho$ . So we have  $w \cdot \lambda = w(\lambda + \rho) - \rho$ . If  $C$  is an alcove for  $W_p$  then its closure  $\bar{C} \cap X(T)$  is a fundamental domain for  $W_p$  operating on  $X(T)$ . The group  $W_p$  permutes the alcoves simply transitively. We set  $C_{\mathbf{Z}} = \{\lambda \in X(T) \otimes_{\mathbb{Z}} \mathbb{R} \mid 0 < \langle \lambda + \rho, \alpha^\vee \rangle < p \ \forall \alpha \in \Phi^+\}$  and call  $C_{\mathbf{Z}}$  the *fundamental alcove*. We also set  $h = \max\{\langle \rho, \beta^\vee \rangle + 1 \mid \beta \in \Phi^+\}$ , the Coxeter number of  $\Phi$ . We have  $C_{\mathbf{Z}} \cap X(T) \neq \emptyset \iff \langle \rho, \beta^\vee \rangle < p, \forall \beta \in \Phi^+ \iff p \geq h$ .

We say that  $\lambda$  and  $\mu$  are *linked* if they belong to the same  $W_p$  orbit on  $X(T)$  (under the dot action). If two irreducible modules  $L(\lambda)$  and  $L(\mu)$  are in the same  $G$  block then  $\lambda$  and  $\mu$  are linked. Linkage gives us another partial order on both  $X(T)$ , and the set of alcoves of  $X(T)$ , denoted  $\uparrow$ . If  $\alpha$  is a positive root and  $m \in \mathbb{Z}$  then we set

$$s_{\alpha, mp} \cdot \lambda \uparrow \lambda \quad \text{if and only if} \quad \langle \lambda + \rho, \alpha^\vee \rangle \geq mp.$$

This then generates an order relation on  $X(T)$ . So  $\mu \uparrow \lambda$  if there are reflections  $s_i \in W_p$  with

$$\mu = s_m s_{m-1} \cdots s_1 \cdot \lambda \uparrow s_{m-1} \cdots s_1 \cdot \lambda \uparrow \cdots \uparrow s_1 \cdot \lambda \uparrow \lambda.$$

The notion  $C_1 \uparrow C_2$  is defined similarly.

For  $\lambda \in X(T)$ , define  $n_\alpha, d_\alpha \in \mathbb{Z}$  by  $\langle \lambda + \rho, \alpha^\vee \rangle = n_\alpha p + d_\alpha$  and  $0 < d_\alpha \leq p$ . Following [Jan03, II.6.6, Remark], we define

$$(1) \quad d(\lambda) = \sum_{\alpha \in \Phi^+} n_\alpha = \sum_{\alpha \in \Phi^+} \left\lfloor \frac{\langle \lambda + \rho, \alpha^\vee \rangle}{p} \right\rfloor,$$

where  $\lfloor r \rfloor$  denotes the greatest integer function for  $r \in \mathbb{R}$ . Recall from [Jan03, II.6.6, Remark], that each  $\lambda \in X(T)$  lies in the upper closure  $\widehat{C}$  of a unique alcove  $C$ . If  $\lambda \in X^+$  and  $d(C)$  is defined as the longest chain  $C_{\mathbf{Z}} \uparrow C_1 \uparrow \cdots \uparrow C_n = C$ , then we have  $d(\lambda) = d(C)$ .

When  $G$  is algebraic or finite, the category of (rational)  $kG$ -modules has enough injectives and so we may define  $\text{Ext}_G^*(-, -)$  as usual by using injective resolutions. We define  $H^i(G, V) := \text{Ext}_G^i(k, V)$ .

### 3. PRELIMINARIES

The following famous result is [Jan03, II.8.19] and is due to Andersen in its full generality.

**Theorem 3.1** (The sum formula). *For each  $\lambda \in X^+$  there is a filtration of  $G$ -modules*

$$V(\lambda) = V(\lambda)^0 \supset V(\lambda)^1 \supset V(\lambda)^2 \supset \dots$$

such that

$$(2) \quad \sum_{i>0} \text{ch } V(\lambda)^i = \sum_{\alpha \in \Phi^+} \sum_{0 < mp < \langle \lambda + \rho, \alpha^\vee \rangle} v_p(mp) \chi(s_{\alpha, mp} \cdot \lambda)$$

and

$$(3) \quad V(\lambda)/V(\lambda)^1 \cong L(\lambda)$$

where  $v_p$  is the usual  $p$  valuation.

Here the definition of formal character  $\chi(\lambda)$  for dominant  $\lambda \in X^+$  is extended to all weights  $\nu \in X(T)$  via the formula  $\chi(w \cdot \nu) = \det(w)\chi(\nu)$  for  $w \in W$ , [Jan03, II.5.9(1)]. If  $w \cdot \nu$  is not dominant for any  $w \in W$ , then  $\chi(\nu) = 0$ . Otherwise we can express  $\chi(\nu)$  as  $\pm\chi(\mu)$  for  $\mu \in X^+$ .

**Lemma 3.2.** *Suppose  $\lambda \in X^+$ , with  $\langle \lambda + \rho, \alpha_0^\vee \rangle \leq b$ .*

(i) *The number of terms arising from expansion of the two summations of the right-hand side of (2) is  $d(\lambda)$ .*

(ii) *The maximum value of  $v_p(mp)$  occurring in the summands of (2) is*

$$\lfloor \log_p(b-1) \rfloor.$$

(iii) *After rewriting each character  $\chi(s_{\alpha, mp} \cdot \lambda)$  of the RHS of (2) in terms of  $\chi(\mu)$  for some  $\mu \in X^+$  and collecting like terms, any remaining  $\chi(\mu)$  with a non-zero coefficient has  $d(\mu) < d(\lambda)$ .*

*Proof.* (i) This is immediate from equation (1).

(ii) As  $\lambda$  is dominant, one has  $\langle \lambda + \rho, \alpha^\vee \rangle \leq \langle \lambda + \rho, \alpha_0^\vee \rangle$  for any  $\alpha \in \Phi^+$ . The result is now clear.

(iii) As noted above we can express the non-zero  $\chi(\nu)$  in the RHS of (2) as  $\pm\chi(\mu)$  for  $\mu \in X^+$ .

Now the strong linkage principle implies that any composition factor  $L(\mu)$  of  $V(\lambda)$  satisfies  $\mu \uparrow \lambda$ . It follows that after complete expansion and rewriting of the terms in (2) as  $\chi(\mu)$  with  $\mu \in X^+$  (as above),  $\mu$  must satisfy  $\mu \uparrow \lambda$ . Now it is immediate that  $d(\mu) < d(\lambda)$ .  $\square$

In [Boe01] the maximum value of  $d(\lambda)$  is calculated for  $\lambda$  satisfying  $\langle \lambda + \rho, \alpha_0^\vee \rangle < (k+1)p$  in terms of  $p$  and  $k$  is an arbitrary positive integer; also when  $\lambda$  is in the Jantzen region. The next lemma calculates something slightly different: we give the maximum value of  $d(\lambda)$  for  $\lambda \in X_1$ . We note that this value is independent of  $p$ .

**Lemma 3.3.** *The maximum value of  $d(\lambda)$  for  $\lambda$  in  $X_1$  is*

- (i)  $\frac{(n-1)n(n+1)}{6}$  for type  $A_n$ ,
- (ii)  $\frac{(n-1)n(4n+1)}{6}$  for types  $B_n$  and  $C_n$ ,
- (iii)  $\frac{2(n-2)(n-1)n}{3}$  for type  $D_n$ ,
- (iv) 120 for type  $E_6$ ,
- (v) 336 for type  $E_7$ ,
- (vi) 1120 for type  $E_8$ ,
- (vii) 86 for type  $F_4$ ,
- (viii) 10 for type  $G_2$ .

*In particular,  $d(\lambda) < \lfloor h^3/6 \rfloor - 1$ .*

*Proof.* By (1) we must calculate the maximum value of  $\sum_{\alpha \in \Phi^+} \left\lfloor \frac{\langle \lambda + \rho, \alpha^\vee \rangle}{p} \right\rfloor$  over all  $\lambda \in X_1$ . This maximum is clearly achieved by the Steinberg weight  $\sigma = (p-1)\rho \in X_1$  and the calculation is straightforward using the data in [Bou82].

For the exceptional types we used the program `minexp` from computer package `Dynkin`.<sup>4</sup>  $\square$

#### 4. BOUNDING THE LENGTH OF WEYL MODULES

Recall from [Jan03, II.5.8(a)] that the  $\text{ch } L(\nu)$  with  $\nu \in X^+$  are a basis of  $\mathbb{Z}[X(T)]^W$ . It follows that if  $M$  is a  $G$ -module and one writes  $\text{ch } M = \sum_{\nu \in X^+} a_\nu \text{ch } L(\nu)$ , one has  $[M : L(\nu)] = a_\nu$ , the multiplicity of  $L(\nu)$  as a composition factor of  $M$ . We write  $\ell(M)$  for  $\sum_{\nu \in X^+} a_\nu$ .

**Theorem 4.1.** *Let  $\lambda \in X^+$  with  $\langle \lambda + \rho, \alpha_0^\vee \rangle \leq b$ . Then the length  $\ell(V)$  (i. e. number of composition factors) of the Weyl module  $V = V(\lambda)$  is bounded by a constant*

$$1 + z_p + z_p^2 + \cdots + z_p^{d(\lambda)} = \frac{z_p^{d(\lambda)+1} - 1}{z_p - 1},$$

where  $z_p = d(\lambda) \lfloor \log_p(b-1) \rfloor$ . In particular  $\ell(V) \leq \frac{z_2^{d(\lambda)+1} - 1}{z_2 - 1}$ .

*Proof.* We prove this by induction on the value of  $d(\lambda)$ . If  $d(\lambda) = 0$  then the strong linkage principle implies that  $V(\lambda) = H^0(\lambda) = L(\lambda)$ , thus  $\ell(V) = 1$  and we are done.

For  $\mu \in X(T)$  (not necessarily dominant) let  $X(\mu) \in \mathbb{N}$  be the number of simple characters counted with multiplicity appearing in a decomposition of  $\chi(\mu)$  into simple characters. When  $X(\mu) \neq 0$ ,  $\pm\chi(\mu)$  is the character of  $V(w \cdot \mu)$  with  $w \cdot \mu \in X^+$  for some  $w \in W$  and we have  $X(\mu) = \ell(V(w \cdot \mu))$ . Consider equation (2) applied to  $V(\lambda)$ . Clearly, we have

$$\begin{aligned} \ell(V(\lambda)) &\leq 1 + \sum_{\alpha \in \Phi^+} \sum_{0 < mp < \langle \lambda + \rho, \alpha^\vee \rangle} v_p(mp) X(s_{\alpha, mp} \cdot \lambda) \\ \text{(by Lemma 3.2(i))} &\leq 1 + d(\lambda) \max_{\alpha \in \Phi^+} \max_{0 < mp < \langle \lambda + \rho, \alpha^\vee \rangle} v_p(mp) X(s_{\alpha, mp} \cdot \lambda) \\ \text{(by Lemma 3.2(ii))} &\leq 1 + d(\lambda) \max_{\alpha \in \Phi^+} \max_{0 < mp < \langle \lambda + \rho, \alpha^\vee \rangle} \lfloor \log_p(b-1) \rfloor X(s_{\alpha, mp} \cdot \lambda) \\ \text{(by Lemma 3.2(iii))} &\leq 1 + d(\lambda) \lfloor \log_p(b-1) \rfloor \max_{d(\mu) < d(\lambda)} X(\mu) \\ \text{(by inductive hypothesis)} &\leq 1 + z_p \frac{z_p^{d(\lambda)} - 1}{z_p - 1} \\ &\leq \frac{z_p^{d(\lambda)+1} - 1}{z_p - 1} \end{aligned}$$

as required. The remaining statement is clear, since  $z_p \leq z_2$ .  $\square$

*Remark 4.2.* Using [Boe01, Theorem 1.1] one can find the maximum value  $d = \max d(\lambda)$  where the maximum is over all  $\lambda$  satisfying  $\langle \lambda, \alpha^\vee \rangle < (k+1)p$  for each positive integer  $k$ . Therefore, if one knows that  $\lambda$  satisfies this condition for some specific value of  $k$ , one may replace  $d(\lambda)$  with the function  $d$  of  $k$  in the conclusion of Theorem 4.1.

<sup>4</sup>See <http://www.math.rutgers.edu/~asbuch/dynkin/>

Using the estimates for  $d(\lambda)$  in Lemma 3.3 we can now get bounds for the lengths of Weyl modules with restricted high weights in each type. We state as a corollary a coarse version of this bound which is valid for each type of root system.

**Corollary 4.3.** *Suppose  $\lambda \in X_1$ . Then the length  $\ell(V)$  of the Weyl module  $V = V(\lambda)$  is bounded by a constant*

$$\frac{z_p^{\lfloor h^3/6 \rfloor} - 1}{z_p - 1},$$

where  $z_p = \lfloor h^3/6(1 + \log_p(h-1)) \rfloor \leq \lfloor h^3/6(1 + \log_2(h-1)) \rfloor$ ; if  $p \geq h$  then we may take  $z_p = \lfloor h^3/6 \rfloor$ .

*Proof.* Lemma 3.3 tells us that  $d(\lambda) \leq \lfloor h^3/6 \rfloor - 1$ . Note that the maximum value of  $b = b(\lambda)$  for  $\lambda \in X_1$  occurs when  $\lambda = (p-1)\rho$  is the first Steinberg weight. Then  $\langle \lambda + \rho, \alpha_0^\vee \rangle = p(h-1)$ ; so that

$$\lfloor \log_p(b-1) \rfloor \leq \lfloor \log_p(p(h-1)) \rfloor = 1 + \lfloor \log_p(h-1) \rfloor \leq 1 + \lfloor \log_2(h-1) \rfloor. \quad \square$$

## 5. BOUNDING THE FIRST COHOMOLOGY GROUPS FOR ALGEBRAIC GROUPS

The point of this section is to reduce the problem of bounding the dimensions of the spaces  $H^1(G, L(\lambda))$  to a question about the composition factors of Weyl modules with restricted high weights.

**Proposition 5.1.** *Let  $\lambda = \lambda_0 + p\lambda'$  with  $\lambda_0 \in X_1$  and  $\lambda' \in X^+$ . Then the following inequality holds:*

$$\dim H^1(G, L(\lambda)) \leq \dim \text{Hom}_G((L(\lambda')^*)^{[1]}, H^0(\lambda_0)/L(\lambda_0)) + 1.$$

*Proof.* We may as well assume  $H^1(G, L(\lambda)) \neq 0$ . The five term exact sequence arising from the Lyndon–Hochschild–Serre spectral sequence applied to  $G_1 \triangleleft G$  implies that

$$(4) \quad \dim H^1(G, L(\lambda)) \leq \dim \text{Hom}_G(L(\lambda')^*, H^1(G_1, L(\lambda_0))^{[-1]}) + \dim H^1(G, H^0(G_1, L(\lambda_0))^{[-1]} \otimes L(\lambda')).$$

(In fact we have equality by [Don82, Corollary 2].)

Suppose first that  $\lambda_0 = 0$ . If  $p \neq 2$  or  $G$  is not of type  $C_n$  then  $H^1(G_1, k) = 0$ , by [Jan03, II.12.2] so that the first term on the right-hand side of (4) is zero and  $\dim H^1(G, L(\lambda)) = \dim H^1(G, L(\lambda')) = \dim H^1(G, L(\lambda)^{[-1]})$ . Thus, by induction on the length of the  $p$ -adic expansion of  $\lambda$ , we may assume that  $\lambda_0 \neq 0$ . If  $p = 2$  and  $G$  is of type  $C_n$  then  $H^1(G_1, k)^{[-1]} \cong L(\omega_1)$ , the natural module for  $\text{Sp}_{2n}$ , by [Jan03, loc. cit.]. Suppose the first term on the right-hand side of (4) is non-zero. Then  $L(\lambda')^* \cong L(\lambda') \cong L(\omega_1)$ . Then the second term on the right-hand side of (4) is  $\dim H^1(G, L(\omega_1))$ , which vanishes (for instance) by the linkage principle, since  $\omega_1$  is miniscule and not linked to 0. Thus  $\dim H^1(G, L(\lambda)) = \dim H^1(G, L(\omega_1)^{[1]}) = 1$  and we are done. Otherwise the first term on the right-hand side of (4) is 0 and we have  $\dim H^1(G, L(\lambda)) = \dim H^1(G, L(\lambda')) = \dim H^1(G, L(\lambda)^{[-1]})$  as before.

Thus we may assume  $\lambda_0 \neq 0$ . Then this time the second term on the right-hand side of (4) vanishes and we have

$$(5) \quad \dim H^1(G, L(\lambda)) = \dim \text{Hom}_G(L(\lambda')^*, H^1(G_1, L(\lambda_0))^{[-1]}).$$

Consider the right-hand side of (5). We have a short exact sequence of  $G$ -modules

$$H^0(G_1, M)^{[-1]} \rightarrow H^1(G_1, L(\lambda_0))^{[-1]} \rightarrow H^1(G_1, H^0(\lambda_0))^{[-1]},$$



where  $M = H^0(\lambda_0)/L(\lambda_0)$ . Applying  $\text{Hom}_G(L(\lambda')^*, ?)$  to this sequence yields a long exact sequence containing

$$(*) \quad \text{Hom}_G(L(\lambda')^*, H^0(G_1, M)^{[-1]}) \rightarrow \text{Hom}_G(L(\lambda')^*, H^1(G_1, L(\lambda_0))^{[-1]}) \\ \rightarrow \text{Hom}_G(L(\lambda')^*, H^1(G_1, H^0(\lambda_0))^{[-1]})$$

as a subsequence.

Now the first term is isomorphic to  $\text{Hom}_G(k, \text{Hom}_{G_1}((L(\lambda')^*)^{[1]}, M)) \cong \text{Hom}_G((L(\lambda')^*)^{[1]}, M)$ . Furthermore, looking at the various cases from [BNP04b, §3 Theorems (A), (B), (C)] one sees that  $H^1(G_1, H^0(\lambda))^{[-1]}$  is a direct sum of distinct  $H^0(\omega_i)$ , where  $\omega_i$  is a fundamental dominant weight.<sup>5</sup> Thus the third term is always at most 1. Now, since the dimension of the middle term in (\*) is bounded by the sum of the dimensions of the outer terms, we are done by (5).  $\square$

**Corollary 5.2.** *For  $\lambda \in X^+$ , the dimension of  $H^1(G, L(\lambda))$  is bounded above by the length  $\ell(V)$  of a Weyl module  $V = V(\lambda_0)$  with  $\lambda_0 \in X_1$ .*

Hence

$$\dim H^1(G, L(\lambda)) \leq \frac{z_p^{\lfloor h^3/6 \rfloor} - 1}{z_p - 1},$$

where  $z_p = \lfloor h^3/6(1 + \log_p(h-1)) \rfloor \leq \lfloor h^3/6(1 + \log_2(h-1)) \rfloor$ ; if  $p \geq h$  then we may take  $z_p = \lfloor h^3/6 \rfloor$ .

*Proof.* This is immediate from Proposition 5.1: one has

$$\text{Hom}_G((L(\lambda')^*)^{[1]}, H^0(\lambda_0)/L(\lambda_0)) + 1 \leq [H^0(\lambda_0)/L(\lambda_0) : (L(\lambda')^*)^{[1]}] + 1 \leq \ell(H^0(\lambda_0)) = \ell(V(\lambda_0)).$$

The remaining conclusion now follows from Corollary 4.3  $\square$

## 6. FROM $G$ -COHOMOLOGY TO $G_\sigma$ -COHOMOLOGY

Let  $\sigma : G \rightarrow G$  be a surjective endomorphism of the simply-connected, simple algebraic group  $G$  which is *strict*, i.e. the set of fixed points  $G_\sigma = \{g \in G(k) : \sigma(g) = g\}$  is finite. Then  $G_\sigma$  is a finite group of Lie type. We have [GLS98, Theorem 2.2.3] that  $\sigma = \tau \circ F^r$  for some  $r \in \mathbb{N}$ , where  $F$  is a standard Frobenius map and  $\tau$  is a graph automorphism normalising  $B$  and  $T$  in  $G$ ; or when  $p = 2$  and  $G$  is of type  $C_2$  or  $F_4$ , or  $p = 3$  and  $G$  is of type  $G_2$  then  $\tau$  is the identity, or a fixed, purely inseparable isogeny satisfying  $\tau^2 = F$ . When  $\tau = 1$  (so  $\sigma$  is standard), we say  $G_\sigma$  is a Chevalley group; when  $\tau$  is a non-trivial graph automorphism, we say  $G_\sigma$  is a Steinberg group; in the exceptional cases,  $G_\sigma$  is a Ree or Suzuki group. All the finite groups of Lie type featuring in the classification of finite simple groups arise by factoring out the centre of some  $G_\sigma$ .<sup>6</sup>

By a result of Steinberg, the simple  $kG_\sigma$ -modules can all be identified with the restrictions of  $\sigma$ -restricted simple  $G$ -modules  $L(\lambda)$  with  $\lambda \in X_\sigma$ . When  $G_\sigma$  is a Chevalley or Steinberg group,  $X_\sigma = X_r$ . For the Ree and Suzuki groups, one must expand  $X_r$  slightly: set  $X_\sigma \subset X^+$  with  $\lambda \in X_\sigma$  if  $\langle \lambda, \alpha^\vee \rangle < p^{r+1}$  for  $\alpha \in \Pi$  short, and  $< p^r$  in case  $\alpha \in \Pi$  is long, where  $\Pi$  denotes the set of simple roots of  $G$ .

We need the following lemma.

<sup>5</sup>The largest number of such summands is three and occurs only in the case where  $p = 2$ ,  $G = D_4$  and  $\lambda = \omega_2$ ; under these assumptions  $H^1(G_1, H^0(\omega_2))^{[-1]} \cong H^0(\omega_1) \oplus H^0(\omega_3) \oplus H^0(\omega_4)$ .

<sup>6</sup>If  $G = {}^2F_2(2)$ , then the simple Tits group is an index 2 subgroup  $G'$  of  $G$ , though this is often thought of as a sporadic group.

**Lemma 6.1.** *Let  $G$  be a simply-connected simple algebraic group over an algebraically closed field  $k$  of characteristic  $p$ . Let  $\lambda \in X_\sigma$ . Then the dimensions of the Weyl module  $V(\lambda)$  and its simple head  $L(\lambda)$  are bounded by  $p^{r|\Phi^+|}$  when  $X_\sigma = X_r$  and by  $p^{(r+\frac{1}{2})|\Phi^+|}$  otherwise.*

*Thus the simple  $kG_\sigma$ -module of largest dimension is the Steinberg module of the given dimension.*

*Proof.* For the first part it clearly suffices to bound the dimension of  $V(\lambda)$ , which is given by the formula [FH91, Corollary 24.6]

$$\frac{\prod_{\alpha \in \Phi^+} \langle \lambda + \rho, \alpha^\vee \rangle}{\prod_{\alpha \in \Phi^+} \langle \rho, \alpha^\vee \rangle}.$$

For  $\lambda \in X_r$  and  $\alpha$  a simple root

$$\langle \lambda + \rho, \alpha^\vee \rangle \leq p^r = \langle (p^r - 1)\rho + \rho, \alpha^\vee \rangle.$$

In other words, among the Weyl modules with  $p^r$ -restricted heads, the  $r$ th Steinberg module ( $= V((p^r - 1)\rho)$ ) maximises all the inner products with the simple roots, and hence any non-negative linear combination of simple roots, and thus in particular, the positive roots. Thus the dimension of a Weyl module with  $p^r$ -restricted head is bounded by the dimension of the  $r$ th Steinberg module, which is clearly  $p^{r|\Phi^+|}$  by the above formula. The same estimates work when  $X_\sigma \neq X_r$  using  $\lambda = (p^r - 1)\rho + (p^r(p - 1))\tilde{\rho}$  where  $\tilde{\rho}$  is the sum of the fundamental dominant weights corresponding to short roots. See [Hum06, §20.3] for more details (including the dimension).

The second part is clear since any simple  $kG_\sigma$ -module is obtained as the restriction to  $G_\sigma$  of a  $\sigma$ -restricted simple  $G$ -module.  $\square$

When  $G_\sigma$  is a Chevalley or Steinberg group, the following result relates  $G$ -cohomology to  $G_\sigma$ -cohomology. We give an abridged statement; in fact, the weight  $\tilde{\lambda}$  below is determined constructively in [*loc. cit.*].

**Theorem 6.2** ([BNP06, Thm. 5.5]). *Let  $\sigma = \tau \circ F^r$ , such that  $\tau$  is a graph automorphism. Suppose  $r \geq 2$  and let  $s = \lfloor \frac{r}{2} \rfloor$ . Assume  $p^{s-1}(p - 1) > h$ . Given  $\lambda \in X_r$ , there exists  $\tilde{\lambda} \in X_r$  with*

$$H^1(G_\sigma, L(\lambda)) \cong H^1(G, L(\tilde{\lambda})).$$

The above result excludes dealing with the Ree and Suzuki groups ‘for simplicity’ (see [BNP06, §1.2]) since, unless  $G_\sigma = {}^2F_4(2^r)$ , these extensions are known by results of Sin. The following lemma addresses the missing cases as a modification of [BNP06].

**Lemma 6.3.** *The theorem above holds in the case that  $G_\sigma$  is a Ree group of type  $F_4$ .*

*Proof.* Here we have  $h = 12$  and  $p = 2$ , so the hypotheses demand  $s - 1 \geq 4$ . Thus  $r \geq 10$ . Under these assumptions, the modifications of the proofs of [BNP06] are straightforward using recent results of [BNP<sup>+</sup>12]. See [Ste13] for a full account.  $\square$

**Proposition 6.4.** *Let  $L$  be a simple  $kG_\sigma$ -module and let  $h$  be the associated Coxeter number of  $G$ . Suppose further that  $\dim H^1(G, L') \leq b$  for all simple  $G$ -modules  $L'$ . Then*

$$\dim H^1(G_\sigma, L) \leq \max \left\{ b, \frac{1}{2} (h^2(3h - 3)^{\frac{h-2}{2}}) \right\}.$$

*Proof.* If  $G_\sigma$  is a Suzuki or Ree group of type  $G_2$  then  $\dim H^1(G, L) \leq 2$  by [Sin92] or [Sin93] so the theorem holds.

Recall that  $\sigma = \tau \circ F^r$  for  $F$  a standard Frobenius automorphism.

If  $G$  is a Ree group of type  $F_4$  then Lemma 6.3 proves the result when  $r \geq 10$ . If  $r < 10$  then by Lemma 6.1 we have  $\dim L \leq 2^{(9+\frac{1}{2})24} = 2^{228}$ . Thus by [GH98], we have  $\dim H^1(G_\sigma, L) \leq 2^{227}$  and it is easy to check that this satisfies the bound given (with  $h = 12$ ).

Now assume  $G$  is either a Chevalley or Steinberg group, let  $q = p^r$  and set  $s = \lfloor \frac{r}{2} \rfloor$ . The case where  $G$  is of type  $A_1$  follows from [AJL83, Corollary 4.5], so assume otherwise. Now, if either (i):  $r \geq 2$  and  $p^{s-1}(p-1) > h$ ; or (ii):  $p \geq 3h-3$ , we have  $\dim H^1(G_\sigma, L) = \dim H^1(G, L')$  for some simple  $G$ -module  $L'$  by Theorem 6.2 and [BNP04a, Theorem 5.1] respectively. Thus we are done in either of these cases.

Therefore we may assume that  $p < 3h-3$  and either (a)  $r = 1$  or (b)  $p^{s-1}(p-1) \leq h$ . In case (a), this implies that  $q < 3h-3$ . In case (b),  $s \leq \log_p h$  so that  $r \leq 2 \log_p h + 3$ . Then  $q = p^r \leq h^2 p^3 \leq h^2(3h-3)^3$ .

Now by [GH98], we have  $\dim H^1(G_\sigma, L) \leq 1/2 \cdot \dim L$  and by Lemma 6.1,  $\dim L$  is at most  $q^{|\Phi^+|}$ . It is easy to check the tables in [Bou82] to see that  $|\Phi^+|$  is no bigger than  $h^2/2$  (often with agreement). Thus in case (a) we have  $\dim H^1(G_\sigma, L) \leq 1/2 (3h-3)^{h^2/2}$  and in case (b) we have

$$\dim H^1(G_\sigma, L) \leq 1/2 (h^2(3h-3)^3)^{\frac{h^2}{2}}.$$

In either case, the theorem holds. □

We may now tackle the proof of our first two main theorems.

*Proof of Theorem A.* In all cases,  $G$  is the quotient of some  $H_\sigma$  by  $Z(H_\sigma)$  with  $H$  a simply connected algebraic group. Hence we can lift  $V$  to a simple module for  $H_\sigma$ . Then a Lyndon–Hochschild–Serre spectral sequence argument gives that  $\dim H^1(H_\sigma, V) = \dim H^1(G, V)$ . Now the result follows from Proposition 6.4 in combination with Corollary 5.2. □

*Proof of Theorem B.* The case in cross-characteristic using [GT11] is easy: by that result one has  $\dim H^1(G, V) \leq |W| + e$  for  $e$  the twisted rank of  $G$ . Now  $|W| > e$  is no bigger than  $2^l l!$ , so  $O(\log(|W| + e)) = O(\log |W|) = O(l \log l)$ . For the defining characteristic case, note that the Coxeter number of  $h$  is linear with the rank of  $G$ . The theorem then follows by taking logs of both sides of the inequality in Theorem A. □

## 7. IF THE LUSZTIG CHARACTER FORMULA HOLDS

In [CPS09], the authors prove a bound on  $\dim H^1(G, V)$  for  $G$  a semisimple algebraic group. This is generalised in [PS11, Lemma 5.2] (which is also easier to read). They show

$$\dim \text{Ext}_G^1(L(\lambda), L(\mu)) \leq p^{|\Phi|} \mathfrak{P}(2(p-1)\rho),$$

where  $\mathfrak{P}$  denotes the Kostant partition function. It follows that the same bound will work in the case of cohomology (the case  $\lambda = 0$ ). Suppose  $p \geq p_0$  such that the Lusztig character formula holds (recall that such a  $p_0$  is guaranteed to exist by [AJS94]). Then one may replace  $p$  in the above

expression by the fixed value  $p_0$ . Thus the authors can assert the existence of an (implicit) bound on the dimensions of 1-cohomology.

Approaching the problem from the point of view of bounding composition factors of Weyl modules, we improve this bound in the case of cohomology; in particular, we can remove the dependency on the Kostant function. This gives us a second bound on the dimension of cohomology groups which is conceivably better than that in Theorem A.

**Proposition 7.1.** *Let  $V = V(\lambda)$  be a Weyl module with  $p$ -restricted head. Then  $\ell(V) \leq p^{|\Phi^+|}$ .*

*If, for all  $p \geq p_0$ , the Lusztig character formula holds on an ideal of weights containing the restricted ones, then  $\ell(V) \leq p_0^{|\Phi^+|} \leq p_0^{h^2/2}$ .*

*Proof.* The first part follows from Lemma 6.1: certainly the dimension of  $V$  must bound its length.

For the remainder of the proposition, Lusztig's character formula implies a decomposition of the character of a Weyl module in the principal block (i.e. those with high weights of the form  $w \cdot 0$  for  $w \in W_p$ ) into characters of simple modules corresponding to a fixed finite collection of elements of the affine Weyl group; moreover, this decomposition is independent of  $p$ . Using translation functors, one gets a bound for all  $\ell(V)$  for  $V$  of the stated type which is independent of  $p \geq p_0$ . The first part of the proposition implies that  $\ell(V) \leq p_0^{|\Phi^+|}$  works when  $p = p_0$ , thus it works in general. The remaining inequality is clear.  $\square$

*Proof of Theorem C.* Let  $H$  be a simply connected, simple algebraic group over  $k$  with the same root system as  $G$ . Then  $V$  is obtained as the restriction of a simple module  $V$  for  $H$ . Since the Lusztig character formula is assumed to hold, we have, by Corollary 7.1, that  $\ell(W) \leq (p_0)^{h^2/2}$  for any Weyl module  $W$  with a restricted head. Now, Corollary 5.2 implies that  $\dim H^1(H, V) \leq (p_0)^{h^2/2}$ .

Proposition 6.4 implies that  $\dim H^1(H_\sigma, V) \leq \max \left\{ (p_0)^{h^2/2}, (h^2(3h-3)^3)^{\frac{h^2}{2}} \right\}$  for any strictly surjective endomorphism  $\sigma$  of  $H$ . The arguments of the proof of Theorem A go through as before, and the first part of Theorem C follows.

For the second part, as in Theorem B, the cross-characteristic case follows from [GT11]. The defining characteristic case is immediate from the above by noting that  $h$  is linear with the rank of  $G$  and taking logs of both sides of the inequality in the Theorem C.  $\square$

## REFERENCES

- [AJL83] Henning Haahr Andersen, Jens Jørgensen, and Peter Landrock, *The projective indecomposable modules of  $SL(2, p^n)$* , Proc. London Math. Soc. (3) **46** (1983), no. 1, 38–52. MR MR684821 (84f:20044)
- [AJS94] Henning Andersen, Jens Jantzen, and Wolfgang Soergel, *Representations of quantum groups at a  $p$ th root of unity and of semisimple groups in characteristic  $p$* , vol. 220, Astérisque, 1994.
- [BNP04a] Christopher P. Bendel, Daniel K. Nakano, and Cornelius Pillen, *Extensions for finite groups of Lie type: twisted groups*, Finite groups 2003, Walter de Gruyter GmbH & Co. KG, Berlin, 2004, pp. 29–46. MR 2125064 (2006a:20024)
- [BNP04b] ———, *Extensions for Frobenius kernels*, J. Algebra **272** (2004), no. 2, 476–511. MR 2028069 (2004m:20089)
- [BNP06] ———, *Extensions for finite groups of Lie type. II. Filtering the truncated induction functor*, Representations of algebraic groups, quantum groups, and Lie algebras, Contemp. Math., vol. 413, Amer. Math. Soc., Providence, RI, 2006, pp. 1–23. MR 2262362 (2007i:20074)
- [BNP<sup>+</sup>12] Christopher P. Bendel, Daniel K. Nakano, Brian Parshall, Cornelius Pillen, Leonard L. Scott, and David. I. Stewart, *Bounding cohomology for finite groups and Frobenius kernels*, arxiv:1208.6333 (2012).

- [Boe01] Brian D. Boe, *Geometry of the Jantzen region in Lusztig's conjecture*, Math. Comp. **70** (2001), no. 235, 1265–1280. MR 1709146 (2001j:20065)
- [Bou82] Nicolas Bourbaki, *Éléments de mathématique: groupes et algèbres de Lie*, Masson, Paris, 1982, Chapitre 9. Groupes de Lie réels compacts. [Chapter 9. Compact real Lie groups]. MR 682756 (84i:22001)
- [CPS09] Ed Cline, Brian Parshall, and Leonard Scott, *Reduced standard modules and cohomology*, Trans. Amer. Math. Soc. **361** (2009), no. 10, 5223–5261. MR 2515810 (2010g:20092)
- [Don82] Stephen Donkin, *On  $\text{Ext}^1$  for semisimple groups and infinitesimal subgroups*, Math. Proc. Cambridge Philos. Soc. **92** (1982), no. 2, 231–238. MR 671179 (84i:20037)
- [FH91] William Fulton and Joe Harris, *Representation theory*, Graduate Texts in Mathematics, vol. 129, Springer-Verlag, New York, 1991, A first course, Readings in Mathematics. MR 1153249 (93a:20069)
- [Fie] Peter Fiebig, *An upper bound on the exceptional characteristics for Lusztig's character formula*, J. Reine Angew. Math., to appear.
- [GH98] Robert M. Guralnick and Corneliu Hoffman, *The first cohomology group and generation of simple groups*, Groups and geometries (Siena, 1996), Trends Math., Birkhäuser, Basel, 1998, pp. 81–89. MR 1644977 (99h:20087)
- [GLS98] Daniel Gorenstein, Richard Lyons, and Ronald Solomon, *The classification of the finite simple groups. Number 3. Part I. Chapter A*, Mathematical Surveys and Monographs, vol. 40, American Mathematical Society, Providence, RI, 1998, Almost simple  $K$ -groups. MR 1490581 (98j:20011)
- [GT11] Robert M. Guralnick and Pham Huu Tiep, *First cohomology groups of Chevalley groups in cross characteristic*, Ann. of Math. (2) **174** (2011), no. 1, 543–559. MR 2811608 (2012f:20136)
- [Gur86] Robert M. Guralnick, *The dimension of the first cohomology group*, Representation theory, II (Ottawa, Ont., 1984), Lecture Notes in Math., vol. 1178, Springer, Berlin, 1986, pp. 94–97. MR 842479 (87h:20095)
- [Hum06] James E. Humphreys, *Modular representations of finite groups of Lie type*, London Mathematical Society Lecture Note Series, vol. 326, Cambridge University Press, Cambridge, 2006. MR 2199819 (2007f:20023)
- [Jan03] Jens Carsten Jantzen, *Representations of algebraic groups*, second ed., Mathematical Surveys and Monographs, vol. 107, American Mathematical Society, Providence, RI, 2003. MR MR2015057 (2004h:20061)
- [PS11] Brian J. Parshall and Leonard L. Scott, *Bounding Ext for modules for algebraic groups, finite groups and quantum groups*, Advances in Mathematics **226** (2011), no. 3, 2065 – 2088.
- [PSS] Brian J. Parshall, Leonard L. Scott, and David I. Stewart, *Shifted generic cohomology*, Compos. Math., to appear.
- [Sco03] Leonard L. Scott, *Some new examples in 1-cohomology*, J. Algebra **260** (2003), no. 1, 416–425, Special issue celebrating the 80th birthday of Robert Steinberg. MR 1976701 (2004f:20092)
- [Sin92] Peter Sin, *Extensions of simple modules for  $\text{Sp}_4(2^n)$  and  $\text{Suz}(2^m)$* , Bull. London Math. Soc. **24** (1992), no. 2, 159–164. MR 1148676 (93b:20025)
- [Sin93] ———, *Extensions of simple modules for  $G_2(3^n)$  and  ${}^2G_2(3^m)$* , Proc. London Math. Soc. (3) **66** (1993), no. 2, 327–357. MR 1199070 (94a:20020)
- [Ste13] David I. Stewart, *On extensions for Ree groups of type  $F_4$* , arxiv:1304.2544 (2013).
- [Wil13] Geordie Williamson, *Schubert calculus and torsion*, <http://people.mpim-bonn.mpg.de/geordie/Torsion.pdf> (2013).

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