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Abstract

The analysis of nonlinear systems in the frequency domain is studied and a new class of filters, called energy transfer filters, is introduced. While conventional linear filter design procedures are based on the principle of attenuating unwanted effects the new energy transfer filter design concept exploits nonlinearity to allow energy to be moved to new frequency locations. The ability to design nonlinear filters that can move energy to designed locations in the frequency domain introduces new degrees of freedom into filter design and offers new solution possibilities to many filtering problems.

1. Introduction

Nonlinear systems have been widely studied by many authors and significant progress towards understanding these systems has been made. Many of these studies have been based in the time domain with many results relating to Volterra series, NARMAX models, neural networks, fuzzy systems and classical nonlinear models including the Duffing equation, and the Van der Pol oscillator. Bifurcations, limit cycles and chaotic regimes have been investigated, categorised and analysed, and numerous important results have been obtained. Nonlinear systems have also been studied in the frequency domain where it is necessary to supplement the classical linear frequency response function with higher order frequency functions called generalised frequency response functions (George 1959). Several authors have investigated the analysis of nonlinear systems based on the generalised frequency response functions and several algorithms have been derived to estimate these functions from input output data (Tick 1961, Kim and Powers 1988, Nam and Powers 1994, Peyton-Jones and Billings 1989, Billings and Peyton-Jones 1990).

However, while there have been studies which include the term nonlinear filter in the title the majority of these investigations relate to designing low order, typically second order, Volterra series models that minimise a cost function or which implement channel equalisation or other similar time domain objectives (Sicuranza 1992, Mathews 1991, Zelniker and Taylor 1994, Heredia et al. 2000). There appear to have been very few if any attempts to design nonlinear filters based on frequency domain objectives. This is surprising given the ubiquitous nature of classical linear low, band pass and band stop filter designs.

Conventional linear filter design is based on the principle that energy in unwanted frequency bands is attenuated. The traditional low pass and band pass filter designs are examples of linear solutions to this problem whereas the Dolby filter (Amos 1977), which varies the amplitude of the input signal as a function of the level and frequency of the input, is an example of a nonlinear implementation of this solution. There are many different filter designs, including Butterworth, Chebyshev and various others (Zelniker and Taylor 1994), but the concept of the design is always based on attenuation. Stochastic filters based on the Kalman and Wiener methods are also available. These are based on prediction of the signal through the noise and assume knowledge of the statistics of the unwanted noise.

We have recently derived a totally new approach. This employs recent theoretical developments derived by the authors and based on the NARMAX method and estimation in the frequency domain. The new approach is based on the principle that energy in one frequency band can be moved or transferred to other frequency locations. This is achieved by exploiting the properties of nonlinear effects. Other energy transfer effects including splitting the unwanted responses and moving these to new frequency bands can also be achieved. Energy can be moved to higher frequencies or lower frequencies, or it can be focused around one frequency location. There are many design possibilities, and subject to realisability constraints these general principles can be applied in many areas. But in every case nonlinearity is exploited in the design such that energy can be transferred to desired frequency locations and consequently the new class of filters will be referred to as Energy Transfer Filters or ETF's.

This is our first paper to addresses the new ETF concept, and only the basic design principles for specified inputs, and the solutions to some relatively simple design problems, will be described. More sophisticated designs and associated analysis will be presented in a series of later publications. In the present paper therefore the design of energy transfer filters which can be described by the NARX (Nonlinear AutoRegressive with eXogenous input) model with input nonlinearities will be investigated. The paper begins with a brief introduction to the concept of energy transfer filters. This is followed by sections which describe theoretical results relating to the output frequency response of nonlinear systems, and finally to the introduction of the energy transfer filter design concept. Several simple design examples are included to illustrate the potential of the new method.

2. The ETF Concept

The output frequency response of a nonlinear system is determined by a complex combination of effects that is dependent on the system characteristics and the input. The possible output frequencies of nonlinear systems are much richer than the frequencies in the input, and as a result of this, signal energy in the system input can be transferred to a different frequency location in the output. These are physical phenomena which have been known for a very long time and which are often observed in engineering systems (Popovic et al. 1995, Szczepanski and Schaumann 1993 and 1989). However, in most cases, researchers and engineers regard such nonlinear effects negatively, and often try to take measures to prevent or to linearise out such phenomena. Even the names which are used, nonlinear interference and distortion, suggest that these are effects that are undesirable. There are two main reasons for this situation. First, the complicated composition of nonlinear output frequency responses means that the analysis and design of nonlinear systems in the frequency domain is generally difficult. Second, people usually attempt to avoid nonlinearity rather than exploit it. But nonlinearity can be a benefit. Rather than linearising nonlinearity out, if it is designed into the system in an appropriate way additional degrees of freedom are introduced and if used correctly this can be a benefit.

Energy transfer filters are just one example where nonlinearity can be designed in to provide additional benefits compared to purely linear designs.

Conventional linear filter design is based on the principle that energy in unwanted frequency bands is attenuated. Fig.1(a) illustrates this effect and shows the power spectrum of a signal before and after filtering. The unwanted response labelled '101' is attenuated as much as possible to produce the response labelled '108' while preserving the response labelled '102'. Low pass, band pass, and band stop filter designs are well known examples and many different designs are available including Butterworth, Chebyshev and many others. The Dolby filter, which varies the amplitude of the output as a function of the level and frequency of the input, behaves nonlinearly but the effect of the Dolby filter is simply to attenuate the unwanted frequency domain effects. All these designs therefore are based on the principle that unwanted effects are attenuated out.

We have recently derived a totally new approach which is based on the principle that energy in one frequency band can be moved or transferred to other designed frequency locations. Fig.1(b) shows an example of this effect where the unwanted portion of the response labelled '204' is moved to a new frequency location labelled '206'. This is achieved by exploiting the properties of nonlinear effects. We believe that this concept is totally new and we refer to this class of filters as Energy Transfer Filters (ETF's). Fig.1(b) shows just one mode of energy transfer that can be achieved with the new design. A wide range of other designed energy transfer effects have been developed and most of them will be discussed in later publications. In order to introduce the new design philosophy the analysis of nonlinear systems in the frequency domain is first investigated in the next section.

3 Output Frequency Responses of Nonlinear Systems

3.1 The output frequency response of a nonlinear system

It is well known that the output frequency response of a stable linear system is given by

$$Y(j\omega) = H(j\omega)U(j\omega) \tag{3.1}$$

In (3.1), $Y(j\omega)$ and $U(j\omega)$ are the system input and output spectrum which are the Fourier Transforms of the system time domain input $u(t)$ and output $y(t)$ respectively, and $H(j\omega)$ represents the linear frequency response function.

Consider a nonlinear system which is stable at the zero equilibrium point and which can be described in the neighbourhood of the equilibrium point by the Volterra series

$$y(t) = \sum_{n=1}^{N} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1,\ldots,\tau_n) \prod_{i=1}^{n} u(t-\tau_i) d\tau_i \tag{3.2}$$

where $h(\tau_1,\ldots,\tau_n)$ is the nth order Volterra kernel, and N denotes the maximum order of the system nonlinearity.

The output frequency response of a nonlinear system, which can be described by equation 3.2, and is subject to a general input, is given by (Lang and Billings 1996)

$$\begin{cases} Y(j\omega) = \sum_{n=1}^{N} Y_n(j\omega) \quad for\ \forall\omega \\ Y_n(j\omega) = \dfrac{1/\sqrt{n}}{(2\pi)^{n-1}} \displaystyle\int_{\omega_1+\ldots+\omega_n=\omega} H_n(j\omega_1,\ldots,j\omega_n) \prod_{i=1}^{n} U(j\omega_i) d\sigma_{n\omega} \end{cases} \tag{3.3}$$

In (3.3), $Y_n(j\omega)$ represents the nth order output frequency response of the system,

$$H_n(j\omega_1,...,j\omega_n) = \int_{-\infty}^{\infty}...\int_{-\infty}^{\infty} h_n(\tau_1,...,\tau_n)e^{-(\omega_1\tau_1+,...,+\omega_n\tau_n)j}d\tau_1...d\tau_n \qquad (3.4)$$

is known as the nth-order Generalised Frequency Response Function (GFRF), and

$$\int_{\omega_1+,...,+\omega_n=\omega} H_n(j\omega_1,...,j\omega_n)\prod_{i=1}^{n} U(j\omega_i)d\sigma_{n\omega}$$

denotes the integration of $H_n(j\omega_1,...,j\omega_n)\prod_{i=1}^{n} U(j\omega_i)$ over the n-dimensional hyperplane $\omega_1 + \cdots + \omega_n = \omega$.

Equation (3.3) is a natural extension of the well-known linear relationship (3.1) to the nonlinear case and indicates that the relationship between the system output in the frequency domain with the system frequency domain characteristics and the input is much more complicated than in the linear case. This is reflected in two aspects. First, the output spectrum of a nonlinear system is a summation of a possibly infinite number of terms, each representing the effect of system nonlinearities of different orders. The effect produced by this summation is referred to as inter-kernel interference. Second, each of the terms such as $Y_n(j\omega)$ is, when $n \geq 2$, the result of an integration of a multivariable complex function over an n-dimensional hyper-plane. This integration is known as the nth order intra-kernel interference. The output frequency response of nonlinear systems is therefore determined by both intra and inter kernel interference of the involved system nonlinearities (Peyton-Jones and Billings 1990). The complicated composition of the output spectrum of a nonlinear system produces much more complex output frequency responses compared to the linear case. The most distinct effect is that the output frequencies of nonlinear systems can be much richer than the frequencies in the input.

3.2 Output frequencies of nonlinear systems

In linear systems, the possible output frequencies at steady state are exactly the same as the frequencies of the input. This can be observed directly from equation (3.1). For nonlinear systems under a general input, equation (3.3) indicates that the possible output frequencies at steady state can be generally described as

$$f_Y = \bigcup_{n=1}^{N} f_{Y_n} \qquad (3.5)$$

where f_Y denotes the non-negative frequency range of the system output, and f_{Y_n} represents the non-negative frequency range produced by the nth-order system nonlinearity.

From (3.5) an explicit expression for the output frequency range f_Y of nonlinear systems under a general input with a spectrum given by

$$U(j\omega) = \begin{cases} U(j\omega) & \text{when} \quad |\omega| \in (a,b) \\ 0 & \text{otherwise} \end{cases} \qquad (3.6)$$

can be derived as (Lang and Billings 1997)

$$
\begin{cases}
f_Y = f_{Y_N} \bigcup f_{Y_{N-(2p^*-1)}} \\
f_{Y_n} = \begin{cases} \bigcup_{k=0}^{i^*-1} I_k & \text{when } \frac{nb}{(a+b)} - \left\lfloor \frac{na}{(a+b)} \right\rfloor < 1 \\ \bigcup_{k=0}^{i^*} I_k & \text{when } \frac{nb}{(a+b)} - \left\lfloor \frac{na}{(a+b)} \right\rfloor \geq 1 \end{cases} \\
i^* = \left\lfloor \frac{na}{(a+b)} \right\rfloor + 1 \\
\text{where } \lfloor . \rfloor \text{ means to take the integer part} \\
I_k = \left(na - k(a+b), nb - k(a+b)\right) \text{ for } k = 0, \ldots, i^* - 1, \\
I_{i^*} = \left(0, nb - i^*(a+b)\right)
\end{cases} \tag{3.7}
$$

In (3.7) p^* can be taken as $1, 2, \cdots, \lfloor N/2 \rfloor$, the specific value of which depends on the system nonlinearities. If the system GFRFs $H_{N-(2i-1)}(.) = 0$, for $i = 1, \cdots, q-1$, and $H_{N-(2q-1)}(.) \neq 0$ then $p^* = q$.

Equation (3.7) is the first analytical description for the output frequencies of nonlinear systems which extends the well-known relationship between the input and output frequencies of linear systems to the nonlinear case. A similar result was developed one year later by Raz and Veen (1998) from a different perspective.

It is worth pointing out that f_Y is actually the frequency range where the frequency components may exist in the output of a nonlinear system. But this does not mean that the output frequency components of a nonlinear system are definitely available over the whole range of f_Y.

4 Output Frequency Response of the NARX Model with Input Nonlinearity

4.1 The NARX model with input nonlinearity

The discrete NARX model is given by

$$
y(k) = \sum_{n=1}^{N} y_n(k) \tag{4.1}
$$

where $y_n(k)$ is a 'NARX nth order output' given by

$$
y_n(k) = \sum_{p=0}^{n} \sum_{l_1, l_{p+q}=1}^{K_n} c_{pq}(l_1, \cdots, l_{p+q}) \prod_{i=1}^{p} y(k - l_i) \prod_{i=p+1}^{p+q} u(k - l_i) \tag{4.2}
$$

with $p + q = n$, $l_i = 1, \cdots, K_n$, $i = 1, \cdots, p + q$, and

$$
\sum_{l_1, l_{p+q}=1}^{K_n} = \sum_{l_1=1}^{K_n} \cdots \sum_{l_{p+q}=1}^{K_n}
$$

K_n is the maximum lag and $y(.), u(.)$ and $c_{pq}(.)$ are the output, input and model coefficients respectively. The NARX model is a special case of the NARMAX model where the MA noise model terms, which are included in identification studies to avoid biased estimates, have been discarded to give the process or NARX model (Pearson 1999). A specific instance of the NARX model such as

$$\begin{aligned} y(k) = 0.3u(k-1) + 0.7y(k-1) - 0.02u(k-1)u(k-1) - 0.04u(k-2)u(k-1) \\ - 0.06y(k-1)u(k-3) - 0.08y(k-2)y(k-3) \end{aligned} \tag{4.3}$$

may be obtained from (4.1) and (4.2) with

$c_{01}(1) = 0.3$, $c_{10}(1) = 0.7$, $c_{02}(1,1) = -0.02$,
$c_{02}(2,1) = -0.04$, $c_{11}(1,3) = -0.06$, $c_{20}(2,3) = -0.08$, else $c_{pq}(.) = 0$

The NARX model with input nonlinearity is a specific case of the NARX model and is given by equation (4.1) where

$$y_n(k) = \begin{cases} \sum_{l_1,l_n=1}^{K_{nu}} c_{0n}(l_1,\cdots,l_n)\prod_{i=1}^{n} u(k-l_i) & \text{for } n \geq 2 \\ \sum_{l_1=1}^{K_y} c_{10}(l_1)y(k-l_1) + \sum_{l_1=1}^{K_{1u}} c_{01}(l_1)u(k-l_1) & \text{for } n = 1 \end{cases} \tag{4.4}$$

and $K_{nu}, n = 1,\cdots,N$, and K_y are the corresponding maximum lags.

The NARX model with input nonlinearity (4.1) (4.4) is, under certain conditions, an equivalent description to the well-known discrete possibly infinite Volterra systems of the form

$$y(k) = \sum_{n=0}^{\infty}\sum_{i_1=0}^{\infty}\cdots\sum_{i_n=0}^{\infty} a_{i_1 i_2\cdots i_n} u(k-i_i)\cdots u(k-i_n) \tag{4.5}$$

This follows from the NARX model derivation by Leoritaritis and Billings (1985) and more recently from Kotsios (1997) who showed that under certain conditions the discrete infinite Volterra system (4.5) can be transformed to the finite input/output form (4.1)(4.4) via the notion of linear factorisation of δ-series introduced by Kotsios and Kalouptsidis (1993). These results justify the choice of the NARX model with input nonlinearity rather than the widely used truncated Volterra series as a basic filter structure for the initial energy transfer filter designs.

The truncated Volterra series is an approximate description of the general expression (4.5) but the NARX model with input nonlinearity (4.1)(4.4) can be an exact alternative to the infinite Volterra series model. In addition, the practical advantage of the model (4.1)(4.4) is also obvious. Firstly, stability can easily be checked, due to the existence of a number of useful theorems, which is very important in filter designs. Secondly the finite expression can easily be transformed to a linear-in-the-parameters form. This is convenient for the filter design in either the time or frequency domain.

4.2 The output frequency response

For nonlinear systems which can be described by the NARX model with input nonlinearity (4.1) (4.4), the GFRFs can readily be obtained using the recursive computation algorithm introduced by Peyton Jones and Billings (1989) to yield

$$H_n(j\omega_1,\ldots,j\omega_n)=\frac{\sum_{l_1,l_n=1}^{K_{nu}} c_{0n}(l_1,\cdots,l_n)\exp[-j(\omega_1 l_1+\cdots+\omega_n l_n)]}{\left\{1-\sum_{l_1=1}^{K_y} c_{10}(l_1)\exp[-j(\omega_1+\cdots+\omega_n)l_1]\right\}}\qquad \text{n=1,...,N} \tag{4.6}$$

Equation (4.6) directly maps the time domain model (4.1) (4.4) into the frequency domain and produces an expression for the system GFRFs in terms of the parameters in the system time domain model.

Substituting equation (4.6) into (3.3) yields the output frequency response for a nonlinear system described by the model (4.1) (4.4) as

$$Y(j\omega)=\frac{1}{\left\{1-\sum_{l_1=1}^{K_Y} c_{10}(l_1)\exp(-j\omega l_1)\right\}}\sum_{n=1}^{N}\frac{1/\sqrt{n}}{(2\pi)^{(n-1)}}\sum_{l_1,l_n=1}^{K_{nu}} c_{0n}(l_1,\cdots,l_n)\int_{\omega_1+\cdots+\omega_n=\omega}\exp[-j(\omega_1 l_1+\cdots+\omega_n l_n)]\prod_{i=1}^{n}U(j\omega_i)d\sigma_{n\omega} \tag{4.7}$$

This is an expression for the system output spectrum in terms of the parameters in the system time domain model and will form the basis for the design of the energy transfer filters based on equations (4.1) and (4.4).

The output frequency range of nonlinear systems described by model (4.1)(4.4) is also given by (3.7) if the system is subject to an input with the spectrum given by (3.6). But, if $H_{N-1}(.)$ of the system is not zero, which as will be shown later is the normal case for the present filter design, then $H_{N-(2q-1)}(.)\neq 0$ when q=1. So $p^*=q=1$, and the first expression in (3.7) becomes

$$f_Y=f_{Y_N}\bigcup f_{Y_{N-1}} \tag{4.8}$$

The maximum order N, of nonlinearity in the model (4.1)(4.4), will be determined using this relationship for the ETF design

5 The Design of Frequency Domain Energy Transfer Filters

5.1 Description of the design problem

The Energy Transfer Filter (ETF) design problem based on the model (4.1)(4.4) which is considered in this paper can be generally stated as follows: Given one (several) specific inputs the frequency components of which are over a frequency band (a,b) and a corresponding desired output spectrum (spectra) over a frequency band (c,d) which is different from (a,b), design a nonlinear filter of the form (4.1)(4.4) to implement the signal energy transfer from the input frequency band (a,b) to the output frequency band (c,d).

The design will involve the following steps

(i) detemine the filter structure

(ii) map the time domain model description into the frequency domain

(iii) express the output spectrum in terms of the input spectrum and the filter time domain model parameters

(iv) conduct the design based on the expression obtained in Step (iii) and

(v) realise the design

The mapping between the time and frequency domain model, Step (ii), has already been given in equation (4.6); The output spectrum, Step (iii), of model (4.1)(4.4) has been expressed in terms of the model parameters in equation (4.7); Many methods are available (Mathews and Sicuranza 2000) for the practical implementation of the design which is normally domain specific. Therefore the major issues to be addressed for the design are the determination of the filter structure and the design based on equation (4.7).

Equations (4.1) and (4.4) can be rewritten in a more compact and general form as

$$y(k)-\sum_{l_1=1}^{K_y} c_{10}(l_1)y(k-l_1)=\sum_{n=N_0}^{N}\sum_{l_1=1}^{K_{nu}}\cdots\sum_{l_n=l_{n-1}}^{K_{nu}}\bar{c}_{0n}(l_1,\cdots,l_n)\prod_{i=1}^{n}u(k-l_i) \tag{5.1}$$

In (5.1), N_0 is the minimum order of the system nonlinearity and

$$\bar{c}_{0n}(l_1,\cdots,l_n)=|\pi(l_1,\cdots,l_n)|\bar{\bar{c}}_{0n}(l_1,\cdots,l_n) \tag{5.2}$$

where

$$\bar{\bar{c}}_{0n}=\frac{\sum_{\pi(.)}c_{0n}(l_1,\cdots,l_n)}{|\pi(l_1,\cdots,l_n)|} \tag{5.3}$$

and the summation is over all distinct permutations $\pi(.)$ of the indices $l_1,\cdots,l_n$ and $|\pi(l_1,\cdots,l_n)|$ represents the number of such permutations. In order to evaluate $|\pi(l_1,\cdots,l_n)|$, denote the number of distinct values in a specific set $(l_1,\cdots,l_n)$ as r. Let $k_1,\cdots,k_r$ denote the number of times these values appear in $(l_1,\cdots,l_n)$. Then

$$|\pi(l_1,\cdots,l_n)|=\frac{n!}{k_1!k_2!\cdots k_r!} \tag{5.4}$$

Consider the ETF design based on (5.1). The structure of (5.1) is defined by the values of N, N_0, K_y and $K_{nu}, n=1,2,\cdots,N$, and involves terms such as

$$\bar{c}_{0n}(l_1,\cdots,l_n)\prod_{i=1}^{n}u(k-l_i) \qquad l_1=1,\cdots,K_{nu},\cdots\cdots,l_n=l_{n-1},\cdots,K_{nu}$$

for $n=N_0,\cdots,N$ and

$$c_{10}(l_1)y(k-l_1), l_1=1,\cdots,K_y$$

The structure parameters N and N_0 are associated with the feasibility of the model to give effect to the required signal energy transfer and this feasibility can be determined from the relationship between the input and output frequencies of nonlinear systems given by equation (3.7). The structure parameters K_y and $K_{nu}, n=1,2,\cdots,N$ are associated with the extent to which specific design requirements for the magnitude and/or phase of the output spectrum over the desired output frequency band (c,d) can be satisfied.

Once these structure parameters having been determined, equation (5.1) can be expressed using the backward shift operator q^{-1} as

$$y(k)=G(q^{-1})\sum_{n=N_0}^{N}\sum_{l_1=1}^{K_{nu}}\cdots\sum_{l_n=l_{n-1}}^{K_{nu}}\bar{c}_{0n}(l_1,\cdots,l_n)\prod_{i=1}^{n}u(k-l_i) \tag{5.5}$$

where

$$G(q^{-1}) = \frac{1}{1-\sum_{l_1=1}^{K_y} c_{10}(l_1)q^{-1}} \tag{5.6}$$

The filter model (5.5) consists of a nonlinear subsystem described by a finite Volterra series and a traditional linear filter, the design therefore involves determining the parameters in these subsystems

Notice that a more general form of (5.6) would be the rational form

$$G(q^{-1}) = \frac{1}{1-\sum_{l_1=1}^{K_y} c_{10}(l_1)q^{-1}} = \frac{\alpha(q^{-1})}{\beta(q^{-1})} \tag{5.7}$$

where $\alpha(q^{-1})$ and $\beta(q^{-1})$ are finite order polynomials in terms of q^{-1}. Therefore

$$1-\sum_{l_1=1}^{K_y} c_{10}(l_1)q^{-1} = \frac{\beta(q^{-1})}{\alpha(q^{-1})} \tag{5.8}$$

and consequently K_y could take an infinite value.

5.2 ETF design for a specified input

Given one specified input, the relationship between the output spectrum of the system (4.1) (4.4) and the spectrum of the input is given by equation (4.7).Using the more compact and general form (5.1) of the model (4.1) (4.4) and equation (5.6), this relationship can be written as

$$Y(j\omega) = G(j\omega)\sum_{n=N_0}^{N}\frac{1/\sqrt{n}}{(2\pi)^{(n-1)}}\sum_{l_1=1}^{K_{nu}}\cdots\sum_{l_n=l_{n-1}}^{K_{nu}}\bar{c}_{0n}(l_1,\cdots,l_n)\int_{\omega_1+\cdots+\omega_n=\omega}\exp[-j(\omega_1 l_1+\cdots+\omega_n l_n)]\prod_{i=1}^{n}U(j\omega_i)d\sigma_{n\omega} \tag{5.9}$$

The ETF design for one specified input consists of three steps: First, determine the orders of nonlinearity which are needed to ensure that the required frequency domain energy transfer can be achieved. This defines N_0 and N, the minimum and maximum order of the system nonlinearity. Second, determine the parameters in the nonlinear subsystem

$$\bar{c}_{0n}(l_1,\cdots,l_n),\ l_1 = 1,\cdots,K_{nu},\cdots\cdots,l_n = l_{n-1},\cdots,K_{nu};\ n = N_0,\cdots,N$$

to make the output spectrum of the nonlinear subsystem

$$\bar{Y}(j\omega) = \sum_{n=N_0}^{N}\frac{1/\sqrt{n}}{(2\pi)^{(n-1)}}\sum_{l_1=1}^{K_{nu}}\cdots\sum_{l_n=l_{n-1}}^{K_{nu}}\bar{c}_{0n}(l_1,\cdots,l_n)\int_{\omega_1+\cdots+\omega_n=\omega}\exp[-j(\omega_1 l_1+\cdots+\omega_n l_n)]\prod_{i=1}^{n}U(j\omega_i)d\sigma_{n\omega} \tag{5.10}$$

approach the desired spectrum as closely as possible. Finally, design a suitable linear filter $G(j\omega)$ to improve the approximation to the desired spectrum obtained in the second step such that

$$Y(j\omega) = G(j\omega)\bar{Y}(j\omega) \tag{5.11}$$

This linear design produces the parameters for the linear subsystem K_y and $c_{10}(l_1), l_1 = 1,\cdots,K_y$. Details regarding the implementation of these steps are given below.

Step 1

Determining N_0 involves finding the minimum value of the orders of system nonlinearities which make a contribution to the output spectrum over the desired output frequency band (c,d). Given the input frequency band (a,b), this can be achieved by evaluating the output frequency range f_{Y_n} for $n=1, n=2,\cdots$, until $n=\overline{n}$ such that at least part of the specified output frequency range (c,d) falls into $f_{Y_{\overline{n}}}$. The value N_0 is then taken as $N_0 = \overline{n}$.

The value N has to be determined so as to find the minimum value of n such that the specified output frequency range (c,d) completely falls into $f_{Y_n} \bigcup f_{Y_{n-1}}$. To achieve this, the frequency range $f_{Y_n} \bigcup f_{Y_{n-1}}$ should be evaluated for $n=\overline{n}, n=\overline{n}+1,\cdots$, until $n=\overline{\overline{n}}$ such that (c,d) completely falls into $f_{Y_{\overline{\overline{n}}}} \bigcup f_{Y_{\overline{\overline{n}}-1}}$. Then N is taken as $N = \overline{\overline{n}}$.

Step 2

Denote the desired output spectrum as $Y^{\#}(j\omega)$. Then, in the second step of the design, the filter parameters

$$\overline{c}_{0n}(l_1,\cdots,l_n),\ l_1 = 1,\cdots,K_{nu},\cdots\cdots,l_n = l_{n-1},\cdots,K_{nu};\ n = N_0,\cdots,N$$

are determined based on the equations

$$Y^{\#}(j\omega(p)) = \sum_{n=N_0}^{N} \frac{1/\sqrt{n}}{(2\pi)^{(n-1)}} \sum_{l_1=1}^{K_{nu}} \cdots \sum_{l_n=l_{n-1}}^{K_{nu}} \overline{c}_{0n}(l_1,\cdots,l_n) \int_{\omega_1+\cdots+\omega_n=\omega(p)} \exp[-j(\omega_1 l_1+\cdots+\omega_n l_n)] \prod_{i=1}^{n} U(j\omega_i)\, d\sigma_{n\omega}$$

$$p = 1,\cdots,M \tag{5.12}$$

In (5.12), M is an a priori given integer and $\omega(p) \in (c,d), p = 1,\cdots,M$. The objective of this is to make the right hand side of the equation approach the desired output spectrum as closely as possible over the M discrete frequency points $\omega(1),\cdots,\omega(M)$. For this purpose, equation (5.12) can be written as

$$Y^{\#}(j\omega(p)) = \sum_{n=N_0}^{N} \frac{1/\sqrt{n}}{(2\pi)^{(n-1)}} \sum_{l_1=1}^{K_{nu}} \cdots \sum_{l_n=l_{n-1}}^{K_{nu}} \overline{c}_{0n}(l_1,\cdots,l_n)\, g_{l_1\cdots l_n}(j\omega(p)) \qquad p = 1,\cdots,M$$

$$\tag{5.13}$$

where

$$g_{l_1\cdots l_n}(j\omega(p)) = \int_{\omega_1+\cdots+\omega_n=\omega(p)} \exp[-j(\omega_1 l_1+\cdots+\omega_n l_n)] \prod_{i=1}^{n} U(j\omega_i)\, d\sigma_{n\omega} \tag{5.14}$$

which is the Fourier Transform of the time series $u(k-l_1)u(k-l_2)\cdots u(k-l_n)$. Therefore, for a specific set of $(l_1,\cdots,l_n)$, $g_{l_1\cdots l_n}(\omega)$ can readily be evaluated from the given input. Consequently, the filter parameters can be determined based on the equations

$$\begin{cases} \mathrm{Re}\left[Y^{\#}(j\omega(p))\right] = \sum_{n=N_0}^{N} \frac{1/\sqrt{n}}{(2\pi)^{(n-1)}} \sum_{l_1=1}^{K_{nu}} \cdots \sum_{l_n=l_{n-1}}^{K_{nu}} \bar{c}_{0n}(l_1,\cdots,l_n)\,\mathrm{Re}\left[g_{l_1\cdots l_n}(j\omega(p))\right] \\ \qquad\qquad p = 1,2,\cdots,M \\ \mathrm{Im}\left[Y^{\#}(j\omega(p))\right] = \sum_{n=N_0}^{N} \frac{1/\sqrt{n}}{(2\pi)^{(n-1)}} \sum_{l_1=1}^{K_{nu}} \cdots \sum_{l_n=l_{n-1}}^{K_{nu}} \bar{c}_{0n}(l_1,\cdots,l_n)\,\mathrm{Im}\left[g_{l_1\cdots l_n}(j\omega(p))\right] \\ \qquad\qquad p = 1,2,\cdots,M \end{cases} \tag{5.15}$$

using a least squares routine to make the right hand side of the equations approach the left hand side as closely as possible. Denote the results as

$$\hat{\bar{c}}_{0n}(l_1,\cdots,l_n),\ l_1 = 1,\cdots,K_{nu},\cdots\cdots,l_n = l_{n-1},\cdots,K_{nu};\ n = N_0,\cdots,N$$

Then the nonlinear subsystem of the filter will be given by

$$\bar{y}(k) = \sum_{n=N_0}^{N} \frac{1/\sqrt{n}}{(2\pi)^{(n-1)}} \sum_{l_1=1}^{K_{nu}} \cdots \sum_{l_n=l_{n-1}}^{K_{nu}} \hat{\bar{c}}_{0n}(l_1,\cdots,l_n)u(k-l_1)\cdots u(k-l_n) \tag{5.16}$$

where $\bar{y}(k)$ represents the time domain output of this subsystem. Notice that M is the number of discrete frequency points over the desired output frequency band (c,d) considered in the design. Generally speaking, the bigger the a priori given integer M, the closer the output spectrum of the designed nonlinear subsystem can approach the desired spectrum over the frequency band (c,d). But a larger value of M may lead to a more complicated nonlinear subsystem than the case where a smaller value of M is selected. In addition, in order to obtain a unique solution to the parameters of the nonlinear subsystem, the selection of M must also be subject to a constraint associated with the number of the parameters of the nonlinear subsystem.

The number of equations in (5.15) is 2M. The number of parameters to be determined for each n is $K_{nu}(K_{nu}^{n-1}+1)/2$. Hence the total number of parameters which need to be evaluated using least squares is

$$n_p = \sum_{n=N_0}^{N} K_{nu}(K_{nu}^{n-1}+1)/2 \tag{5.17}$$

and if $K_{nu} = K_u$ for $n = N_0,\cdots,N$, this result becomes

$$n_p = \sum_{n=N_0}^{N} K_u(K_u^{n-1}+1)/2 = \frac{(K_u^{N+1} - K_u^{N_0}) + (K_u - 1)K_u(N - N_0 + 1)}{2(K_u - 1)} \tag{5.18}$$

Obviously to obtain a unique least squares solution to the n_p model parameters, it is necessary to choose an M such that

$$M \geq n_p/2 \tag{5.19}$$

Step 3

Having determined the parameters in the nonlinear subsystem, the output spectrum of the nonlinear subsystem over the frequency set $\{\omega(1),\cdots,\omega(M)\}$ can be obtained as

$$\bar{Y}(j\omega(p)) = \sum_{n=N_0}^{N} \frac{1/\sqrt{n}}{(2\pi)^{(n-1)}} \sum_{l_1=1}^{K_{nu}} \cdots \sum_{l_n=l_{n-1}}^{K_{nu}} \hat{\bar{c}}_{0n}(l_1,\cdots,l_n) g_{l_1\cdots l_n}(j\omega(p))\quad p = 1,\ldots,M \tag{5.20}$$

In the third step of the design, a linear filter with frequency response function $G_1(j\omega)$ is determined to improve the approximation effect of $\overline{Y}(j\omega)$ on the desired output spectrum $Y^{\#}(j\omega)$ over the frequency set $\{\omega(1),\cdots,\omega(M)\}$. This is achieved by designing $G_1(j\omega)$ subject to a stability constraint to minimise the criterion

$$J(G_1)=\sum_{p=1}^{M}\left\{Y^{\#}[j\omega(p)]-G_1[j\omega(p)]\overline{Y}[j\omega(p)]\right\}^{*}\left\{Y^{\#}[j\omega(p)]-G_1[j\omega(p)]\overline{Y}[j\omega(p)]\right\} \tag{5.21}$$

The frequency response $G_1(j\omega)\overline{Y}(j\omega)$, which is the output spectrum of the system composed of a cascade of the nonlinear subsystem and the linear filter, should have a better approximation to the desired output spectrum $Y^{\#}(j\omega)$ over the desired output frequency band (c,d). A band pass filter, with frequency response function $G_2(j\omega)$, can then be designed to remove any unwanted residual frequency components in $G_1(j\omega)\overline{Y}(j\omega)$ which are outside the output frequency band (c,d) and, as a result of this, $G_1(j\omega)G_2(j\omega)\overline{Y}(j\omega)$ should approach $Y^{\#}(j\omega)$ as required by the design. Finally

$$G(j\omega)=G_1(j\omega)G_2(j\omega) \tag{5.22}$$

and both the structure and parameters of the linear subsystem can be determined.

The criterion (5.21) can be rewritten as

$$J(G_1)=\sum_{p=1}^{M}\left|G_1[j\omega(p)]-\frac{Y^{\#}[j\omega(p)]}{\overline{Y}[j\omega(p)]}\right|^2\left|\overline{Y}[j\omega(p)]\right|^2 \tag{5.23}$$

Denote $G_1(j\omega)$ as

$$G_1(j\omega)=\left.\frac{\overline{b}_1+\overline{b}_2q^{-1}+\cdots+\overline{b}_{n_{b1}+1}q^{-n_{b1}}}{\overline{a}_1+\overline{a}_2q^{-1}+\cdots+\overline{a}_{n_{a1}+1}q^{-n_{a1}}}\right|_{q=e^{j\omega}} \tag{5.24}$$

Then the optimisation problem is to find $\overline{b}_1,\overline{b}_2,\cdots,\overline{b}_{n_{b1}+1}$ and $\overline{a}_1,\overline{a}_2,\cdots,\overline{a}_{n_{a1}+1}$ for the a priori given structure parameters n_{b1} and n_{a1} to make $J(G_1)$ reach a minimum under the constraint that the filter is stable. This can be solved using for example the MATLAB function "invfreqz.m" which uses the damped Gauss-Newton method for iterative search (Dennis and Schnabel 1983).

The design of the band pass filter $G_2(j\omega)$ can be achieved using one of the many standard filter design methods. If the specification for the design only involves the magnitude of the output frequency response, denote $G_2(j\omega)$ as

$$G_2(j\omega)=\left.\frac{\overline{\overline{b}}_1+\overline{\overline{b}}_2q^{-1}+\cdots+\overline{\overline{b}}_{n_{b2}+1}q^{-n_{b2}}}{\overline{\overline{a}}_1+\overline{\overline{a}}_2q^{-1}+\cdots+\overline{\overline{a}}_{n_{a2}+1}q^{-n_{a2}}}\right|_{q=e^{j\omega}} \tag{5.25}$$

and apply a typical band pass filter design method such as Butterworth, Chebyshev, etc. The parameters $\overline{\overline{b}}_1,\overline{\overline{b}}_2,\cdots,\overline{\overline{b}}_{n_{b2}+1}$ and $\overline{\overline{a}}_1,\overline{\overline{a}}_2,\cdots,\overline{\overline{a}}_{n_{a2}+1}$ for a given choice of n_{b2} and n_{a2} can then be obtained. If the specification of the design is for both the magnitude and phase of the output frequency response, $G_2(j\omega)$ should be described as a linear phase FIR filter of the form

$$G_2(j\omega) = \bar{\bar{b}}_1 + \bar{\bar{b}}_2 q^{-1} + \cdots + \bar{\bar{b}}_{n_{b2}+1} q^{-n_{b2}} \Big|_{q=e^{j\omega}} \tag{5.26}$$

Again many methods can be used for this purpose (Zelniker and Taylor 1994) and the design produces the filter parameters $\bar{\bar{b}}_1, \bar{\bar{b}}_2, \cdots, \bar{\bar{b}}_{n_{b2}+1}$.

Obviously the three design steps depend on the filter structure parameters which have to be given a priori. Specifically these parameters are $K_{nu}, n = N_0, \cdots, N$, M, together with the structure parameters for $G_1(j\omega)$ and $G_2(j\omega)$. The selection of M should satisfy the constraint (5.19) but generally a much larger value of M is usually needed to ensure a sufficiently good approximation accuracy between the spectrum of the filter output and the desired output spectrum over the frequency band (c, d). The structure of $G_1(j\omega)$ and $G_2(j\omega)$ is relatively less important and this is therefore usually fixed during the design. $K_{nu}, n = N_0, \cdots, N$, are normally all taken to be equal to K_u. The specific value of K_u can be determined in an iterative way until a satisfactory filtering effect is achieved. For example, K_u can initially be set as $K_u = 1$. The design using this K_u is completed and the performance of the resulting filter is checked to see if this is satisfactory or not. If the result is satisfactory, then the design is finished. Otherwise, take $K_u = 2$ and repeat the procedure again. This process can be continued with $K_u = 3, 4, \cdots$ until a satisfactory result is achieved.

For a given M, the maximum value of K_u which can be taken for the design is limited by the inequality (5.19). Substituting (5.18) into (5.19) yields a clear description of this constraint

$$M \geq \frac{(K_u^{N+1} - K_u^{N_0}) + (K_u - 1)K_u(N - N_0 + 1)}{4(K_u - 1)} = \left[K_u^N + K_u^{N-1} +, \cdots, + K_u^{N_0} + (N - N_0 + 1)K_u\right]/4 \tag{5.27}$$

If the value of K_u reaches the upper limit but the performance of the filter is still not satisfactory, the design has to stop without a satisfactory solution. Our results suggest that such problems are rare but an improved design method has been developed to overcome this problem and will be presented in a later publication.

5.3 ETF design for several specified inputs

Consider the design of an ETF filter which can transfer energy of n_s specific input signals from an input frequency band (a,b) to a desired output frequency band (c,d) and shape the corresponding output frequency responses as required.

Denote the spectra of the n_s specific input signals as $U_\kappa(j\omega), \kappa = 1, 2, \cdots, n_s$. The output spectrum of the general filter model (5.1) under each of these inputs is, when taking $K_{nu} = K_u$, for $n = N_0, \cdots, N$, given by

$$Y^\kappa(j\omega) = G(j\omega) \sum_{n=N_0}^{N} \frac{1/\sqrt{n}}{(2\pi)^{(n-1)}} \sum_{l_1=1}^{K_u} \cdots \sum_{l_n=l_{n-1}}^{K_u} \bar{c}_{0n}(l_1, \cdots, l_n) g^\kappa_{l_1 \cdots l_n}(j\omega) \quad \kappa = 1, 2, \cdots, n_s \tag{5.28}$$

where

$$g^{\kappa}_{l_1\cdots l_n}(j\omega)=\int_{\omega_1+\cdots+\omega_n=\omega}\exp[-j(\omega_1 l_1+\cdots+\omega_n l_n)]\prod_{i=1}^{n}U_\kappa(j\omega_i)d\sigma_{n\omega} \tag{5.29}$$

The design procedure again consists of three steps which are: determination of N_0 and N; design of the nonlinear subsystem; and design of the linear subsystem. The first step is exactly the same as Step 1 described in Section 5.2 since this design also involves transferring energy from an input frequency band (a,b) to an output frequency band (c,d).

Denote the desired output spectrum corresponding to the κth specific input as $Y^{\#\kappa}(j\omega)$. Then, in the second step of the design, the filter parameters are determined based on the equations

$$Y^{\#\kappa}(j\omega(p))=\sum_{n=N_0}^{N}\frac{1/\sqrt{n}}{(2\pi)^{(n-1)}}\sum_{l_1=1}^{K_u}\cdots\sum_{l_n=l_{n-1}}^{K_u}\bar{c}_{0n}(l_1,\cdots,l_n)g^{\kappa}_{l_1\cdots l_n}(j\omega(p)) \qquad p=1,\cdots,M;\ \kappa=1,\cdots,n_s \tag{5.30}$$

The objective is to make the right hand side of the equation approach the desired output spectrum as closely as possible over the M discrete frequency points $\omega(1),\cdots,\omega(M)$ for all the n_s specific inputs considered in the design. This can be achieved using a least squares routine to make the right hand side of the equations

$$\begin{cases}\mathrm{Re}[Y^{\#\kappa}(j\omega(p))]=\sum_{n=N_0}^{N}\frac{1/\sqrt{n}}{(2\pi)^{(n-1)}}\sum_{l_1=1}^{K_u}\cdots\sum_{l_n=l_{n-1}}^{K_u}\bar{c}_{0n}(l_1,\cdots,l_n)\mathrm{Re}[g^{\kappa}_{l_1\cdots l_n}(j\omega(p))] \\ \qquad p=1,2,\cdots,M,\ \kappa=1,2,\cdots,n_s \\ \mathrm{Im}[Y^{\#\kappa}(j\omega(p))]=\sum_{n=N_0}^{N}\frac{1/\sqrt{n}}{(2\pi)^{(n-1)}}\sum_{l_1=1}^{K_{nu}}\cdots\sum_{l_n=l_{n-1}}^{K_{nu}}\bar{c}_{0n}(l_1,\cdots,l_n)\mathrm{Im}[g^{\kappa}_{l_1\cdots l_n}(j\omega(p))] \\ \qquad p=1,2,\cdots,M,\ \kappa=1,2,\cdots,n_s\end{cases} \tag{5.31}$$

approach the left hand side as closely as possible. Denote the results obtained as $\hat{\bar{c}}_{0n}(l_1,\cdots,l_n)$. Then the nonlinear subsystem of the filter to be designed is also of the form given by equation (5.16) and the output spectrum of the subsystem under each of the n_s specific inputs is given by

$$\bar{Y}^{\kappa}(j\omega)=\sum_{n=N_0}^{N}\frac{1/\sqrt{n}}{(2\pi)^{(n-1)}}\sum_{l_1=1}^{K_{nu}}\cdots\sum_{l_n=l_{n-1}}^{K_{nu}}\hat{\bar{c}}_{0n}(l_1,\cdots,l_n)g^{\kappa}_{l_1\cdots l_n}(j\omega) \qquad \kappa=1,\cdots,n_s \tag{5.32}$$

In the third step, a linear filter with frequency response function $G_1(j\omega)$ is determined to improve the approximation effect of the n_s output spectra $\bar{Y}^{\kappa}(j\omega),\kappa=1,\cdots,n_s$ of the nonlinear subsystem on the corresponding desired results $Y^{\#\kappa}(j\omega),\kappa=1,\cdots,n_s$ over the frequency set $\{\omega(1),\cdots,\omega(M)\}$. This is achieved by designing $G_1(j\omega)$ subject to a constraint on stability to minimise the criterion

$$J(G_1)=\sum_{\kappa=1}^{n_s}\sum_{p=1}^{M}\{Y^{\#\kappa}[j\omega(p)]-G_1[j\omega(p)]\bar{Y}^{\kappa}[j\omega(p)]\}^{*}\{Y^{\#\kappa}[j\omega(p)]-G_1[j\omega(p)]\bar{Y}^{\kappa}[j\omega(p)]\} \tag{5.33}$$

The resulting frequency responses

$$G_1(j\omega)\overline{Y}^{\kappa}(j\omega), \kappa = 1,\cdots,n_s$$

should produce a better approximation to the corresponding desired output spectra

$$Y^{\#\kappa}(j\omega), \kappa = 1,\cdots,n_s$$

over the specified output frequency band (c,d). Next $G_2(j\omega)$ is designed, as described in Section 5.2, to remove any extraneous frequency components in

$$G_1(j\omega)\overline{Y}^{\kappa}(j\omega), \kappa = 1,\cdots,n_s$$

which are beyond the output frequency band (c,d) so that

$$G_1(j\omega)G_2(j\omega)\overline{Y}^{\kappa}(j\omega), \kappa = 1,\cdots,n_s$$

may approach $Y^{\#\kappa}(j\omega), \kappa = 1,\cdots,n_s$, as required by the design. The linear subsystem part of the filter is then given by $G(j\omega) = G_1(j\omega)G_2(j\omega)$.

Rewrite the criterion (5.33) as

$$J(G_1) = \sum_{p=1}^{M}\sum_{\kappa=1}^{n_s}\left|G_1[j\omega(p)] - \frac{Y^{\#\kappa}[j\omega(p)]}{\overline{Y}^{\kappa}[j\omega(p)]}\right|^2 \left|\overline{Y}^{\kappa}[j\omega(p)]\right|^2 \tag{5.34}$$

Then, with $G_1(j\omega)$ defined by (5.24), the optimisation problem consists of minimising $J(G_1)$ using for example "invfreqz.m" in MATLAB to yield a solution for $\overline{b}_1,\overline{b}_2,\cdots,\overline{b}_{n_{b1}+1}$ and $\overline{a}_1,\overline{a}_2,\cdots,\overline{a}_{n_{a1}+1}$ for given values of n_{b1} and n_{a1}.

The design for the band pass filter $G_2(j\omega)$ follows the description in Section 5.2. This produces $\overline{\overline{b}}_1,\overline{\overline{b}}_2,\cdots,\overline{\overline{b}}_{n_{b2}+1}$ and $\overline{\overline{a}}_1,\overline{\overline{a}}_2,\cdots,\overline{\overline{a}}_{n_{a2}+1}$ for choices of n_{b2} and n_{a2}, when the specification of the design is only for the magnitudes of the output frequency responses. When both the magnitude and phase of the output frequency responses are specified, a linear phase FIR filter is required and consequently $\overline{\overline{b}}_1,\overline{\overline{b}}_2,\cdots,\overline{\overline{b}}_{n_{b2}+1}$ for a choice of n_{b2} are determined.

The iterative computation of the structure parameter K_u is also needed in this case. But the limit for K_u is determined by the inequality

$$M \geq \left[K_u^N + K_u^{N-1} +,\cdots,+ K_u^{N_0} + (N - N_0 + 1)K_u\right]/4n_s \tag{5.35}$$

rather than (5.27) because the number of equations in (5.31) is $2 \times M \times n_s$. Satisfactory designs will normally be achieved before K_u reaches its upper limits, but as noted in Section 5.2 more complex designs which overcome these limitations are also available for the multiple input case and will be presented in a later study.

The design of the ETF filters to transfer energy from an input signal having a spectrum over a frequency band (a,b) to an output signal having a spectrum over another frequency band (c,d) has been investigated above both for one specific and several specific inputs. The ETF design procedure is a natural extension of the well-known linear filter designs to the nonlinear case. In the designs presented above the filters are composed of two subsystems, a linear and a nonlinear subsystem. The nonlinear subsystem mainly moves the input signal energy from the frequency band (a,b) to the new frequency band (c,d) and shapes the output frequency response over this frequency band. The linear subsystem has two functions. $G_1(j\omega)$ improves the approximation effect of the output frequency response of the nonlinear subsystem on the desired spectrum over the frequency band (c,d), and $G_2(j\omega)$ is a band pass filter which removes any residual frequency components which are beyond the desired output frequency band. If (c,d)=(a,b), then the design reduces to a simple linear case where $N = N_0 = 1$ in the nonlinear subsystem design. Otherwise nonlinearity with an appropriate degree has to be introduced to ensure that the required frequency domain energy transfer is feasible.

Although only energy transfer from one single input frequency band (a,b) to another single output frequency band (c,d) is considered in the present study, the same design principle can be readily extended to more complicated cases. For example, energy can be transferred from two input signals with frequency components over the frequency bands (a1,b1) and (a2,b2) respectively to two corresponding but different output frequency locations (c1,d1) and (c2,d2), and shape the output frequency responses as required.

Notice that apart from stability, no other constraints are necessary in the above designs. This implies that the structure and parameters of the filters thus designed are allowed to be arbitrarily determined to achieve design specifications provided that the filter stability is guaranteed. The filters could be implemented using DSP chips or dedicated processors. The implementation is analogous to the classical linear digital filter case except that in the energy transfer filter case nonlinearies are deliberately introduced.

6 Design Examples

Three examples will be used to illustrate the design of energy transfer filters. The design in Example 1 involves a design where the objective is to move the signal energy from a lower input frequency band to a higher output frequency location. The designs in Examples 2 and 3 consider the situation for two specific inputs. A frequency domain energy transfer filter is designed to move the energy of two signals from a higher input frequency band to a lower output frequency location in Example 2, and from a lower input frequency band to a higher output frequency location in Example 3.

All the examples have been designed to represent a practical design problem. A continuous time signal is to be processed. The signal is sampled by an A/D converter and is passed to a digital signal processor that is coded to implement the ETF design. The output of the digital signal processor is then transformed by a D/A converter back to a continuous time signal, and the frequency spectrum of this signal should approach the desired spectrum specified by the design.

Example 1

Consider a continuous time signal $u(t)$ which is generated from a white noise uniformly distributed over (0,4) and band-limited within the frequency range (5.6,7.6) rads/sec. The sampling interval was set as $T_s = 0.01s$. Fig.2 shows the signal in the time domain which has been padded out with zero's at the end so that the FFT can be applied. Fig.3 shows the magnitude of the spectrum of $u(t)$ which is obtained by evaluating the discrete Fourier Transform of $u(t)$ from the sampled values. From Fig.3 it can be observed that the frequency range of $u(t)$ is approximately $(\bar{a},\bar{b}) = (5.364, 8.582)$ rads/sec.

The objective is to design a frequency domain energy transfer filter to transfer the energy of $u(t)$ to the higher frequency band $(\bar{c},\bar{d}) = (11.6, 13.6)$ rads/sec and shape the magnitude of the filter output frequency response as specified by the desired spectrum

$$Y^d(j\omega_c) = \begin{cases} \dfrac{\exp(-500\omega_c) + j(600\omega_c^2)}{100000} & \omega_c \in (11.6, 13.6) \\ 0 & \text{otherwise} \end{cases} \qquad (6.1)$$

where ω_c denotes the continuous frequency in radians.

The ETF design is performed using the procedure described in Section 5.2 where

$$(a,b) = T_s(\bar{a},\bar{b}) = (0.05364, 0.08582) \qquad (6.2)$$

$$(c,d) = T_s(\bar{c},\bar{d}) = (0.116, 0.136) \qquad (6.3)$$

$$Y^{\#}(j\omega) = \frac{1}{T_s}Y^d\left(j\frac{\omega}{T_s}\right) = \begin{cases} \dfrac{\exp(-500\,\omega/T_s) + j(600\,\omega^2/T_s^2)}{100000} & \omega \in T_s(11.6, 13.6) \\ 0 & \text{otherwise} \end{cases}$$

(6.4)

In Step 1, N and N_0 are determined to be $N = N_0 = 2$ because, in this case, $\bar{n} = \bar{\bar{n}} = 2$.

In Step 2, M is taken as

$$M = i_d - i_c + 1 \qquad (6.5)$$

where

$$i_c = \lfloor c\overline{M}/2\pi \rfloor \qquad (6.6)$$

$$i_d = \lfloor d\overline{M}/2\pi \rfloor \qquad (6.7)$$

and $\overline{M} = 4100$ is the length of data used to evaluate the input spectrum $U(j\omega)$ for the design. $\omega(1),\cdots,\omega(M)$ are taken as

$$\omega(p) = 2(p + i_c - 1)\pi/\overline{M} \qquad p = 1,2,\cdots,M \qquad (6.8)$$

The result obtained after six iterative selections of K_u for $K_u = 1,\cdots,6$ is the nonlinear subsystem

$\bar{y}$(k) =+(-3.767e+02) u(k-1)u(k-1)+(-4.902e+02) u(k-1)u(k-2)+(7.779e+03) u(k-1)u(k-3)
+(-1.418e+04) u(k-1)u(k-4)+(1.015e+04) u(k-1)u(k-5)+(-2.51e+03) u(k-1)u(k-6)
+(2.423e+03) u(k-2)u(k-2) +(-1.677e+04) u(k-2)u(k-3) +(3.019e+04) u(k-2)u(k-4)
+(-2.407e+04) u(k-2)u(k-5) +(6.316e+03) u(k-2)u(k-6) +(5.65e+03) u(k-3)u(k-3)
+(-1.43e+04) u(k-3)u(k-4) +(1.709e+04) u(k-3)u(k-5) +(-5.143e+03) u(k-3)u(k-6)
+(9.691e+02) u(k-4)u(k-4)+(-5.149e+03) u(k-4)u(k-5) +(1.526e+03) u(k-4)u(k-6)
+(8.717e+02) u(k-5)u(k-5) +(2.298e+02) u(k-5)u(k-6) +(-2.087e+02) u(k-6)u(k-6)

In Step 3, the structure of the first linear filter $G_1(j\omega)$ was chosen to be

$$n_{b1} = n_{a1} = 2$$

and the parameters were determined as

$$[\bar{b}_1,\cdots,\bar{b}_{n_{b1}+1}] = [\bar{b}_1,\bar{b}_2,\bar{b}_3]$$
$$= [1.00126596767599, -1.98644370512015, 1.00121872524601]$$

$$[\bar{a}_1,\cdots,\bar{a}_{n_{a1}+1}] = [\bar{a}_1,\bar{a}_2,\bar{a}_3]$$
$$= [1.00000000000000, -1.98394066601323, 0.99996154063006]$$

The structure of the second linear filter $G_2(j\omega)$ was configured as

$$G_2(j\omega) = G_2(j\omega)^{1/2} G_2(j\omega)^{1/2} \tag{6.9}$$

and $G_2(j\omega)^{1/2}$ is designed as the required band pass filter. This is to enhance the performance of the band pass filter $G_2(j\omega)$. The structure of $G_2(j\omega)^{1/2}$ was chosen to be

$$n'_{b2} = n'_{a2} = 8$$

and the parameters were determined as

$$[\bar{\bar{b}}'_1,\cdots,\bar{\bar{b}}'_{n'_{b2}+1}] = [\bar{\bar{b}}'_1,\cdots,\bar{\bar{b}}'_9]$$
$$= [0.0974336613923,\ 0.00000000888178,\ -0.38973468008408,\ 0.00000007105427$$
$$0.5846020202611,\ -0.00000007105427,\ -0.38973460902980,\ 0$$
$$0.0974336635968]$$

and

$$[\bar{\bar{a}}'_1,\cdots,\bar{\bar{a}}'_{n'_{a2}+1}] = [\bar{\bar{a}}'_1,\cdots,\bar{\bar{a}}'_9]$$
$$= [1.00000000000000,\ -7.88512643636851,\ 27.26377679465395,\ -53.98949989228983$$
$$66.97190890674418, -53.28866123207629,\ 26.56054905178533,\ -7.58202584692060$$
$$0.949078714507871]$$

Consequently the structure of $G_2(j\omega)$ is

$$n_{b2} = n_{a2} = 16$$

and the parameters can be determined as

$$[\bar{\bar{b}}_1,\cdots,\bar{\bar{b}}_{n_{b2}+1}] = [\bar{\bar{b}}_1,\cdots,\bar{\bar{b}}_{17}] = Conv\{[\bar{\bar{b}}'_1,\cdots,\bar{\bar{b}}'_9],[\bar{\bar{b}}'_1,\cdots,\bar{\bar{b}}'_9]\}$$

and

$$[\bar{\bar{a}}_1,\cdots,\bar{\bar{a}}_{n_{a2}+1}] = [\bar{\bar{a}}_1,\cdots,\bar{\bar{a}}_{17}] = Conv\{[\bar{\bar{a}}'_1,\cdots,\bar{\bar{a}}'_9],[\bar{\bar{a}}'_1,\cdots,\bar{\bar{a}}'_9]\}$$

where *Conv* (x,y) denotes the convolution of vectors x and y.

Figs 4 and 5 show the output response of the filter in the time and frequency domain respectively. The performance of this design can be assessed from Fig.5 where a comparison between the real output spectrum of the filter and the desired result can be observed. Clearly, a very good result has been achieved by the design and the energy of the specified input has been moved to the frequency band (c,d)=(11.6,13.6) and the shape of the magnitude matches the desired spectrum defined by equation (6.1).

Example 2

In this example the design of an energy transfer filter is illustrated where the requirement is to move energy from two specific signals from a higher input frequency band to a lower output frequency location. Figs 6 and 7 show the two signals $u_1(t)$ and $u_2(t)$ in the time and frequency domain, respectively. The two signals were generated from a white uniformly distributed noise sequence over (0,4) but banded-limited over the frequency range (11.6,13.6) rads/sec. $T_s = 0.01s$. The frequency range of $u_1(t)$ and $u_2(t)$, are from Fig.7, approximately $(\bar{a},\bar{b}) = (10.88, 14.87)$ rads/sec.

The objective of this design is to transfer energy of $u_1(t)$ and $u_2(t)$ to a lower frequency band $(\bar{c},\bar{d}) = (5.6, 7.6)$ rads/sec and to shape the magnitudes of the corresponding output frequency responses as specified by the desired spectrum

$$Y^d(j\omega_c) = \begin{cases} \dfrac{\exp(-500\omega_c) + j(600\omega_c^2)}{100000} & \omega_c \in (5.6, 7.6) \\ 0 & \text{otherwise} \end{cases} \tag{6.10}$$

The design is performed using the procedure described in Section 5.3 where

$$(a,b) = T_s(\bar{a},\bar{b}) = (0.1088, 0.1487) \tag{6.11}$$

$$(c,d) = T_s(\bar{c},\bar{d}) = (0.056, 0.076) \tag{6.12}$$

$$Y^{\#}(j\omega) = \frac{1}{T_s} Y^d\left(j\frac{\omega}{T_s}\right) = \begin{cases} \dfrac{\exp(-500\,\omega/T_s) + j(600\,\omega^2/T_s^2)}{100000} & \omega \in T_s(5.6, 7.6) \\ 0 & \text{otherwise} \end{cases} \tag{6.13}$$

In Step 1, N and N_0 are determined to be $N = 5$ and $N_0 = 3$.

In Step 2, $\bar{M} = 4100$ and the formulas used to determine i_c, i_d, M, and $\omega(p), p = 1,\ldots,M$ are the same as in Example 1, that is (6.6), (6.7), (6.5), and (6.8) but with $c = 0.056$ and $d = 0.076$ in this case.

The nonlinear subsystem is obtained after three iterative selections of K_u for $K_u = 1,2,3$ and is given by

$\bar{y}(k)$ = +(-7.321e+01) u(k-1)u(k-1)u(k-1) +(4.974e+02) u(k-1)u(k-1)u(k-2)
+(-3.113e+02) u(k-1)u(k-1)u(k-3) +(-1.136e+03) u(k-1)u(k-2)u(k-2)
+(1.415e+03) u(k-1)u(k-2)u(k-3) +(-4.321e+02) u(k-1)u(k-3)u(k-3)
+(8.694e+02) u(k-2)u(k-2)u(k-2) +(-1.612e+03) u(k-2)u(k-2)u(k-3)
+(9.765e+02) u(k-2)u(k-3)u(k-3) +(-1.936e+02) u(k-3)u(k-3)u(k-3)
+(1.539e+04) u(k-1)u(k-1)u(k-1)u(k-1) +(-8.271e+04) u(k-1)u(k-1)u(k-1)u(k-2)
+(3.644e+04) u(k-1)u(k-1)u(k-1)u(k-3) +(1.404e+05) u(k-1)u(k-1)u(k-2)u(k-2)
+(-1.209e+05) u(k-1)u(k-1)u(k-2)u(k-3) +(2.75e+04) u(k-1)u(k-1)u(k-3)u(k-3)
+(-6.036e+04) u(k-1)u(k-2)u(k-2)u(k-2) +(7.066e+04) u(k-1)u(k-2)u(k-2)u(k-3)

+(-3.409e+04) u(k-1)u(k-2)u(k-3)u(k-3) +(7.027e+03) u(k-1)u(k-3)u(k-3)u(k-3)
+(-2.535e+04) u(k-2)u(k-2)u(k-2)u(k-2) +(5.143e+04) u(k-2)u(k-2)u(k-2)u(k-3)
+(-3.008e+04) u(k-2)u(k-2)u(k-3)u(k-3) +(3.905e+03) u(k-2)u(k-3)u(k-3)u(k-3)
+(7.333e+02) u(k-3)u(k-3)u(k-3)u(k-3)
+(3.766e+08) u(k-1)u(k-1)u(k-1)u(k-1)u(k-1)
+(-3.003e+09) u(k-1)u(k-1)u(k-1)u(k-1)u(k-2)
+(1.52e+09) u(k-1)u(k-1)u(k-1)u(k-1)u(k-3)
+(8.977e+09) u(k-1)u(k-1)u(k-1)u(k-2)u(k-2)
+(-9.083e+09) u(k-1)u(k-1)u(k-1)u(k-2)u(k-3)
+(2.297e+09) u(k-1)u(k-1)u(k-1)u(k-3)u(k-3)
+(-1.192e+10) u(k-1)u(k-1)u(k-2)u(k-2)u(k-2)
+(1.809e+10) u(k-1)u(k-1)u(k-2)u(k-2)u(k-3)
+(-9.146e+09) u(k-1)u(k-1)u(k-2)u(k-3)u(k-3)
+(1.542e+09) u(k-1)u(k-1)u(k-3)u(k-3)u(k-3)
+(5.927e+09) u(k-1)u(k-2)u(k-2)u(k-2)u(k-2)
+(-1.198e+10) u(k-1)u(k-2)u(k-2)u(k-2)u(k-3)
+(9.084e+09) u(k-1)u(k-2)u(k-2)u(k-3)u(k-3)
+(-3.062e+09) u(k-1)u(k-2)u(k-3)u(k-3)u(k-3)
+(3.874e+08) u(k-1)u(k-3)u(k-3)u(k-3)u(k-3)
+(1.004e+07) u(k-2)u(k-2)u(k-2)u(k-2)u(k-2)
+(-3.066e+07) u(k-2)u(k-2)u(k-2)u(k-2)u(k-3)
+(3.366e+07) u(k-2)u(k-2)u(k-2)u(k-3)u(k-3)
+(-1.571e+07) u(k-2)u(k-2)u(k-3)u(k-3)u(k-3)
+(2.601e+06) u(k-2)u(k-3)u(k-3)u(k-3)u(k-3)
+(2.28e+04) u(k-3)u(k-3)u(k-3)u(k-3)u(k-3)

Note that a large number of terms in the filter is usual for linear FIR designs and is evident here since only nonlinear input terms are allowed in these simple ETF designs.

In Step 3, the structures of the first and second linear filters $G_1(j\omega)$ and $G_2(j\omega)$ are the same as the structures used in Example 1. The parameters for $G_1(j\omega)$ were determined as

$$[\bar{b}_1, \bar{b}_2, \bar{b}_3] = [1.00292849060320 \;\; -2.00126160798959 \;\; 1.00260541942764]$$

$$[\bar{a}_1, \bar{a}_2, \bar{a}_3] = [1.00000000000000 \;\; -1.99549358009987 \;\; 0.99975285293475]$$

and the parameters for $G_2(j\omega)^{1/2}$ are

$$[\bar{\bar{b}}_1', \cdots, \bar{\bar{b}}_9']$$
$$= [0.09743366113923, \; 0, \; -0.38973450244839, \; -0.00000035527137, \; 0.58460230434321, \; -0.00000042632564, \; -0.38973432481271, \; -0.00000010658141, \; 0.09743367668236]$$

$[\bar{\bar{a}}_1', \cdots, \bar{\bar{a}}_9']$
$= [1.00000000000000,\ -7.93083055178438,\ 27.53485102349816,\ -54.66075365766915$
$67.86019765416364,\ -53.95120135423146,\ 26.82463128925107,\ -7.62597311741651$
$0.94907871450787]$

Figs 8 and 9 show the output responses of the filter designed for the two specific inputs in the time and the frequency domain respectively. A comparison between the real output spectra of the filter and the desired results can be observed in Fig.9 which clearly shows that the design satisfies the initial specifications.

Example 3

In this example the design of an energy transfer filter is used to illustrate how energy from two specific signals can be moved from a lower input frequency band to a higher output frequency location. The two signals, $u_1(t)$ and $u_2(t)$, are shown in Figs 10 and 11 in the time and the frequency domain respectively. The signals were produced in exactly the same way as in Example 1 and $T_s = 0.01s$. Fig.11 indicates that the frequency range of $u_1(t)$ and $u_2(t)$ is approximately $(\bar{a}, \bar{b}) = (5.057, 8.275)$ rads/sec.

The design objective was to transfer the energy from $u_1(t)$ and $u_2(t)$ to a higher frequency band $(\bar{c}, \bar{d}) = (11.6, 13.6)$ rads/sec and to shape the magnitudes of the corresponding output frequency responses as specified by the desired spectrum $Y^d(j\omega_c)$ defined by (6.1).

The design was performed in the same way as the design in Example 2 except that in this case

$$(a,b) = T_s(\bar{a}, \bar{b}) = (0.05057, 0.08275) \text{ rads/sec} \tag{6.14}$$

and (c,d) and $Y^{\#}(j\omega)$ are given by (6.3) and (6.4) rather than (6.12) and (6.13).

In Step 1, N and N_0 were determined to be $N = N_0 = 2$.

In Step 2, $\overline{M} = 4100$, i_c, i_d, M, and $\omega(p), p = 1, \ldots, M$ were determined using equations (6.6), (6.7), (6.5), and (6.8) respectively.

The nonlinear subsystem was determined after seven iterative selections of $K_u = 1, \ldots, 7$.

$\bar{y}$(k) =+(2.941e+03) u(k-1)u(k-1) +(-9.614e+03) u(k-1)u(k-2)
+(9.457e+03) u(k-1)u(k-3) +(-8.39e+03) u(k-1)u(k-4)
+(1.195e+04) u(k-1)u(k-5) +(-9.406e+03) u(k-1)u(k-6)
+(2.886e+03) u(k-1)u(k-7) +(-7.934e+03) u(k-2)u(k-2)
+(2.865e+04) u(k-2)u(k-3) +(-1.132e+04) u(k-2)u(k-4)
+(-2.51e+04) u(k-2)u(k-5)+(2.567e+04) u(k-2)u(k-6)
+(-8.92e+03) u(k-2)u(k-7) +(5.627e+03) u(k-3)u(k-3)
+(-4.667e+04) u(k-3)u(k-4)+(5.661e+04) u(k-3)u(k-5)
+(-2.899e+04) u(k-3)u(k-6) +(1.075e+04) u(k-3)u(k-7)
+(8.101e+03) u(k-4)u(k-4)+(8.42e+03) u(k-4)u(k-5)
+(-7.388e+03) u(k-4)u(k-6) +(-5.504e+03) u(k-4)u(k-7)

+(-8.096e+03) u(k-5)u(k-5) +(-9.905e+02) u(k-5)u(k-6)

+(6.303e+03) u(k-5)u(k-7)+(3.513e+03) u(k-6)u(k-6)

+(-2.369e+03) u(k-6)u(k-7) +(-1.969e+02) u(k-7)u(k-7)

In Step 3, $G_1(j\omega)$ and $G_2(j\omega)$ were chosen to be of the same structure as in Examples 1 and 2. The parameters for $G_1(j\omega)$ and $G_2(j\omega)^{1/2}$ were determined as

$$[\bar{b}_1, \bar{b}_2, \bar{b}_3] = [1.00570641091412 \;\; -1.99325418031704 \;\; 1.00264296176239]$$

$$[\bar{a}_1, \bar{a}_2, \bar{a}_3] = [1.00000000000000 \;\; -1.98194955548426 \;\; 0.99697773546151]$$

and

$$[\bar{\bar{b}}_1', \cdots, \bar{\bar{b}}_9']$$
$$= [0.09743366113923, \; 0.00000000888178, \; -0.38973468008408, \; 0.00000007105427 \; 0.58460202012611, \; -0.00000007105427, \; -0.38973460902980, \; 0 \; 0.09743366335968 \;]$$

$$[\bar{\bar{a}}_1', \cdots, \bar{\bar{a}}_9']$$
$$= [1.0000000000000, \; -7.88512643636851, \; 27.26377679465395, \; -53.98949989228983 \; 66.97190890674418, \; -53.28866123207629, \; 26.56054905178533, \; -7.58202584692060 \; 0.94907871450787 \;]$$

The output responses of the filter for the specified inputs are shown in Figs.12 and 13 in the time and the frequency domain respectively. Fig.13 clearly indicates that an excellent result has again been achieved by the ETF design.

In all the examples the scaling of $\bar{y}(k)$ has been reduced and the gain of $G_1(j\omega)G_2(j\omega)$ has been increased by the same factor. This is simply to avoid either large or small coefficients. But the overall effect cancels when the combined nonlinear filter followed by $G_1(j\omega)G_2(j\omega)$ is implemented and as such this scaling is only used as a convenience in the computation.

The three design examples demonstrate some potential designs that can be achieved using energy transfer filters. Although only two specific inputs were considered in Examples 2 and 3, there is no limitation on the number n_s of inputs which can be dealt with by the designs. In principle, the design procedure in Section 5.3 can, as demonstrated by these examples, work equally well when more than two inputs have to be considered by the design.

7 Conclusions

The application of classical attenuating filter designs is pervasive in most branches of science and engineering. Energy transfer filters introduce a completely new family of filters that provide additional degrees of freedom and significant new design possibilities. In electronic circuits or filters energy can be moved to frequency bands where the system has minimal response. In mechanical systems the new filters can be employed to transfer energy of vibration to other frequencies, or the energy can be focused to a band where we have tuned attenuation devices that can suppress the transmission of vibrations. Control systems can be designed based on these ideas, and there are many other application domains where energy transfer filters would be a useful addition to current methods.

The energy transfer filters described above are just one class of filters that can be designed based on the general principles which have been introduced in this paper. Many other novel designs are possible, including designs that work for all inputs within a frequency bound, and some of these will be described in future publications.

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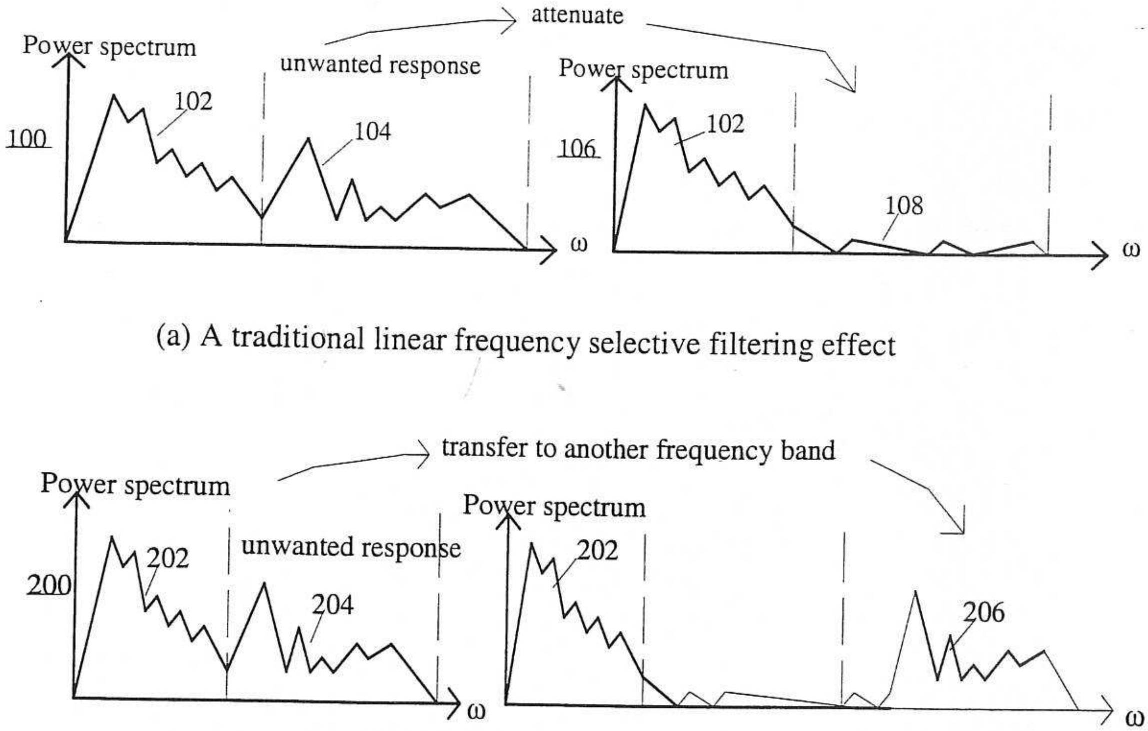

(a) A traditional linear frequency selective filtering effect

(b) An alternative solution using the ETF design

Figure 1 A comparison of traditional frequency selective filtering and the energy transfer filter concept

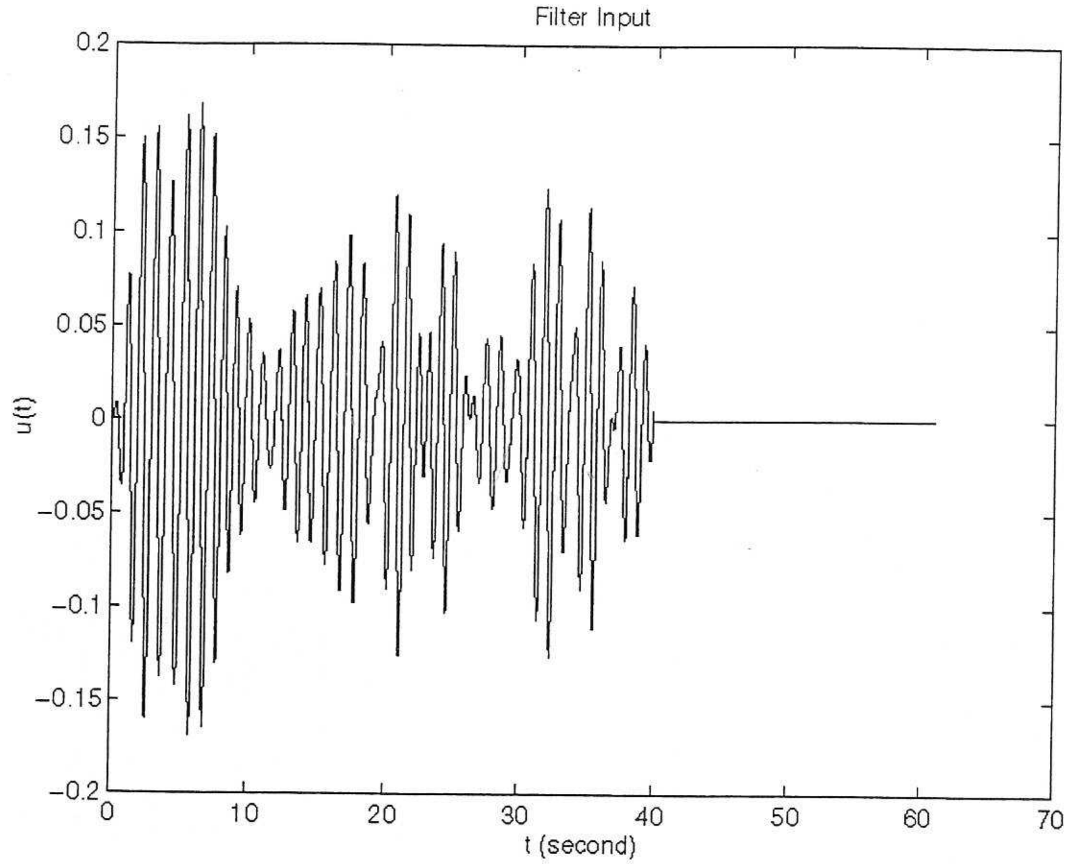

Figure 2 The input signal to be processed in Example 1

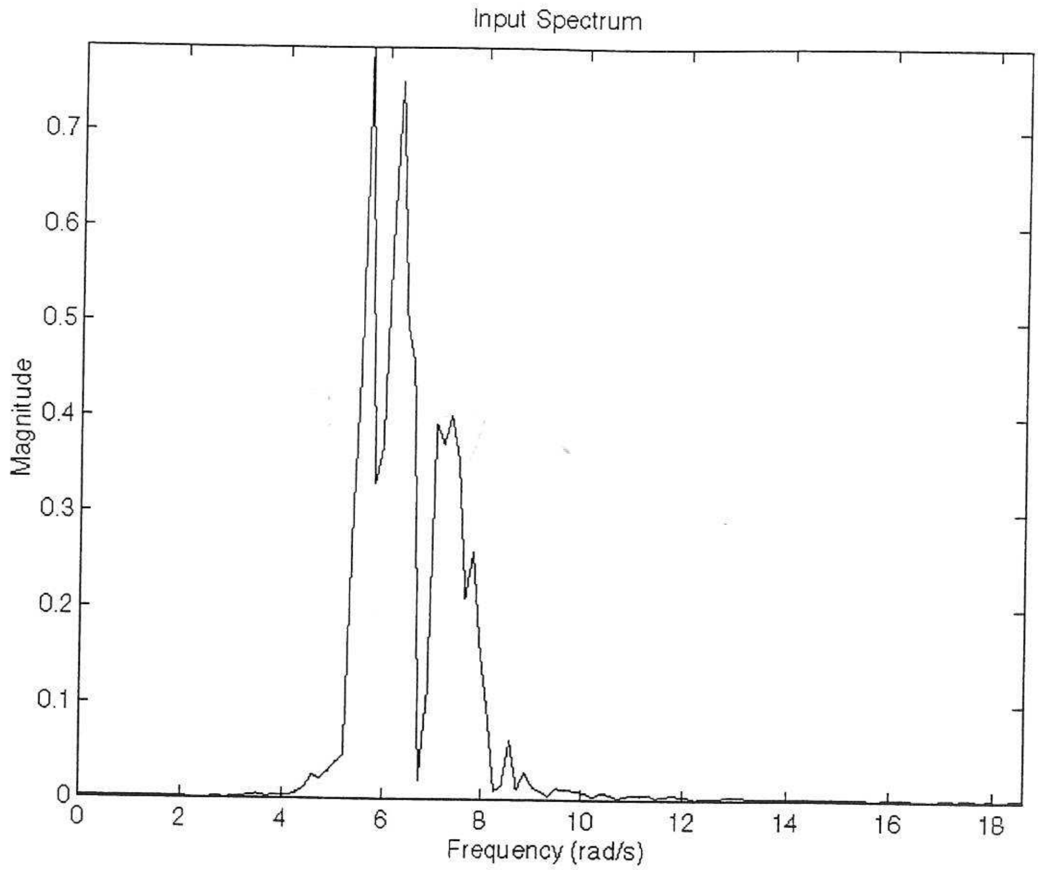

Figure 3 The spectrum of the input signal in Example 1

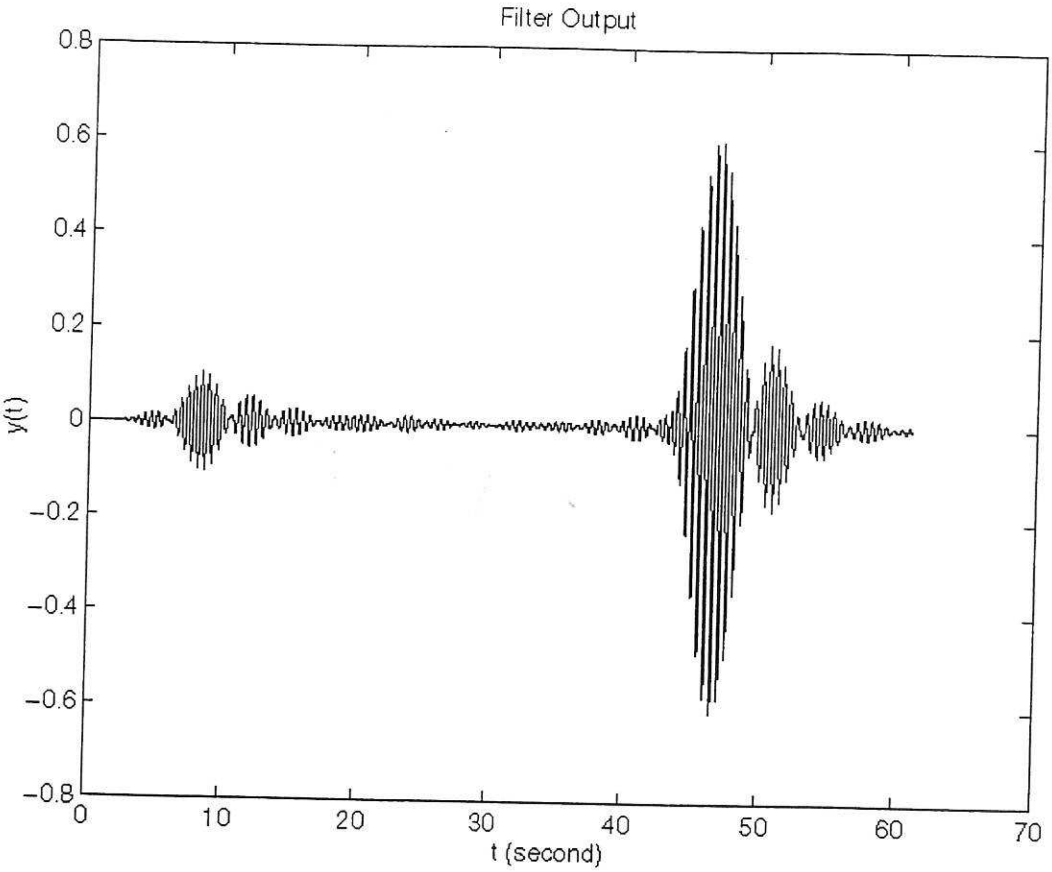

Figure 4 The time domain output of the filter designed in Example 1

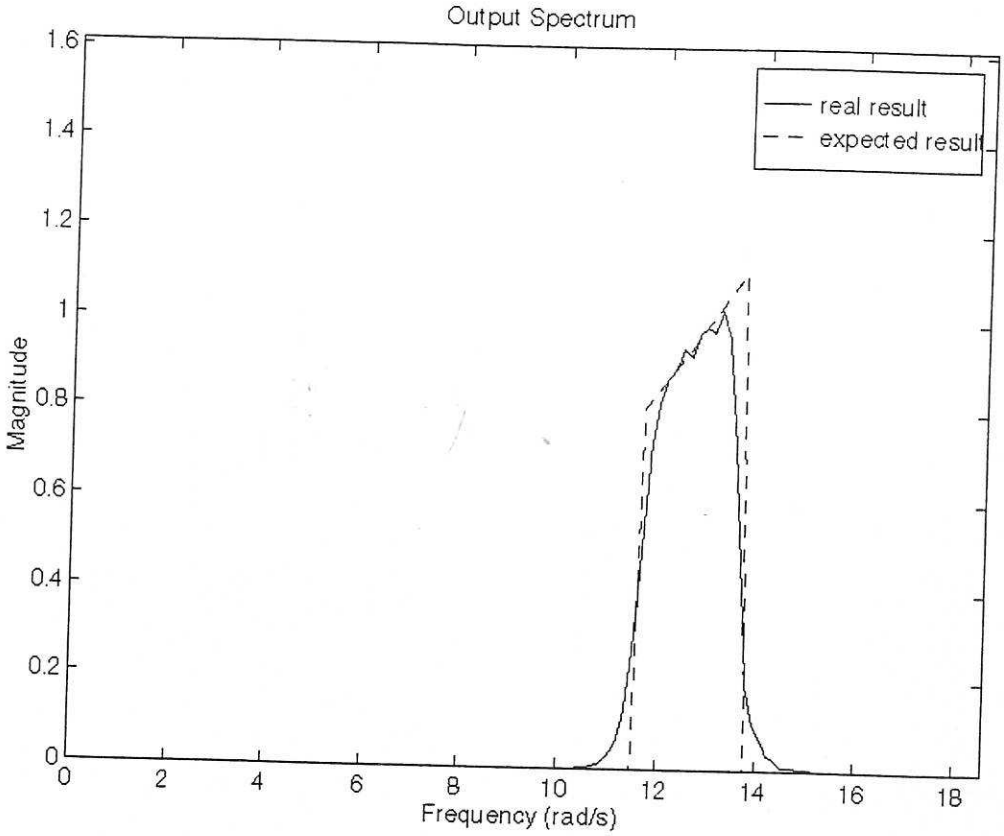

Figure 5 A comparison between the output spectrum of the filter designed in Example 1 and the desired spectrum specified for the design

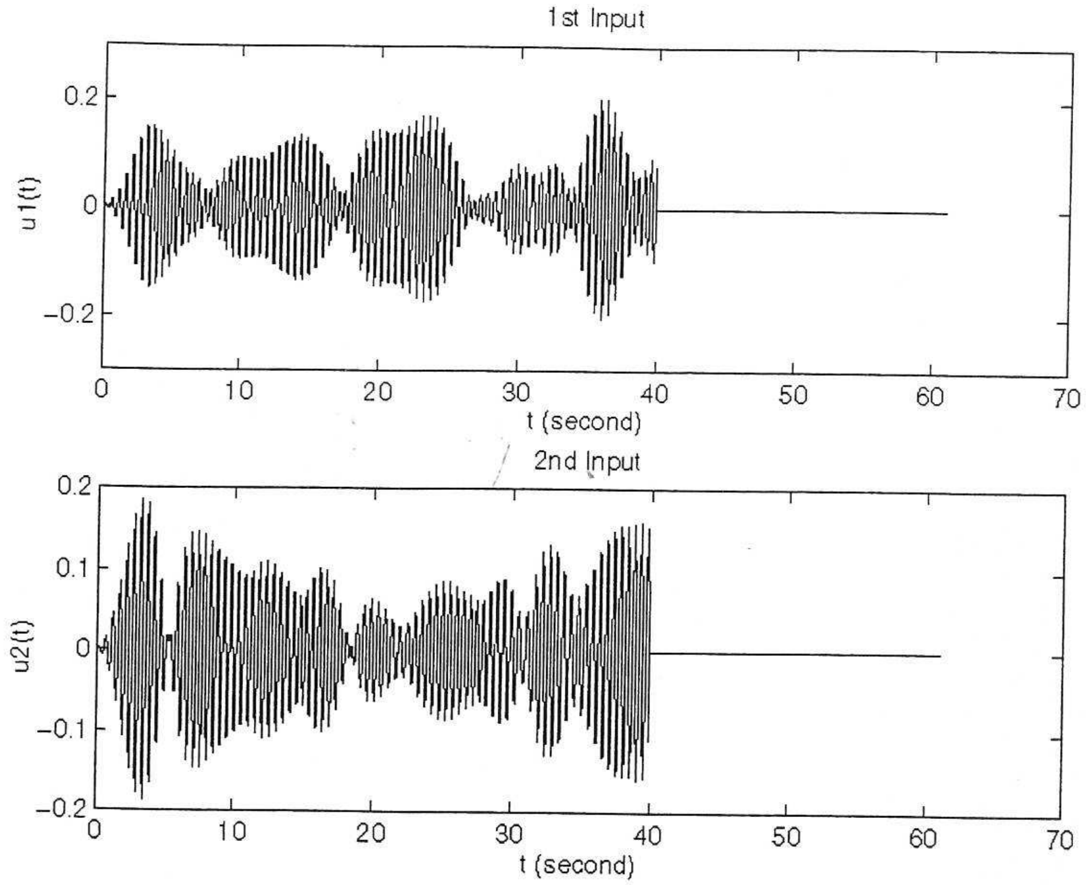

Figure 6 The two input signals to be processed in Example 2

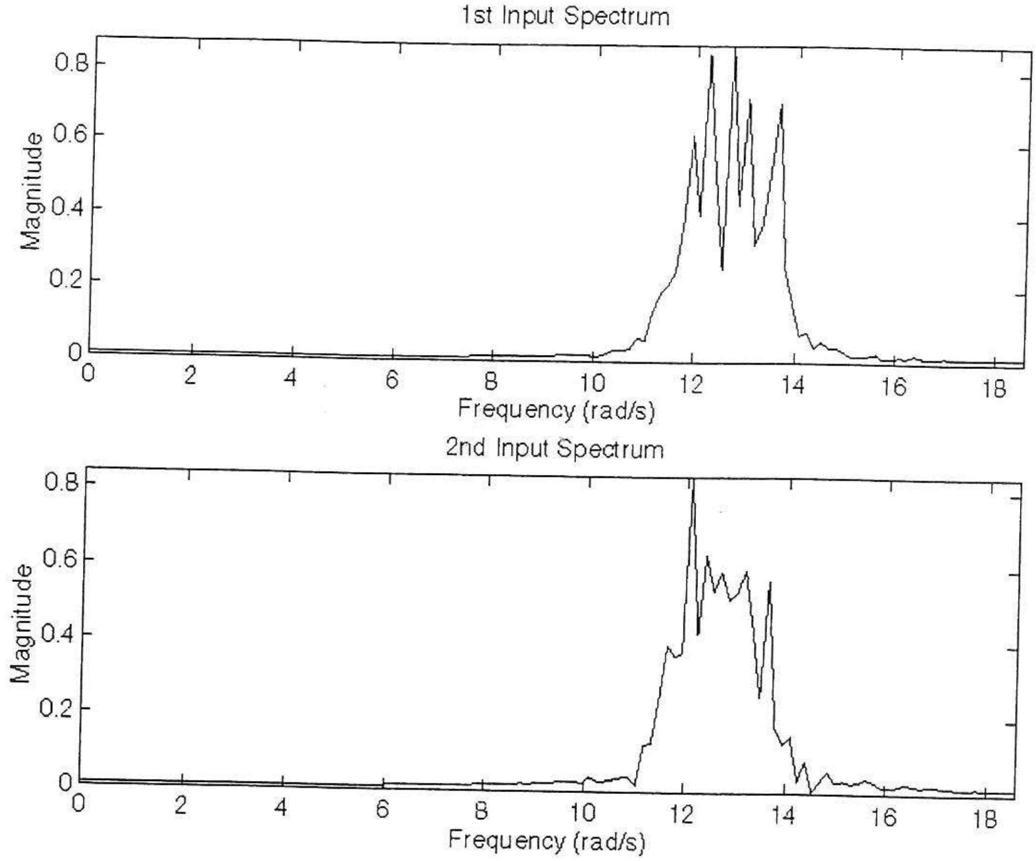

Figure 7 The spectra of the two input signals in Example 2

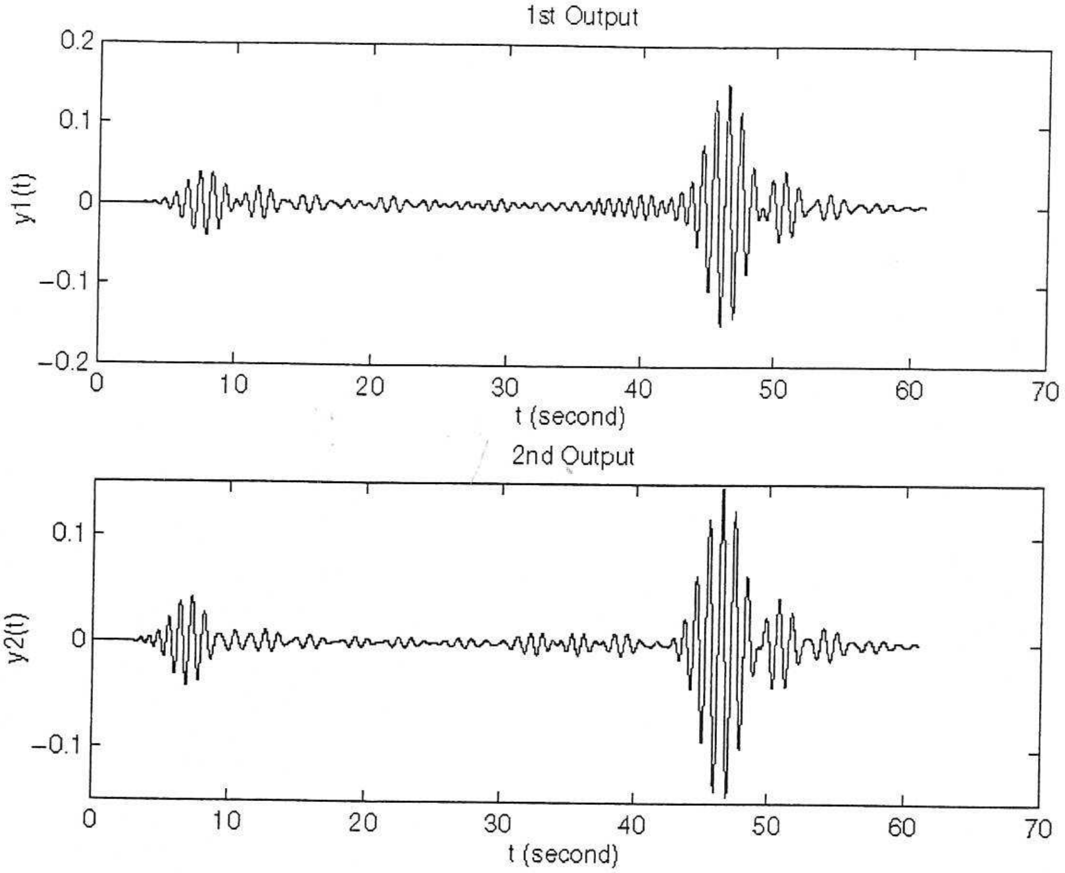

Figure 8 The time domain outputs of the filter designed in Example 2

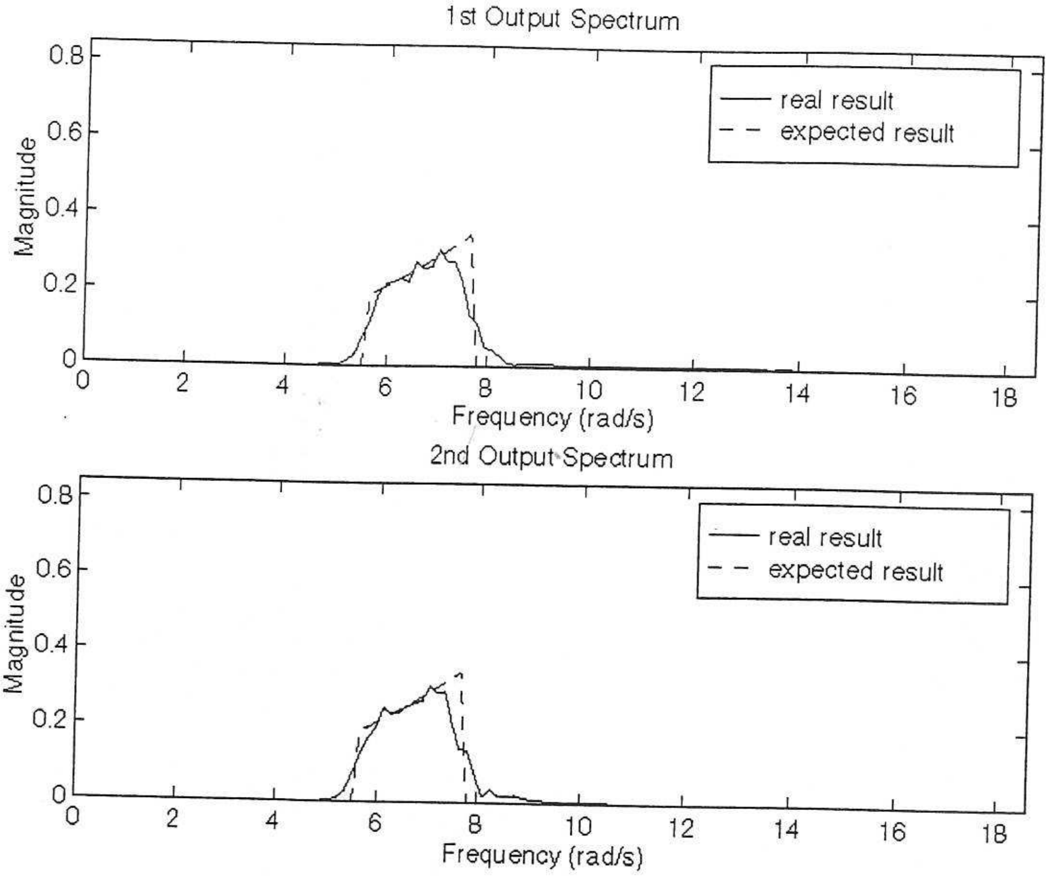

Figure 9 A comparison between the output spectra of the filter designed in Example 2 and the desired spectra specified for the design

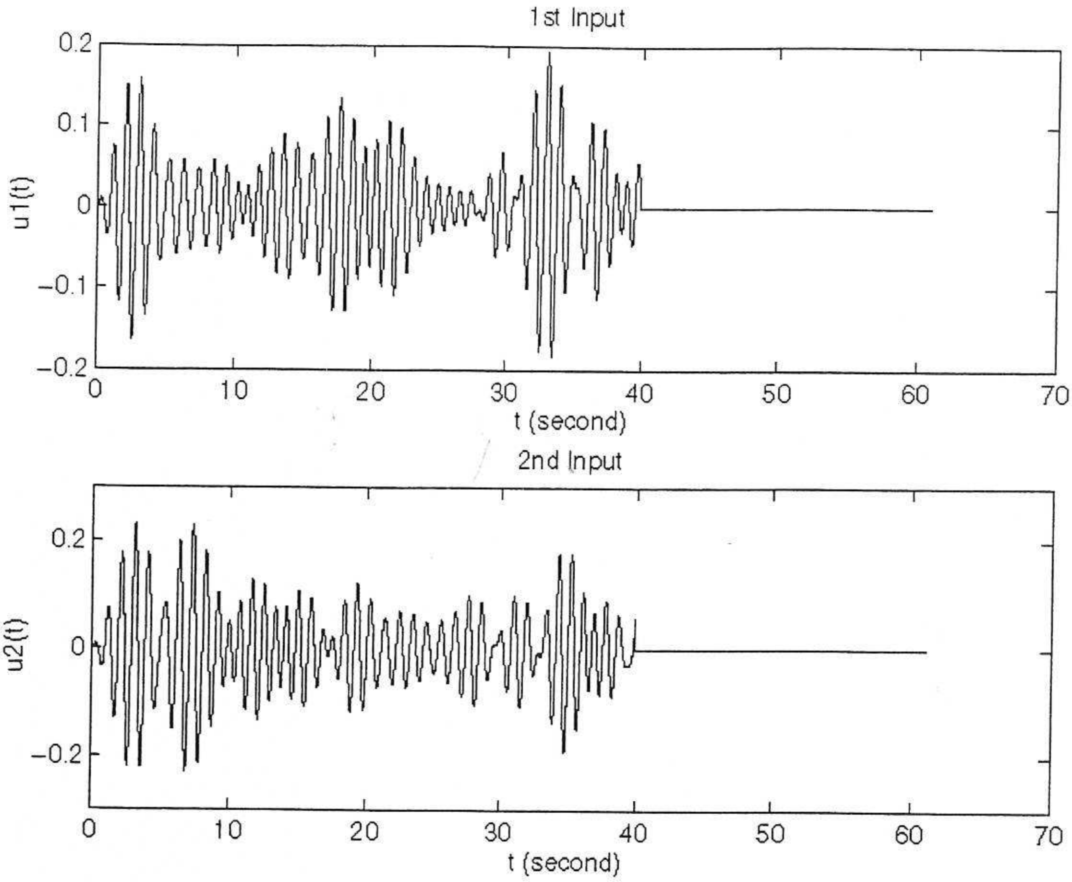

Figure 10 The two input signals to be processed in Example 3

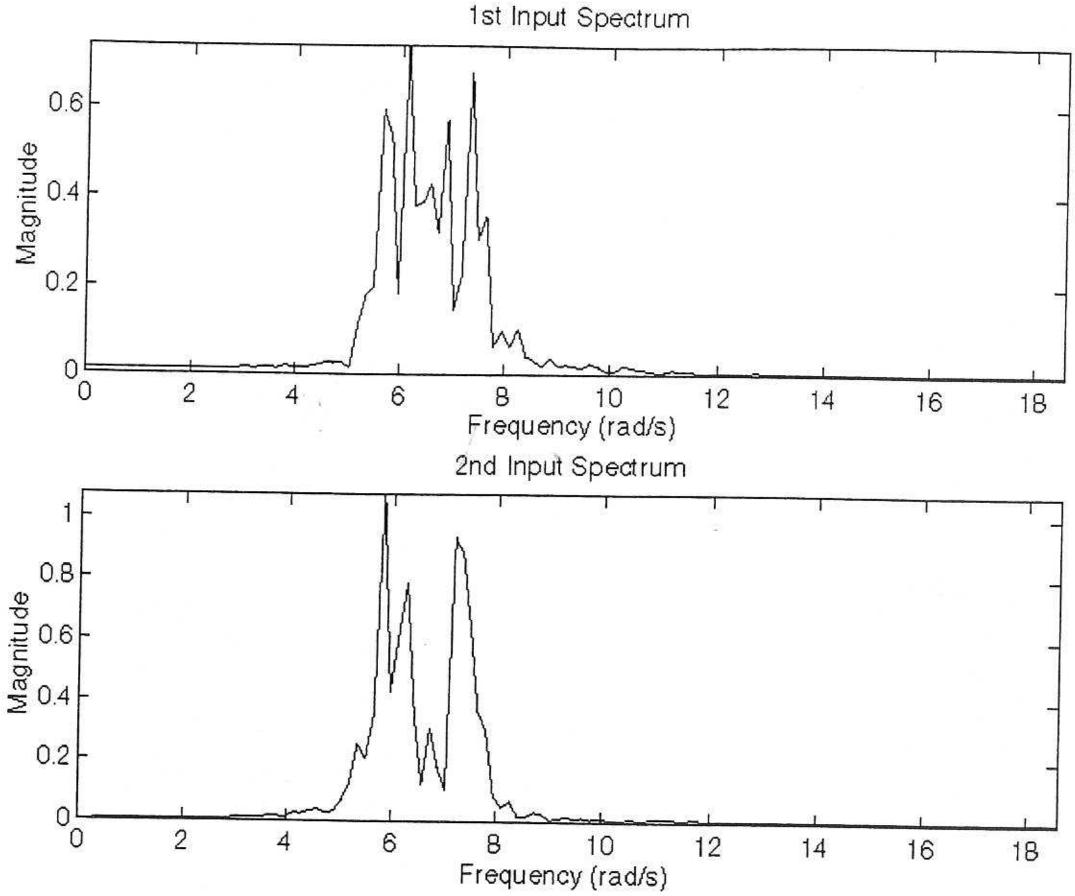

Figure 11 The spectra of the two input signals in Example 3

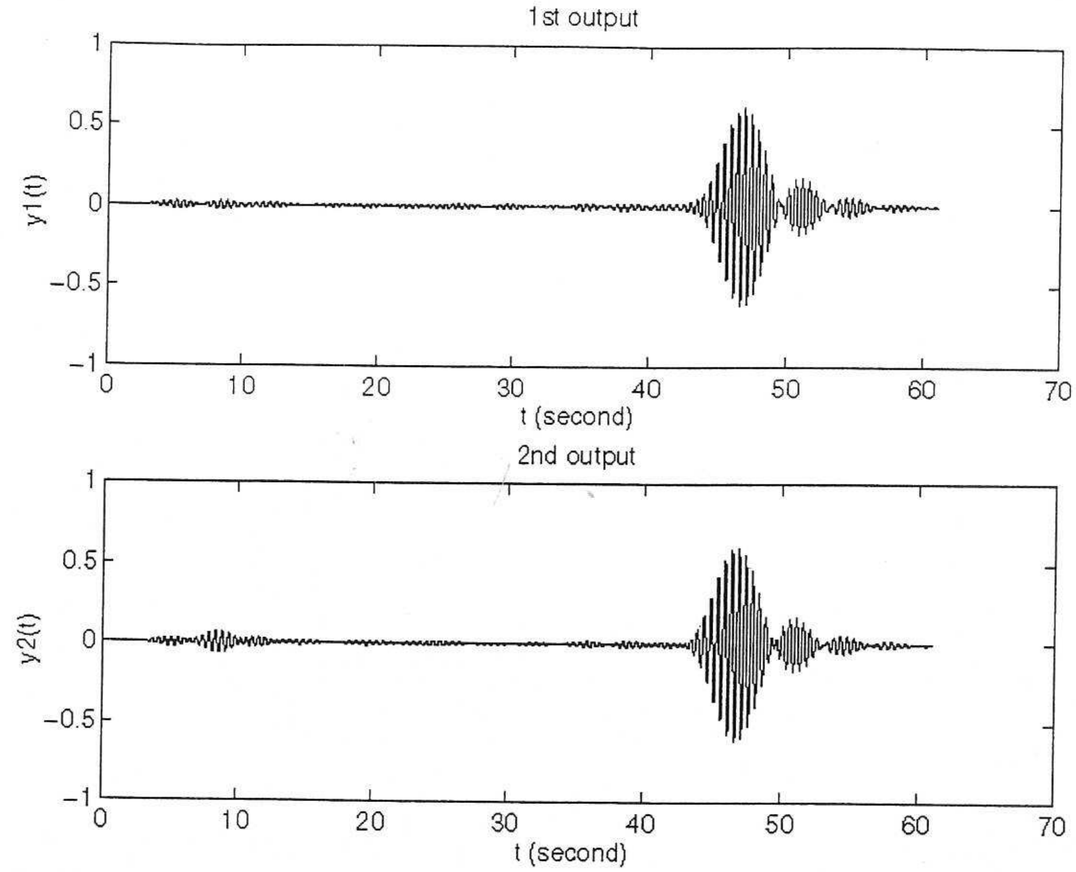

Figure 12 The time domain outputs of the filter designed in Example 3

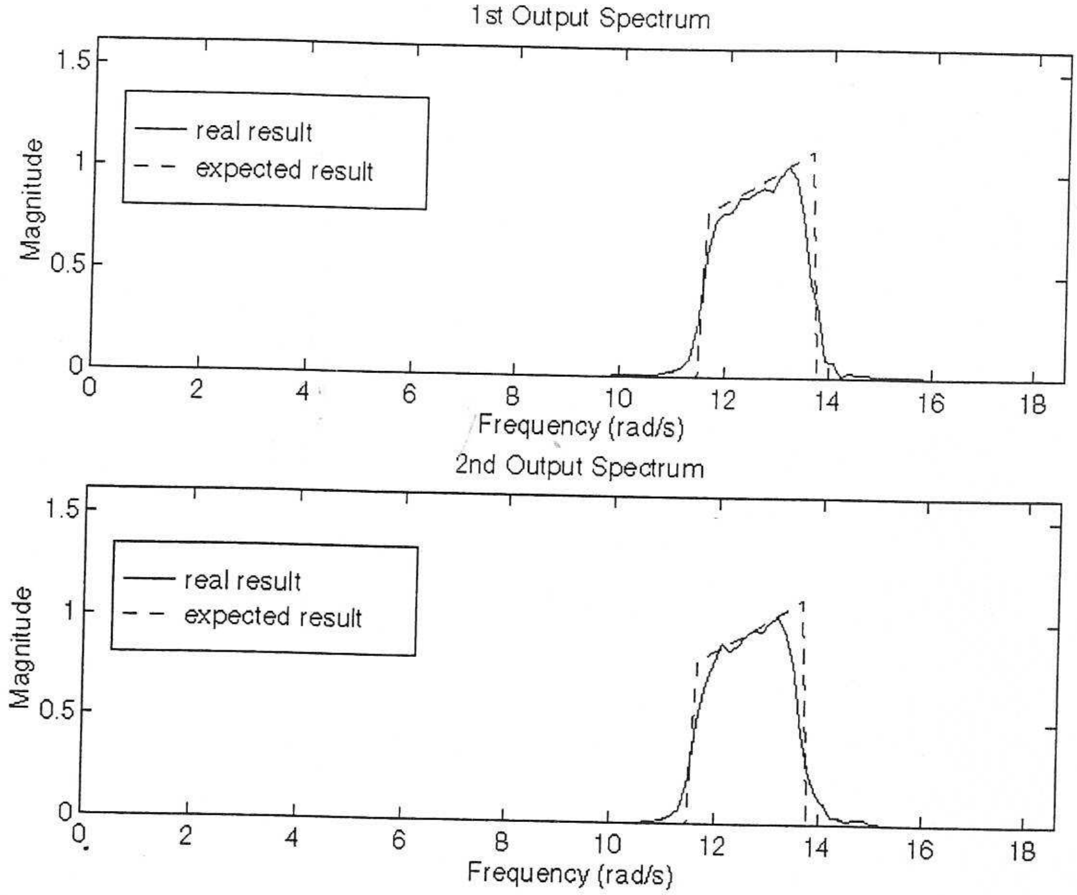

Figure 13 A comparison between the output spectra of the filter designed in Example 3 and the desired spectra specified for the design