This is a repository copy of *Bilinear Systems in the "Frequency Domain" and an Application to Optimal Control*.

White Rose Research Online URL for this paper:
http://eprints.whiterose.ac.uk/83547/

---

**Monograph:**

---

**Reuse**
Unless indicated otherwise, fulltext items are protected by copyright with all rights reserved. The copyright exception in section 29 of the Copyright, Designs and Patents Act 1988 allows the making of a single copy solely for the purpose of non-commercial research or private study within the limits of fair dealing. The publisher or other rights-holder may allow further reproduction and re-use of this version - refer to the White Rose Research Online record for this item. Where records identify the publisher as the copyright holder, users can verify any specific terms of use on the publisher's website.

**Takedown**
If you consider content in White Rose Research Online to be in breach of UK law, please notify us by emailing eprints@whiterose.ac.uk including the URL of the record and the reason for the withdrawal request.
Bilinear Systems in the 'Frequency Domain' and an

Application to Optimal Control

by

S.P. Banks and M. Iddir

University of Sheffield
Department of Control Engineering
Mappin Street
Sheffield S13JD

1987

Research Report No. 320
Abstract

A 'frequency domain' approach to the optimal control of bilinear systems is presented. This is based on a new representation of the input-output map of the system in terms of a basis of $L^2(0,T)$. 
1. Introduction

The theory of linear systems has been developed at different times using one of two basic approaches - (i) the frequency domain, transfer function method and (ii) the state space (time domain) technique. These two methods can be regarded as being equivalent because of the isomorphism between the time domain and the frequency domain provided by the Fourier transform. This isomorphism maps \( L^2[0, \omega] \) onto \( L^2[-\omega, \omega] \) or more generally from (some) space of distributions onto another. The Fourier transformed system then exhibits the frequency response behaviour of the original system.

In the theory of nonlinear systems it is no longer of much significance what the response of the system to a sine wave may be, since this response will not be a sine wave and the original system has no superposition principle, in general. However, if we consider such a nonlinear system to be defined on a finite time interval \([0, T]\) (merely for convenience, the infinite time interval can be considered similarly) and the inputs belong to \( L^2[0, T] \) then each input may be regarded as an infinite sum of 'special' functions. For if \( \{e_i\}_{i=1}^\infty \) is a basis of \( L^2[0, T] \), then any input \( u \) may be written

\[
u = \sum_{i=1}^\infty u_i e_i.
\]

Note, however, that as we have stated above, there is now no 'canonical' basis since the sine and cosine functions have no special significance for general nonlinear systems.

If the nonlinear system \( S \) is input-output stable in the usual sense, then \( S: L^2[0, T] \rightarrow L^2[0, T] \), where \( S \) is the input-output map of the system. Thus,

\[
y = S(u)
\]

and so
\[ y_i = s_i(u_1, u_2, \ldots) \]  

(1.1)

where

\[ s_i(u_1, u_2, \ldots) = \langle S(Iu_j e_j), e_i \rangle. \]

We shall call the sequence \( \{s_i\}_i \) the generalized frequency response of \( S \).

In [1] and [2] we have presented a realization theory based on the generalized frequency response. This paper will be concerned mainly with the application of this approach to the optimal control of bilinear systems. The bilinear-quadratic regulator problem was solved recently [3],[4] with the solution being given as a power series with tensor-operator coefficients. This solution has two drawbacks - first it is only valid for sufficiently small initial states and secondly the tensor-operator valued 'Riccati-like' equations have expanding dimension making computation very difficult. In this paper we shall show that, if we take a finite number of basis vectors, say \( e_1, \ldots, e_K \), then the solution of the optimal control problem is given in terms of a set of polynomial equations.

Note finally that there have been other approaches to the 'frequency domain' behaviour of nonlinear systems ([5-7]) but these are based mainly on the Volterra series and the generalized Laplace transform. The method proposed here is a direct generalization of the linear frequency domain approach, since it is based on an isomorphism between the 'time' and 'frequency' domains.
2. Notation and Terminology

In this paper we shall denote by $L^2([0,T])$ the space of square integrable functions $x:[0,T] \rightarrow \mathbb{R}^n$; i.e. the integral

$$\int_0^T \|x(t)\|^2 \, dt$$

is finite, where $\| \cdot \|$ is the usual Euclidean norm on $\mathbb{R}^n$. If $n=1$ we shall write simply $L^2([0,T])$. If $\{e_i\}$ is a basis of $L^2([0,T])$, then there is an obvious isomorphism $\mathcal{F}: L^2([0,T]) \rightarrow \ell^2$ induced by this basis, where $\ell^2$ is the usual space of square summable sequences. Let $\ell^{2,n}$ denote the space of square summable sequences with values in $\mathbb{R}^n$, so that if $(\xi(i)) \in \ell^{2,n}$, where $\xi(i) \in \mathbb{R}^n$, for each $i$, then

$$\sum_{i=1}^{\infty} \|\xi(i)\|^2 < \infty.$$ 

Then $\mathcal{F}$ induces an isomorphism $\mathcal{F}^n: L^2([0,T]) \rightarrow \ell^{2,n}$ given by

$$\mathcal{F}^n(x) = \mathcal{F}^n\left(\sum_{i=1}^{\infty} x(i)e_i\right) = (\mathcal{F}x_1, \ldots, \mathcal{F}x_n)^T = (x(i))$$

where

$$x(i) = (x_1(i), \ldots, x_n(i))$$

and

$$x(t) = (x_1(t), \ldots, x_n(t)).$$

We shall then denote the sequence $(x(i))$ by $x$. Finally, I will denote the identity operator without further comment. It will be clear from the context on which space this operates.
3. 'Frequency Domain' Representation of Bilinear Systems

Consider a bilinear system

\[
\dot{x} = Ax + \sum_{i=1}^{m} u_i E_i x, \quad x(0) = x_0 \in \mathbb{R}^n
\]  

(3.1)

defined on the interval \([0, T]\), where \(T\) is arbitrary, but finite. This system gives rise to an input-output relation given by the well-known Volterra series. In this section we shall present an alternative representation of the input-output relation which has a connection with the frequency domain transfer function for linear systems. Let

\[
x = h(u)
\]

(3.2)
denote the input-output relation of (3.1). Then we require

Lemma 3.1 As a nonlinear operator, \(h\) maps \(L^2, \mathbb{R}^n [0, T] \cap L^\infty [0, T]\) into \(L^2, \mathbb{R}^n [0, T]\).

Proof Define the sequence of functions \(\xi_i(t)\) by

\[
\xi_0(t) = e^{At} x_0
\]

\[
\xi_1(t) = \int_0^t e^{A(t-s)} \sum_{j=1}^{m} u_j(s) B_j \xi_{i-1}(s) ds, \quad i \geq 1
\]

We can write

\[
\|e^{At}\| \leq Ke^{\omega t}
\]

for some real constants \(K, \omega\) and so

\[
\|\xi_1(t)\| \leq \int_0^t Ke^{\omega(t-s)} \sum_{j=1}^{m} \|u_j(s)\| \|B_j\| \|\xi_{i-1}(s)\| ds
\]

Thus, if \(\beta = \max \|B_j\|\) and \(\gamma = \max \|u_j(t)\|\), then

\[
\|\xi_1(t)\| \leq \int_0^t Ke^{\omega(t-s)} \beta \|\xi_{i-1}(s)\| ds
\]

\[
\leq (K\gamma \beta)^{i-1} \frac{e^{\omega t}}{i!} \|x_0\|
\]
by iteration. Hence, if \( x(t) = \sum_{i=0}^{\infty} \xi_i(t) \), then
\[
\| \xi(t) \| \leq e^{(\omega + K \gamma) t} \| x_0 \|
\]
and the series is uniformly convergent. It is easy to check that
\[
h(u) = \sum_{i=0}^{\infty} \xi_i(t)
\]
and the result follows. \( \square \)

Let \( \{ e_i \}_{i=1}^{\infty} \) be an orthonormal basis of \( L^2[0,T] \). From (3.1) we have
\[
x(t) = e^{At} x_0 + \sum_{i=1}^{m} \int_{0}^{t} e^{A(t-s)} u_i(s) B_i x(s) ds,
\]
and by lemma 3.1, if \( u \in L^2, L^\infty[0,T] \cap L^\infty, L^m[0,T] \) then \( x \in L^2, L^m[0,T] \) and so we can write
\[
u = \sum_{i=1}^{\infty} u(i) e_i, \quad x = \sum_{i=1}^{\infty} x(i) e_i
\]
where
\[u \in \ell^2, L^m, \quad x \in \ell^2, L^m.
\]
Thus,
\[
\sum_{i=1}^{\infty} x(i) e_i = e^{At} x_0 + \sum_{i=1}^{m} \int_{0}^{t} e^{A(t-s)} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} u(j) B_i x(j) e_i(s) e_j(s) ds.
\]
Now assume that the basis \( \{ e_i \} \) satisfies the assumption

(A) the function \( t \rightarrow \int_{0}^{t} e^{A(t-s)} e_i(s) e_j(s) ds \in L^2[0,T] \) for each \( i, j \).

Then, under assumption (A) we can write
\[
\sum_{i=1}^{\infty} x(i) e_i = e^{At} x_0 + \sum_{i=1}^{m} \sum_{j=1}^{\infty} \int_{0}^{t} e^{A(t-s)} e_i(s) e_j(s) ds u(j) B_i x(j)
\]
Again, by assumption (A), we have
\[ \int_0^t e^{A(t-s)} e_i(s) e_j(s) ds = \sum_{k=0}^{\infty} \alpha_{ijk} e_k(t) . \]

Hence

\[ x(k) = \xi(k) + \sum_{\ell=1}^{m} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \alpha_{ijk} u_{\ell}^i B_{\ell} x(j) \]

where \( \xi(k) = \langle e^A x_0, e_k \rangle . \) (3.3)

If we define the operator \( K(u) : \ell^{2,n} \oplus \ell^{2,n} \) by

\[ (K(u)x)(k) = \sum_{\ell=1}^{m} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \alpha_{ijk} u_{\ell}^i B_{\ell} x(j) \] (3.4)

where \( u = \{ u(i) \} , \ x = \{ x(i) \} , \) then we have

\[ (I-K(u))x = \xi \]

where \( \xi = \{ \xi(k) \} . \) Hence we have

Lemma 3.2 If \( u \in \ell^{2,m} \) then the solution of the bilinear system (3.1) with input \( u \) is given by

\[ x = (I-K(u))^{-1} \xi \] (3.5)

where \( \xi \) and \( K(u) \) are given by (3.3) and (3.4), respectively. \( \Box \)

The 'inverse system' (3.5) can be written in a different way which is convenient for applications. We have

\[ (K(u)x)(k) = \sum_{\ell=1}^{m} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \alpha_{ijk} u_{\ell}^i B_{\ell} x(j) \]

Define the operators \( L_1 : \ell^{2,n} \oplus \ell^{2,n} \) by

\[ (L_1 x)(k) = ( \sum_{j=1}^{\infty} \alpha_{ij1} B_{1} x(j) , \ldots , \sum_{j=1}^{\infty} \alpha_{ijk} B_{k} x(j) ) \]

Then
\[ K(u) = \sum_{i=1}^{\infty} u(i) L_i. \]

Hence, by lemma 3.2 we have

\[ x = (I - \sum_{i=1}^{\infty} u(i) L_i)^{-1} x. \]  

(3.6)

In particular, if \( u \in P_k L^2, m[0, T] \), where \( P_k \) is the projection of \( L^2, m[0, T] \) onto the subspace generated by the basis \( \{e_1, \ldots, e_k\} \), then

\[ x = (I - \sum_{i=1}^{k} u(i) L_i)^{-1} x. \]  

(3.7)

Theorem 3.3 If the input \( u \) satisfies the inequality

\[ \sum_{i=1}^{k} \| u(i) \| L_i < 1 \]

then we have the expansion

\[ x = (I + \sum_{i=1}^{k} u(i) L_i + \sum_{i=1}^{k} \sum_{j=1}^{k} u(i) u(j) L_i L_j + \ldots). \]  

(3.8)

Proof This follows directly from the Neumann series. \( \square \)

Remark In contrast to the Volterra series, which follows from lemma 3.1 and is valid for \( u \in L^\infty, m[0, T] \), the expansion (3.8) is only valid for sufficiently small \( \| u \| \). However, lemma 3.1 shows that (3.5) and (3.7) are valid for \( u \in L^2, m[0, T] \cap L^\infty, m[0, T] \) and so (3.7) is an analytic continuation of (3.8).
4. Application to Optimal Control

The solution of the linear-quadratic regulator problem has recently been extended to bilinear systems ([3],[4]). The major drawback with the solution of the bilinear-quadratic regulator problem is that the feedback control is a tensorial power series in the state with tensor operator coefficients, the ranks of which tend to infinity. Hence, even for low-order approximations, we must perform considerable amounts of computation and to increase the accuracy of the control we must compute tensor operators of higher rank.

We can obviate this difficulty by expressing the problem in the 'frequency domain'. Thus, if we consider the optimal control problem:

\[ \dot{x} = Ax + \sum_{i=1}^{m} u_i B_i x, \]

minimize \[ J = \int_{0}^{T} (\|x\|^2 + \|u\|^2) dt, \]

then we can write

\[ J = \|x(\cdot)\|_{L^2,m[0,T]}^2 + \|u(\cdot)\|_{L^2,m[0,T]}^2 \]

\[ = \sum_{i=1}^{\infty} \|x(i)\|^2 + \|u(i)\|^2 \]

\[ = \|x\|^2 + \|u\|^2. \]

By lemma 3.2 we have

\[ x = (I-K(u))^{-1} x_0 \]

and so

\[ J = ((I-K(u))^{-1} x_0)^T ((I-K(u))^{-1} x_0) + \|u\|^2. \]

Hence

\[ \frac{\partial J}{\partial u_j(k)} = 2(\frac{\partial}{\partial u_j(k)} (I-K(u)^{-1} x_0)^T ((I-K(u)^{-1} x_0) + 2u_j(k) \]
However,
\[
\frac{\partial}{\partial u_j(k)} (I-K(u))^{-1} y = -\frac{1}{2} \frac{\partial}{\partial u_j(k)} (I-K(u))(I-K(u))^{-1} \frac{\partial}{\partial u_j(k)} (I-K(u))^T
\]

and so
\[
\frac{\partial J}{\partial u_j(k)} = -2 \left( (I-K(u))^{-1} \frac{\partial}{\partial u_j(k)} (I-K(u))(I-K(u))^{-1} \frac{\partial}{\partial u_j(k)} \right)^T (I-K(u))^{-1} y + 2 u_j(k).
\]

Also, we have
\[
\frac{\partial}{\partial u_j(k)} (I-K(u)) = -\frac{\partial}{\partial u_j(k)} K(u) = -L_{j,k}
\]

where
\[
L_{j,k} = P(k)L_j
\]

and
\[
P(k) = \bigoplus_{\ell=1}^m Q^2, n + L^2, n
\]
is the projection on the \(k^{th}\) factor. We have therefore proved

**Theorem 4.1** A necessary condition for \(u\) to be an optimal solution of the bilinear-quadratic regulator control problem

\[
\dot{x} = Ax + \sum_{i=1}^m u_i B_i x
\]

\[
\min J = \int_0^T (\|x\|^2 + \|u\|^2) dt
\]
is that \(u\) satisfies the equations

\[
u_j(k) = -((I-K(u))^{-1} L_{j,k} (I-K(u))^{-1} \xi^T (I-K(u))^{-1} \xi)
\]

for \(1 \leq j \leq m, 1 \leq k \leq \infty\), where
\[
u_j(k) = \langle u_j, e_k \rangle, \quad \xi(k) = \langle e^A x_0, e_k \rangle.
\]

Of course, we have an infinite number of equations in an infinite number of variables and so the computational problem is no easier than before. However,
since we are considering Fourier series representations of functions, we have an obvious approximation by applying the projections $P_k$ defined above. Thus, we define the operator $K_k(u): L^2, n[0,T] \rightarrow L^2, n[0,T]$ by

$$K_k(u) = P_k K(u) P_k$$

for each $u \in P_k L^2, n[0,T]$. Then we have

Lemma 4.2 The solution of the bilinear system (3.1) (specified in (3.5)) is given by

$$\lim_{k \to \infty} x_k^k = (\mathbb{I} - K_k(u_k))^{-1} \xi_k,$$

where

$$x_k^k = (\mathbb{I} - K_k(u_k))^{-1} \xi_k,$$

and

$$u_k^k = P_k u_k, \quad \xi_k^k = P_k \xi_k,$$

and $1: P_k L^2, n[0,T] \rightarrow P_k L^2, n[0,T]$ is the identity map.

Proof We have seen that (3.5) is equivalent to the equation

$$x(\varphi) = \xi(\varphi_1 \xi + \sum_{\ell=1}^{m} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \alpha_{ij} \varphi_{\ell}^i B_{\ell} x(j), \varphi_1 \xi + \varphi_1), \varphi_2 \xi + \varphi_2.$$  

Hence,

$$x(\varphi) = \sum_{\ell=1}^{m} \sum_{i=1}^{x_k} \sum_{j=1}^{x_k} \alpha_{ij} \varphi_{\ell}^i B_{\ell} x(j) + \xi(\varphi_1 \xi + \varphi_1), \varphi_2 \xi + \varphi_2,$$

where

$$\xi(\varphi) = \sum_{\ell=1}^{m} \sum_{i=1}^{x_k} \sum_{j=1}^{x_k} \alpha_{ij} \varphi_{\ell}^i B_{\ell} x(j).$$

Hence

$$P_k x = (\mathbb{I} - K_k(u_k))^{-1} \xi_k + (\mathbb{I} - K_k(u_k))^{-1} \xi_k$$

where $\xi_k^k = P_k \xi_k$, $\xi_k = (\varphi(1), \varphi(2), \ldots)$. The result now follows from the fact that $\mathbb{I} - K(u)$ is invertible, $\mathbb{I} - K_k(u_k)$ is arbitrarily close to $\mathbb{I} - K(u)$ for large $k$ and $\xi_k^k$ is arbitrarily small for large $k$. □
It follows from lemma 4.2 that we can approximate the cost functional $J$ arbitrarily closely by writing

$$J_k = \|P_k x\|^2 + \|P_k u\|^2.$$  

From theorem 4.1 we have

**Corollary 4.3** A necessary condition for $u$ to be an optimal solution of the approximate bilinear-quadratic control problem

$$\dot{P_k} = P_k A_r x + \sum_{i=1}^{m} (P_k u_i^*) P_k B_i P_k x \tag{4.2}$$

$$\min J = \int_0^T (\|P_k x\|^2 + \|P_k u\|^2) dt \text{ is that } P_k u \text{ satisfies the equations}$$

$$u_j(\ell) = -(I - K_k(u^k)^{-1} L_j (I - K_k(u^k)^{-1} \Xi_k)^T (I - K_k(u^k)^{-1} \Xi_k)) \tag{4.3}$$

for $1 \leq k \leq m$. □

Consider the operator-valued function $(I - K_k(u^k))$ in the $km$ variables $u_j(\ell)$, $1 \leq \ell \leq k, 1 \leq j \leq m$. This function has a matrix representation on $\mathbb{R}^{nk}$, the diagonal elements of which are affine functions of $u^k$. Hence the determinant of this matrix representation of $(I - K_k(u^k))$ is a polynomial in $u^k$ of order $(nk)$. It follows that the equations (4.3) can be written in the form

$$\begin{align*}
p_i(u_j(1), \ldots, u_m(1), \ldots, u_j(k), \ldots, u_m(k)) &= 0 \\
\ldots \\
p_{km}(u_j(1), \ldots, u_m(1), \ldots, u_j(k), \ldots, u_m(k)) &= 0 \tag{4.4}
\end{align*}$$

where $p_i$, $1 \leq i \leq km$ is a polynomial of order $3nk+1$. An explicit expression for $p_i$ is given by

$$p_i = u_j(\ell) D_{k,j} + (A_k L_j, -k \Xi_k)^T (A_k \Xi_k),$$

where $i=(\ell-1)k+j$, and $D_k$ and $A_k$ are, respectively, the determinant and adjugate matrix of the matrix representation of $(I - K_k(u^k))$. 
5. Example

In this section we shall consider the simple example
\[ \dot{x} = \begin{pmatrix} 0 & 1 \\ 0.75 & 1 \end{pmatrix} x + u \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} x , \quad x_0 = (1 \ 1)^T \]
where \( u(.) \in \mathbb{R} \), \( x = (x_1, x_2)^T \in \mathbb{R}^2 \), and evaluate the polynomial equations (4.4) in the cases \( k=1 \), \( k=2 \) to illustrate the method. Let \( T=2\pi \) and choose the standard basis \( \{(1/2\pi), (1/\pi)\cos t, (1/\pi)\sin t, (1/\pi)\cos 2t, (1/\pi)\sin 2t, \ldots \} \) of \( L^2[0,2\pi] \).

We have
\[ e^{At} = \begin{pmatrix} (1/4)e^{(3/2)t} + (3/4)e^{-(1/2)t} & (1/2)e^{(3/2)t} - (1/2)e^{-(1/2)t} \\ (3/8)e^{(3/2)t} - (3/8)e^{-(1/2)t} & (3/4)e^{(3/2)t} + (1/4)e^{-(1/2)t} \end{pmatrix} \]
and
\[ \alpha_{ij;k} = \int_0^{2\pi} \int_0^t e^A(t-s) e_i(s) e_j(s) ds \ e_k(t) \ dt . \]

Thus,
\[ \alpha_{111} = \frac{1}{(2\pi)^3} \int_0^{2\pi} \int_0^t e^A(t-s) ds \ dt \approx \begin{pmatrix} 5.6 & 11 \\ 8.3 & 17 \end{pmatrix} . \]

Also,
\[ \mathcal{X}(1) = \frac{1}{2\pi} \int_0^{2\pi} e^{At} x_0 dt \approx \begin{pmatrix} 986 \\ 1479 \end{pmatrix} \]
and
\[ K_1(u^1) = u(1) \begin{pmatrix} 5.6 & 11 \\ 8.3 & 17 \end{pmatrix} = u(1) \begin{pmatrix} 11 \\ 17 \end{pmatrix} \]

Hence,
\[ x(1) = \begin{pmatrix} 1-11u(1) & -11u(1) \\ -17u(1) & 1-17u(1) \end{pmatrix}^{-1} \begin{pmatrix} 986 \\ 1479 \end{pmatrix}. \]

The cost functional in (4.2) becomes
\[ J = \alpha^T (P^{-1}(u)) T (P^{-1}(u) \alpha) + u(1)^2 \]
where
\[ P(u) = \begin{pmatrix} 1-11u(1) & -11u(1) \\ -17u(1) & 1-17u(1) \end{pmatrix}, \quad \alpha = \begin{pmatrix} 986 \\ 1479 \end{pmatrix}. \]

Hence,
\[ \frac{dJ}{du(1)} = 2\alpha^T (P^{-1}(u)) T \left( \frac{dP^{-1}(u)}{du(1)} \alpha \right) + u(1) \]
\[ = -2\alpha^T (P^{-1}(u)) T \begin{pmatrix} -11 & -11 \\ -17 & -17 \end{pmatrix} P^{-1}(u) \alpha + u(1) \]
\[ = 0 \]
when \( u(1) \) satisfies the polynomial equation
\[ \begin{pmatrix} 986 & 1479 \\ 1-17u & 17u \end{pmatrix} \begin{pmatrix} 1-17u & 11u \\ 11u & 1-11u \end{pmatrix} \begin{pmatrix} -11 & -11 \\ -17 & -17 \end{pmatrix} \begin{pmatrix} 1-17u & 11u \\ 11u & 1-11u \end{pmatrix} \begin{pmatrix} 986 \\ 1479 \end{pmatrix} + u(1-28u)^3 = 0. \]

This has a real root when \( u = u(1)^2 = -6.54 \).

Now consider the case \( k=2 \), so that the basis of \( P_2[0,2\pi] \) is
\[ ((1/2\pi), (1/\pi) \cos t) \]. Then
\[ K_2(u^2) = \begin{pmatrix} u(1) \begin{pmatrix} 11 & 11 \\ 17 & 17 \end{pmatrix} + u(2) \begin{pmatrix} 15 & 15 \\ 23 & 23 \end{pmatrix} & u(1) \begin{pmatrix} 15 & 15 \\ 23 & 23 \end{pmatrix} + u(2) \begin{pmatrix} 30 & 30 \\ 45 & 45 \end{pmatrix} \\ u(1) \begin{pmatrix} 15 & 15 \\ 23 & 23 \end{pmatrix} + u(2) \begin{pmatrix} 21 & 21 \\ 32 & 32 \end{pmatrix} & u(1) \begin{pmatrix} 21 & 21 \\ 32 & 32 \end{pmatrix} + u(2) \begin{pmatrix} 42 & 42 \\ 63 & 63 \end{pmatrix} \end{pmatrix} \]
and
\[ \alpha \triangleq \begin{pmatrix} \xi(1) \\ \xi(2) \end{pmatrix} = \begin{pmatrix} 986 \\ 1479 \\ 1365 \\ 2048 \end{pmatrix}. \]

Hence,

\[ \begin{pmatrix} x(1) \\ x(2) \end{pmatrix} = P^{-1}(u) \begin{pmatrix} \xi(1) \\ \xi(2) \end{pmatrix}, \]

where

\[ P(u) = 1 - K_2(u^2). \]

The cost function becomes

\[ J = \alpha^T (P^{-1}(u)) (P^{-1}(u) \alpha + u(1)^2 + u(2)^2) \]

and so

\[ \frac{\partial J}{\partial u(1)} = -2 \alpha^T (P^{-1}(u)) \left( (P^{-1}(u) \frac{\partial P}{\partial u(1)}) + 2u(1) \right) = 0 \]

\[ \frac{\partial J}{\partial u(2)} = -2 \alpha^T (P^{-1}(u)) \left( (P^{-1}(u) \frac{\partial P}{\partial u(2)}) + 2u(2) \right) = 0. \]

In this case we obtain \( u(1) \approx -1.7 \) and \( u(2) \approx -1.95 \) and it is easily checked that the cost is reduced.

6. Conclusions: In this paper we have given a 'frequency domain' approach to the study of bilinear systems and applied it to the optimal control problem. The method is based on the isomorphism of \( L^2[0,T] \) with \( \ell^2 \), defined by a basis of \( L^2[0,T] \), and the fact that a quadratic cost may be written in the form

\[ J = \| x \|_{L^2[0,T;\mathbb{R}^n]} + \| u \|_{L^2[0,T;\mathbb{R}^m]} \]

\[ = \| x \|_{\ell^2,n} + \| u \|_{\ell^2,m}. \]

If we approximate the system in terms of a finite number of basis elements, then it has been seen that the optimal control problem reduces to the solution of a system of polynomial equations.
7. References


