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ASYMPTOTICALLY OPTIMAL NONLINEAR FILTERING

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ABSTRACT. In this note we present a computationally simple algorithm for non-linear filtering. the algorithm involves solving, at a given point in state space, an algebraic Riccati equation. The coefficients of this equation vary with the given point in state space. We investigate conditions under which the state estimate given by this algorithm converges asymptotically to the first order minimum variance estimate given by the extended Kalman filter. We also investigate conditions for determining a region of stability for the filter given by this algorithm. The analysis is based on stable manifold theory and Hamilton-Jacobi-Bellman (HJB) equations. The motivation for introducing HJB equations is given by reference to the maximum likelihood approach to deriving the extended Kalman filter.

1. INTRODUCTION

In this note we present a computationally simple algorithm for non-linear filtering. the algorithm involves solving, at a given point in state space, an algebraic Riccati equation. The coefficients of this equation vary with the given point in state space. We investigate conditions under which the state estimate given by this algorithm converges asymptotically to the first order minimum variance estimate given by the extended Kalman filter. We also investigate conditions for determining a region of stability for the filter given by this algorithm. The analysis is based on stable manifold theory and Hamilton-Jacobi-Bellman (HJB) equations. The motivation for introducing HJB equations is given by reference to the maximum likelihood approach to deriving the extended Kalman filter.

2. ASYMPTOTIC MINIMUM VARIANCE FILTER

Suppose we have a nonlinear, autonomous system driven by white noise with white noise corrupted observations

$$\dot{\mathbf{x}}(t) = \mathbf{f}[\mathbf{x}(t)] + \mathbf{G}[\mathbf{x}(t)]\mathbf{w}(t) \tag{1}$$

$$\mathbf{z}(t) = \mathbf{h}[\mathbf{x}(t)] + \mathbf{v}(t) \tag{2}$$

where $\mathbf{w}(t)$ and $\mathbf{v}(t)$ are zero-mean, white and gaussian and uncorrelated with themselves and with $\mathbf{x}(t_0)$ such that for $t > t_0$,

$$\mathrm{cov}\{\mathbf{w}(t), \mathbf{w}(\tau)\} = \mathbf{Q}\delta(t-\tau)$$
$$\mathrm{cov}\{\mathbf{v}(t), \mathbf{v}(\tau)\} = \mathbf{R}\delta(t-\tau).$$

Define the processes $\mathbf{d}\omega(t) = \mathbf{w}(t)dt$, $\mathbf{d}\nu(t) = \mathbf{v}(t)dt$ and $\mathbf{dy}(t) = \mathbf{z}(t)dt$. Then the above system can be written more properly as the following Ito sense stochastic

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differential equations

$$\mathbf{dx}(t) = \mathbf{f}[\mathbf{x}(t)]dt + \mathbf{G}[\mathbf{x}(t)]\mathbf{d}\boldsymbol{\omega}(t)$$
$$\mathbf{dy}(t) = \mathbf{h}[\mathbf{x}(t)]dt + \mathbf{d}\boldsymbol{\nu}(t)$$

where $\boldsymbol{\omega}(t)$ and $\boldsymbol{\nu}(t)$ are independent Browian motions uncorrelated with $\mathbf{x}(t_0)$ such that

$$\operatorname{cov}\{\boldsymbol{\omega}(t), \boldsymbol{\omega}(\tau)\} = \mathbf{Q}\min(t,\tau)$$
$$\operatorname{cov}\{\boldsymbol{\nu}(t), \boldsymbol{\nu}(\tau)\} = \mathbf{R}\min(t,\tau).$$

Let $\mathbf{Y}(t) = \{\mathbf{y}(\tau) : t_0 \le \tau \le t\}$ denote the observations up to time t. Let $\hat{\mathbf{x}}(t) = \mathcal{E}\{\mathbf{x}(t)|\mathbf{Y}(t)\}$ denote the conditional mean, i.e. the minimum variance optimal estimate, and $\mathbf{V}(t) = \operatorname{var}\{\mathbf{x}(t) - \hat{\mathbf{x}}(t)|\mathbf{Y}(t)\}$ denote the conditional error variance. Then to a first order approximation the solution to the filtering problem is given by the extended Kalman filter ([4], Chapter 9)

$$\begin{aligned}
\mathbf{d}\hat{\mathbf{x}}(t) &= \mathbf{f}[\hat{\mathbf{x}}(t)]dt + \mathbf{V}(t)\frac{\partial \mathbf{h}^T[\hat{\mathbf{x}}(t)]}{\partial \hat{\mathbf{x}}(t)}\mathbf{R}^{-1}\left\{\mathbf{dy}(t) - \mathbf{h}[\hat{\mathbf{x}}(t)]dt\right\} \\
\mathbf{dV}(t) &= \left\{\frac{\partial \mathbf{f}[\hat{\mathbf{x}}(t)]}{\partial \hat{\mathbf{x}}(t)}\mathbf{V}(t) + \mathbf{V}(t)\frac{\partial \mathbf{f}^T[\hat{\mathbf{x}}(t)]}{\partial \hat{\mathbf{x}}(t)}\right. \\
&\quad \left. + \mathbf{G}[\hat{\mathbf{x}}(t)]\mathbf{Q}\mathbf{G}^T[\hat{\mathbf{x}}(t)] - \mathbf{V}(t)\frac{\partial \mathbf{h}^T[\hat{\mathbf{x}}(t)]}{\partial \hat{\mathbf{x}}(t)}\mathbf{R}^{-1}\frac{\partial \mathbf{h}[\hat{\mathbf{x}}(t)]}{\partial \hat{\mathbf{x}}(t)}\mathbf{V}(t)\right\}dt.
\end{aligned}$$

The initial conditions for the extended Kalman filter are $\hat{\mathbf{x}}(t_0) = \mathcal{E}\{\mathbf{x}(t_0)\}$ and $\mathbf{V}(t_0) = \operatorname{var}\{\mathbf{x}(t_0)|\mathbf{Y}(t_0)\}$.

Suppose now that $\mathbf{f}(\mathbf{0}) = \mathbf{0}$ and $\mathbf{h}(\mathbf{0}) = \mathbf{0}$, i.e. there is an equilibrium at $\mathbf{x} = \mathbf{0}$. Let

$$\mathbf{A}(\mathbf{0}) = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{0}) \quad \mathbf{H}(\mathbf{0}) = \frac{\partial \mathbf{h}}{\partial \mathbf{x}}(\mathbf{0})$$

and suppose that the linear system $(\mathbf{A}(\mathbf{0}), \mathbf{G}(\mathbf{0}), \mathbf{H}(\mathbf{0}))$ is completely controllable and completely observable.

If the state variable $\mathbf{x}$ is close to the equilibrium $\mathbf{x} = \mathbf{0}$, then the extended Kalman filter reduces to the linear Kalman filter around $\mathbf{x} = \mathbf{0}$

$$\mathbf{d}\hat{\mathbf{x}}(t) = \mathbf{A}(\mathbf{0})\hat{\mathbf{x}}(t)dt + \mathbf{V}(t)\mathbf{H}^T(\mathbf{0})\mathbf{R}^{-1}\left\{\mathbf{dy}(t) - \mathbf{H}(\mathbf{0})\hat{\mathbf{x}}(t)dt\right\} \tag{3}$$

$$\begin{aligned}
\mathbf{dV}(t) &= \Big\{\mathbf{A}(\mathbf{0})\mathbf{V}(t) + \mathbf{V}(t)\mathbf{A}^T(\mathbf{0}) \\
&\quad + \mathbf{G}(\mathbf{0})\mathbf{Q}\mathbf{G}^T(\mathbf{0}) - \mathbf{V}(t)\mathbf{H}^T(\mathbf{0})\mathbf{R}^{-1}\mathbf{H}(\mathbf{0})\mathbf{V}(t)\Big\}dt.
\end{aligned} \tag{4}$$

By the hypotheses on $(\mathbf{A}(\mathbf{0}), \mathbf{G}(\mathbf{0}), \mathbf{H}(\mathbf{0}))$, the linear filter is asymptotically stable (Theorem 13.18, [3]). That is, the homogeneous part of the above linear system

$$\mathbf{d}\hat{\mathbf{x}}(t) = \mathbf{A}(\mathbf{0})\hat{\mathbf{x}}(t)dt - \mathbf{V}(t)\mathbf{H}^T(\mathbf{0})\mathbf{R}^{-1}\mathbf{H}(\mathbf{0})\hat{\mathbf{x}}(t)dt$$

is asymptotically stable with Lyapunov function $S(\hat{\mathbf{x}}, t) = \frac{1}{2}\hat{\mathbf{x}}^T\mathbf{V}^{-1}(t)\hat{\mathbf{x}}$. In addition, by Theorem 13.33 of [3], $\mathbf{V}(t)$ tends to the unique positive definite solution of

$$\mathbf{0} = \mathbf{A}(\mathbf{0})\mathbf{V} + \mathbf{V}\mathbf{A}^T(\mathbf{0}) + \mathbf{G}(\mathbf{0})\mathbf{Q}\mathbf{G}^T(\mathbf{0}) - \mathbf{V}\mathbf{H}^T(\mathbf{0})\mathbf{R}^{-1}\mathbf{H}(\mathbf{0})\mathbf{V}. \tag{5}$$

By the assumptions on $(\mathbf{A}(\mathbf{0}), \mathbf{G}(\mathbf{0}), \mathbf{H}(\mathbf{0}))$, it follows that $(\mathbf{A}(\mathbf{0}), \mathbf{G}(\mathbf{0}))$ is stabilizable and $(\mathbf{H}(\mathbf{0}), \mathbf{A}(\mathbf{0}))$ is observable. Also $\mathbf{Q}$ is clearly positive definite. So by Lemma 3 of [1] (or Theorem 12.2 of [6]), H has no purely imaginary eigenvalues and the unstable eigenspace of H is spanned by the columns of the $2n \times n$ matrix

$$\begin{bmatrix} \mathbf{I} \\ \mathbf{V}^{-1} \end{bmatrix} \tag{12}$$

where $\mathbf{V}^{-1}$ is the positive definite solution of (6), thus showing again that (7) has a positive solution $\frac{1}{2}\hat{\mathbf{x}}\mathbf{V}^{-1}\hat{\mathbf{x}}$.

Note, we have to apply the phase-space transformation $\bar{\lambda} \to -\bar{\lambda}$ to get H in the correct form to apply Lemma 3 of [1] and so their result, which concerns the stable eigenspace, becomes a result about the unstable eigenspace. Note also that, as can easily be checked,

$$\begin{bmatrix} \mathbf{I} \\ \mathbf{V} \end{bmatrix}$$

spans the stable eigenspace for the Hamiltonian matrix corrsponding to (5) and so asymptotic stability for the Kalman filter (3), (4) corresponds to asymptotic instability for the $\hat{\mathbf{x}}$-dynamics of (7) with the feedback $\lambda = \mathbf{V}^{-1}\hat{\mathbf{x}}$.

So, from the paragraph before the previous one, the equilibrium at $\hat{\mathbf{x}} = \bar{\lambda} = \mathbf{0}$ for the flow given by (10) is hyperbolic. By the stable manifold theorem there therefore exist n-dimensional stable and unstable manifolds for this equilibrium. Furthermore, the columns of (12) span the tangent space to the unstable manifold at the origin and so the unstable manifold can be parameterised by the $\hat{\mathbf{x}}$ coordinates in a neighbourhood U_1 of the origin. This parameterisation has the form $\bar{\lambda} = \partial \bar{S}/\partial \hat{\mathbf{x}}$ where $\bar{S}$ is the solution to (10) in U_1 satisfying $\bar{S}(0) = 0$, $\partial \bar{S}(0)/\partial \hat{\mathbf{x}} = \mathbf{0}$ and $\partial^2 \bar{S}(0)/\partial \hat{\mathbf{x}}^2 = \mathbf{V}^{-1}$. This argument comes from [5].

Since $\mathbf{V}^{-1}$ is positive definite and, to second order locally at $\hat{\mathbf{x}} = \mathbf{0}$,

$$\bar{S} = \frac{1}{2}\hat{\mathbf{x}}^T \frac{\partial^2 \bar{S}(0)}{\partial \hat{\mathbf{x}}^2}\hat{\mathbf{x}} = \frac{1}{2}\hat{\mathbf{x}}^T \mathbf{V}^{-1}\hat{\mathbf{x}},$$

if follows that $\bar{S}$ is positive definite in some neighbourhood $U_2 \subset U_1$ of $\hat{\mathbf{x}} = \mathbf{0}$. $\bar{S}$ is then a Lyapunov function for the homogeneous part of (8) provided a certain inequality holds on U_2. For, along the trajectories of the homogeneous part of (8),

$$\begin{aligned}
\frac{d\bar{S}}{dt} &= \frac{\partial \bar{S}}{\partial \hat{\mathbf{x}}}\frac{d\hat{\mathbf{x}}}{dt} \\
&= \bar{\lambda}^T \left(\mathbf{f}(\hat{\mathbf{x}}) - \mathbf{V}(\hat{\mathbf{x}})\mathbf{H}^T(\hat{\mathbf{x}})\mathbf{R}^{-1}\mathbf{H}(\hat{\mathbf{x}})\hat{\mathbf{x}}\right) \\
&= -\frac{1}{2}\bar{\lambda}^T \mathbf{G}(\hat{\mathbf{x}})\mathbf{Q}\mathbf{G}^T(\hat{\mathbf{x}})\bar{\lambda} \\
&\quad + \frac{1}{2}\hat{\mathbf{x}}^T\mathbf{H}^T(\hat{\mathbf{x}})\mathbf{R}^{-1}\mathbf{H}(\hat{\mathbf{x}})\hat{\mathbf{x}} - \bar{\lambda}^T\mathbf{V}(\hat{\mathbf{x}})\mathbf{H}^T(\hat{\mathbf{x}})\mathbf{R}^{-1}\mathbf{H}(\hat{\mathbf{x}})\hat{\mathbf{x}} \\
&= -\frac{1}{2}\bar{\lambda}^T \mathbf{G}(\hat{\mathbf{x}})\mathbf{Q}\mathbf{G}^T(\hat{\mathbf{x}})\bar{\lambda} \\
&\quad + \frac{1}{2}\left(\mathbf{V}^{-1}(\hat{\mathbf{x}})\hat{\mathbf{x}} - \bar{\lambda}\right)^T \mathbf{V}(\hat{\mathbf{x}})\mathbf{H}^T(\hat{\mathbf{x}})\mathbf{R}^{-1}\mathbf{H}(\hat{\mathbf{x}})\mathbf{V}(\hat{\mathbf{x}})\left(\mathbf{V}^{-1}(\hat{\mathbf{x}})\hat{\mathbf{x}} - \bar{\lambda}\right) \\
&\quad - \frac{1}{2}\bar{\lambda}^T\mathbf{V}(\hat{\mathbf{x}})\mathbf{H}^T(\hat{\mathbf{x}})\mathbf{R}^{-1}\mathbf{H}(\hat{\mathbf{x}})\mathbf{V}(\hat{\mathbf{x}})\bar{\lambda}
\end{aligned}$$

where $\mathbf{V}(\hat{\mathbf{x}})$ is the solution to (9). This is less than zero provided

$$\begin{aligned}
&\left(\mathbf{V}^{-1}(\hat{\mathbf{x}})\hat{\mathbf{x}} - \bar{\lambda}\right)^T \mathbf{V}(\hat{\mathbf{x}})\mathbf{H}^T(\hat{\mathbf{x}})\mathbf{R}^{-1}\mathbf{H}(\hat{\mathbf{x}})\mathbf{V}(\hat{\mathbf{x}})\left(\mathbf{V}^{-1}(\hat{\mathbf{x}})\hat{\mathbf{x}} - \bar{\lambda}\right) \\
&\qquad\qquad < \bar{\lambda}^T\mathbf{V}(\hat{\mathbf{x}})\mathbf{H}^T(\hat{\mathbf{x}})\mathbf{R}^{-1}\mathbf{H}(\hat{\mathbf{x}})\mathbf{V}(\hat{\mathbf{x}})\bar{\lambda}.
\end{aligned} \tag{13}$$

In the steady state case, stability follows since $S(\hat{\mathbf{x}}) = \frac{1}{2}\hat{\mathbf{x}}^T\mathbf{V}^{-1}\hat{\mathbf{x}}$ is clearly greater than zero for $\hat{\mathbf{x}} \neq \mathbf{0}$ and

$$\begin{aligned}\frac{dS}{dt} &= \hat{\mathbf{x}}^T\mathbf{V}^{-1}\left(\mathbf{A}(0) - \mathbf{V}\mathbf{H}^T(0)\mathbf{R}^{-1}\mathbf{H}(0)\right)\hat{\mathbf{x}} \\ &\quad + \hat{\mathbf{x}}^T\left(\mathbf{A}^T(0) - \mathbf{H}^T(0)\mathbf{R}^{-1}\mathbf{H}(0)\mathbf{V}\right)\mathbf{V}^{-1}\hat{\mathbf{x}} \\ &= \hat{\mathbf{x}}^T\left(\mathbf{V}^{-1}\mathbf{A}(0) + \mathbf{A}^T(0)\mathbf{V}^{-1} - 2\mathbf{H}^T(0)\mathbf{R}^{-1}\mathbf{H}(0)\right)\hat{\mathbf{x}} \\ &= \hat{\mathbf{x}}^T\left(-\mathbf{V}^{-1}\mathbf{G}(0)\mathbf{Q}\mathbf{G}^T(0)\mathbf{V}^{-1} - \mathbf{H}^T(0)\mathbf{R}^{-1}\mathbf{H}(0)\right)\hat{\mathbf{x}} < 0.\end{aligned}$$

For the proof of stability when $\mathbf{V}$ depends on t see [3], Theorem 13.18. Since $(\mathbf{A}(0), \mathbf{G}(0))$ is controllable, $\mathbf{V}$ is positive definite and so $\mathbf{V}^{-1}$ exists and satisfies the algebraic Riccati equation

$$\mathbf{0} = \mathbf{V}^{-1}\mathbf{A}(0) + \mathbf{A}^T(0)\mathbf{V}^{-1} + \mathbf{V}^{-1}\mathbf{G}(0)\mathbf{Q}\mathbf{G}^T(0)\mathbf{V}^{-1} - \mathbf{H}^T(0)\mathbf{R}^{-1}\mathbf{H}(0). \tag{6}$$

This corresponds to the Hamilton-Jacobi equation for $S(\hat{\mathbf{x}})$

$$0 = \hat{\mathbf{x}}^T\left(\mathbf{V}^{-1}\mathbf{A}(0) + \mathbf{A}^T(0)\mathbf{V}^{-1} + \mathbf{V}^{-1}\mathbf{G}(0)\mathbf{Q}\mathbf{G}^T(0)\mathbf{V}^{-1} - \mathbf{H}^T(0)\mathbf{R}^{-1}\mathbf{H}(0)\right)\hat{\mathbf{x}}$$

since $\partial S/\partial\hat{\mathbf{x}} = \mathbf{V}^{-1}\hat{\mathbf{x}}$. Setting $\lambda = \partial S/\partial\hat{\mathbf{x}}$ we get

$$0 = \lambda^T\mathbf{A}(0)\hat{\mathbf{x}} + \frac{1}{2}\lambda^T\mathbf{G}(0)\mathbf{Q}\mathbf{G}^T(0)\lambda - \frac{1}{2}\hat{\mathbf{x}}^T\mathbf{H}^T(0)\mathbf{R}^{-1}\mathbf{H}(0)\hat{\mathbf{x}}. \tag{7}$$

We have thus shown that the Hamilton-Jacobi equation (7) has a solution $S(\hat{\mathbf{x}}) \geq 0$.

Consider now the following model for $\hat{\mathbf{x}}$ away from the equilibrium $\hat{\mathbf{x}} = \mathbf{0}$. Assume $(\mathbf{A}(\hat{\mathbf{x}}), \mathbf{G}(\hat{\mathbf{x}}))$ and $(\mathbf{A}^T(\hat{\mathbf{x}}), \mathbf{H}^T(\hat{\mathbf{x}}))$ are stabilizable for all $\hat{\mathbf{x}}$. Note this holds at $\hat{\mathbf{x}} = \mathbf{0}$ by assumption on $(\mathbf{A}(0), \mathbf{G}(0), \mathbf{H}(0))$. Write $\mathbf{f}(\hat{\mathbf{x}}) = \mathbf{A}(\hat{\mathbf{x}})\hat{\mathbf{x}}$ and $\mathbf{h}(\hat{\mathbf{x}}) = \mathbf{H}(\hat{\mathbf{x}})\hat{\mathbf{x}}$ where $\mathbf{A}(\hat{\mathbf{x}}) \to \mathbf{A}(0)$ and $\mathbf{H}(\hat{\mathbf{x}}) \to \mathbf{H}(\mathbf{0})$ as $\hat{\mathbf{x}} \to \mathbf{0}$. Then $\hat{\mathbf{x}}$ is the solution of

$$\mathbf{d}\hat{\mathbf{x}}(t) = \mathbf{f}(\hat{\mathbf{x}}(t))dt + \mathbf{V}(\hat{\mathbf{x}}(t))\mathbf{H}^T(\hat{\mathbf{x}}(t))\mathbf{R}^{-1}\left\{\mathbf{dy}(t) - \mathbf{h}(\hat{\mathbf{x}}(t))dt\right\} \tag{8}$$

with initial condition $\hat{\mathbf{x}}(t_0) = \mathcal{E}\{\mathbf{x}(t_0)\}$ where $\mathbf{V}(\hat{\mathbf{x}})$ satisfies at $\hat{\mathbf{x}}$

$$\mathbf{0} = \mathbf{A}(\hat{\mathbf{x}})\mathbf{V}(\hat{\mathbf{x}}) + \mathbf{V}(\hat{\mathbf{x}})\mathbf{A}^T(\hat{\mathbf{x}}) + \mathbf{G}(\hat{\mathbf{x}})\mathbf{Q}\mathbf{G}^T(\hat{\mathbf{x}}) - \mathbf{V}(\hat{\mathbf{x}})\mathbf{H}^T(\hat{\mathbf{x}})\mathbf{R}^{-1}\mathbf{H}(\hat{\mathbf{x}})\mathbf{V}(\hat{\mathbf{x}}). \tag{9}$$

Note the above assumptions on $(\mathbf{A}(\hat{\mathbf{x}}), \mathbf{G}(\hat{\mathbf{x}}))$ and $(\mathbf{A}^T(\hat{\mathbf{x}}), \mathbf{H}^T(\hat{\mathbf{x}}))$ ensure a positive semi-definite solution to (9) exists for all $\hat{\mathbf{x}}$. This model is simpler to implement than the extended Kalman filter and appears to be insensitive to errors in the initial condition, unlike the extended Kalman filter. We shall attempt to answer the following three questions concerning this model. Firstly, what meaning can be attached to $\hat{\mathbf{x}}$ as derived from this model? Secondly, under what conditions is the model stable and thus insensitive to the initial condition? Thirdly, how does this model relate to the extended Kalman filter?

Clearly, (8) and (9) tend to the same limit as the extended Kalman filter if $\hat{\mathbf{x}} \to \mathbf{0}$, namely the steady state form of the linear filter (3) and (5) which we have assumed is stable. We now give conditions under which (8) and (9) will be stable on some neighbourhood of $\mathbf{0}$. To do this, consider the following Hamilton-Jacobi equation for the function $\bar{S}(\hat{\mathbf{x}})$,

$$0 = \bar{\lambda}^T\mathbf{f}(\hat{\mathbf{x}}) + \frac{1}{2}\bar{\lambda}^T\mathbf{G}(\hat{\mathbf{x}})\mathbf{Q}\mathbf{G}^T(\hat{\mathbf{x}})\bar{\lambda} - \frac{1}{2}\hat{\mathbf{x}}^T\mathbf{H}^T(\hat{\mathbf{x}})\mathbf{R}^{-1}\mathbf{H}(\hat{\mathbf{x}})\hat{\mathbf{x}} \tag{10}$$

where $\bar{\lambda} = \partial\bar{S}/\partial\hat{\mathbf{x}}$. This equation is the nonlinear extension of (7).

The Hamiltonian flow corresponding to (10) has an equilibrium at $\hat{\mathbf{x}} = \bar{\lambda} = \mathbf{0}$ in phase space. Equation (7) is the Hamilton-Jacobi equation corresponding to the linearisation of this flow at $\hat{\mathbf{x}} = \bar{\lambda} = \mathbf{0}$. The Hamiltonian matrix describing this linearised flow is

$$H = \begin{bmatrix}\mathbf{A}(0) & \mathbf{G}(0)\mathbf{Q}\mathbf{G}^T(0) \\ \mathbf{H}^T(0)\mathbf{R}^{-1}\mathbf{H}(0) & -\mathbf{A}^T(0)\end{bmatrix}. \tag{11}$$

This inequality essentially says that the difference between the true feedback and the approximation given by (9) is less than the true feedback as measured with respect to the norm given by $\mathbf{V}(\hat{\mathbf{x}})\mathbf{H}^T(\hat{\mathbf{x}})\mathbf{R}^{-1}\mathbf{H}(\hat{\mathbf{x}})\mathbf{V}(\hat{\mathbf{x}})$. Since $\mathbf{V}(\hat{\mathbf{x}}) \to \mathbf{V}$, the solution to (5), as $\hat{\mathbf{x}} \to \mathbf{0}$, we would expect this inequality to hold on a neighbourhood of the origin.

Note it is enough to test if

$$\frac{1}{2}\hat{\mathbf{x}}^T\mathbf{H}^T(\hat{\mathbf{x}})\mathbf{R}^{-1}\mathbf{H}(\hat{\mathbf{x}})\hat{\mathbf{x}} < \bar{\lambda}^T\mathbf{V}(\hat{\mathbf{x}})\mathbf{H}^T(\hat{\mathbf{x}})\mathbf{R}^{-1}\mathbf{H}(\hat{\mathbf{x}})\hat{\mathbf{x}}$$

which avoids one having to consider whether $\mathbf{V}^{-1}(\hat{\mathbf{x}})$ exists. Note also that the value of $\bar{\lambda} = \partial\bar{S}/\partial\hat{\mathbf{x}}$ can be obtained by following characteristics of (10) along the unstable manifold starting near the origin without having to solve (10) for $\bar{S}$.

So suppose this inequality holds on some neighbourhood U_3 of the origin contained within U_2. Then the homogeneous part of (8) will be asymptotically stable on the largest sublevel set of $\bar{S}$ contained within U_3, i.e. for the largest $\gamma > 0$ such that $\{\hat{\mathbf{x}} : \bar{S}(\hat{\mathbf{x}}) \leq \gamma\} \subset U_3$.

So to summarise we have the following result which answers the second and third questions posed above.

Proposition 2.1. *Suppose* $\mathbf{f}(\mathbf{0}) = \mathbf{0}$ *and* $\mathbf{h}(\mathbf{0}) = \mathbf{0}$. *Suppose also that the linear system* $(\mathbf{A}(\mathbf{0}), \mathbf{G}(\mathbf{0}), \mathbf{H}(\mathbf{0}))$ *is completely controllable and completely observable. Then equations (8) and (9) give a model for* $\hat{\mathbf{x}}$ *which agrees asymptotically as* $\hat{\mathbf{x}} \to \mathbf{0}$ *with the optimal (minimum variance) estimate given by the steady state limit of the extended Kalman filter (should the extended Kalman filter attain this limit). If* U *is the largest neighbourhood of the origin on which both a positive solution* $\bar{S}$ *to the Hamilton-Jacobi equation (10) exists and the inequality (13) holds, then the algorithm will be asymptotically stable on the largest sublevel set of* $\bar{S}$ *contained within* U.

What about the first question posed above on what meaning can be attached to the estimate $\hat{\mathbf{x}}$ produced by (8) and (9) away from the origin? We make no claim about this in the above result as it is not clear that away from $\hat{\mathbf{x}} = \mathbf{0}$ there is any relationship between $\hat{\mathbf{x}}$ and $\mathbf{x}$. Probably the most that can be said is that if the extended Kalman filter attains a steady state $\mathbf{dV}(t) = \mathbf{0}$ away from $\hat{\mathbf{x}} = \mathbf{0}$, then (8) and (9) provide an approximate solution of this steady state, the approximation being that $\mathbf{A}(\hat{\mathbf{x}})$ and $\mathbf{H}(\hat{\mathbf{x}})$ are used instead of $\partial\mathbf{f}(\hat{\mathbf{x}})/\partial\hat{\mathbf{x}}$ and $\partial\mathbf{h}(\hat{\mathbf{x}})/\partial\hat{\mathbf{x}}$. We know of no conditions, however, for determining whether the extended Kalman filter will attain a steady state. Thus a meaning can only be attached to $\hat{\mathbf{x}}$ as $\hat{\mathbf{x}} \to \mathbf{0}$, namely $\hat{\mathbf{x}}$ in this limit gives the minimum variance estimate of $\mathbf{x}$. However, we do have conditions which imply that the dynamics of $\hat{\mathbf{x}}$ are stable in a region around the origin and so we can expect this interpretation to hold eventually for trajectories which start in and remain in this region. These conditions also imply that errors in the initial condition $\hat{\mathbf{x}}(t_0)$ eventually disappear, unlike in the case of the extended Kalman filter where there are no guarantees that errors in $\hat{\mathbf{x}}(t_0)$ and $\mathbf{V}(t_0)$ are insignificant.

3. Maximum likelihood approach

In the above analysis, the central equations were those for the extended Kalman filter. These are derived from a first order approximate solution to the modified Fokker-Plank equation which describes the evolution of the conditional probability density of $\mathbf{x}(t)$ (see [4], Chapter 9). The Hamilton-Jacobi equations (7) and (10) which were used to prove stability were introduced in a formal way above and are not directly related to, or required in, the derivation of the equations for the filter dynamics. In this section we outline the maximum likelihood approach to estimating $\mathbf{x}(t)$. This approach does lead directly to the Hamilton-Jacobi equations from which (7) and (10) come.

We start by formulating a least squares version of the problem of estimating $\mathbf{x}(t)$ in (1) given the observations $\mathbf{z}(t)$ from (2). Minimising an appropriate error function subject to the dynamical constraint of (1) can be shown to be equivalent to maximising the conditional probability density function of $\mathbf{x}(t)$. The corresponding estimate $\hat{\mathbf{x}}(t)$ so obtained is the peak or mode of the conditional probability density function and constitutes the maximum likelihood (Bayesian) estimate of $\mathbf{x}(t)$. A Hamilton-Jacobi equation is obtained by using dynamical programming to solve the least squares problem.

Let $\bar{\mathbf{x}}(t_0) = \mathcal{E}\{\mathbf{x}(t_0)\}$, $\bar{\mathbf{V}}(t_0) = \mathrm{var}\{\mathbf{x}(t_0)\}$ and consider first the linear-quadratic estimation problem (around the equilibrium $\mathbf{x} = \mathbf{0}$) of minimising

$$\begin{aligned} J = {} & \frac{1}{2}\left(\mathbf{x}(t_0) - \bar{\mathbf{x}}(t_0)\right)^T \bar{\mathbf{V}}^{-1}(t_0)\left(\mathbf{x}(t_0) - \bar{\mathbf{x}}(t_0)\right) \\ & + \frac{1}{2}\int_{t_0}^{t}\left[\left(\mathbf{z}(\tau) - \mathbf{H}(\mathbf{0})\mathbf{x}(\tau)\right)^T \mathbf{R}^{-1}\left(\mathbf{z}(\tau) - \mathbf{H}(\mathbf{0})\mathbf{x}(\tau)\right) + \mathbf{w}^T(\tau)\mathbf{Q}^{-1}\mathbf{w}(\tau)\right] d\tau \end{aligned}$$

with respect to $\mathbf{x}(\tau)$ and $\mathbf{w}(\tau)$ subject to the constraint

$$\dot{\mathbf{x}}(t) = \mathbf{A}(\mathbf{0})\mathbf{x}(t) + \mathbf{G}(\mathbf{0})\mathbf{w}(t).$$

This can be thought of as attempting to determine (estimate) $\mathbf{x}(\tau)$ for $t_0 \leq \tau \leq t$ so that, simultaneously, the errors in the dynamical system and in the observations are small. In view of the constraint it is enough to minimise J with respect to $\mathbf{x}(t_0)$ and $\mathbf{w}(\tau)$ since $\mathbf{x}(\tau)$ is then determined for $t_0 \leq \tau \leq t$. Let

$$S(\mathbf{x}, t) = \min_{\mathbf{w}(\tau)} J.$$

Then we get the dynamic programming equation

$$\begin{aligned} -\frac{\partial S}{\partial t} = \min_{\mathbf{w}}\Bigg\{ & \frac{\partial S}{\partial \mathbf{x}}^T \mathbf{A}(\mathbf{0})\mathbf{x} + \frac{\partial S}{\partial \mathbf{x}}^T \mathbf{G}(\mathbf{0})\mathbf{w} \\ & - \frac{1}{2}\left(\mathbf{z} - \mathbf{H}(\mathbf{0})\mathbf{x}\right)^T \mathbf{R}^{-1}\left(\mathbf{z} - \mathbf{H}(\mathbf{0})\mathbf{x}\right) - \frac{1}{2}\mathbf{w}^T\mathbf{Q}^{-1}\mathbf{w}\Bigg\}. \end{aligned}$$

This is minimised by $\mathbf{w} = \mathbf{Q}\mathbf{G}^T(\mathbf{0})\partial S/\partial \mathbf{x}$ giving

$$\begin{aligned} -\frac{\partial S}{\partial t} = {} & \frac{\partial S}{\partial \mathbf{x}}^T \mathbf{A}(\mathbf{0})\mathbf{x} + \frac{1}{2}\frac{\partial S}{\partial \mathbf{x}}^T \mathbf{G}(\mathbf{0})\mathbf{Q}\mathbf{G}^T(\mathbf{0})\frac{\partial S}{\partial \mathbf{x}} \\ & - \frac{1}{2}\left(\mathbf{z} - \mathbf{H}(\mathbf{0})\mathbf{x}\right)^T \mathbf{R}^{-1}\left(\mathbf{z} - \mathbf{H}(\mathbf{0})\mathbf{x}\right). \end{aligned} \tag{14}$$

This is essentially the finite time version of (7) with the observation term $\mathbf{z}$ included. The linear filter is obtained by supposing there is a solution of the form

$$S(\mathbf{x}, t) = \frac{1}{2}\left(\mathbf{x} - \hat{\mathbf{x}}(t)\right)^T \mathbf{V}^{-1}(t)\left(\mathbf{x} - \hat{\mathbf{x}}(t)\right) + a(t).$$

It can be shown that $-S(\mathbf{x}, t)$ is the exponent of the conditional probability density function of $\mathbf{x}(t)$. Thus $-S(\mathbf{x}, t)$ can be interpreted as the likelihood of the state trajectory passing through $\mathbf{x}$ at time t, given the observations $\mathbf{z}$ made up to time t. This is clearly maximised by $\mathbf{x} = \hat{\mathbf{x}}(t)$ which is therefore the maximum likelihood filtering solution. The equations for $\hat{\mathbf{x}}$ and $\mathbf{V}$ are obtained by calculating $\partial S/\partial t$ and $\partial S/\partial \mathbf{x}$ and substituting in (14). Since the dynamics are linear, $\mathbf{x}(t)$ is normal and so the conditional mean coincides with the conditional mode. In other words, in the linear case, the minimum variance and maximum likelihood solutions coincide and, indeed, it turns out that the equations obtained from (14) are the same as those for the linear Kalman filter (3) and (4). Details of the above are contained in [2], Section 5.3 and Examples 7.11 and 7.12 and will essentially be given in the derivation of the first order nonlinear solution below.

Consider now the nonlinear estimation problem away from $\mathbf{x} = \mathbf{0}$. We seek to minimise

$$\bar{J} = \frac{1}{2}\left(\mathbf{x}(t_0) - \bar{\mathbf{x}}(t_0)\right)^T \bar{\mathbf{V}}^{-1}(t_0)\left(\mathbf{x}(t_0) - \bar{\mathbf{x}}(t_0)\right) + \frac{1}{2}\int_{t_0}^{t}\left[\left(\mathbf{z}(\tau) - \mathbf{h}(\mathbf{x}(\tau))\right)^T \mathbf{R}^{-1}\left(\mathbf{z}(\tau) - \mathbf{h}(\mathbf{x}(\tau))\right) + \mathbf{w}^T(\tau)\mathbf{Q}^{-1}\mathbf{w}(\tau)\right] d\tau$$

subject to

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) + \mathbf{G}(\mathbf{x}(t))\mathbf{w}(t).$$

Letting

$$\bar{S}(\mathbf{x}, t) = \min_{\mathbf{w}(\tau)} \bar{J}$$

and repeating the above analysis leads to the following finite time version of (10)

$$-\frac{\partial \bar{S}}{\partial t} = \frac{\partial \bar{S}}{\partial \mathbf{x}}^T \mathbf{f}(\mathbf{x}) + \frac{1}{2}\frac{\partial \bar{S}}{\partial \mathbf{x}}^T \mathbf{G}(\mathbf{x})\mathbf{Q}\mathbf{G}^T(\mathbf{x})\frac{\partial \bar{S}}{\partial \mathbf{x}} - \frac{1}{2}\left(\mathbf{z} - \mathbf{h}(\mathbf{x})\right)^T \mathbf{R}^{-1}\left(\mathbf{z} - \mathbf{h}(\mathbf{x})\right) \tag{15}$$

again with the observation term $\mathbf{z}$ included. The value of $\mathbf{x}$ which maximises $-\bar{S}$ is the maximum likelihood estimate and is denoted $\hat{\mathbf{x}}(t)$. However it is no longer true that this coincides with the minimum variance estimate. Also it is not possible to derive an exact equation for $\hat{\mathbf{x}}(t)$ from (15).

We can however derive a first order approximate equation for $\hat{\mathbf{x}}(t)$ by expanding the various terms in (15) in Taylor series around $\hat{\mathbf{x}}(t)$ in terms up to order 1. For the sake of brevity, we will omit the dependence of $\hat{\mathbf{x}}$ on t in the following where $\hat{\mathbf{x}}$ appears inside another function. Since $\hat{\mathbf{x}}(t)$ minimises $\bar{S}(\mathbf{x}, t)$, we get that, close to $\hat{\mathbf{x}}$,

$$\frac{\partial \bar{S}(\mathbf{x}, t)}{\partial \mathbf{x}} = \frac{\partial^2 \bar{S}(\hat{\mathbf{x}}, t)}{\partial \mathbf{x}^2}(\mathbf{x} - \hat{\mathbf{x}}(t)) = \mathbf{V}^{-1}(t)(\mathbf{x} - \hat{\mathbf{x}}(t))$$

where we have written

$$\mathbf{V}^{-1}(t) = \frac{\partial^2 \bar{S}(\hat{\mathbf{x}}(t), t)}{\partial \mathbf{x}^2}.$$

To a first order approximation we then get

$$\bar{S}(\mathbf{x}, t) = \frac{1}{2}(\mathbf{x} - \hat{\mathbf{x}}(t))^T \mathbf{V}^{-1}(t)(\mathbf{x} - \hat{\mathbf{x}}(t)) + a(t) \tag{16}$$

and so

$$\frac{\partial \bar{S}(\mathbf{x}, t)}{\partial t} = \dot{a}(t) - (\mathbf{x} - \hat{\mathbf{x}}(t))^T \mathbf{V}^{-1}(t)\frac{d\hat{\mathbf{x}}}{dt} + \frac{1}{2}(\mathbf{x} - \hat{\mathbf{x}}(t))^T \frac{\mathbf{dV}^{-1}(t)}{dt}(\mathbf{x} - \hat{\mathbf{x}}(t)).$$

For the other terms in (15) we get

$$\begin{aligned}
\mathbf{f}(\mathbf{x}) &= \mathbf{f}(\hat{\mathbf{x}}) + \frac{\partial \mathbf{f}(\hat{\mathbf{x}})}{\partial \mathbf{x}}(\mathbf{x} - \hat{\mathbf{x}}(t)) \\
\mathbf{G}(\mathbf{x}) &= \mathbf{G}(\hat{\mathbf{x}}) + \frac{\partial \mathbf{G}(\hat{\mathbf{x}})}{\partial \mathbf{x}}(\mathbf{x} - \hat{\mathbf{x}}(t)) \\
\mathbf{h}(\mathbf{x}) &= \mathbf{h}(\hat{\mathbf{x}}) + \frac{\partial \mathbf{h}(\hat{\mathbf{x}})}{\partial \mathbf{x}}(\mathbf{x} - \hat{\mathbf{x}}(t)).
\end{aligned}$$

Substituting these into (15) and ignoring terms of order higher than two in $\mathbf{x}$ we get

$$\begin{aligned}
-\dot{a}(t) &+ (\mathbf{x}-\hat{\mathbf{x}}(t))^T \mathbf{V}^{-1}(t)\frac{d\hat{\mathbf{x}}}{dt} - \frac{1}{2}(\mathbf{x}-\hat{\mathbf{x}}(t))^T \frac{\mathbf{dV}^{-1}(t)}{dt}(\mathbf{x}-\hat{\mathbf{x}}(t)) \\
&= (\mathbf{x}-\hat{\mathbf{x}}(t))^T \mathbf{V}^{-1}(t)\mathbf{f}(\hat{\mathbf{x}}) + (\mathbf{x}-\hat{\mathbf{x}}(t))^T\mathbf{V}^{-1}(t)\frac{\partial \mathbf{f}(\hat{\mathbf{x}})}{\partial \mathbf{x}}(\mathbf{x}-\hat{\mathbf{x}}(t)) \\
&+ \frac{1}{2}(\mathbf{x}-\hat{\mathbf{x}}(t))^T \mathbf{V}^{-1}(t)\mathbf{G}(\hat{\mathbf{x}})\mathbf{Q}\mathbf{G}^T(\hat{\mathbf{x}})\mathbf{V}^{-1}(t)(\mathbf{x}-\hat{\mathbf{x}}(t)) \\
&- \frac{1}{2}\mathbf{z}^T\mathbf{R}^{-1}\mathbf{z} + (\mathbf{x}-\hat{\mathbf{x}}(t))^T \frac{\partial \mathbf{h}^T(\hat{\mathbf{x}})}{\partial \mathbf{x}}\mathbf{R}^{-1}(\mathbf{z}-\mathbf{h}(\hat{\mathbf{x}})) \\
&- \frac{1}{2}(\mathbf{x}-\hat{\mathbf{x}}(t))^T \frac{\partial \mathbf{h}^T(\hat{\mathbf{x}})}{\partial \mathbf{x}}\mathbf{R}^{-1}\frac{\partial \mathbf{h}(\hat{\mathbf{x}})}{\partial \mathbf{x}}(\mathbf{x}-\hat{\mathbf{x}}(t)) \\
&+ \mathbf{h}^T(\hat{\mathbf{x}})\mathbf{R}^{-1}\left(\mathbf{z}-\frac{1}{2}\mathbf{h}(\hat{\mathbf{x}})\right).
\end{aligned}$$

Now, equating terms of order 1 and 2 in $(\mathbf{x}-\hat{\mathbf{x}}(t))$ gives the first order approximate equations for $\hat{\mathbf{x}}$ and $\mathbf{V}^{-1}$

$$\frac{\mathbf{d}\hat{\mathbf{x}}}{dt} = \mathbf{f}(\hat{\mathbf{x}}) + \mathbf{V}(t)\frac{\partial \mathbf{h}^T(\hat{\mathbf{x}})}{\partial \hat{\mathbf{x}}}\mathbf{R}^{-1}(\mathbf{z}(t)-\mathbf{h}(\hat{\mathbf{x}})) \tag{17}$$

$$\begin{aligned}
-\frac{\mathbf{dV}^{-1}(t)}{dt} &= \mathbf{V}^{-1}(t)\frac{\partial \mathbf{f}(\hat{\mathbf{x}})}{\partial \hat{\mathbf{x}}} + \frac{\partial \mathbf{f}^T(\hat{\mathbf{x}})}{\partial \hat{\mathbf{x}}}\mathbf{V}^{-1}(t) \\
&+ \mathbf{V}^{-1}(t)\mathbf{G}(\hat{\mathbf{x}})\mathbf{Q}\mathbf{G}^T(\hat{\mathbf{x}})\mathbf{V}^{-1}(t) - \frac{\partial \mathbf{h}^T(\hat{\mathbf{x}})}{\partial \hat{\mathbf{x}}}\mathbf{R}^{-1}\frac{\partial \mathbf{h}(\hat{\mathbf{x}})}{\partial \hat{\mathbf{x}}}.
\end{aligned} \tag{18}$$

The initial conditions are obtained from the boundary condition for (15)

$$\bar{S}(\mathbf{x}(t_0),t_0) = \frac{1}{2}(\mathbf{x}(t_0)-\bar{\mathbf{x}}(t_0))^T \bar{\mathbf{V}}^{-1}(t_0)(\mathbf{x}(t_0)-\bar{\mathbf{x}}(t_0)).$$

For $\bar{S}$ of the form (16), this is satisfied by $\hat{\mathbf{x}}(t_0) = \bar{\mathbf{x}}(t_0)$ and $\mathbf{V}^{-1}(t_0) = \bar{\mathbf{V}}^{-1}(t_0)$. Implementing the algorithm (17) and (18) involves inverting $\mathbf{V}^{-1}(t)$ at each step. An explicit equation for $\mathbf{V}(t)$ is obtained from (18) by noting that $\mathbf{V}\mathbf{V}^{-1} = \mathbf{I}$ implies that

$$\frac{d\mathbf{V}}{dt} = -\mathbf{V}\left(\frac{d\mathbf{V}^{-1}}{dt}\right)\mathbf{V}.$$

and so

$$\begin{aligned}
\frac{\mathbf{dV}(t)}{dt} &= \frac{\partial \mathbf{f}(\hat{\mathbf{x}})}{\partial \hat{\mathbf{x}}}\mathbf{V}(t) + \mathbf{V}(t)\frac{\partial \mathbf{f}^T(\hat{\mathbf{x}})}{\partial \hat{\mathbf{x}}} \\
&+ \mathbf{G}(\hat{\mathbf{x}})\mathbf{Q}\mathbf{G}^T(\hat{\mathbf{x}}) - \mathbf{V}(t)\frac{\partial \mathbf{h}^T(\hat{\mathbf{x}})}{\partial \hat{\mathbf{x}}}\mathbf{R}^{-1}\frac{\partial \mathbf{h}(\hat{\mathbf{x}})}{\partial \hat{\mathbf{x}}}\mathbf{V}(t).
\end{aligned} \tag{19}$$

Equations (17) and (19) thus constitute a first order approximation to the maximum likelihood solution to the filtering problem (1) and (2). We note that these equations are the same as those for the extended Kalman filter which gives the minimum variance solution. As noted above these solutions do not generally coincide and so one can expect the higher order approximations to diverge from one another. Also, as noted in Section 2, the algorithm given in (8) and (9) can be thought of as an approximation to the steady state form of (17) and (19), should this exist. It is pointed out in [2], Section 5.3, that maximum likelihood estimation is of questionable value unless one knows in advance that the conditional probability density function of $\mathbf{x}(t)$ is unimodal and concentrated near the mode. The point of this section, however, was to indicate where the Hamilton-Jacobi equations (7) and (10) come from and how they are related to the derivation of the equations for the extended Kalman filter.

References

[1] J.C. Doyle, K. Glover, P.P. Khargonekar and B.A. Francis, State-space solutions to standard H_2 and H_∞ control problems, *IEEE Trans. Automatic Control*, **34(8)**, 831-847, 1989.

[2] A.H. Jazwinski, *Stochastic Processes and Filtering Theory*, Academic Press, New York, 1970.

[3] R.E. Kalman, New methods in Wiener filtering theory, in *Proceedings of the First Symposium on Engineering Applications of Random Function Theory and Probability*, J.L. Bogdanoff and F. Kozin eds., John Wiley and Sons, New York, 1963.

[4] A.P. Sage and J.L. Melsa, *Estimation Theory with Applications to Communications and Control*, McGraw-Hill, New York, 1971.

[5] A.J. van der Schaft, On a state space approach to nonlinear H_∞ control, *Systems & Control Lett.*, **16**, 1-8, 1991.

[6] W.M. Wonham, *Linear Multivariable Control: A Geometric Approach*, 3rd Ed., Springer-Verlag, New York, 1985.