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# Dynamic Wavelet and Equivalent Models

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**Abstract**

The representation of nonlinear dynamic wavelet models in the form of an equivalent global model which is valid over the operating range of the system is investigated. The results are used to analyse and interpret the nonlinear wavelet models using nonlinear frequency response functions.

## 1 Introduction

Discrete-time multiresolution wavelet models can be used as a basis for the identification of complex nonlinear systems. However these models are piece-wise polynomial and many of the well established procedures of nonlinear model analysis and interpretation, which are available only for global models defined over an operating range, can no longer be applied. This inhibits the interpretation of these models in both the time and frequency domain, makes it difficult to compare the results with other model forms, and complicates attempts to relate the models to the physical systems under study.

In the class of global models, polynomial models have been extensively studied and many procedures for system analysis and interpretation have been derived [Marmarelis and Marmarelis, 1978], [Schetzen, 1980], [Rugh, 1981]. The main objective of the present paper is to investigate the existence of global equivalents for discrete wavelet models which can then be interpreted using existing techniques.

The paper is organised as follows: section 2 provides a brief overview of the discrete-time wavelet model developed by Billings and Coca [1997]. In section 3 the wavelet model is analysed and a global approximation is derived. The existence of nonlinear transfer functions for the wavelet model is discussed in section 4. An experimental example is described in section 5 and the results are summarised in section 6.

## 2 The discrete-time wavelet model - brief overview

System modelling is an important prerequisite to nonlinear systems analysis. There are many ways of modelling but it can be shown that under two very mild assumptions, the input/output behaviour of a wide class of nonlinear discrete-time systems can be represented by a NARX (Nonlinear AutoRegressive model with eXogenous inputs) model of the form [Leontaritis and Billings, 1985]:

$$y(k) = F[y(k-1), \dots, y(k-n_y), u(k-1), \dots, u(k-n_u)] \tag{1}$$

where $F[\cdot]$ is some nonlinear function, $u$ and $y$ are the system input and output, and $n_u$ and $n_y$ represent the associated lags.

A means of approximating $F[\cdot]$ in equation (1) using known functions is desirable and the polynomial functions have been extensively studied in terms of interpretation and identification.

More recently many authors have proposed wavelet expansions [Chui and Wang, 1992], [Zhang and Beneviste, 1992], [Zhang, 1994]. In the present study an expansion of $F[\cdot]$ as a B-spline wavelet series, which was introduced by Billings and Coca [1997], will be investigated. In this section the wavelet expansion will be briefly presented with an emphasis on the type and constituents of the model.

The discrete-time multiresolution wavelet model proposed by Billings and Coca [1997] for a discrete-time dynamical system $\Sigma = (\mathcal{T}, \mathcal{U}, \mathcal{Y}, f)$, where $\mathcal{T} \in \mathbb{Z}$ is a discrete set, $\mathcal{U}$ and $\mathcal{Y}$ are the discrete input and output spaces and $f : (\mathcal{T}, \mathcal{U}, \mathcal{Y}) \rightarrow \mathcal{Y}$ is the response map of $\Sigma$, is represented as:

$$f(X) = \sum_{i=1}^{n_\Sigma} S_i^{s_i}(X_i) = \sum_{i=1}^{n_\Sigma} \left[ \sum_k c_{j_i,k}^i \phi_{j_i,k}(X_i) + \sum_k \sum_{l=j_i}^{j_f} d_{l,k}^i \psi_{l,k}(X_i) \right] \tag{2}$$

where $n_\Sigma$ is the number of $s_i$-dimensional additive submodels and $j_i$ and $j_f$ are the initial and final scale respectively. The $s_i$-dimensional regression vector $X_i \in \mathcal{U}$ can be represented as $X_i = \prod_{dim=1}^{s_i} x_{dim}$, where $x_{dim} \in \{y(k-1), \ldots, y(k-n_y), u(k-1), \ldots, u(k-n_u)\}$ is the vector of regression variables consisting of past outputs and inputs. The $s_i$-dimensional submodel $S_i^{s_i}(X_i)$ is implemented as the tensor-product of one-dimensional submodels [Billings and Coca, 1997]:

$$S_i^{s_i}(X_i) = \prod_{dim=1}^{s_i} S_i^{s_i}(x_{dim}) \cdot \tag{3}$$

The function $\phi_{j_i,k}(x)$ is the B-spline scaling function of degree $n$, given by $\phi_{j_i,k}(x) = 2^{j_i/2}\phi(2^{j_i}x - k)$ and the function $\psi_{l,k}(x)$ is the wavelet function of degree $n$ given by $\psi_{l,k}(x) = 2^{l/2}\psi(2^l x - k)$. The functions $\phi(x)$ and $\psi(x)$ are the corresponding mother scaling and wavelet functions which define the multiresolution approximation.

The explicit B-spline function $\phi^n(x)$ of degree $n$ is given by [Chui, 1992]:

$$\phi^n(x) = \sum_{j=0}^{n+1} \frac{(-1)^j}{n!} \begin{pmatrix} n+1 \\ j \end{pmatrix} [x-j]_+^n \tag{4}$$

$$[x]_+^n = \begin{cases} x^n & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

where the truncated powers $[x]_+^n$ are defined by (5). The scaling function $\phi^n(x)$ is symmetric, has compact support $[0, n+1]$ and consists of $n+1$ polynomial pieces of degree $n$ for every interval $[j, j+1)$, for $j < n$. The wavelet function $\psi^n(x)$ of degree $n$ is defined in terms of the scaling function $\phi^n(x)$ of degree $n$, by:

$$\psi^n(x) = \sqrt{2} \sum_{k=0}^{3n+1} q_k^n \phi^n(2x - k) \tag{6}$$

where

$$q_k^n = \frac{(-1)^k}{2^n} \sum_{l=0}^{n+1} \begin{pmatrix} n+1 \\ l \end{pmatrix} \phi^{2n}(k+1-l), \text{ where } k = 0, \ldots, 3n+1 \tag{7}$$

The wavelet function $\psi^n(x)$ is symmetric for $n$ even and anti-symmetric for $n$ odd, it has compact support $[0, 2n+1]$ and consist of $3n+4$ polynomial pieces of degree $n$ for every interval $[\frac{k}{2}, \frac{k+1}{2}]$, for $k < 2(2n-1)$.

# 3 Wavelet model analysis

The wavelet model has been successfully used to model various dynamical systems including both simulated and real systems [Billings and Coca, 1997]. The results show [Billings and Coca, 1997] that the wavelet models obtained produce excellent predictions and are qualitatively valid. They can also faithfully reproduce the bifurcation and Poincare map of the original dynamical system. The largest Lyapunov exponent and the correlation dimension are also closely matched by the estimated model. These results would be further enhanced if the wavelet model could be analysed using existing nonlinear analysis methods.

In this section it is shown that the wavelet model is equivalent to a spline function and therefore the wavelet model is a local model. In the present context the terminology local model will be used to mean a model identified for a particular system which is valid for a sub-range of the operational values, as opposed to a global model which will be used to refer to a model which is valid over the operating range used in the system identification. The local character of the wavelet model is the price paid for the flexibility and accuracy of the model. A global equivalent of the wavelet model will also be derived, expressed as a polynomial approximation.

The definition of a spline function is reviewed first:

**Definition 3.1** [Braess, 1980] A polynomial spline of order $n$ with simple nodes on the interval $[\alpha;\beta]$ is defined by:

$$s(x) = \sum_{j=0}^{n} b_j x^j + \sum_{i=1}^{k} a_i [x-\tau_i]_+^n \tag{8}$$

where $\alpha < \tau_1 < \tau_2 \ldots < \tau_k < \beta$ and the truncated powers are defined by (5). The restriction of the spline function (8) to every interval $(\alpha, \tau_1)$, $(\tau_1, \tau_2)$ , ..., $(\tau_k, \beta)$ is a polynomial of order $n$, $n-1$ times continuously differentiable at the nodes. If $\mathcal{C}^n_{[\alpha;\beta]}$ denotes the class of functions that are continuous and have $n$ continuous derivatives on the domain $[\alpha;\beta]$, the family of spline functions of order $n$, defined on the interval $[\alpha;\beta]$ is included in $\mathcal{C}^{n-1}_{[\alpha;\beta]}$.

An equivalent representation, which shows explicitly the continuity of the first $n-1$ derivatives at the nodes, is given by:

$$\begin{aligned} s(x) &= \sum_{j=0}^{n} b_j x^j + \sum_{i=1}^{k} a_i [x-\tau_i]_+^n \\ &= \begin{cases} \sum_{j=0}^{n} b_j x^j & x \in (-\infty;\alpha) \\ \sum_{j=0}^{n} b_j x^j + c_\alpha (x-\alpha)^n & x \in [\alpha;\tau_1) \\ \sum_{j=0}^{n} b_j x^j + c_\alpha (x-\alpha)^n + (c_{\tau_1} - c_\alpha)(x-\tau_1)^n & x \in [\tau_1;\tau_2) \\ \cdots\cdots\cdots & \\ \sum_{j=0}^{n} b_j x^j + c_\alpha (x-\alpha)^n + \ldots + (c_{\tau_k} - c_{\tau_{k-1}})(x-\tau_k)^n & x \in [\tau_k;\beta] \\ \sum_{j=0}^{n} b_j x^j & x \in (\beta;\infty) \end{cases} \end{aligned} \tag{9}$$

where $c_{\tau_i} - c_{\tau_{i-1}} = a_i$ for $i \in \{2; k\}$ and $c_\alpha = a_1$. This representation provides more insight and is useful in spline function generation because the coefficients $\{c_j | j = \alpha, \tau_1, \ldots, \tau_k, \beta\}$ are independent, while the coefficients $\{a_i\}$ in Definition 3.1 are connected through continuity conditions for the first $(n-1)$-order derivatives.

The wavelet model (2) is a linear combination of B-spline scaling and wavelet functions. Therefore the wavelet model is also expected to be a spline model. The following theorem shows the relationship between the equivalent spline model and the corresponding wavelet model.

**Theorem 3.1** Consider the discrete-time dynamical system $\Sigma = (\mathcal{T}, \mathcal{U}, \mathcal{Y}, f)$, where $\mathcal{T} \in \mathbb{Z}$ is a discrete set, $\mathcal{U}$ and $\mathcal{Y}$ are the discrete input and output spaces and $f : (\mathcal{T}, \mathcal{U}, \mathcal{Y}) \to \mathcal{Y}$ is the response map of $\Sigma$. If the response map representation $f$ is given by the multivariable multiresolution wavelet model:

$$f(X) = \sum_{i=1}^{n_\Sigma} S_i^{s_i}(X_i) = \sum_{i=1}^{n_\Sigma} \left[ \sum_k c^i_{j_i,k} \phi_{j_i,k}(X_i) + \sum_k \sum_{l=j_i}^{j_f} d^i_{l,k} \psi_{l,k}(X_i) \right] \tag{10}$$

consisting of $n_\Sigma$ additive $s_i$-dimensional submodels, with the notation defined in section 2, then the submodels $S_i^{s_i}(X_i)$ are represented by tensor-product polynomial spline functions:

$$S_i^{s_i}(X_i) = \prod_{dim=1}^{s_i} S_i^{s_i}(x_{dim}) = \prod_{dim=1}^{s_i} \sum_{r \in \mathbb{Z}} \alpha^i_{j_i,r} \left[ x_{dim} - \frac{r}{2^{j_f}} \right]_+^n \tag{11}$$

where $r$ represents the nodes of the spline model, $j_f$ is the final scale, $s_i$ is the dimension of the submodel $S_i^{s_i}(X_i)$, $X_i = \prod_{dim=1}^{s_i} x_{dim}$ and $x_{dim} \in \{y(k-1), \ldots, y(k-n_y), u(k-1), \ldots, u(k-n_u)\}$ as defined for equation (2). The proof is given in Appendix A.

Theorem 3.1 provides a new description of the wavelet model (10), which now can be seen as a sum of spline functions (11) defined for each regression vector $X_i$. In other words the model is a family of locally polynomial models. Each of the individual locally polynomial models is independent and has no influence on any other polynomial model. There is a fairly large number of locally polynomial models which may be necessary to characterise the nonlinear system. This number increases exponentially as the number of nodes of the spline functions is increased. For example, a 4-dimensional regression variable $x_{dim}$ and spline functions with 4 intervals (or 3 nodes) results in $4^4 = 256$ locally polynomial models.

The following theorem derives a very compact, equivalent model for the wavelet model (10) based on the modulus function of the nodes. The proof for this theorem is given in Appendix B.

**Theorem 3.2** If the $s_i$-dimensional submodel $S_i^{s_i}(x)$ is represented by the tensor-product polynomial spline functions:

$$S_i^{s_i}(X_i) = \prod_{dim=1}^{s_i} S_i^{s_i}(x_{dim}) = \prod_{dim=1}^{s_i} \sum_{r=m}^{M} \alpha^i_{j_i,r} \left[ x_{dim} - \frac{r}{2^{j_f+1}} \right]_+^n \tag{12}$$

then:

$$S_i^{s_i}(X_i) = \prod_{dim=1}^{s_i} S_i^{s_i}(x_{dim}) = \prod_{dim=1}^{s_i} \left[ \sum_{i=0}^{n} a_i x^i_{dim} + \sum_{k=m}^{M} b_k \left| x_{dim} - \frac{k}{2^{j_f+1}} \right| \left( x_{dim} - \frac{k}{2^{j_f+1}} \right)^{n-1} \right] \tag{13}$$

where $b_k = \frac{1}{2}\alpha^i_{j_i,k}$ and $a_i = \frac{(-1)^{n-i}}{2t^{n-i}} \begin{pmatrix} n \\ n-i \end{pmatrix} \sum_{k=m}^{M} k^{n-i}\alpha^i_{j_i,k}$.

Theorem 3.2 states that for each submodel and each regression variable $x_{dim} \in \{y(k-1),\ldots,y(k-n_y),u(k-1),\ldots,u(k-n_u)\}$, the equivalent model consists of a sum of an $n$-order polynomial and a sum of modulus terms of the type $|x - \frac{k}{2^{j_f+1}}|(x - \frac{k}{2^{j_f+1}})^{n-1}$ for each node of the spline model. For an $n$th-order spline function with $N-1$ distinct nodes the model will have $n+N-1+1 = n+N$ terms for each regression variable $x_{dim}$ in the submodel.

In conclusion, Theorem 3.1 gives the mathematical description of the wavelet model (2) as a sum of locally polynomial models and Theorem 3.2 finds an equivalent for this as a sum of polynomial and modulus functions.

In fact there is a high resemblance between the wavelet model (2) expressed in the form of (13) and the extended model set introduced by Billings and Chen [1989]. For the extended model set, the nonlinear function $F[\cdot]$ in the NARX model (1) contains, besides monomials of the lagged input and output, other nonlinear functions such as exponentials, hyperbolic functions $sinh(\cdot)$, $cosh(\cdot)$ and $tanh(\cdot)$, the inverse trigonometric function $atan(\cdot)$, coulomb friction $sgn(\cdot)$, saturation, etc. The extended model set was found to give a richer description of non-linear systems and a more effective modelling was achieved.

The idea of local modelling for nonlinear systems is not new. There are several possible ways in which nonlinear systems can be approximated by locally linear models (e.g. [Billings and Voon, 1987]), or locally nonlinear models (e.g. [Billings and Chen, 1989]). Billings and Chen [1989] also discuss the local-global modelling alternative. A nonlinear global model is often desirable for analysing the dynamic behaviour of the system using existing methods over a large range of operation and to design a control law that is valid for the whole operating range. One simple and practical way to obtain a global description of the nonlinear system from locally linear models is to use a least-squares polynomial approximation [Billings and Voon, 1987].

Theorem 3.2 states that the wavelet model consists of a sum of a polynomial and modulus functions. A global approximation of the $s_i$-dimensional submodel $S_i^{s_i}$, given by equation (13), is obtained by approximating the modulus functions $|x_{dim} - \frac{k}{2^{j_f+1}}|(x_{dim} - \frac{k}{2^{j_f+1}})^{n-1}$ with a polynomial least-squares approximator. The approximation is applied to all $N-1$ nodes, for each regression variable $x_{dim} \in \{y(k-1),\ldots,y(k-n_y),u(k-1),\ldots,u(k-n_u)\}$ in the submodel $S_i^{s_i}(x_{dim})$ in equation (13).

For example, in a wavelet model defined in terms of B-spline functions (4) of 2nd-degree ($n=2$), the least-squares polynomial approximating model on the interval $[-1;1]$ is obtained by replacing $|x_{dim} - \frac{k}{2^{j_f+1}}|(x_{dim} - \frac{k}{2^{j_f+1}})$ modulus functions in (13) with the approximation:

$$\left|x_{dim} - \frac{k}{2^{j_f+1}}\right|\left(x_{dim} - \frac{k}{2^{j_f+1}}\right) \approx 0.3075\left(x_{dim} - \frac{k}{2^{j_f+1}}\right) + \\ +0.8461\left(x_{dim} - \frac{k}{2^{j_f+1}}\right)^3 - 0.1834\left(x_{dim} - \frac{k}{2^{j_f+1}}\right)^5 + 0.0196\left(x_{dim} - \frac{k}{2^{j_f+1}}\right)^7 \quad (14)$$

As noted in [Billings and Chen, 1989], some nonlinear systems with strong nonlinearities cannot always be modelled by a globally unique model. The model approximation presented above will therefore be applied if it passes the model validity tests.

In conclusion the existence of a global model valid over the operating range, which is equivalent to the wavelet model, has been investigated in this section. The global equivalent is a polynomial approximating function, which is easier to interpret and to analyse using the nonlinear transfer function, which are introduced for the wavelet model in the next section. The theoretical results derived above are analysed and assessed for an experimental example in the Section 5.

# 4 The wavelet model and the nonlinear transfer function

The frequency domain representation of nonlinear systems can be very useful for analysis and interpretation. The system representations in the frequency domain are associated with nonlinear transfer functions and the existence of a these for the wavelet model will be studied in this section.

The nonlinear transfer functions are defined as the Laplace transform of the kernels in the Volterra series representation of the nonlinear system. The question of existence of a nonlinear transfer function is therefore intimately connected to the question of existence and convergence of the Volterra series representation. During the 1980's issues such as the existence and uniqueness of Volterra series were theoretically and practically debated [Lesiak and Krener, 1978], [Sandberg, 1983], [Boyd et al, 1984], [Boyd and Chua, 1985].

Sandberg [1983] established that for a wide class of systems a Volterra series representation will exist providing the nonlinearities are analytic. A later result obtained by Boyd and Chua [1985] led to the introduction of the fading memory concept. Boyd and Chua [1985] showed that even if a system has non-analytic nonlinearities but has the property of fading memory, the response can be approximated with arbitrary precision for all bounded input functions, by a Volterra series operator. They also proved that the Volterra series representation is unique.

Boyd and Chua give an example of a control system containing an ideal saturator, which is not analytic [Boyd and Chua, 1985]:

$$Sat(a) = \begin{cases} sgn(a) & |a| > 1 \\ a & |a| < 1 \end{cases} \qquad (15)$$

The nonlinearity in (15) is not analytic and therefore this system does not have an exact Volterra representation. However, it can be shown that (15) has a fading memory and therefore a Volterra series approximation can be found via a polynomial approximation [Boyd and Chua, 1985]. Not all systems have a fading memory, for example systems with multiple equilibria or systems with subharmonics are not members of this class [Boyd et al, 1984].

In fact the concept of a Volterra series representation is related to the concept of a unique or global polynomial model discussed in the previous section. If a nonlinear system has fading memory, there is a polynomial approximating Volterra series which models the system, in other words the system can be represented by a unique, global polynomial model. Therefore fading memory is a sufficient condition for the global model existence. It is not a necessary condition though.

The wavelet model, which is composed of a sum of polynomial and modulus functions, is not analytic because the modulus functions are not analytic. If however the system, for which the wavelet model has been derived or identified, has fading memory, the existence theorem stated by Boyd and Chua [1985] ensures the existence of an approximating Volterra series which is unique.

The existence of a Volterra series expansion determines the existence of the nonlinear transfer functions which are also unique. Therefore, a wavelet model can be mapped to the nonlinear transfer functions only if the system has fading memory. In this case, the nonlinear transfer functions of the wavelet model should be computed for an approximating global model.

The global polynomial approximating model derived in the previous section for the wavelet model corresponds to a polynomial expansion of $F[\cdot]$. The method of Peyton-Jones and Billings [1989] can therefore be applied to directly compute the nonlinear frequency response functions.

# 5 A case study

In the previous section it was shown that a dynamic wavelet model can be represented by a dynamic spline model, which can be further approximated by a dynamic polynomial model. The main application of this result will be in the interpretation of the identified wavelet models. Previous studies have shown that identified wavelet models have excellent qualitative validation properties and that they can capture the dynamical properties of the underlying system [Billings and Coca, 1997]. However in order to analyse and interpret these results using the well known methods derived mainly for analytic models, they have to be mapped into an analytic polynomial form. In this section the polynomial approximation of the wavelet model is exemplified for an experimental example.

## 5.1 Experimental example: Nonlinear Wave Force Data

An accurate prediction of wave forces on offshore structures that are subjected to random ocean waves is essential for safety and design. Recently extensive studies have been applied on a variety of experimental data, in order to assess traditional analysis methods and to derive new modelling procedures. Wave forces on structures composed of slender members are traditionally calculated on the basis of Morison's equation which was introduced by Morison et al [1950]:

$$F(t) = K_i \frac{du(t)}{dt} + K_d u(t)|u(t)| \tag{16}$$

where $F(t)$ is the force per unit axial length, $u(t)$ is the instantaneous flow velocity, and $K_i$ and $K_d$ are parameters which depend on the flow. Various attempts to determine new model structures to predict wave forces have been made. Stansby et al [1992] introduced the Morison-Duffing equation:

$$\alpha_1 \frac{d^2F(t)}{dt^2} + \alpha_2 \frac{dF(t)}{dt} + \alpha_3 F(t)|F(t)| + F(t) = K_i \frac{du(t)}{dt} + K_d u(t)|u(t)| \tag{17}$$

Recently Swain et al [1998] proposed the Dynamic Morison equation, which has a simpler form than the Morison-Duffing equation and is capable of generating all the relevant features of the wave force mechanism:

$$\alpha_1 \frac{d^2F(t)}{dt^2} + \alpha_2 \frac{dF(t)}{dt} + F(t) = K_i \frac{du(t)}{dt} + K_d u(t)|u(t)| \tag{18}$$

All the wave force models proposed (16), (17) and (18) contain the term $u(t)|u(t)|$. The same type of term is present in the equivalent representation of the B-spline wavelet model. It should be possible therefore to identify a very good B-spline wavelet model from the wave force data. In all previous attempts to identify a wave force model [Worden et al, 1994], [Swain et al, 1998] polynomial NARX models were estimated.

In this case study the Salford data set was identified and analysed in both the time and frequency domain and the results obtain by Swain et al [1998] are used for comparison. The Salford data set relates to a fixed vertical cylinder in random waves where the force and velocity data are obtained for a small spanwise element. The data was obtained in a laboratory wave flume for unidirectional waves with rectangular velocity spectra.

The input and output data represent horizontal water particle velocity and the inline force. The data were decimated by a factor of 2, giving an effective sampling frequency of 25 Hz and 1000 points of input-output data were generated. The wavelet identification procedure was applied by selecting $n_u = 3$, $n_y = 3$ with 2nd-order wavelet and scaling functions and scale values $j_i = j_f = 0$. The estimated model was given by $4^6 = 4096$ locally polynomial models. The equivalent representation of the wavelet model from equation (13) was:

$$
\begin{aligned}
y(k) &= 0.0073 \\
&+ 0.8216u(k-1) - 6.8499u(k-1)^2 + 7.4894(u(k-1)+0.0786)|u(k-1)+0.0786| \\
&- 5.7309(u(k-1)+0.0168)|u(k-1)+0.0168| \\
&+ 1.9424(u(k-1)-0.0451)|u(k-1)-0.0451| \\
&+ 3.1200u(k-2) + 12.5462u(k-2)^2 - 17.3698(u(k-2)+0.0786)|u(k-2)+0.0786| \\
&+ 12.2655(u(k-2)+0.0168)|u(k-2)+0.0168| \\
&- 6.8426(u(k-2)-0.0451)|u(k-2)-0.0451| \\
&- 3.9041u(k-3) - 6.7669u(k-3)^2 + 9.3520(u(k-3)+0.0786)|u(k-3)+0.0786| \\
&- 3.5621(u(k-3)+0.0168)|u(k-3)+0.0168| \\
&+ 4.0900(u(k-3)-0.0451)|u(k-3)-0.0451| \\
&+ 0.6145y(k-1) + 0.1915y(k-1)^2 + 0.2126(y(k-1)+0.4604)|y(k-1)+0.4604| \\
&+ 0.1220(y(k-1)-0.0648)|y(k-1)-0.0648| \\
&+ 0.3987(y(k-1)-0.5901)|y(k-1)-0.5901| \\
&+ 0.9417y(k-2) - 0.2469y(k-2)^2 - 0.4025(y(k-2)+0.4604)|y(k-2)+0.4604| \\
&- 0.1341(y(k-2)-0.0648)|y(k-2)-0.0648| \\
&- 0.6351(y(k-2)-0.5901)|y(k-2)-0.5901| \\
&- 0.5919y(k-3) + 0.0836y(k-3)^2 + 0.1664(y(k-3)+0.4604)|y(k-3)+0.4604| \\
&+ 0.0218(y(k-3)-0.0648)|y(k-3)-0.0648| \\
&+ 0.2340(y(k-3)-0.5901)|y(k-3)-0.5901| \qquad (19)
\end{aligned}
$$

The polynomial approximation equation (14) was applied and the relevant terms were selected using the Error Reduction Rate (ERR) [Korenberg et al, 1988], which measures the contribution of each term to the overall output variance. After selecting the most relevant terms the final model found was given by:

$$
\begin{aligned}
y(k) &= 0.0033 + 1.8262u(k-1) + 0.5636u(k-2) - 2.3631u(k-3) \qquad (20) \\
&\quad + 13.3654u(k-3)^3 + 1.2532y(k-1) + 0.3300y(k-1)^3 - 0.1068y(k-2)
\end{aligned}
$$

Figure 1 shows the model predicted output generated over the test data set, for the wavelet model and the polynomial approximating model (20). A metric which measures the closeness of fit between the predicted output and the measured output is the normalised root mean square error defined as:

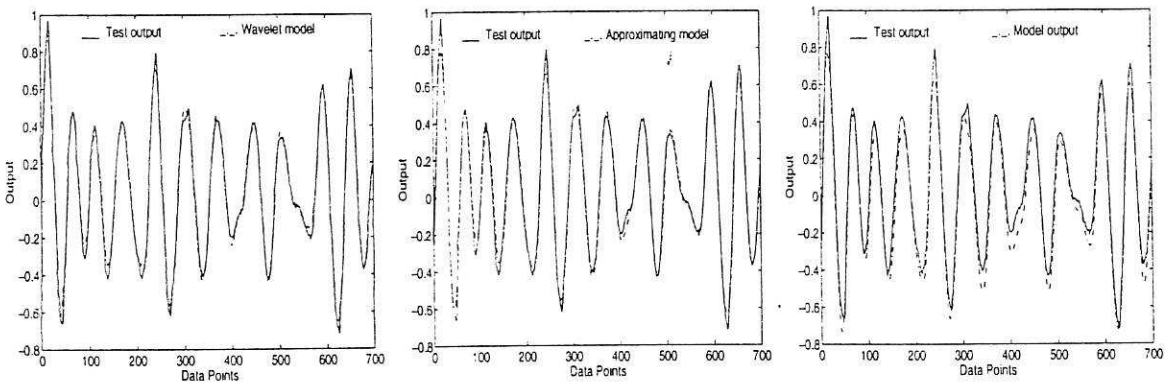
$$NMSE = \sqrt{\frac{\sum(y_{est}(k) - y(k)^2}{\sum(y(k) - y_{mean}(k))^2}} \tag{21}$$

where $y_{est}(k)$ is the model predicted output and $y_{mean}(k)$ is the mean value of the data set $y(k)$. The normalised mean square errors were 0.0115 for the wavelet model and 0.0177 for the polynomial approximation (20).

The polynomial NARX model identified by Swain et al [1998] for the same data set was:

$$\begin{aligned} y(k) &= 1.2829u(k-1) - 1.1957u(k-3) + 4.8262u(k-3)^3 \\ &\quad +1.5593y(k-1) - 0.44738y(k-2) - 0.15585y(k-3) \end{aligned} \tag{22}$$

Figure 1-(c) illustrates the predicted output of the model equation (22) over the same



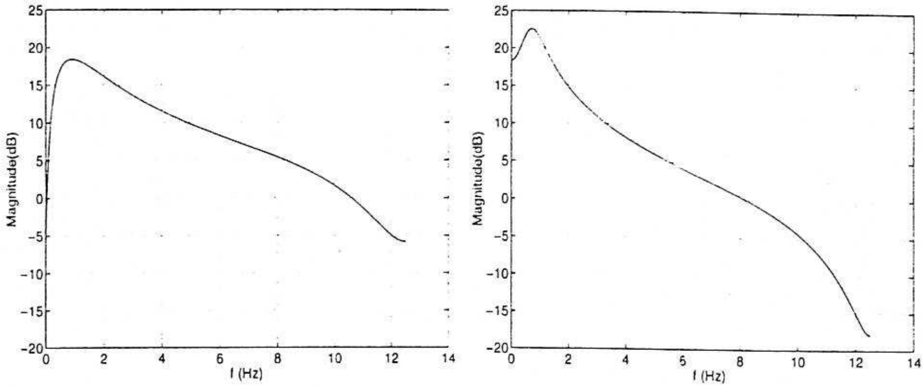

(a) (b) (c)

Figure 1: Model predicted output over the test set for (a) the wavelet model (b) the approximating polynomial model (20) and (c) the estimated model (22) [Swain et al, 1998]

test data set. The normalised mean square error was 0.0575. The polynomial approximating model equation (20) therefore predicts new data extremely well, even better than the model found by Swain et al [1998]. Although both models have a similar structure, model (20) found via the B-spline wavelet model identification gives significantly improved predictions over the test data set.

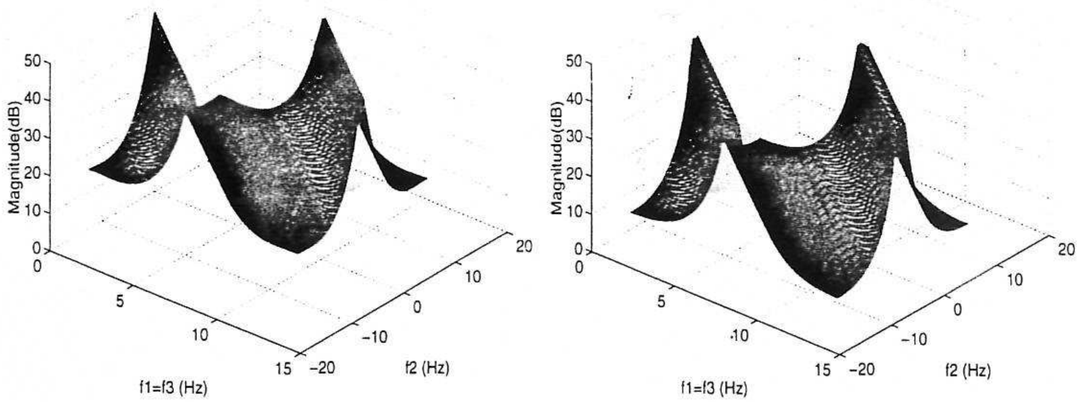
This result confirms the initial expectations, that the identification of a B-spline wavelet model is more suitable in wave force modelling. Another example of a system with a modulus type of nonlinearity is Chua's circuit, where the B-spline wavelet identification method also provided very good results [Billings and Coca, 1997].

The nonlinear transfer functions can now be computed for the model (20) and (22) by directly applying the method of Peyton-Jones and Billings [1989]. The first order and third order transfer functions are represented in Figures 2 and 3. The second order transfer functions are equal to zero in both cases. The first order transfer functions in Figure 2 show that the system exhibits a single resonance. The third order transfer functions in Figure 3 exhibit two highly dominant ridges which reveal the energy transfer phenomena associated with wave force systems.




(a) (b)

Figure 2: The linear transfer function for (a) polynomial approximating model (20) and (b) model (22) [Swain et al, 1998]




(a) (b)

Figure 3: The third order transfer function for (a) polynomial approximating model (20) and (b) model (22) [Swain et al, 1998]

In this section the main theoretical issues presented in Section 3 have been analysed for sampled experimental wave force data. The results show that very accurate B-spline wavelet models of real systems can be readily identified and mapped into the frequency domain using the new results derived in Section 3. The equivalent representation for the wavelet model provides a compact model expression. Because the B-spline wavelet model is composed of polynomial and modulus type terms, it gives very good results in the identification of systems with a polynomial or modulus type model terms.

# 6 Conclusions

The existence of a global model which is valid over the system operating range and which is equivalent to identified nonlinear wavelet models has been investigated. The advantages of wavelet model identification, which include excellent predictive and qualitative model properties can therefore be augmented by interpretation of the model in both the time and frequency domain using traditional nonlinear model analysis procedures.

The equivalent expression which was derived for the B-spline wavelet model gives a clear representation of the type of nonlinearities which are present in the model. The polynomial approximation of the B-spline wavelet model therefore provides an alternative to polynomial model identification of nonlinear systems.

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# Appendix A

Proof (Theorem 3.1): The one-dimensional case will be analysed initially. When $S_i = 1$, the submodel $S_i^1$, or simply $S_i$, is a linear combination of B-spline scaling and wavelet functions, in a one-dimensional regression vector $X_i$:

$$S_i(X_i) = \sum_k c^i_{j_i,k}\phi_{j_i,k}(X_i) + \sum_k \sum_{l=j_i}^{j_f} d^i_{l,k}\psi_{l,k}(X_i) = S_i^I(X_i) + S_i^{II}(X_i) \tag{23}$$

By using the B-spline scaling $\phi_{j_i,k}(x)$ and wavelet function $\psi_{l,k}(x)$ expressions given in section 2, a general formula is given for the sub-model $S_i(X_i)$ in (23). The linear combination of B-Spline scaling functions $S_i^I(X_i)$ is given by:

$$\begin{aligned} S_i^I(X_i) &= \sum_k c^i_{j_i,k}\phi_{j_i,k}(X_i) = \\ &= \sum_k c^i_{j_i,k} 2^{\frac{j_i}{2}} \sum_{l=0}^{n+1} \frac{(-1)^l}{n!} \begin{pmatrix} n+1 \\ l \end{pmatrix} [2^{j_i} X_i - k - l]_+^n \end{aligned} \tag{24}$$

where it can easily be shown that:

$$[2_{j_i} X_i - k - l]_+^n = 2^{j_i n} \left[ X_i - \frac{k+l}{2^{j_i}} \right]_+^n \tag{25}$$

The linear combination of spline functions is therefore given by a polynomial spline function with the node values determined by the translations $k$ and the initial resolution $j_i$:

$$\begin{aligned} S_i^I(X_i) &= \sum_k c^i_{j_i,k} 2^{j_i(\frac{1}{2}+n)} \sum_{l=0}^{n+1} \frac{(-1)^l}{n!} \begin{pmatrix} n+1 \\ l \end{pmatrix} \left[ X_i - \frac{k+l}{2^{j_i}} \right]_+^n \\ &= \sum_p s^i_{j_i,p} \left[ X_i - \frac{p}{2^{j_i}} \right]_+^n \end{aligned} \tag{26}$$

$$\text{where} \quad p = k + l; \tag{27}$$

$$\text{and} \quad s^i_{j_i,p} = 2^{j_i(\frac{1}{2}+n)} \sum_{l=0}^{n} c^i_{j_i,(p-l)} \frac{(-1)^l}{n!} \begin{pmatrix} n+1 \\ l \end{pmatrix} \tag{28}$$

The spline function corresponding to the wavelet function is found in a similar manner. The linear combination of B-Spline wavelet functions $S_i^{II}(X_i)$ is defined as:

$$\begin{aligned} S_i^{II}(X_i) &= \sum_m \sum_{l=j_i}^{j_f} d^i_{l,m}\psi_{l,m}(X_i) \\ &= \sum_m \sum_{l=j_i}^{j_f} d^i_{l,m} \left( \sum_{k=0}^{3n+1} q_k^n \left( \sum_{j=0}^{n+1} \frac{(-1)^j}{n!} \begin{pmatrix} n+1 \\ j \end{pmatrix} 2^{(l+1)(\frac{1}{2}+n)} \left[ X_i - \frac{1}{2^l} \left( \frac{k+j}{2} + m \right) \right]_+^n \right) \right) \\ &= \sum_r t^i_{j_i,r} \left[ X_i - \frac{r}{2^{j_f+1}} \right]_+^n \end{aligned} \tag{29}$$

$$\text{where} \quad r = k + j + 2m; \tag{30}$$

$$\text{and} \quad t^i_{j_i,r} = \sum_{l=j_i}^{j_f} 2^{(l+1)(\frac{1}{2}+n)} \sum_m d^i_{l,m} \sum_{j=0}^{min(r-2m;n+1)} q^n_{r-2m-j} \frac{(-1)^j}{n!} \begin{pmatrix} n+1 \\ j \end{pmatrix} \tag{31}$$

The coefficients $a_i$ and $b_k$, where $i \in \{0;\ldots;n\}$ and $k \in \{m;\ldots;M\}$, can be determined by equating coefficients of every $x^i$ from the expansions (35) and (36) on each interval $\left[\frac{k}{2^{j_f+1}};\frac{k+1}{2^{j_f+1}}\right)$. In this manner a homogeneous system of linear $(n+1)\times(M-m+2)$ equations with $n+1+M-m+1$ unknowns is generated. It can easily be verified that $(n+1)\times(M-m+1+1) = n+1+M-m+1+n(M-m+1)+1 > n+1+M-m+1$ and therefore the system of equations is over-determined. This system always has a unique solution.

By subtracting any two consecutive lines in (35) and (36) respectively, the coefficients $b_k$ are determined as:

$$b_k = \frac{1}{2}\alpha^i_{j_i,k} \tag{37}$$

Continuity conditions for $n-1$ derivatives for every node are also imposed. These conditions transform the over-determined system of equations into a determined system with a unique solution. The coefficients $a_i$ are also uniquely determined. By equating the coefficients of every $x^i$ on the interval $\left(-\infty;\frac{m}{2^{j_f+1}}\right)$:

$$a_i = \frac{(-1)^{n-i}}{2t^{n-i}}\begin{pmatrix} n \\ n-i \end{pmatrix}\sum_{k=m}^{M} k^{n-i}\alpha^i_{j_i,k} \tag{38}$$

By equating the coefficients of every $x^i$ on subsequent intervals, the values found for the coefficients $a_i$ are also given by (38), providing the continuity conditions are fulfilled.

Consider now the multi-dimensional case, when $S_i > 1$. The $s_i$-dimensional spline model is given by a tensor-product of one-dimensional sub-models:

$$S_i^{s_i}(X_i) = \prod_{dim=1}^{s_i} S_i^{s_i}(x_{dim}) = \prod_{dim=1}^{s_i}\left[\sum_{i=0}^{n} a_i x^i_{dim} + \sum_{k=m}^{M} b_k\left|x_{dim} - \frac{k}{2^{j_f+1}}\right|\left(x_{dim} - \frac{k}{2^{j_f+1}}\right)^{n-1}\right] \tag{39}$$

Note that for the $s_i$-dimensional case, the exact order of approximation does not change when the approximations for all nodes $\frac{k}{2^{j_f+1}}$, where $k \in \{m; M\}$ are added together or when the submodels are multiplied using the tensor-product, providing the approximation is made on the interval $[-1;1]$.

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