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The Jacobian algebras

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Abstract

A new class of algebras (the Jacobian algebras) is introduced and studied in detail. The Jacobian algebras are obtained from the Weyl algebras by inverting (not in the sense of Ore) of certain elements. Surprisingly, the Jacobian algebras and the Weyl algebras have little in common. Moreover, they have almost opposite properties.

Key Words: the Jacobian algebras, prime ideal, prime spectrum, unique factorization of ideal, minimal primes, group of units, commutant, integro-differential operators.

Mathematics subject classification 2000: 16D25, 16S99, 16U60, 16U70, 16S60.

1 Introduction

The aim of this paper is to introduce a new class of algebras, the so-called Jacobian algebras. They arose in my study of the group of polynomial automorphisms and the Jacobian Conjecture, which is a conjecture that makes sense only for polynomial algebras in the class of all commutative algebras [6]. In order to solve the Jacobian Conjecture, it is reasonable to believe that one should create technique which makes sense only for polynomials; the Jacobian algebras are a step in this direction (they exist for polynomials but make no sense even for Laurent polynomials). In this Introduction, we describe the main results of the paper.

Throughout, ring means an associative ring with 1. Let $K$ be a commutative $\mathbb{Q}$-algebra, $K^*$ be its group of units, $P_n := K[x_1, \ldots, x_n]$ be a polynomial algebra over $K$; $\partial_1 := \frac{\partial}{\partial x_1}, \ldots, \partial_n := \frac{\partial}{\partial x_n}$ be the partial derivatives ($K$-linear derivations) of $P_n$.

Definition: The Jacobian algebra $A_n$ is the subalgebra of $\text{End}_K(P_n)$ generated by the Weyl algebra $A_n := K\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \rangle$ and the elements $H_1^{-1}, \ldots, H_n^{-1} \in \text{End}_K(P_n)$ where $H_1 := \partial_1 x_1, \ldots, H_n := \partial_n x_n$.

Clearly, $A_n = A_1(1) \otimes \cdots \otimes A_1(n) \simeq A_1^\otimes n$ where $A_1(i) := K\langle x_i, \partial_i, H_i^{-1} \rangle \simeq A_1$. The algebra $A_n$ contains all the integrations $\int_i : P_n \to P_n, p \mapsto \int p \, dx_i$, since $\int_i = x_i H_i^{-1}$. In particular, the algebra $A_n$ contains all (formal) integro-differential operators with polynomial coefficients. This fact explains (i) the significance of the algebras $A_n$ for Algebraic Geometry and the theory of integro-differential operators; (ii) why the algebras $A_n$ and $A_1$ have different properties, and (iii) why the group $A_n^*$ of units of the algebra $A_n$ is huge (there are many invertible integro-differential operators).
General properties of the Jacobian algebras.

Until the end of this section, $K$ is a field of characteristic zero. When $n = 1$ the group $A_1^*$ is found explicitly, $A_1^* \simeq K^* \times (\mathbb{Z}^{(2)} \ltimes \text{GL}_\infty(K))$ (Theorem 4.2) as well as an inversion formula $u^{-1}$ for $u \in A_1^*$. This gives explicitly polynomial solutions for all invertible integro-differential operators on an affine line: $uy = f \Rightarrow y = u^{-1}f$ where $f \in K[x_1]$ and $y$ is an unknown. For $n \geq 2$, a description of the group $A_n^*$ is given (Theorem 4.4), it looks like it is a challenging problem to find an inversion formula for $u \in A_n^*$ (one should go far beyond the Dieudonné determinant). Though, a criterion of invertibility is found (Theorem 4.5).

Moreover, the group $A_n^*$ contains the subgroup $K^* \times ((\mathbb{Z}^n)^{(2)} \ltimes \text{GL}_\infty(K))$, elements of which are called minimal integro-differential operators. For each such an operator $u$ one can write down an inversion formula $u^{-1}$ in the same manner as in the case $n = 1$, and, therefore, one obtains explicitly polynomial solutions for all minimal integro-differential equations $uy = f$ where $f \in P_n$.

The Weyl algebra $A_n = A_n(K)$ is a simple, Noetherian domain of Gelfand-Kirillov dimension $\text{GK}(A_n) = 2n$. The Jacobian algebra $A_n$ is neither left nor right Noetherian, it contains infinite direct sums of nonzero left and right ideals. This means that the concept of the left (and right) Krull dimension makes no sense for $A_n$ but the classical Krull dimension of $A_n$ is $n$ (Corollary 3.7). The algebra $A_n$ is a central, prime algebra of Gelfand-Kirillov dimension $3n$ (Corollary 2.7).

The canonical involution $\theta$ of the Weyl algebra can be extended to the algebra $A_n$ (see (15)). This means that the algebra $A_n$ is self-dual ($A_n \simeq A_n^{\theta}$), and so its left and right algebraic properties are the same. Note that the Fourier transform on the Weyl algebra $A_n$ can not be lifted to $A_n$. Many properties of the algebra $A_n = A_n^{\theta}$ are determined by properties of $A_1$. When $n = 1$ we usually drop the subscript ‘1’ in $x_1$, $\partial_1$, $H_1$, etc. The algebra $A_1$ contains the only proper ideal $F = \oplus_{i,j \in \mathbb{N}} KE_{ij}$ where

$$E_{ij} := \begin{cases} x^{i-j} (x^j \frac{1}{\partial_j x^j} \partial_j - x^{j+1} \frac{1}{\partial_j x^j} \partial_j^{j+1}) & \text{if } i \geq j, \\ \left(\frac{1}{\partial_i x^i}\right)^{j-i} (x^j \frac{1}{\partial_j x^j} \partial_j - x^{j+1} \frac{1}{\partial_j x^j} \partial_j^{j+1}) & \text{if } i < j. \end{cases}$$

As a ring without 1, the ring $F$ is canonically isomorphic to the ring $M_\infty(K) := \lim M_d(K) = \oplus_{i,j \in \mathbb{N}} KE_{ij}$ of infinite-dimensional matrices where $E_{ij}$ are the matrix units ($F \twoheadrightarrow M_\infty(K)$, $E_{ij} \mapsto E_{ij}$). This is a very important fact as we can apply concepts of finite-dimensional linear algebra (like trace, determinant, etc) to integro-differential operators which is not obvious from the outset. This fact is crucial in finding an inversion formula for elements of $A_1$.

The algebra $A_n = \oplus_{\alpha \in \mathbb{Z}^n} A_{n,\alpha}$ is a $\mathbb{Z}^n$-graded algebra where $A_{n,\alpha} := \otimes_{k=1}^n A_{1,\alpha_k}(k)$ and, for $n = 1$, (Theorem 2.3)

$$A_{1,i} = \begin{cases} x^i \mathcal{D}_1 & \text{if } i \geq 1, \\ \mathcal{D}_1 & \text{if } i = 0, \\ \mathcal{D}_1 \partial^{-i} & \text{if } i \leq -1, \end{cases}$$

where $\mathcal{D}_1 := L \oplus (\oplus_{i,j \geq 1} Kx^i H^{-j} \partial^i)$ is a commutative, non-Noetherian algebra and $L = K[H^1, (H + 1)^{-1}, (H + 2)^{-1}, \ldots]$. This gives a ‘compact’ $K$-basis for the algebra $A_1$ (and
\( A_n \). This basis ‘behaves badly’ under multiplication. A more conceptual (‘multiplicatively friendly’) basis is given in Theorem 2.5.

- (Corollary 2.7.(10)) \( P_n \) is the only faithful, simple \( A_n \)-module.

**Spec** (\( A_n \)). 0 is a prime ideal of \( A_n \).

\[
p_1 := F \otimes A_{n-1}, p_2 := A_1 \otimes F \otimes A_{n-2}, \ldots, p_n := A_{n-1} \otimes F,
\]

are precisely the prime ideals of height 1 of \( A_n \). Let \( \text{Sub}_n \) be the set of all subsets of \( \{1, \ldots, n\} \).

- (Corollary 3.5) The map \( \text{Sub}_n \rightarrow \text{Spec}(A_n) \), \( I \mapsto p_I := \sum_{i \in I} p_i, \emptyset \mapsto 0 \), is a bijection, i.e. any nonzero prime ideal of \( A_n \) is a unique sum of primes of height 1; \( |\text{Spec}(A_n)| = 2^n \); the height of \( p_I \) is \( |I| \); and

- (Lemma 3.6) \( p_I \subseteq p_J \) iff \( I \subseteq J \).

- (Corollary 3.15) \( a_n := p_1 + \cdots + p_n \) is the only prime ideal of \( A_n \) which is completely prime; \( A_n \) is the only ideal \( a \) of \( A_n \) such that \( a \neq A_n \) and \( A_n/a \) is a Noetherian (resp. left Noetherian, resp. right Noetherian) ring.

**Ideals of** \( A_n \) **and their unique factorization.** The ideal theory of \( A_n \) is ‘very arithmetic.’ Let \( B_n \) be the set of all functions \( f : \{1, 2, \ldots, n\} \rightarrow \{0, 1\} \). For each function \( f \in B_n \), \( I_f := I_{f(1)} \otimes \cdots \otimes I_{f(n)} \) is the ideal of \( A_n \) where \( I_0 := F \) and \( I_1 := A_1 \). Let \( C_n \) be the set of all subsets of \( B_n \) all distinct elements of which are incomparable (two distinct elements \( f \) and \( g \) of \( B_n \) are incomparable if neither \( f(i) \leq g(i) \) nor \( f(i) \geq g(i) \) for all \( i \)). For each \( C \in C_n \), let \( I_C := \sum_{f \in C} I_f \), the ideal of \( A_n \). The next result classifies all the ideals of \( A_n \).

- (Theorem 3.1) The map \( C \mapsto I_C := \sum_{f \in C} I_f \) from the set \( C_n \) to the set of ideals of \( A_n \) is a bijection where \( I_0 := 0 \). In particular, there are only finitely many ideals, say \( s_n \), of \( A_n \). Moreover, \( 2 - n + \sum_{i=1}^n 2^n \leq s_n \leq 2^{2^n} \) (Corollary 3.4).

- Each ideal \( I \) of \( A_n \) is an idempotent ideal, i.e. \( I^2 = I \).

- Ideals of \( A_n \) commute \( (IJ = JI) \).

- (Theorem 3.11) The lattice of ideals of \( A_n \) is distributive.

- (Corollary 2.7.(4,7)) The ideal \( a_n \) is the largest (hence, the only maximal) ideal of \( A_n \) distinct from \( A_n \), and \( F \otimes A_n \) is the smallest nonzero ideal of \( A_n \).

- (Corollary 2.7.(11)) \( \text{GK}(A_n/a) = 3n \) for all ideals \( a \) of \( A_n \) such that \( a \neq A_n \).

For each ideal \( a \) of \( A_n \), \( \text{Min}(a) \) denotes the set of minimal primes over \( a \). Two distinct prime ideals \( p \) and \( q \) are called incomparable if neither \( p \subseteq q \) nor \( p \supseteq q \). The algebras \( A_n \) have beautiful ideal theory as the following unique factorization properties demonstrate.
• (Theorem 3.8) 1. Each ideal \( a \) of \( \mathbb{A}_n \) such that \( a \neq \mathbb{A}_n \) is a unique product of incomparable primes, i.e. if \( a = q_1 \cdots q_s = t_1 \cdots t_t \) are two such products then \( s = t \) and \( q_1 = r_{\sigma(1)}, \ldots, q_s = r_{\sigma(s)} \) for a permutation \( \sigma \) of \( \{1, \ldots, n\} \).

2. Each ideal \( a \) of \( \mathbb{A}_n \) such that \( a \neq \mathbb{A}_n \) is a unique intersection of incomparable primes, i.e. if \( a = q_1 \cap \cdots \cap q_s = t_1 \cap \cdots \cap t_t \) are two such intersections then \( s = t \) and \( q_1 = r_{\sigma(1)}, \ldots, q_s = r_{\sigma(s)} \) for a permutation \( \sigma \) of \( \{1, \ldots, n\} \).

3. For each ideal \( a \) of \( \mathbb{A}_n \) such that \( a \neq \mathbb{A}_n \), the sets of incomparable primes in statements 1 and 2 are the same, and so \( a = q_1 \cdots q_s = q_1 \cap \cdots \cap q_s \).

4. The ideals \( q_1, \ldots, q_s \) in statement 3 are the minimal primes of \( a \), and so \( a = \prod_{p \in \text{Min}(a)} p = \cap_{p \in \text{Min}(a)} p \).

• (Corollary 3.10) \( a \cap b = ab \) for all ideals \( a \) and \( b \) of \( \mathbb{A}_n \).

The next theorem gives all decompositions of an ideal as a product or intersection of ideals.

• (Theorem 3.12) Let \( a \) be an ideal of \( \mathbb{A}_n \), and \( \mathcal{M} \) be the minimal elements with respect to inclusion of a set of ideals \( a_1, \ldots, a_k \) of \( \mathbb{A}_n \). Then

1. \( a = a_1 \cdots a_k \) iff \( \text{Min}(a) = \mathcal{M} \).

2. \( a = a_1 \cap \cdots \cap a_k \) iff \( \text{Min}(a) = \mathcal{M} \).

This is a rare example of a non-commutative algebra of Krull dimension \( > 1 \) where one has a complete picture of decompositions of ideals.

**The group \( \mathbb{A}_1^* \) of units of \( \mathbb{A}_1 \).** For each integer \( i \geq 1 \), consider the element of \( \mathbb{A}_1^* \):

\[
(H - i)^{-1} := \begin{cases} 
\frac{1}{\frac{1}{i!} \partial^i} (1 + x) \frac{1}{H \partial x} & \text{if } i = 1, \\
\frac{x^i}{\frac{1}{i!} \partial^i} (1 + x) \sum_{j=0}^{i-2} \frac{1}{j!} \partial^{j+1} & \text{if } i \geq 2,
\end{cases}
\]

where \( \pi_j := x^j \frac{1}{\partial x^j} \partial^j - x^{j+1} \frac{1}{\partial x^{j+1}} \partial^{j+1} \). Consider the following subgroup of \( \mathbb{A}_1^* \):

\[
\mathcal{H} := \prod_{i \geq 0} (H + i)^{n_i} \times \prod_{i \geq 1} (H - i)^{n_i} \mid (n_i) \in \mathbb{Z}^{(\mathbb{Z})}
\]

where \( \mathbb{Z}^{(\mathbb{Z})} \) is the direct sum of \( \mathbb{Z} \) copies of the group \( \mathbb{Z} \), see (32) for detail. Let \( \text{GL}_\infty(K) := \{ u \in 1 + M_\infty(K) \mid \det(u) \neq 0 \} \). The group \( (1 + F)^* \) of units of the multiplicative monoid \( 1 + F \) is equal to \( (1 + F)^* = (1 + M_\infty(K))^* = \text{GL}_\infty(K) \). Note that \( (1 + F)^* \subseteq \mathbb{A}_1^* \).

• (Theorem 4.2) 1. \( \mathbb{A}_1^* = K^* \times (\mathcal{H} \times (1 + F)^*) \), each unit \( a \) of \( \mathbb{A}_1 \) is a unique product \( a = \lambda \alpha(1 + f) \) for some elements \( \lambda \in K^* \), \( \alpha \in \mathcal{H} \), and \( f \in F \) such that \( \det(1 + f) \neq 0 \).

2. \( \mathbb{A}_1^* = K^* \times (\mathcal{H} \times \text{GL}_\infty(K)) \).

3. The centre of the group \( \mathbb{A}_1^* \) is \( K^* \).

4. The commutant \( \mathbb{A}_1^{(2)} := [\mathbb{A}_1^*, \mathbb{A}_1^*] \) of the group \( \mathbb{A}_1^* \) is equal to \( \text{SL}_\infty(K) := \{ v \in (1 + F)^* = M_\infty(K) \mid \det(v) = 1 \} \), and \( \mathbb{A}_1/[\mathbb{A}_1^*, \mathbb{A}_1^*] \simeq K^* \times \mathcal{H} \times K^* \).

5. All the higher commutants \( \mathbb{A}_1^{(i)} := [\mathbb{A}_1^*, \mathbb{A}_1^{(i-1)}] \), \( i \geq 2 \), are equal to \( \mathbb{A}_1^{(2)} \).
The group of units $A_n^*$ of $A_n$.

- (Theorem 4.4) 1. $A_n^* = K^* \times (H_n \ltimes (1 + a_n)^*)$ where $H_n := \prod_{i=1}^n H(i)$ and $a_n := p_1 + \cdots + p_n$.
  
2. The centre of the group $A_n^*$ is $K^*$.

Theorem 4.5 is a criterion of when an element of the monoid $1 + a_n$ belongs to its group $(1 + a_n)^*$ of units.

**Question.** Is the global dimension of $A_n$ equal to $2n$ (or $\infty$)?

## 2 The Jacobian algebras and localizations of the Weyl algebras

In this section, two $K$-bases for the Jacobian algebras are found (Theorems 2.3 and 2.5), and several properties of the algebras $A_n$ are proved: $A_n$ is a central, prime, self-dual, non-Noetherian algebra.

We start by recalling some properties of generalized Weyl algebras. Some of these algebras are factor algebras of the Jacobian algebras.

**Generalized Weyl Algebras.** Let $D$ be a ring, $\sigma = (\sigma_1, ..., \sigma_n)$ be an $n$-tuple of commuting automorphisms of $D$, and $a = (a_1, ..., a_n)$ be an $n$-tuple of (non-zero) elements of the centre $Z(D)$ of $D$ such that $\sigma_i(a_j) = a_j$ for all $i \neq j$.

The **generalized Weyl algebra** $A = D(\sigma, a)$ (briefly GWA) of degree $n$ with the base ring $D$ is a ring generated by $D$ and $2n$ indeterminates $x_1, ..., x_n, y_1, ..., y_n$ subject to the defining relations [3], [4]:

$$
y_i x_i = a_i, \quad x_i y_i = \sigma_i(a_i),
$$

$$
x_i \alpha = \sigma_i(\alpha)x_i, \quad y_i \alpha = \sigma_i^{-1}(\alpha)y_i, \quad \alpha \in D,
$$

$$
[x_i, x_j] = [y_i, y_j] = [x_i, y_j] = 0, \quad i \neq j,
$$

where $[x, y] = xy - yx$. We say that $a$ and $\sigma$ are the sets of defining elements and automorphisms of $A$ respectively. The GWAs are also known as hyperbolic rings, see the book of Rosenberg [20]. For a vector $k = (k_1, ..., k_n) \in \mathbb{Z}^n$, let $v_k = v_{k_1}(1) \cdots v_{k_n}(n)$ where, for $1 \leq i \leq n$ and $m \geq 0$: $v_m(i) = x_i^m$, $v_{-m}(i) = y_i^m$, $v_0(i) = 1$. It follows from the definition of the GWA that

$$
A = \bigoplus_{k \in \mathbb{Z}^n} A_k
$$

is a $\mathbb{Z}^n$-graded algebra ($A_k A_e \subseteq A_{k+e}$, for all $k, e \in \mathbb{Z}^n$), where $A_k = Dv_k = v_k D$.

The tensor product (over the base field) $A \otimes A'$ of generalized Weyl algebras of degree $n$ and $n'$ respectively is a GWA of degree $n + n'$:

$$
A \otimes A' = D \otimes D'((\sigma, \sigma'), (a, a')).
$$
Let $P_n$ be a polynomial algebra $K[H_1, \ldots, H_n]$ in $n$ indeterminates and let $\sigma = (\sigma_1, \ldots, \sigma_n)$ be an $n$-tuple of commuting automorphisms of $P_n$ such that $\sigma_i(H_i) = H_i - 1$ and $\sigma_i(H_j) = H_j$, for $i \neq j$. Let $A_n = K \langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \rangle$ be the Weyl algebra. The algebra homomorphism

$$A_n \rightarrow P_n((\sigma_1, \ldots, \sigma_n), (H_1, \ldots, H_n)), \ x_i \mapsto x_i, \ \partial_i \mapsto y_i, \ i = 1, \ldots, n,$$

is an isomorphism. We identify the Weyl algebra $A_n$ with the GWA above via this isomorphism. Note that $H_i = \partial_i x_i = x_i \partial_i + 1$. Denote by $S_n$ the multiplicative submonoid of $P_n$ generated by the elements $H_i + j$, $i = 1, \ldots, n$, and $j \in \mathbb{Z}$. It follows from the above presentation of the Weyl algebra $A_n$ as a GWA that $S_n$ is an Ore set in $A_n$, and, using the $\mathbb{Z}^n$-grading, that the (two-sided) localization $A_n := S_n^{-1} A_n$ of the Weyl algebra $A_n$ at $S_n$ is the skew Laurent polynomial ring

$$A_n = S_n^{-1} P_n[x_1^{\pm 1}, \ldots, x_n^{\pm 1}; \sigma_1, \ldots, \sigma_n]$$

with coefficients from

$$S_n^{-1} P_n = K[H_1^{\pm 1}, (H_1 \pm 1)^{-1}, (H_1 \pm 2)^{-1}, \ldots, H_n^{\pm 1}, (H_n \pm 1)^{-1}, (H_n \pm 2)^{-1}, \ldots],$$

the localization of $P_n$ at $S_n$. We identify the Weyl algebra $A_n$ with the subalgebra of $A_n$ via the monomorphism,

$$A_n \rightarrow A_n, \ x_i \mapsto x_i, \ \partial_i \mapsto H_i x_i^{-1}, \ i = 1, \ldots, n.$$

Let $k_n$ be the $n$'th Weyl skew field, that is the full ring of quotients of the $n$'th Weyl algebra $A_n$ (it exists by Goldie’s Theorem since $A_n$ is a Noetherian domain). Then the algebra $A_n$ is a $K$-subalgebra of $k_n$ generated by the elements $x_i$, $x_i^{-1}$, $H_i$ and $H_i^{-1}$, $i = 1, \ldots, n$ since, for all natural $j$,

$$(H_i \mp j)^{-1} = x_i^{\mp j} H_i^{-1} x_i^{\mp j}, \ i = 1, \ldots, n.$$

Clearly, $A_n \cong A_1 \otimes \cdots \otimes A_1 \ (n \text{ times}).$

**Definition:** A $K$-algebra $R$ has the endomorphism property over $K$ if, for each simple $R$-module $M$, $\text{End}_R(M)$ is algebraic over $K$.

**Theorem 2.1** [5] Let $K$ be a field of characteristic zero.

1. The algebra $A_n$ is a simple, affine, Noetherian domain.
2. The Gelfand-Kirillov dimension $\text{GK}(A_n) = 3n \ (\neq 2n = \text{GK}(A_n))$.
3. The (left and right) global dimension $\text{gl.dim}(A_n) = n$.
4. The (left and right) Krull dimension $\text{K.dim}(A_n) = n$. 


Let $d = \text{gl.dim or } d = K \text{dim}. \text{ Let } R$ be a Noetherian $K$-algebra with $d(R) < \infty$ such that $R[t]$, the polynomial ring in a central indeterminate, has the endomorphism property over $K$. Then $d(A_1 \otimes R) = d(R) + 1$. If, in addition, the field $K$ is algebraically closed and uncountable, and the algebra $R$ is affine, then $d(A_n \otimes R) = d(R) + n$.

$\text{GK}(A_1) = 3$ is due to A. Joseph [12], p. 336; see also [16], Example 4.11, p. 45.

It is an experimental fact that many small quantum groups are GWAs. More about GWAs and their generalizations the interested reader can find in [1, 2, 6, 7, 8, 11, 13, 14, 15, 17, 18, 19, 21].

**Projections.** The polynomial algebra $P_n = \bigoplus_{\alpha \in \mathbb{N}^n} Kx^\alpha$ is a left $A_n$-module and $\text{End}_K(P_n)$-module. For each $i = 1, \ldots, n$ and $\alpha \in \mathbb{N}^n$, $H_i(x^\alpha) = (\alpha_i + 1)x^\alpha$, and so $H_i$ is an invertible map with $H_i^{-1}(x^\alpha) = (\alpha_i + 1)^{-1}x^\alpha$. Let $h_i := x_i \partial_i$. Then, in $P_n$, $h_i(x^\alpha) = \alpha_i x^\alpha$, and so $\text{ker}(h_i) = K[x_1, \ldots, \widehat{x_i}, \ldots, x_n]$.

Note that (in $\text{End}_K(P_n)$)

$$
(H^{-1}(\partial))^i(xH^{-1})^i = \frac{1}{H(H + 1) \cdots (H + i - 1)}, \quad i \geq 1. \tag{3}
$$

For each $\alpha \in \mathbb{N}^n$, the following element of $\text{End}_K(P_n)$ is invertible,

$$
(-\alpha, \alpha) := \partial x^\alpha = \prod_{i=1}^n H_i(H_i + 1) \cdots (H_i + \alpha_i - 1). \tag{4}
$$

**Lemma 2.2** Let $K$ be a commutative $\mathbb{Q}$-algebra and $\alpha \in \mathbb{N}_n$. Then $x^\alpha(-\alpha, \alpha)^{-1}\partial x^\alpha$ is the projection onto the ideal $(x^\alpha)$ of $P_n$ in the decomposition $P_n = (\bigoplus_{x^\beta \not\in (x^\alpha)} Kx^\beta) \oplus (x^\alpha)$.

**Proof.** If $x^\beta \not\in (x^\alpha)$ then $\partial x^\alpha(\partial x^\beta) = 0$, and so $x^\alpha \frac{1}{\partial x^\beta} \partial x^\alpha(\partial x^\beta) = 0$. If $x^\beta \in (x^\alpha)$ then $x^\alpha \frac{1}{\partial x^\beta} \partial x^\alpha(\partial x^\beta) = x^\alpha \frac{1}{\partial x^\beta} \partial x^\alpha(x^{\beta - \alpha}) = x^\alpha(x^{\beta - \alpha}) = x^\beta$. $\square$

Lemma 2.2 is useful in producing various projections onto homogeneous $K$-submodules of $P_n$. Let $S$ be a subset of $\mathbb{N}^n$ and $S'$ be its complement. Then $P_n = P_{n,S} \oplus P_{n,S'}$ where $P_{n,S} := \bigoplus_{\alpha \in S} Kx^\alpha$ and $P_{n,S'} := \bigoplus_{\alpha \in S'} Kx^\alpha$. Then $\pi_S := \sum_{\alpha \in S} \pi_\alpha$ is the projection onto $P_{n,S}$.

**Example.** Let $a, b \in \mathbb{N}^n$ with $a \leq b$, i.e. $a_1 \leq b_1, \ldots, a_n \leq b_n$; and $C := \{\alpha \in \mathbb{N}^n \mid a \leq \alpha \leq b\}$ be the discrete cube in $\mathbb{N}^n$ and $C'$ be its complement. Then $\pi_C := \sum_{\alpha \in C} \pi_\alpha$ is the projection onto $P_{n,C}$ in the decomposition $P_n = P_{n,C} \oplus P_{n,C'}$. Note that $\pi_C = \prod_{i=1}^n (x_i^{a_i} \frac{1}{\partial x_i^{b_i}} \partial_{x_i} - x_i^{b_i} \frac{1}{\partial x_i^{a_i}} \partial_{x_i})$, by Lemma 2.2. In more detail, for each $i = 1, \ldots, n$, $x_i^{a_i} \frac{1}{\partial x_i^{b_i}} \partial_{x_i}$ is the projection onto the ideal $x_i^{a_i} K[x_i]$ in the decomposition $K[x_i] = (\bigoplus_{j=0}^{a_i-1} Kx_i^j) \oplus x_i^{a_i} K[x_i]$. Therefore, $p_i := x_i^{a_i} \frac{1}{\partial x_i^{b_i}} \partial_{x_i} - x_i^{b_i} \frac{1}{\partial x_i^{a_i}} \partial_{x_i}$ is the projection onto $Kx_i^{a_i} \oplus Kx_i^{a_i+1} \oplus \cdots \oplus Kx_i^{b_i}$. Now, it is obvious that the product $p_1 \cdots p_n$ is equal to $\pi_C$.

**The Jacobian algebra $A_n$.** Let $K$ be a commutative $\mathbb{Q}$-algebra.
Definition: The Jacobian algebra $A_n$ is the subalgebra of $\text{End}_K(P_n)$ generated by the Weyl algebra $A_n$ and the elements $H^{-1}_1, \ldots, H^{-1}_n$.

Surprisingly, the Weyl algebras $A_n$ and the Jacobian algebras $A_n$ have little in common. For example, the algebra $A_n$ contains the infinite direct sum $K^{(n)}$ of rings $K$. In particular, $A_n$ is not a domain, and we will see that $A_n$ is not left or right Noetherian algebra.

By the very definition,

$$A_n = A_1(1) \otimes A_1(2) \otimes \cdots \otimes A_1(n) \simeq A_1^{\otimes n},$$

where $A_1(i) := K\langle x_i, \partial_i, H^{-1}_i \rangle$ and $\otimes := \otimes_K$. The algebra $A_n$ contains all the integrations $\int : x_i H^{-1}_i, 1 \leq i \leq n$. In the algebra $A_n$, each element $\partial_i$ has a right inverse, $\int : \partial_i \int = \text{id}_{P_n}$; and each element $x_i$ has a left inverse, $H^{-1}_i \partial_i : H^{-1}_i \partial_i x_i = \text{id}_{P_n}$. So, the algebra $A_n$ contains all necessary operations of Analysis (like integrations and differentiations) to deal with polynomials. The algebra $A_n$ contains all integro-differential operators. By (5), properties of the algebra $A_n$ is mainly determined by properties of the algebra $A_1$.

We pointed out already that the multiplicative submonoid $S_n$ of $K[H_1, \ldots, H_n]$ generated by the elements $H_i + j, 1 \leq i \leq n, j \in \mathbb{Z}$, is a (left and right) Ore set of the Weyl algebra $A_n$ and $S_n^{-1}A_n = A_n$. Using the $\mathbb{Z}^n$-grading of the Weyl algebra $A_n$ coming from its presentation as a generalized Weyl algebra one can easily verify that the multiplicative submonoid $S_{n,+}$ of $K[H_1, \ldots, H_n]$ generated by the elements $H_i + j, 1 \leq i \leq n, j \in \mathbb{N}$, is not a (left and right) Ore set of the Weyl algebra $A_n$. This also follows from the fact that the algebra $A_n$ is a domain but $A_n$ is not (if $S_{n,+}$ were a left or right Ore set then $A_n \subseteq A_n$, a contradiction).

Consider the case $n = 1$. Let $2^\mathbb{N}$ be the Boolean algebra of all subsets of $\mathbb{N}$ and $B_1$ be the Boolean subalgebra generated by all the finite subsets of $\mathbb{N}$. So, a subset $S$ of $\mathbb{N}$ is an element of $B_1$ iff either $S$ is finite or co-finite (that is, its complement is finite). Note that the Jacobian algebra $A_1$ contains all the projections $\pi_S, S \in B_1$.

In order to make formulae more readable, we drop the subscript 1. So, let, for a moment, $x := x_1, \partial := \partial_1$, and $H := H_1$. Since $(H^{-1}\partial)^i H^{-1}x^i = (H^{-1}\partial)^i x^i (H+i)^{-1} = (H+i)^{-1}$, $i \geq 1$, the algebra $A_1$ contains the subalgebra $L := K[H, H^{-1}, (H+1)^{-1}, \ldots, (H+i)^{-1}, \ldots]$. For each $i \geq 1$, $\frac{1}{x^i} = \frac{1}{H(H+1)\cdots(H+i-1)} \in L$. Let $D_1 := L + \sum_{i,j \geq 1} Kx^i H^{-j} \partial^j$ and $V := \oplus_{j \geq 1} K^H \partial^j$. The next theorem gives a $K$-basis for the algebra $A_1$.

Theorem 2.3 Let $K$ be a commutative $\mathbb{Q}$-algebra. Then the Jacobian algebra $A_1 = \bigoplus_{i \in \mathbb{Z}} A_{1,i}$ is a $\mathbb{Z}$-graded algebra ($A_{1,i}A_{1,j} \subseteq A_{1,i+j}$ for all $i, j \in \mathbb{Z}$) where $A_{1,0} = D_1$, $D_1 = L \oplus (\oplus_{i,j \geq 1} Kx^i H^{-j} \partial^j)$; and, for each $i \geq 1$, $A_{1,i} = x^i D_1$ and $A_{1,-i} = D_1 \partial^i$.

Proof. First, let us prove that the sum in the definition of $D_1$ is the direct one. Suppose that $r := l + xv_1 \partial + x^2 v_2 \partial^2 + \cdots + x^s v_s \partial^s = 0$ is a nontrivial relation for some elements $l \in L$ and $v_i \in V := \bigoplus_{i \geq 1} K^H \partial^j$. We seek a contradiction. Since $L$ is a subalgebra of $\text{End}_K(P_n)$, the relation $r$ is not of the type $r = l$. So, we can assume that $v_s \neq 0$ and the natural number $s \geq 1$ is called the degree of the relation $r$. Let $r$ be a nontrivial relation of the least degree. The rational function $l \in K(H)$ can be written as $\frac{p}{q}$ where $p$ and $q$ are co-prime.
polynomials and the polynomial \( q \) is a finite product of the type \( \prod_{i \geq 0} (H + i)^{n_i} \). Evaluating the relation \( r \) at 1: \( 0 = r(1) = l(1) = \frac{p(1)}{q(1)} \), we see that the polynomial \( p \) is equal to \( (H - 1)p' \) for some polynomial \( p' \in K[H] \). Suppose that \( n = 1 \), then \( 0 = H^{-1}(\partial r)x = H\sigma^{-1}(\frac{p'}{q}) + v_1H \) where \( \sigma : H \to H - 1 \) is the \( K \)-automorphism of the polynomial algebra \( K[H] \) (and of its field of fractions \( K(H) \)). It follows that \( v_1 = -\sigma^{-1}(\frac{p'}{q}) \in V \cap \sigma^{-1}(L) = 0 \), a contradiction. Therefore, \( s \geq 2 \).

The \( K[H] \)-module \( (V + K[H])/K[H] \approx V \) has the \( K \)-basis \( \{ H^{-i}, i \geq 1 \} \). In this basis, the matrix of the \( K \)-linear map \( v \mapsto (H + \lambda)v \) (where \( \lambda \in K \)) is an upper triangular infinite matrix with \( \lambda \) on the diagonal. In particular, the matrices of the maps \( H + 1, H + 2, \ldots, H + s - 1 \), are invertible, upper triangular. It follows from this fact that the relation

\[
H^{-1}(\partial r)x = H\sigma^{-1}(\frac{p'}{q}) + v_1H + xv_2(H+1)\partial + \cdots + x^{i-1}v_i(H+i-1)\partial^i - 1 + \cdots + x^{s-1}v_s(H+s-1)\partial^s
\]

has degree \( s - 1 \) since \( s \geq 2 \). Then, by induction on \( s \), and the fact that each matrix of the map \( H + i - 1, 2 \leq i \leq s \), is an upper triangular, invertible matrix with \( i - 1 \neq 0 \) on the diagonal, we have \( v_2 = \cdots = v_s = 0 \) and \( H\sigma^{-1}(\frac{p'}{q}) + v_1H = 0 \). Then, \( v_1 = -\sigma^{-1}(\frac{p'}{q}) \in V \cap \sigma^{-1}(L) = 0 \). This means that the relation \( r \) is a trivial one, a contradiction. This finishes the proof of the claim.

Let \((G, +)\) be an additive group (not necessarily commutative) and \( U = \oplus_{\alpha \in G} U_\alpha \) be a \( G \)-graded \( K \)-module, i.e. a direct sum of \( K \)-modules \( U_\alpha \). A \( K \)-linear map \( f : U \to U \) has degree \( \beta \in G \) if \( f(U_\alpha) \subseteq U_{\beta + \alpha} \) for all \( \alpha \in G \). The set \( E_\beta \) of all \( K \)-linear maps of degree \( \beta \) is a \( K \)-submodule of \( \text{End}_K(U) \). Clearly, \( E_\beta = \prod_{\beta \in G} \text{Hom}_K(U_\alpha, U_{\beta + \alpha}) \). In particular, \( E_0 = \prod_{\alpha \in G} \text{End}_K(U_\alpha) \). It follows at once that the sum \( E := \sum_{\beta \in G} E_\beta \subseteq \text{End}_K(U) \) is a direct one,

\[
E = \oplus_{\beta \in G} E_\beta \quad \text{and} \quad E_\beta E_\gamma \subseteq E_{\beta + \gamma}, \quad \beta, \gamma \in G.
\]

(6)

So, \( E := E(U) \) is a \( G \)-graded ring.

The \( K \)-module \( K[x] = \oplus_{i \geq 0} Kx^i \) is naturally \( \mathbb{Z} \)-graded (even \( \mathbb{N} \)-graded). By (6), the sum \( S := \sum_{i \geq 1} D_1 x^i + D_1 + \sum_{i \geq 1} x^i D_1 \) is a direct sum since the maps \( D_1 x^i, D_1, \) and \( x^i D_1 \) have degree \( -i, 0, \) and \( i \) respectively. In order to prove that \( A_1 = \oplus_{i \in \mathbb{Z}} A_{1,i} \) it suffices to show that \( A_1 \subseteq S \). It follows directly from the inclusions:

\[
D_1 x^i \subseteq x^i D_1, \quad \partial^i D_1 \subseteq D_1 \partial^i, \quad i \geq 0, \quad x^i \partial^i \subseteq D_1 \partial^{i-i}, \quad x^i \partial^i \subseteq x^{j-i} D_1, \quad j \geq i,
\]

that \( A_1 = \sum_{i,j \geq 0} x^i D_1 \partial^j = \sum_{s \in \mathbb{Z}} (\sum_{i-j=s} x^i D_1 \partial^j) \). It remains to show that, for \( s \geq 0 \), \( \sum_{i-j=s} x^i D_1 \partial^j \subseteq x^s D_1 \); and, for \( s < 0 \), \( \sum_{i-j=s} x^i D_1 \partial^j \subseteq D_1 \partial^{-s} \).

Consider the case \( s = 0 \). We have to show that \( \sum_{i \geq 0} x^i D_1 \partial^j \subseteq D_1 \). By the very definition of \( D_1 \), this is equivalent to the inclusions \( x^i L \partial^j \subseteq D_1, \ i \geq 0, \) and, by the very definition of \( L \), this is equivalent to the inclusions \( x^i (H + j)^{-k} \partial^j \in D_1, \ i, j, k \geq 0 \).

If \( i \leq j \) then \( x^i (H + j)^{-k} \partial^j = (H + j - i)^{-k} x^i \partial^j \in L \subseteq D_1 \).

If \( i > j \) then \( x^i (H + j)^{-k} \partial^j = x^{i-j} H^{-k} x^j \partial^j = x^{i-j} H^{-k} (H-1)(H-2) \cdots (H-j) \partial^{i-j} \in x^{i-j} (V + K[H]) \partial^{i-j} \subseteq D_1 \). This proves the case \( s = 0 \).
If $s \geq 1$ then $\sum_{i-j=s} x^i \partial^j = x^s (\sum_{k \geq 0} x^k \partial^k) \subseteq x^s \mathbb{D}_1$, by the case $s = 0$.

If $s \leq -1$ then $\sum_{i-j=s} x^i \partial^j = (\sum_{k \geq 0} x^k \partial^k) \partial^{-s} \subseteq \mathbb{D}_1 \partial^{-s}$, by the case $s = 0$.

Thus, the equality $\mathbb{A}_1 = \oplus_{i \in \mathbb{Z}} \mathbb{A}_{1,i}$ is established. By (6), this is a $\mathbb{Z}$-graded algebra since the maps $\mathbb{A}_{1,i}$ have degree $i$. $\square$

Despite the fact that Theorem 2.3 provides a cute $K$-basis for the algebra $\mathbb{D}_1$ it is unsuitable for computations: to write down the product of the type $x^i H^{-j} \partial^i \cdot x^k H^{-i} \partial^k$ literally takes half a page. Later, in Corollary 2.4 a more conceptual $K$-basis is introduced, and which is more important we interpret elements of $\mathbb{D}_1$ as functions from $\mathbb{N}$ to $K$. We will see that the ring $\mathbb{D}_1$ is large and has analytic flavour.

The polynomial algebra $K[x] = \oplus_{i \geq 0} Kx^i$ is naturally a $\mathbb{Z}$-graded algebra. Let $E = \oplus_{i \in \mathbb{Z}} E_i$ be the algebra from (6) for $K[x]$. The $E$-module $K[x]$ is simple. Note that the map

$$E_0 = \{ f \in \text{End}_K(K[x]) \mid f(x^i) = f_i, i \geq 0, f_i \in K \} \rightarrow K^N, \ f \mapsto (f_i),$$

is an isomorphism of $K$-algebras. In particular, $E_0$ is a commutative algebra, and so $\mathbb{D}_1$ is a commutative $K$-algebra, and so $\mathbb{D}_1$ is a commutative algebra, and so $\mathbb{D}_1 \subseteq E_0 (K[x]$ is the faithful $\mathbb{A}_1$-module). It is obvious that, for $i \geq 0$, $E_i = x^i E_0$ and $E_{-i} = E_0 \partial^i$. The algebra $K^N$ is the algebra of all functions from $\mathbb{N}$ to $K$. When we identify the set of monomials $M := \{ x^i \}_{i \in \mathbb{N}}$ and $\mathbb{N}$ via $x^i \mapsto i$ the algebras $E_0$ and $K^N$ are identified. So, each element of $E_0$ can be seen as a function. This is a very nice observation indeed as we can use facts and terminology of Analysis. For a function $\varphi : \mathbb{N} \rightarrow K$, the set $\text{supp}(\varphi) := \{ i \in \mathbb{N} \mid \varphi(i) \neq 0 \}$ is called the support of $\varphi$. The set of functions $F_0$ with finite support is an ideal of the algebra $E_0$. Clearly, $F_0 = \oplus_{i \in \mathbb{N}} K \pi_i$ where

$$\pi_i := x^i \frac{1}{\partial} x^i - x^{i+1} \frac{1}{\partial+i+1} x^{i+1} : K[x] \rightarrow K[x]$$

is the projection onto $Kx^i$ (Lemma 2.2), i.e. $\pi_i (x^j) = \delta_{ij} x^i$ where $\delta_{ij}$ is the Kronecker delta.

Let $\pi_{-i} := 0$; then

$$x \pi_i = \pi_{i+1}, \ \partial \pi_i = \pi_{i-1} \partial, \ i \geq 0.$$

The concept of support can be extended to an arbitrary element $f$ of the algebra $E$ as $\text{supp}(f) := \{ i \in \mathbb{N} \mid f(x^i) \neq 0 \}$. The set $F$ of all maps $f \in E$ with finite support is a $\mathbb{Z}$-graded algebra $F = \oplus_{i \in \mathbb{Z}} F_i$ without 1 where $F_i = F \cap E_i$. For $i \geq 1$, $F_i = x^i F_0 = \oplus_{j \in \mathbb{N}} K x^j \pi_j = \oplus_{j \in \mathbb{N}} K \pi_{j+i} x^i = F_0 x^i$ and $F_{-i} = F_0 \partial^i = \oplus_{j \in \mathbb{N}} K \pi_j \partial^i = \oplus_{j \in \mathbb{N}} K \partial^i \pi_{j+i} = \partial^i F_0$. Note that $F = \{ f \in \text{End}_K(K[x]) \mid f(x^i K[x]) = 0 \text{ for some } i \in \mathbb{N} \}$. It is obvious that $F$ is an ideal of the algebra $E$, and so $F$ is also an ideal of $\mathbb{A}_1$ since $F \subseteq \mathbb{A}_1 \subseteq E$.

Note that $h := x \partial \in E_0$ and $h(x^i) = ix^i, i \geq 0$. So, $h$ can be identified with the function $\mathbb{N} \rightarrow K, i \mapsto i$. Note that $H = h + 1$. When $h$ runs through 0, 1, 2, . . ., $H$ runs through 1, 2, . . . Under the identification $E_0 = K^N$, for $i, j \geq 1$,

$$x^i H^{-j} \partial^i = \begin{cases} \frac{(H-1)(H-2)\cdots(H-i+1)}{(H-i)^{i-1}} & \text{if } H = i + 1, i + 2, \ldots, \\ 0 & \text{if } H = 1, 2, \ldots, i. \end{cases}$$

(7)

This means that the element $x^i H^{-j} \partial^i$ is a function of the discrete argument $H = 1, 2, \ldots$ which takes zero value for $H = 1, 2, \ldots, i$; and $\frac{(H-1)(H-2)\cdots(H-i+1)}{(H-i)^{i-1}}$ for $H = i + 1, i + 2, \ldots$.
Before the identification this simply means that

\[ x^i H^{-j} \partial^i (x^k) = \begin{cases} 
\frac{(k+1-i)(k+1-2i):(k+1-i+1)}{(k+1-i)\cdot i} x^k & \text{if } k \geq i, \\
0 & \text{if } k = 0, 1, \ldots, i-1.
\end{cases} \]

So, the function \( x^i H^{-j} \partial^i \) is almost a rational function. The case \( i = j = 1 \) is rather special, it yields almost a constant function

\[ xH^{-1} \partial = \begin{cases} 
1 & \text{if } H = 2, 3, \ldots, \\
0 & \text{if } H = 1.
\end{cases} \]

Similarly, for each \( i = 1, 2, \ldots \), the element \( \rho_i := x^i \frac{1}{\partial x^i} \partial^i \in D_1 \) is the function

\[ \rho_i = \begin{cases} 
1 & \text{if } h = i, i + 1, \ldots, \\
0 & \text{if } h = 0, 1, \ldots, i - 1.
\end{cases} \] (8)

For each \( i \geq 0 \), \( \pi_i = \rho_i - \rho_{i+1} \in D_1 \) where \( \rho_0 := 1 \), and so \( F_0 = \oplus_{i \geq 0} K \pi_i \subseteq D_1 \). More generally, for each \( i = 1, 2, \ldots \) and \( j \in \mathbb{N} \), let

\[ \rho_{ji} := x^i \frac{1}{(H - i)^j} \partial^j = \frac{1}{(H - i)^j} \rho_i = \begin{cases} 
\frac{1}{(H - i)^j} & \text{if } H = i + 1, i + 2, \ldots, \\
0 & \text{if } H = 1, 2, \ldots, i.
\end{cases} \] (9)

Note that all \( \rho_{ji} \in D_1 \) and \( \rho_{0i} = \rho_i \). For \( \lambda \in \mathbb{Z} \), the element \( H + \lambda \) is invertible in \( D_1 \) iff \( \lambda \neq -1, -2, \ldots \) if \( H + \lambda \) is invertible in \( E_0 \). For \( i = 1, 2, \ldots \), \( \ker D_1 (H - i)^j = \ker E_0 (H - i)^j = K \pi_i \) where \( \ker D_1 (H - i)^j \) is the kernel of the map \( D_1 \rightarrow D_1, d \mapsto (H - i)^j d \). Similarly, \( \ker E_0 (H - i)^j \) is defined.

For each \( i \geq 0 \), let \( \pi'_i := 1 - \pi_i \). For natural numbers \( i, j \geq 1 \), consider the element \( \frac{1}{(H - i)^j} \pi'_{i-1} \) of \( E_0 \) which is as a linear map defined by the rule

\[ \frac{1}{(H - i)^j} \pi'_{i-1}(x^k) = \begin{cases} 
\frac{1}{(H - i)^j} x^k & \text{if } k \neq i - 1, \\
0 & \text{if } k = i - 1.
\end{cases} \]

As a function, it is almost the rational function \( \frac{1}{(H - i)^j} \) but at \( H = i \) it takes value 0 rather than \( \infty \) as the usual function \( \frac{1}{(H - i)^j} \pi'_{i-1} \in D_1 \) since

\[ \frac{1}{(H - i)^j} \pi'_{i-1} = \begin{cases} 
\rho_{ji} & \text{if } i = 1, \\
\rho_{ji} + \sum_{k=0}^{i-2} \frac{1}{(H - i)^j} \pi_k & \text{if } i \geq 2.
\end{cases} \]

For \( i, j, n, m \geq 1 \), \( \frac{1}{(H - i)^j} \pi'_{i-1} \cdot \frac{1}{(H - j)^m} \pi'_{j-1} = \frac{1}{(H - i)^j} \pi'_{i-1} \cdot (H - i)^n \cdot \frac{1}{(H - j)^m} \pi'_{j-1} = \frac{1}{(H - i)^j} \pi'_{i-1} + K \pi_{i-1} + K \pi_{j-1} \).

and for \( j \geq 1 \) such that \( j \neq i \)

\[ \frac{1}{(H - i)^j} \pi'_{i-1} \cdot \frac{1}{(H - j)^m} \pi'_{j-1} \subseteq \sum_{s=1}^n K \frac{1}{(H - i)^s} \pi'_{i-1} + \sum_{t=1}^m K \frac{1}{(H - j)^t} \pi'_{j-1} + K \pi_{i-1} + K \pi_{j-1}. \]
Clearly, the set
\[ \mathbb{I}_1 := L \oplus \oplus_{i,j \geq 1} K^{\frac{1}{(H-i)^j}} \pi_{i-1}^{j} \]  
(10)
is a \( K \)-submodule of \( \mathbb{D}_1 \). \( \mathbb{I}_1 \) is not an algebra, though it is an algebra modulo \( F_0 \) which is isomorphic to the algebra \( K[H^\pm, (H \pm 1)^{-1}, (H \pm 2)^{-1}, \ldots] \) (see Corollary 2.4 and (11)).

**Corollary 2.4** Let \( K \) be a commutative \( \mathbb{Q} \)-algebra, \( \rho_{ji} := x^i \frac{1}{H^j \partial x^i} \partial^i \), \( j \geq 0 \), \( i \geq 1 \). Then \( \mathbb{D}_1 = \mathbb{I}_1 \oplus F_0 = L \oplus (\oplus_{i,j \geq 1} K \rho_{ji}) \).

**Proof.** By Theorem 2.3, \( \mathbb{D}_1 = L \oplus (\oplus_{i,j \geq 1} K \pi_j \partial^j) \). By (7) and \( F_0 \subseteq \mathbb{D}_1 \), we have \( \mathbb{D}_1 \subseteq \mathbb{L}_1 + F_0 \). By (9) and \( F_0 \subseteq \mathbb{D}_1 \), we have the opposite inclusion, and so \( \mathbb{D}_1 = \mathbb{L}_1 + F_0 = \mathbb{L}_1 \oplus F_0 \) since \( \mathbb{L}_1 \cap F_0 = 0 \).

The \( K \)-module \( M := L \oplus \oplus_{i \geq 1} K \rho_{ji} \) contains \( F_0 = \oplus_{i \geq 0} \pi_i \) since \( \pi_i = \rho_i - \rho_{i+1} \) for all \( i \geq 0 \) where \( \rho_0 := 1 \); and \( M = L' + F_0 \) where \( L' := L + \oplus_{i,j \geq 1} K \rho_{ji} \). Consider the factor module \( M/F_0 \). By (9), \( \rho_{ji} \equiv \frac{1}{(H-i)^j} \pi_{i-1}^j \mod F_0 \), hence \( L' \equiv L_1 \mod F_0 \). Since \( \mathbb{D}_1 = \mathbb{L}_1 + F_0 \) and \( M = L' + F_0 \), we must have the equality \( \mathbb{D}_1 = M \). To finish the proof of the second equality of the corollary it suffices to show that \( L' + F_0 = L \oplus (\oplus_{i,j \geq 1} K \rho_{ji}) \oplus F_0 \) since then the equality \( M = L \oplus (\oplus_{i,j \geq 1} K \rho_{ji}) \) follows as \( F_0 = \oplus_{i \geq 0} K \pi_i \) and \( \pi_i = \rho_i - \rho_{i+1} \).

Let \( l + \sum_{i,j \geq 1} \lambda_{ji} \rho_{ji} + f = 0 \) for some \( l \in \mathbb{L} \), \( \lambda_{ji} \in K \), and \( f \in F_0 \). Taking this equality modulo \( F_0 \) yields \( l = 0 \) and \( \lambda_{ji} = 0 \) since \( \rho_{ji} \equiv \frac{1}{(H-i)^j} \pi_{i-1}^{j} \mod F_0 \). This implies \( f = 0 \), and we are done. \( \square \)

Note that \( F_0 \) is an ideal of \( \mathbb{D}_1 \) such that \( F_0^2 = F_0 \) and
\[ \mathbb{D}_1/F_0 \simeq K[H^\pm, (H \pm 1)^{-1}, (H \pm 2)^{-1}, \ldots]. \]  
(11)
The equality \( \mathbb{D}_1 = \mathbb{L}_1 \oplus F_0 \) (Corollary 2.4) means that the set
\[ \{ H^i, \frac{1}{(H+j)^x} (H-j)^k \pi_{i-1}^j, \pi_i \mid i \in \mathbb{Z}, l \in \mathbb{N}, j \geq 1 \} \]
is a \( K \)-basis for \( \mathbb{D}_1 \). Clearly, the set
\[ \{ H^i, \frac{1}{(H+j)^x} (H-j)^k \pi_{i-1}^j \mid i \in \mathbb{Z}, j,k \geq 1 \} \]  
(12)
is a \( K \)-basis for \( \mathbb{L}_1 \). Similarly, the equality \( \mathbb{D}_1 = L \oplus (\oplus_{i,j \geq 1} K \rho_{ji}) \) means that the set
\[ \{ H^i, \frac{1}{(H+j)^x} \rho_{ji} \mid j \in \mathbb{N}, i,k \geq 1 \} \]
is a \( K \)-basis for \( \mathbb{D}_1 \).

Clearly, \( F' = \oplus_{i,j \geq 0} K E_{ij} \) where \( E_{ij}(x^k) := \delta_{jk} x^i \), i.e. \( \{ E_{ij} \} \) are the ‘elementary matrices’ \( (E_{ij} E_{kl} = \delta_{jk} E_{il}) \), and
\[ E_{ij} = \begin{cases} x^{i-j} \pi_j & \text{if } i \geq j, \\ \frac{1}{(H-i)^{i-j}} \pi_j & \text{if } i < j. \end{cases} \]  
(13)
For $k \in \mathbb{N}$, $F_{\pm k} = \oplus_{i=j=\pm k} KE_{ij}$. Note that $E_{ij} = E_{ik} \pi_k E_{kj}$ for all $i, j, k \geq 0$. Therefore, $F$ is a simple ring such that $F^2 = F$, and $K[x]$ is a simple faithful $F$-module. The ring $F$ is neither left nor right Noetherian as the next arguments show: for each natural $k \geq 0$, let $L_k := \oplus_{i\in\mathbb{N},0\leq j\leq k} KE_{ij}$ and $R_k := \oplus_{0\leq i\leq k,j\in\mathbb{N}} KE_{ij}$ then $L_0 \subset L_1 \subset \cdots$ and $R_0 \subset R_1 \subset \cdots$ are strictly ascending sequences of left and right $F$-modules respectively. This is the main reason why the Jacobian algebra $A_1$ is also neither left nor right Noetherian (Theorem 2.5.(3)). The ring $F$ is neither left nor right Artinian: for each natural $k \geq 1$, let $L'_k := \oplus_{i,j\in\mathbb{N}} KE_{i,j} 2^k$ and $R_k := \oplus_{i,j\in\mathbb{N}} KE_{i,j} 2^k$ then $L'_0 \subset L'_1 \subset \cdots$ and $R'_0 \subset R'_1 \subset \cdots$ are strictly descending sequences of left and right $F$-modules respectively.

**Theorem 2.5** Let $K$ be a commutative $\mathbb{Q}$-algebra. Then

1. $A_1 = \oplus_{i\geq 1} L_i \partial^i \oplus L_1 \oplus (\oplus_{i\geq 1} x^i L_1) \oplus F$.

2. The set $\{H^i \partial^j, \frac{1}{(H+j)^{m}} \partial^j, \frac{1}{(H-j)^{m}} \pi_j^{i-1} \partial^j, x^m H^i, x^m \frac{1}{(H-j)^{m}} \pi_j^{i-1}, E_{st} | i \in \mathbb{Z}; j, k, l \geq 1; m, s, t \in \mathbb{N}\}$ is a $K$-basis for $A_1$.

3. The algebra $A_1$ is neither a left nor right Noetherian algebra.

4. $F$ is an ideal of $A_1$, $F^2 = F$, and the factor algebra $A_1/F$ is canonically isomorphic to the algebra $A_1$ (the localization of the Weyl algebra $A_1$ at $S_1$, the multiplicative monoid generated by $H + i$, $i \in \mathbb{Z}$).

**Proof.** 1. By Corollary 2.4, $D_1 = L_1 \oplus F_0$. For each natural number $i \geq 1$, $D_i \partial^i = L_i \partial^i \oplus F_{-i}$ and $x^i D_1 = x^i L_1 \oplus F_i$. Using these equalities together with the equalities $A_1 = \oplus_{i \geq 1} D_i \partial^i \oplus D_1 \oplus (\oplus_{i \geq 1} x^i D_1)$ (Theorem 2.3) and $F = \oplus_{i \geq 1} F_0 \partial^i \oplus F_0 \oplus (\oplus_{i \geq 1} x^i F_0)$, one obtains the equality of statement 1 and (12).

2. Since, for all $i \geq 1$, the maps $L_1 \rightarrow L_1 \partial^i$, $u \mapsto u \partial^i$, and $L_1 \rightarrow x^i L_1$, $u \mapsto x^i u$, are isomorphisms of $K$-modules, statement 2 follows from statement 1.

3. For each $i \in \mathbb{N}$, the sum $I_i := \oplus_{j=0}^i K \pi_j$ is an ideal of $E_0$. Since $I_i \subseteq D_i \subseteq E_0$, one has the strictly ascending chain of ideals of $D_i$: $I_0 \subset I_1 \subset \cdots$. The ascending chain $A_1 I_0 \subset A_1 I_1 \subset \cdots$ of left homogeneous ideals of the algebra $A_1$ is strictly ascending since the zero component of the left ideal $A_1 I_j = \oplus_{i \geq 1} D_i \partial^i I_j \oplus I_j \oplus (\oplus_{i \geq 1} x^i D_1)$ is $I_j$ (note that $D_1 \partial^i I_j \subseteq F_{-i}$ and $x^i I_j \subseteq F_i$). Therefore, $A_1$ is not a left Noetherian algebra.

Similarly, the ascending chain $I_0 A_1 \subset I_1 A_1 \subset \cdots$ of right homogeneous ideals of the algebra $A_1$ is strictly ascending since the zero component of the right ideal $I_j A_1 = \oplus_{i \geq 1} I_j \partial^i \oplus I_j \oplus (\oplus_{i \geq 1} I_j x^i D_1)$ is $I_j$, and so $A_1$ is not a right Noetherian algebra.

4. We proved already that $F$ is an ideal of $A_1$ such that $F^2 = F$. The $K$-module $K[x]$ is a topological $K$-module (even a topological $K$-algebra) with respect to the $m$-adic topology determined by the $m$-adic filtration $\{m^i\}_{i \geq 0}$ on $K[x]$ where $m := (x)$. Let $\text{End}_{K,c}(K[x])$ be the algebra of all continuous $K$-endomorphisms of $K[x]$. Then $E \subseteq \text{End}_{K,c}(K[x])$. Let $G$ be the algebra of germs of continuous $K$-endomorphisms of $K[x]$ at 0. An element of $G$ is an equivalence class $[f]$ of a continuous $K$-linear map of the type $f : m^i \rightarrow K[x]$, and two such maps are equivalent, $f \sim f'$, if they have the same restriction $f|_{m^j} = f'|_{m^j}$ for
Corollary 2.6  Let $K$ be a field of characteristic zero. Then

1. $F$ is the only proper ideal of the algebra $A_1$, hence $F$ is a maximal ideal.

2. $A_1/F \cong A_1$ is a simple Noetherian domain.

3. $\text{GK}(A_1) = \text{GK}(A_1) = 3$.

4. $A_1$ is a prime ring.

5. The algebra $A_1$ is central, i.e. the centre of $A_1$ is $K$.

Proof. 2. Statement 2 follows from Theorem 2.5.(4) and Theorem 2.1.(1).

1. By statement 2, $F$ is a maximal ideal of $A_1$. Let $I$ be a proper ideal of $A_1$. We have to show that $I = F$. Let $a$ be a nonzero element of $I$. Then $0 \neq FaF \subseteq F \cap I$, and so $FaF = F$ since $F$ is a simple algebra (i.e. a simple $F$-bimodule). Now, $F \subseteq I$ implies $F = I$ by the maximality of $F$. So, $F$ is the only proper ideal of the algebra $A_1$.

3. Since $A_1/F \cong A_1$, we have $\text{GK}(A_1) \geq \text{GK}(A_1) = 3$ (Theorem 2.1.(2)). Since $A_1 = \oplus_{i \geq 1}L_i \partial^i \oplus L_i \oplus (\oplus_{i \geq 1}x^iL_1) \oplus F$ and $A_1 = \oplus_{i \geq 1}S_1^{-1}K[H] \partial^i \oplus S_1^{-1}K[H] \oplus (\oplus_{i \geq 1}x^iS_1^{-1}K[H])$, the reverse inequality $\text{GK}(A_1) \leq 3$ follows from Theorem 2.5.(1,2) using the same sort of estimates as in the proof of the inequality $\text{GK}(A_1) \leq 3$ (see [5] for details and the fact that the elements of $F$ do not contribute to the growth of degree 3).

4. $F$ is the only proper ideal of $A_1$, and so $F^2 = F$, hence $A_1$ is a prime ring.

5. The field $K$ belongs to the centre of $A_1$. Let $z$ be a central element of $A_1$. We have to show that $z \in K$. The algebra $A_1/F \cong A_1$ is central, hence $z = \lambda + f$ for some $\lambda \in K$ and $f \in F$. Then $z - \lambda = f$ belongs to the centre of $F$ which is obviously equal to zero. Hence $z = \lambda \in K$, as required. □

For $k \geq 1$,

$$x \frac{1}{(H-j)^k} \pi_j^{\prime -1} = \frac{1}{(H-1-j)^k} \pi_j^{\prime -1} x, \quad j \geq 1,$$

$$\partial \frac{1}{(H-j)^k} \pi_j^{\prime -1} = \frac{1}{(H+1-j)^k} \pi_{j-2}^{\prime -1} \partial, \quad j \geq 2,$$

$$\partial \frac{1}{(H-1)^k} \pi_0^{\prime} = \frac{1}{H} \partial.$$
By (5), the algebra $A_n$ is the tensor product of the $\mathbb{Z}$-graded algebras $A_1(i) = \bigoplus_{j \in \mathbb{Z}} A_1(j)$.

Therefore, the algebra $A_n$ is a $\mathbb{Z}^n$-graded algebra,

$$A_n = \bigoplus_{\alpha \in \mathbb{Z}^n} A_{n,\alpha}, \quad A_{n,\alpha} := \bigotimes_{i=1}^n A_{1,\alpha(i)}.$$ 

The $\mathbb{Z}^n$-grading on $A_n$ is the tensor product of the $\mathbb{Z}$-gradings of the tensor multiples, and an element $a$ of $A_n$ belongs to $A_{n,\alpha}$ iff $a(x^\beta) \in Kx^{\alpha+\beta}$ for all $\beta \in \mathbb{N}^n$. Let

$$D_n := A_{n,0} = D_1(1) \otimes D_1(2) \otimes \cdots \otimes D_1(n).$$

The polynomial algebra $P_n = \bigoplus_{\alpha \in \mathbb{N}^n} P_n,\alpha$ is an $\mathbb{N}^n$-graded, hence a $\mathbb{Z}^n$-graded algebra. Let $E = E(P_n) = \bigoplus_{\alpha \in \mathbb{Z}^n} E_\alpha$ be the $\mathbb{Z}^n$-graded algebra as in (6). The map

$$E_0 = \{ f \in \text{End}_K(P_n) \mid f(x^\alpha) = f_\alpha x^\alpha, f_\alpha \in K, \alpha \in \mathbb{N}^n \} \rightarrow K^{\mathbb{N}^n}, \quad f \mapsto (f_\alpha),$$

is a $K$-algebra isomorphism. Each element $\alpha = \sum_{i=1}^n \alpha_i e_i \in \mathbb{Z}^n = \bigoplus_{i=1}^n Z e_i$ is a unique difference $\alpha = \alpha_+ - \alpha_-$ where $\alpha_+ = \sum_{\alpha_i \geq 0} \alpha_i e_i$ and $\alpha_- = -\sum_{\alpha_i \leq 0} \alpha_i e_i$. For each $\alpha \in \mathbb{Z}^n$, $E_\alpha = x^{\alpha} E_0 \theta^{\alpha_-}$, and so

$$E = \bigoplus_{\alpha \in \mathbb{Z}^n} x^{\alpha} E_0 \theta^{\alpha_-}. \quad (14)$$

For each $\alpha \in \mathbb{Z}^n$, $A_{n,\alpha} = A_n \cap E_\alpha = x^{\alpha} D_n \theta^{\alpha_-}$ where $D_n = A_{n,0} = A_n \cap E_0$.

**The involution $\theta$ on $A_n$.** Let $K$ be a commutative $\mathbb{Q}$-algebra. The Weyl algebra $A_n$ admits the **involution** $\theta : A_n \rightarrow A_n, \quad x_i \mapsto \partial_i, \quad \partial_i \mapsto x_i, \quad i = 1, \ldots, n,$

i.e. it is a $K$-algebra anti-isomorphism $(\theta(ab) = \theta(b)\theta(a))$ such that $\theta^2 = 1_{A_n}$. The involution $\theta$ can be uniquely extended to the involution of $A_n$ by the rule

$$\theta : A_n \rightarrow A_n, \quad x_i \mapsto \partial_i, \quad \partial_i \mapsto x_i, \quad \theta(H_i^{-1}) = H_i^{-1} \quad i = 1, \ldots, n. \quad (15)$$

Uniqueness is obvious: $\theta(H_i) = \theta(\partial_i x_i) = \theta(x_i)\theta(\partial_i) = \partial_i x_i = H_i$ and so $\theta(H_i^{-1}) = H_i^{-1}$. To prove existence recall that each right module over a ring $R$ is a left module over the opposite ring $R^{op}$. The involution $\theta$ on $A_n$ comes from considering the polynomial algebra $P_n$ as the **right** $A_n$-module by the rule $pa := \theta(a)p$ for all $p \in P_n$ and $a \in A_n$. Since $\theta(H_i) = H_i$, $i = 1, \ldots, n$, $P_n$ is the **faithful** right $A_n$-module, and this proves the existence of the involution $\theta : A_n \rightarrow A_n$ ($\theta$ is injective since $P_n$ is a faithful right $A_n$-module, $\theta$ is obviously surjective). So, the algebra $A_n$ is **self-dual** (i.e. it is isomorphic to its opposite algebra, $\theta : A_n \simeq A_n^{op}$). This means that left and right algebraic properties of the algebra $A_n$ are the same.

For $n = 1$, $F$ is the only proper ideal of $A_1$, hence $\theta(F) = F$. Moreover,

$$\theta(E_{ij}) = \frac{i!}{j!} E_{ji}. \quad (16)$$
where $0! := 1$. In more detail, since $\theta(H) = H$ and $E_{ii} = \pi_i = x^i \frac{1}{(i-1)!} \partial^i - x^{i+1} \frac{1}{(i+1)!} \partial^{i+1}$, we have $\theta(E_{ii}) = E_{ii}$. For $i > j$, $E_{ij} = x^{i-j} \pi_j$, and so

$$\theta(E_{ij}) = \pi_j \partial^{i-j} = i(i-1) \cdots (j+1) E_{ji} = \frac{i!}{j!} E_{ji}.$$  

For $i < j$, $E_{ij} = (\frac{1}{H} \partial)^{i-j} \pi_j$, and so $\theta(E_{ij}) = \pi_j (x \frac{1}{H} \partial)^{j-i} = \frac{1}{(i+1)(i+2) \cdots j} E_{ji} = \frac{1}{\Gamma(j-i+1)} E_{ji}$.

For $n = 1$, the ring $F = \oplus_{i,j \in \mathbb{N}} K E_{ij}$ is equal to the matrix ring $M_\infty(K) := \bigcup_{d \geq 1} M_d(K)$ where $M_d(K) := \oplus_{0 \leq i,j \leq d-1} K E_{ij}$. The ring $F = M_\infty(K)$ admits the canonical involution which is the transposition $(\cdot)^t : E_{ij} \mapsto E_{ji}$. Let $D_t$ be the infinite diagonal matrix $\text{diag}(0!, 1!, 2!, \ldots)$. Then, for $u \in F = M_\infty(K)$,

$$\theta(u) = D_t^{-1} u^t D_t. \tag{17}$$

Note that $D_t \notin M_\infty(K)$.

For an arbitrary $n$, $F^\otimes n = \oplus_{\alpha,\beta \in \mathbb{N}^n} K E_{\alpha\beta} = M_\infty(K)^\otimes n$ where $E_{\alpha\beta} := \otimes_{i=1}^n E_{\alpha_i \beta_i}$. By (16),

$$\theta(E_{\alpha\beta}) = \frac{\alpha!}{\beta!} E_{\beta\alpha}, \tag{18}$$

$$\theta(F^\otimes n) = F^\otimes n. \tag{19}$$

Let $D_{n!} := D_t^\otimes n$. Then, for $u \in F^\otimes n$,

$$\theta(u) = D_{n!}^{-1} u^t D_{n!}. \tag{20}$$

where $(\cdot)^t : M_\infty(K)^\otimes n \rightarrow M_\infty(K)^\otimes n$, $E_{\alpha\beta} \mapsto E_{\beta\alpha}$, is the transposition map.

Consider the bilinear, symmetric, non-degenerate form $(\cdot, \cdot) : P_n \times P_n \rightarrow K$ given by the rule $(x^\alpha, x^\beta) := \alpha! \delta_{\alpha,\beta}$ for all $\alpha, \beta \in \mathbb{N}^n$. Then, for all $p, q \in P_n$ and $a \in \mathbb{A}_n$,

$$(p, aq) = (\theta(a)p, q). \tag{21}$$

The Weyl algebra $\mathbb{A}_n$ admits, so-called, the Fourier transform, it is the $K$-algebra automorphism $\mathcal{F} : \mathbb{A}_n \rightarrow \mathbb{A}_n$, $x_i \mapsto \partial_i$, $\partial_i \mapsto -x_i$, $i = 1, \ldots, n$. Since $\mathcal{F}(H_i) = -(H_i - 1)$, $H_i$ is a unit of $\mathbb{A}_n$ and $H_i - 1$ is not, one cannot extend the Fourier transform to $\mathbb{A}_n$.

The algebra $\mathbb{A}_n$ is a prime algebra. Consider the ideals of the algebra $\mathbb{A}_n$:

$$p_1 := F \otimes \mathbb{A}_{n-1}, p_2 := \mathbb{A}_1 \otimes F \otimes \mathbb{A}_{n-2}, \ldots, p_n := \mathbb{A}_{n-1} \otimes F.$$  

Then $\mathbb{A}_n/p_i \simeq (\mathbb{A}_1/F) \otimes \mathbb{A}_{n-1} \simeq \mathbb{A}_1 \otimes \mathbb{A}_{n-1}$ and $\bigcap_{i=1}^n p_i = F^\otimes n$. Let $a_n := p_1 + \cdots + p_n$. Then

$$\mathbb{A}_n/a_n \simeq (\mathbb{A}_1/F)^{\otimes n} \simeq \mathbb{A}_1^\otimes n = \mathbb{A}_n. \tag{22}$$

**Corollary 2.7** Let $K$ be a field of characteristic zero. Then

1. $\text{GK} \left( \mathbb{A}_n \right) = 3n$. 

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2. \( \text{GK}(M) \geq n \) for all nonzero finitely generated (left or right) \( \mathcal{A}_n \)-modules \( M \).

3. The centre of \( \mathcal{A}_n \) is \( K \).

4. \( F^\otimes n \) is the smallest nonzero ideal of the algebra \( \mathcal{A}_n \), \( (F^\otimes n)^2 = F^\otimes n \).

5. The algebra \( \mathcal{A}_n \) is prime.

6. If \( 0 \neq a \in \mathcal{A}_n \) and \( I \) is a nonzero ideal of \( \mathcal{A}_n \) then \( aI \neq 0 \), \( Ia \neq 0 \), and \( IaI \neq 0 \).

7. The ideal \( a_n \) is the largest ideal of \( \mathcal{A}_n \) distinct from \( \mathcal{A}_n \); \( a_n^2 = a_n \); hence \( a_n \) is the only maximal ideal of \( \mathcal{A}_n \).

8. \( \mathcal{A}_n F^\otimes n \simeq P_n^{(N^1)} \) is a faithful, semi-simple, left \( \mathcal{A}_n \)-module; \( F^\otimes n \simeq P_n^{(N^1)} \mathcal{A}_n \) is a faithful, semi-simple, right \( \mathcal{A}_n \)-module; \( \mathcal{A}_n F^\otimes n \simeq P_n^{(N^1)} \) is a faithful, simple \( \mathcal{A}_n \)-bimodule.

9. \( F^\otimes n \) is the socle of \( \mathcal{A}_n \) considered as a left \( \mathcal{A}_n \)-module, or a right \( \mathcal{A}_n \)-module, or an \( \mathcal{A}_n \)-bimodule.

10. \( \mathcal{A}_n P_n \) (resp. \( (P_n)_{\mathcal{A}_n} \)) is the only faithful, simple, left (resp. right) \( \mathcal{A}_n \)-module.

11. \( \text{GK}(\mathcal{A}_n/a) = 3n \) for all ideals \( a \) of \( \mathcal{A}_n \) such that \( a \neq \mathcal{A}_n \).

**Proof.** 1. On the one hand, \( \text{GK}(\mathcal{A}_n) \geq \text{GK}(\mathcal{A}_n/a_n) = \text{GK}(\mathcal{A}_n) = 3n \) (Theorem 2.1.(2)); on the other, \( \text{GK}(\mathcal{A}_n) = \text{GK}(\mathcal{A}_1^{(1)}) \leq n \text{GK}(\mathcal{A}_1) = 3n \). Therefore, \( \text{GK}(\mathcal{A}_n) = 3n \).

2. Statement 2 is an easy corollary of the inequality of Bernstein: \( \text{GK}(N) \geq n \) for all nonzero finitely generated (left or right) \( \mathcal{A}_n \)-modules \( N \). Let \( M \) be a nonzero finitely generated \( \mathcal{A}_n \)-module and \( 0 \neq u \in M \). Then \( \text{GK}_{\mathcal{A}_n}(M) \geq \text{GK}_{\mathcal{A}_n}(\mathcal{A}_n u) \geq \text{GK}_{\mathcal{A}_n}(\mathcal{A}_n u) \geq n \).

3. To prove that the centre \( Z(\mathcal{A}_n) \) of \( \mathcal{A}_n \) is \( K \) we use induction on \( n \). The case \( n = 1 \) is Corollary 2.6.(5). Suppose that \( n > 1 \) and the algebra \( \mathcal{A}_m \) is central for all \( m < n \). The kernel of the algebra homomorphism \( \mathcal{A}_n \rightarrow \prod_{i=1}^n \mathcal{A}_n/p_i \) is \( \cap_{i=1}^n p_i = F^\otimes n \). Since all the algebras \( \mathcal{A}_n/p_i \simeq \mathcal{A}_1 \otimes \mathcal{A}_{n-1} \) are central, we have \( Z(\prod_{i=1}^n \mathcal{A}_n/p_i) = Z(\cap_{i=1}^n Z(\mathcal{A}_n/p_i)) = \prod_{i=1}^n K \) where \( Z(R) \) is the centre of \( R \). If \( z \in Z(\mathcal{A}_n) \) then \( z + F^\otimes n \in Z(\prod_{i=1}^n \mathcal{A}_n/p_i) = \prod_{i=1}^n K \), and so \( z = \lambda + f \) for some \( \lambda \in K \) and \( f \in F^\otimes n \). Now, \( f = z - \lambda \in Z(F^\otimes n) = 0 \), i.e. \( z = \lambda \in K \), and so the algebra \( \mathcal{A}_n \) is central.

4. Clearly, \( (F^\otimes n)^2 = (F^2)^\otimes n = F^\otimes n \). It remains to prove minimality of \( F^\otimes n \). This is obvious for \( n = 1 \) (Corollary 2.6.(1)). To prove the general case we use induction on \( n \). Suppose that \( n > 1 \) and the result is true for all \( n' < n \). Let \( I \) be a nonzero ideal of \( \mathcal{A}_n \). We have to show that \( F^\otimes n \subseteq I \). Choose a nonzero element, say \( a \), of \( I \). Since \( a \in \mathcal{A}_n = \mathcal{A}_1 \otimes \mathcal{A}_{n-1} \), the element \( a \) can be written as a sum \( a = \sum_{i=1}^s a_i \otimes b_i \) for some elements \( a_i \in \mathcal{A}_1 \) and \( b_i \in \mathcal{A}_{n-1} \) such that the elements \( a_1, \ldots, a_s \) and \( b_1, \ldots, b_s \) are \( K \)-linearly independent elements of the algebras \( \mathcal{A}_1 \) and \( \mathcal{A}_{n-1} \) respectively. Choose an element, say \( f \) of \( F \) such that \( fa_i \neq 0 \). Changing \( a \) for \( fa \neq 0 \) (and deleting zero terms of the type \( fa_i \otimes b_i \)) one may assume that all \( a_i \in F \). Note that \( \{E_{kl}\} \) is the \( K \)-basis of \( F \). So, the element \( a \) can be written as a finite sum \( a = E_{kl} \otimes \alpha + E_{st} \otimes \beta + \cdots + E_{pq} \otimes \gamma \) for some
$K$-linearly independent elements $\alpha, \beta, \ldots, \gamma$ of $\mathbb{A}_{n-1}$ and distinct elements $E_{kl}, E_{st}, \ldots, E_{pq}$. Then $b := E_{kk}aE_{ll} = E_{kl} \otimes \alpha \in I$. By induction, $F^{\otimes(n-1)} \subseteq \mathbb{A}_{n-1} \alpha \mathbb{A}_{n-1}$, and so

$$I \supseteq \mathbb{A}_n b \mathbb{A}_n = \mathbb{A}_1 E_{kl} \mathbb{A}_1 \otimes \mathbb{A}_{n-1} \alpha \mathbb{A}_{n-1} \supseteq F \otimes F^{\otimes(n-1)} = F^{\otimes n}.$$

5. Let $I$ and $J$ be nonzero ideals of $\mathbb{A}_n$. By statement 4, they contain the ideal $F^{\otimes n}$. Now, $IJ \supseteq (F^{\otimes n})^2 = F^{\otimes n} \neq 0$. This means that $\mathbb{A}_n$ is a prime ring.

6. Statement 6 follows directly from statement 5. Suppose that $aI = 0$ for some nonzero element $a$ of $\mathbb{A}_n$ and a nonzero ideal $I$. We seek a contradiction. Then $0 = \mathbb{A}_n aI = \mathbb{A}_n a \mathbb{A}_n I \neq 0$ since $\mathbb{A}_n$ is a prime algebra, a contradiction. Similarly, $Ia = 0$ (resp. $IaI = 0$) implies $0 = Ia \mathbb{A}_n a \mathbb{A}_n \neq 0$ (resp. $0 = IaIaI \neq 0$), a contradiction.

7. $a_n \supseteq a_n^2 = (p_1 + \cdots + p_n)^2 \supseteq p_1^2 + \cdots + p_n^2 = p_1 + \cdots + p_n = a_n$ since $p_1^2 = p_1, \ldots, p_n^2 = p_n$, and so $a_n = a_n^2$.

It remains to show that $a_n$ is the largest ideal distinct from $\mathbb{A}_n$, that is $a_n \not\subset a_m$ implies $\mathbb{A}_n a_m \mathbb{A}_n = \mathbb{A}_n$ where $a_m \in \mathbb{A}_n$. Let $B_n$ be the $K$-basis for $\mathbb{A}_n$ that is the tensor product of the $K$-bases from Theorem 2.5.(2). $B_n$ is the disjoint union of its two subsets $\mathcal{M}_n := \{b \in B_n | b \in a_n\}$ and $\mathcal{N}_n := \{b \in B_n | b \not\in a_n\}$. So, $b \in \mathcal{M}_n$ iff the product $b$ contains a matrix unit $E_{st}(i) \in F(i)$. Clearly, $a_n = \bigoplus_{\mu \in \mathcal{M}_n} K\nu$ and $\mathbb{A}_n = a_n' \cap a_n$ where $a_n' := \bigoplus_{\mu \in \mathcal{M}_n} K\mu$. Elements of the set $\mathbb{A}_n \setminus a_n \setminus a_n$ are called generic. So, an element of $\mathbb{A}_n$ is generic iff it has at least one nonzero $\mu$-coordinate for some $\mu \in \mathcal{M}_n$. We have to show that $\mathbb{A}_n a_{n-1} a_n = \mathbb{A}_n$ for all generic elements $a_n, a_{n-1}$ in $\mathbb{A}_n$. This is true when $n = 1$ (Corollary 2.6.(1)). To prove general case we use induction on $n$. So, let $n \geq 2$ and we assume that the claim is true for all $n' < n$. Let $a$ be a generic element of $\mathbb{A}_n$. Then $a = a_1 \otimes b_1 + \cdots + a_s \otimes b_s \in \mathbb{A}_1 \otimes \mathbb{A}_{n-1}$ where $a_i$ are nonzero elements of $\mathbb{A}_1$ such that $a_1$ is generic; $b_i$ are distinct elements of the basis $B_{n-1}$ such that $b_1 \in \mathcal{N}_{n-1}$. By statement 6, $F a_1 F \neq 0$, and so $E_{ij} a_1 E_{kl} \neq 0$ for some $i, j, k, l \in \mathbb{N}$. Then, for each $t = 1, \ldots, s$, $E_{ij} a_1 E_{kl} = \lambda_t E_{il}$ for some $\lambda_t \in K$, necessarily $\lambda_t \neq 0$. Now, $E_{ij} a E_{kl} = E_{il} \otimes u$ where $u := \lambda_t b_1 + \cdots + \lambda_s b_s$ is a generic element of $\mathbb{A}_{n-1}$ since $\lambda_t \neq 0$ and $b_1 \in \mathcal{N}_{n-1}$. By induction, $\mathbb{A}_{n-1} u \mathbb{A}_{n-1} = \mathbb{A}_{n-1}$, and so

$$\mathbb{A}_n a_{n-1} \mathbb{A}_n \subseteq \mathbb{A}_n (E_{il} \otimes u) \mathbb{A}_n = \mathbb{A}_1 E_{il} \mathbb{A}_1 \otimes \mathbb{A}_{n-1} \mathbb{A}_{n-1} = F \otimes \mathbb{A}_{n-1} = p_1.$$

By symmetry, all $p_i \subseteq \mathbb{A}_n a_{n-1}$, and so $a_n = p_1 + \cdots + p_n \subseteq \mathbb{A}_n a_{n-1}$. $a_n$ is the maximal ideal of $\mathbb{A}_n$ that is properly contained in the ideal $\mathbb{A}_n a_{n-1}$ (since $a$ is generic), and so $\mathbb{A}_n a_{n-1} = \mathbb{A}_{n-1}$, as required.

8. Let, for a moment, $n = 1$. One can easily verify that, for all $i, j, k \in \mathbb{N}$,

$$x^k E_{ij} = E_{i+k,j}, \quad \partial^k E_{ij} = i(i-1) \cdots (i-k+1) E_{i-k,j},$$

$$E_{ij} x^k = E_{i,j+k}, \quad E_{ij} \partial^k = (j+k)(j+k-1) \cdots (j+1) E_{i,j+k}.$$

where $E_{st} := 0$ if either $s < 0$ or $t < 0$. By (23), for each $j \in \mathbb{N}$, the left $\mathbb{A}_1$-module $C_j := \bigoplus_{i \in \mathbb{N}} K E_{ij}$ is isomorphic to the left $\mathbb{A}_1$-module $K[x]$, and so $C_j$ is a faithful, simple, left $\mathbb{A}_1$-module. The left $\mathbb{A}_1$-module

$$F = \bigoplus_{i \in \mathbb{N}} C_i \simeq K[x]^{(N)}$$

(25)
is a direct sum of \( \mathbb{N} \) copies of \( K[x] \), and so \( A_n \) is the \textit{faithful, semi-simple, left} \( A_1 \)-module.

For an arbitrary \( n \), the left \( A_n \)-module \( C_{i_1} \otimes \cdots \otimes C_{i_n} \) is isomorphic to \( P_n \). Therefore,

\[
A_n F^\otimes n \cong \bigoplus_{i_1, \ldots, i_n \in \mathbb{N}} C_{i_1} \otimes \cdots \otimes C_{i_n} \cong P_n^{(n)}
\]

is a semi-simple, left \( A_n \)-module which is faithful, by statement 6.

Recall that the structure of the right \( A_n \)-module \( P_n \) is given by the rule: \( p \ast a := \theta(a)p \) where \( p \in P_n \) and \( a \in A_n \). By (19), \( \theta(F^\otimes n) = F^\otimes n \). Then, by (26), the \( K \)-module \( F^\otimes n \cong P_n^{(n)} \) has the natural structure of \textit{right} \( A_n \)-module, namely, \( f \ast a := \theta(a)f \) where \( f \in F^\otimes n \) and \( a \in A_n \). In order to distinguish this structure of \textit{right} \( A_n \)-module from the obvious structure (as a right ideal of \( A_n \)) we write \( F^\otimes_n := P_n^{(n)} \). The map

\[
\theta : F^\otimes_n \rightarrow F^\otimes_n \cong P_n^{(n)}, \ f \mapsto \theta(f),
\]

is an \textit{isomorphism} of \textit{right} \( A_n \)-modules (since \( \theta(fa) = \theta(a)\theta(f) = \theta(f)\ast a \)). Therefore, \( F^\otimes_n \cong F^\otimes_n \cong P_n^{(n)} \) is the faithful, semi-simple, right \( A_n \)-module by the proved left version of this fact.

The \( A_n \)-bimodule \( F^\otimes n \) is simple since the ring \( F^\otimes n \cong M_\infty(K)^\otimes n \cong M_\infty(K) \) is simple. The map \( 1 \otimes \theta : A_n \otimes A_{\text{op}} \rightarrow A_{2n} \) is an isomorphism of \( K \)-algebras such that \( 1 \otimes \theta(F^\otimes n \otimes F^\otimes n) = F^\otimes 2n \). Using this equality and \( A_n F^\otimes n \cong A_n \otimes A_{\text{op}} F^\otimes n \cong A_{2n} F^\otimes n \), we see that the \( A_n \)-bimodule \( F^\otimes n \) is faithful: if \( a F^\otimes n = 0 \) for some nonzero ideal \( a \) of \( A_{2n} \), then \( F^\otimes 2n \subseteq a \), and so \( 0 = a F^\otimes n \supseteq F^\otimes 2n \), \( F^\otimes n = F^\otimes n F^\otimes n = F^\otimes n \neq 0 \), a contradiction.

9. Let soc(\( A_n \)) be the socle of the module \( \kappa_n A_n \) (resp. \( \kappa_n A_{\text{An}} \)). By statement 8, \( F^\otimes n \subseteq \text{soc}(\kappa_n A_n) \). Suppose that \( F^\otimes n \neq \text{soc}(\kappa_n A_n) \). Then \( \text{soc}(\kappa_n A_n) = F^\otimes n \oplus M \) for a nonzero module \( \kappa_n M \) (resp. \( \kappa_n M_{\text{An}} \)). On the one hand, \( F^\otimes n \text{soc}(\kappa_n A_n) \subseteq F^\otimes n \) and so \( F^\otimes n M = 0 \). On the other hand, \( F^\otimes n M \neq 0 \), by statement 6, a contradiction. Therefore, \( \text{soc}(\kappa_n A_n) = F^\otimes n \).

The algebra \( A_n \) admits the involution \( \theta \) such that \( \theta(F^\otimes n) = F^\otimes n \). Therefore, \( \text{soc}(A_n \kappa_n A_n) = F^\otimes n \) since \( \text{soc}(\kappa_n A_n) = F^\otimes n \).

10. Let \( M \) be a faithful, simple, left (resp. right) \( A_n \)-module. Then \( F^\otimes n M \neq 0 \) (resp. \( MF^\otimes n \neq 0 \)). Choose a nonzero element, say \( m \), of \( M \) such that \( F^\otimes n m \neq 0 \) (resp. \( m F^\otimes n \neq 0 \)). Then \( M = F^\otimes n m \) (resp. \( M = m F^\otimes n \)), by simplicity of \( M \). There is the epimorphism \( F^\otimes n \rightarrow F^\otimes n m = M \), \( f \mapsto fm \) (resp. \( F^\otimes n \rightarrow m F^\otimes n = M \), \( f \mapsto mf \)) of left (resp. right) \( A_n \)-modules. Now, the result follows from statement 8.

11. \( 3n = \text{GK}(\kappa_n) \geq \text{GK}(\kappa_n/a) \geq \text{GK}(\kappa_n/a_n) = \text{GK}(\kappa_n) = 3n \), and so \( \text{GK}(\kappa_n/a) = 3n \). \( \square \)

3 \quad \text{Unique factorization of ideals of} \ A_n \text{ and Spec}(\kappa_n)

In this section, all the results on ideals that are mentioned in the Introduction are proved.

Let \( B_n \) be the set of all functions \( f : \{1, 2, \ldots, n\} \rightarrow \mathbb{Z}_2 := \{0, 1\} \) where \( \mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z} \) is a field. \( B_n \) is a commutative ring with respect to addition and multiplication of functions. For \( f, g \in B_n \), we write \( f \geq g \) iff \( f(i) \geq g(i) \) for all \( i = 1, \ldots, n \) where \( 1 > 0 \). Then \( (B_n, \geq) \) is a partially ordered set. For each function \( f \in B_n \), \( I_f \) denotes the ideal \( I_f(1) \otimes \cdots \otimes I_f(n) \).
of \( A_n \) which is the tensor product of the ideals \( I_f(i) \) of the tensor components \( A_1(i) \) in \( A_n = A_1(1) \otimes \cdots \otimes A_1(n) \) where \( I_0 := F \) and \( I_1 := A_1 \). \( f \geq g \) iff \( I_f \supset I_g \). For \( f, g \in B_n \), \( I_f I_g = I_f \cap I_g = I_{fg} \). By induction on the number of functions one immediately proves that, for \( f_1, \ldots, f_s \in B_n \),

\[
\prod_{i=1}^{s} I_{f_i} = \cap_{i=1}^{s} I_{f_i} = I_{f_1 \cdots f_s}.
\]

Let \( C_n \) be the set of all subsets of \( B_n \) all distinct elements of which are incomparable (two distinct elements \( f \) and \( g \) of \( B_n \) are incomparable iff \( f \not\leq g \) and \( g \not\leq f \)). For each \( C \in C_n \), let \( I_C := \sum_{f \in C} I_f \), the ideal of \( A_n \). The next result classifies ideals of the algebra \( A_n \).

**Theorem 3.1** Let \( K \) be a field of characteristic zero. Then

1. The map \( C \mapsto I_C := \sum_{f \in C} I_f \) from the set \( C_n \) to the set of ideals of \( A_n \) is a bijection where \( I_0 := 0 \). In particular, there are only finitely many ideals of \( A_n \).
2. Each ideal \( I \) of \( A_n \) is an idempotent ideal, i.e. \( I^2 = I \).
3. Ideals of \( A_n \) commute \((IJ = JI)\).

**Proof.** 1. Statement 1 follows from Lemma 3.2.

2. The result is obvious for \( I = 0 \). So, let \( I \neq 0 \). By statement 1, \( I = \sum_{f \in C} I_f \) for some \( C \in C_n \). Then \( I^2 = \sum_{f \in C} I_f^2 + \sum_{f \neq g} I_f I_g = \sum_{f \in C} I_f + \sum_{f \neq g} I_f I_g = \sum_{f \in C} I_f = I \).

3. \( I_f I_g = I_{fg} = I_{fg} = I_{g} I_f \) for all \( f, g \in B_n \). The result is obvious if either \( I = 0 \) or \( J = 0 \). So, let \( I \neq 0 \) and \( J \neq 0 \). By statement 1, \( I = I_C \) and \( J = I_D \) for some \( C, D \in C_n \). Then \( IJ = (\sum_{f \in C} I_f)(\sum_{g \in D} I_g) = (\sum_{g \in D} I_g)(\sum_{f \in C} I_f) = JI \). □

Let \( B_n \) be the \( K \)-basis for the algebra \( A_n \) that is the tensor product of the \( K \)-bases from Theorem 2.5.(2) (see the proof of Corollary 2.7.(7) for details). For each element \( b = b_1 \otimes \cdots \otimes b_n \) of \( B_n \), one can attach an element \( f_b \) of \( B_n \) by the rule

\[
f_b(i) := \begin{cases} 1 & \text{if } b_i \not\in F, \\ 0 & \text{if } b_i \in F. \end{cases}
\]

Let \( a = \sum_{b \in B_n} \lambda_b b \) be a nonzero element of \( A_n \) where \( \{\lambda_b\} \) are the coordinates of \( a \) with respect to the basis \( B_n \). One has the well-defined map \( A_n \rightarrow C_n, \ a \mapsto \text{Max}(a) \), where \( \text{Max}(a) \) are the maximal elements (in \( B_n \)) of the subset \( \{f_b \mid \lambda_b \neq 0\} \) of \( B_n \) where \( \text{Max}(0) := \emptyset \).

**Lemma 3.2** Let \( K \) be a field of characteristic zero and \( 0 \neq a = \sum_{b \in B_n} \lambda_b b \in A_n \). Then \( A_n a A_n = \sum_{b \in \text{Max}(a)} I_{f_b} = \sum_{\{b \mid \lambda_b \neq 0\} } I_{f_b} \).

**Proof.** It suffices to prove only the first equality since the second follows from the first: for any \( b \in B_n \) such that \( \lambda_b \neq 0 \) there exists \( c \in \text{Max}(a) \) such that \( \lambda_c \neq 0 \) and \( f_c \geq f_b \), and so \( I_{f_c} \supset I_{f_b} \). Hence, \( \sum_{b \in \text{Max}(a)} I_{f_b} = \sum_{\{b \mid \lambda_b \neq 0\} } I_{f_b} \).
Fix $b \in \text{Max}(a)$. Up to order of the tensor multiples in the tensor product $A_n = \otimes_{i=1}^n A_1(i)$, one may assume that 

$$f_b(i) := \begin{cases} 
1 & \text{if } 1 \leq i \leq s, \\
0 & \text{if } s + 1 \leq i \leq n. 
\end{cases}$$

We have to show that $A_n a A_n \supseteq A_s \otimes F^{\otimes(n-s)}$. $a = \lambda_1 b_1 + \cdots + \lambda_r b_r$ where $b := b_1$, all $\lambda_i \neq 0$, $f_{b_1} = \cdots = f_b$ and $f_{b_j} \neq f_b$ for $j = t + 1, \ldots, r$. By the choice of $b$, for each $j$ such that $t + 1 \leq j \leq r$, either $f_{b_j} < f_b$ or, otherwise, the functions $f_{b_j}$ and $f_b$ are incomparable. $b_1 = c_1 \otimes f_1$ for unique elements $c_1 \in M_s$ and $f_1 = E_{p_1,q_1}(s + 1) \cdots E_{p_{n-s},q_{n-s}}(n) \in N_{n-s}$ where $M_s$ and $N_{n-s}$ were defined in the proof of Corollary 2.7.(7). Let $E_{pp} := E_{p_1,p_1}(s + 1) \cdots E_{p_{n-s},p_{n-s}}(n)$ and $E_{qq} := E_{q_1,q_1}(s + 1) \cdots E_{q_{n-s},q_{n-s}}(n)$. Then $E_{pp} f_1 E_{qq} = f_1$. Note that $E_{pp} A_{n-s} E_{qq} = KE_{pq} = K f_1$. Changing the element $a$ for the element $E_{pp} a E_{qq}$ and deleting zero terms of the type $E_{pp} \lambda_i b_i E_{qq}$, one may assume that $a = \left(\sum_{i=1}^l \mu_i c_i\right) \otimes f_1$ where all $\mu_i \neq 0$ and $\mu_i \in K$; all $c_i \in B_s$; $f_1 = \cdots = f_k = 1$ for some $k$ such that $1 \leq k \leq l$, i.e. $c_1, \ldots, c_k \in M_s$; and $f_{c_j} < f_{c_i}$ where $k + 1 \leq j \leq l$. The element $c := \sum_{i=1}^l \mu_i c_i \not\in A_s$, hence $A_s c A_s = A_s$. Now, $A_n a A_n = A_n(c \otimes f)A_n = A_s c A_s \otimes A_{n-s} f_1 A_{n-s} = A_s \otimes F^{\otimes(n-s)}$, as required. \hfill \Box

The next result is a useful criterion of when one ideal contains another.

**Corollary 3.3** Let $K$ be a field of characteristic zero and $C, C' \in C_n$. Then $I_C \subseteq I_{C'}$ iff $C \subseteq C'$ (this means that, for each $f \in C$, there exists $f' \in C'$ such that $f \leq f'$).

**Proof.** This follows at once from Lemma 3.2 and Theorem 3.1.(1). \hfill \Box

**Corollary 3.4** Let $K$ be a field of characteristic zero and $s_n$ the number of ideals of $A_n$. Then $2 - n + \sum_{i=1}^n \binom{n}{i} \leq s_n \leq 2^n$.

**Proof.** Let $\mathcal{B}_n$ be the set of all subsets of $\{1, \ldots, n\}$. $\mathcal{B}_n$ is a partially ordered set with respect to ‘$\subseteq$’. For each $f \in \mathcal{B}_n$, the subset $\text{supp}(f) := \{i \mid f(i) \neq 0\}$ of $\{1, \ldots, n\}$ is called the support of $f$. The map $\mathcal{B}_n \rightarrow \mathcal{B}_n$, $f \mapsto \text{supp}(f)$, is an isomorphism of posets. Let $\text{SSub}_n$ be the set of all subsets of $\text{Sub}_n$. An element $\{X_1, \ldots, X_j\}$ of $\text{SSub}_n$ is called incomparable if for all $i \neq j$ such that $1 \leq i, j \leq s$ neither $X_i \subseteq X_j$ nor $X_i \supseteq X_j$. An empty set and one element set are called incomparable by definition. Let $\text{Inc}_n$ be the subset of $\text{SSub}_n$ of all incomparable elements of $\text{SSub}_n$. Then the map

$$\text{C}_n \rightarrow \text{Inc}_n, \quad \{f_1, \ldots, f_s\} \mapsto \{\text{supp}(f_1), \ldots, \text{supp}(f_s)\}, \tag{28}$$

is a bijection. For each $i = 1, \ldots, n$, there are precisely $\binom{n}{i}$ subsets of $\{1, \ldots, n\}$ that contain exactly $i$ elements. Any non-empty collection of these is an incomparable set, hence $s_n \geq 2 + \sum_{i=1}^n \binom{n}{i} - 1 = 2 - n + \sum_{i=1}^n \binom{n}{i}$ where 2 ‘represents’ the zero ideal and the ideal $F^{\otimes n}$ which corresponds to an empty set. Clearly, $s_n = |\text{Inc}_n| \leq 2^n$. \hfill \Box

The next corollary classifies all the prime ideals of $A_n$. 

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Corollary 3.5 Let $K$ be a field of characteristic zero. Then the map
\[
\operatorname{Sub}_n \to \operatorname{Spec}(A_n), \ I \mapsto p_I := \sum_{i \in I} p_i, \ \emptyset \mapsto 0,
\]
is a bijection, i.e. any nonzero prime ideal of $A_n$ is a unique sum of height 1 primes; $|\operatorname{Spec}(A_n)| = 2^n$; the height of $p_I$ is $|I|$.

Proof. 0 is the prime ideal since $A_n$ is a prime ring. Let $I$ be a nonzero subset of \{1, \ldots, n\} that contains, say $s$ elements. Then $A_n/p_I \simeq A_s \otimes A_{n-s}$. The ring $A_s$ is a central, simple $K$-algebra, hence the map $a \mapsto A_s \otimes a$ is a bijection from the set of ideals of $A_{n-s}$ to the set of ideals of $A_s \otimes A_{n-s}$ (this is an easy consequence of the Density Theorem). Then, $A_s \otimes F^{\otimes(n-s)}$ is the smallest nonzero ideal of $A_s \otimes A_{n-s}$ and it is idempotent, and so $A_s \otimes A_{n-s}$ is a prime ring. This means that $p_I$ is a prime ideal. By Theorem 3.1.(1), the map $\operatorname{Sub}_n \to \operatorname{Spec}(A_n), I \mapsto p_I$, is an injection. It remains to prove that it is a surjection.

Let $p$ be a prime nonzero ideal of $A_n$. Then $p = \sum_{f \in C} I_f$ for some $C \in \mathcal{C}_n$ (Theorem 3.1.(1)). If $|\operatorname{supp}(f)| = n-1$ for all $f \in C$ then $I_f = p_i$ where $i = i(f)$ is a unique element of the set $\{1, \ldots, n\}$ such that $f(i) = 0$. Thus $p = \sum p_i$.

Suppose that $|\operatorname{supp}(f)| \neq n-1$ for some $f \in C$. Let us show that this case is not possible since then $p$ would not be a prime ideal. One can choose two distinct functions, say $g, h \in B_n$, such that $g > f$, $h > f$, and $gh = f$. Then $I_f = I_{gh} = I_gI_h$. Let $a := I_g + c$ and $b := I_h + c$ where $c := \sum_{f \neq f' \in C} I_{f'}$. The ideals $a$ and $b$ strictly contain the ideal $p$ and
\[
ab = (I_g + c)(I_h + c) = I_gI_h + cI_h + I_g c + c^2 = I_{gh} + cI_h + I_g c + c^2 \subseteq I_f + c = p.
\]
This contradicts to the fact that $p$ is a prime ideal, and we are done.

It is obvious that $|\operatorname{Spec}(A_n)| = 2^n$. The fact that the height of the ideal $p_I$ is $|I|$ follows from Lemma 3.6.(1). \(\square\)

The next criterion of when a prime ideal contains another prime is used in finding the classical Krull dimension of the algebra $A_n$ (Corollary 3.7).

Lemma 3.6 Let $K$ be a field of characteristic zero; $p, q \in \operatorname{Spec}(A_n); p = p_{i_1} + \cdots + p_{i_s}$ and $q = p_{j_1} + \cdots + p_{j_t}$ be their decompositions as in Corollary 3.5. Then

1. $p \subseteq q$ iff $\{p_{i_1}, \ldots, p_{i_s}\} \subseteq \{p_{j_1}, \ldots, p_{j_t}\}$.
2. If $p \subseteq q$ then $pq = p$.
3. The poset $(\operatorname{Spec}(A_n), \subseteq)$ is an isomorphic to the set $\operatorname{Sub}_n$ of all subsets of $\{1, \ldots, n\}$.

Proof. 1. $(\Rightarrow)$ $(q + p)/p$ is the ideal of the algebra $A_n/p \simeq A_s \otimes A_{n-s}$. The algebra $A_s$ is central and simple. By the Density Theorem, each ideal of the algebra $A_s \otimes A_{n-s}$ is of the type $A_s \otimes a$ for some ideal $a$ of $A_{n-s}$. By Corollary 3.5 (applied to the algebra $A_{n-s}$), we have the inclusion $\{p_{i_1}, \ldots, p_{i_s}\} \subseteq \{p_{j_1}, \ldots, p_{j_t}\}$.

$(\Leftarrow)$ If $\{p_{i_1}, \ldots, p_{i_s}\} \subseteq \{p_{j_1}, \ldots, p_{j_t}\}$ then $p = p_{i_1} + \cdots + p_{i_s} \subseteq p_{j_1} + \cdots + p_{j_t} = q$.

2. By statement 1, if $p \subseteq q$ then $q = p + r$ for an ideal $r$. Then $pq = p^2 + pr = p + pr = p$. 22
3. Statement 3 follows from Corollary 3.5 and statement 1. □

For each \( s = 0, 1, \ldots, n \), there are precisely \( \binom{n}{s} \) prime ideals of height \( s \), namely,

\[
\{ p_{i_1} + \cdots + p_{i_s} \mid 1 \leq i_1 < \cdots < i_s \leq n \}.
\]

**Corollary 3.7** Let \( K \) be a field of characteristic zero. Then the classical Krull dimension of \( \mathbb{A}_n \) is \( n \).

**Proof.** By Lemma 3.6.(1), \( 0 \subset p_1 \subset p_1 + p_2 \subset \cdots \subset p_1 + \cdots + p_n \) is a longest chain of primes, and so \( \text{cl.K.dim}(\mathbb{A}_n) = n \). □

\((\text{Spec}(\mathbb{A}_n), \subseteq)\) is a poset. Two primes \( p \) and \( q \) are called *incomparable* if neither \( p \subseteq q \) nor \( p \supseteq q \).

For each ideal \( a \) of \( \mathbb{A}_n \) such that \( a \neq \mathbb{A}_n \), let \( \text{Min}(a) \) be the set of all minimal primes over \( a \). The set \( \text{Min}(a) \) is a non-empty set since the ring \( \mathbb{A}_n \) has only finitely many primes.

For each \( f \in \mathcal{B}_n \), the set \( \text{csupp}(f) := \{ i \mid f(i) = 0 \} \) is called the *co-support* of \( f \). Clearly, \( \text{csupp}(f) = \{1, \ldots, n\} \setminus \text{supp}(f) \).

**Theorem 3.8** Let \( K \) be a field of characteristic zero. Then

1. Each ideal \( a \) of \( \mathbb{A}_n \) such that \( a \neq \mathbb{A}_n \) is a unique product of incomparable primes, i.e. if \( a = q_1 \cdots q_s = r_1 \cdots r_t \) are two such products then \( s = t \) and \( q_1 = r_{\sigma(1)}, \ldots, q_s = r_{\sigma(s)} \) for a permutation \( \sigma \) of \( \{1, \ldots, n\} \).

2. Each ideal \( a \) of \( \mathbb{A}_n \) such that \( a \neq \mathbb{A}_n \) is a unique intersection of incomparable primes, i.e. if \( a = q_1 \cap \cdots \cap q_s = r_1 \cap \cdots \cap r_t \) are two such intersections then \( s = t \) and \( q_1 = r_{\sigma(1)}, \ldots, q_s = r_{\sigma(s)} \) for a permutation \( \sigma \) of \( \{1, \ldots, n\} \).

3. For each ideal \( a \) of \( \mathbb{A}_n \) such that \( a \neq \mathbb{A}_n \), the sets of incomparable primes in statements 1 and 2 are the same, \( a = q_1 \cdots q_s = q_1 \cap \cdots \cap q_s \).

4. The ideals \( q_1, \ldots, q_s \) in statement 3 are the minimal primes of \( a \), and so \( a = \prod_{p \in \text{Min}(a)} p = \cap_{p \in \text{Min}(a)} p. \)

**Proof.** 1. For each ideal \( a \) of \( \mathbb{A}_n \), we have to prove that \( a \) is a product of incomparable primes and that this product is unique. Since the ring \( \mathbb{A}_n \) is prime these two statements are obvious when \( a = 0 \). So, let \( a \neq 0 \).

Existence: Let \( f \in \mathcal{B}_n \); then \( I_f = \prod_{i \in \text{csupp}(f)} p_i \). Let \( b \) be any ideal of \( \mathbb{A}_n \). Since \( b^2 = b \), it follows at once that

\[
I_f + b = \prod_{i \in \text{csupp}(f)} (p_i + b).
\]  

By Theorem 3.1.(1), \( a = I_{f_1} + \cdots + I_{f_s} \) for some \( f_i \in \mathcal{B}_n \). Repeating \( s \) times (29), we see that

\[
a = \prod_{i \in \text{csupp}(f_1), \ldots, i_s \in \text{csupp}(f_s)} (p_{i_1} + \cdots + p_{i_s})
\]
is the product of primes, by Corollary 3.5. Note that ideals of \( \mathbb{A}_n \) commute; each ideal is an idempotent ideal; and if \( p \subseteq q \) is an inclusion of primes then \( pq = p \). Using these three facts and (30), we see that \( a \) is a product of incomparable primes.

Uniqueness follows from the next lemma which will be used several times in the proof of this theorem.

**Lemma 3.9** Let \( \{q_1, \ldots, q_s\} \) and \( \{r_1, \ldots, r_t\} \) be two sets of incomparable ideals of a ring such that each ideal from the first set contains an ideal from the second and each ideal from the second set contains an ideal from the first. Then \( s = t \) and \( q_1 = r_{\sigma(1)}, \ldots, q_s = r_{\sigma(s)} \) for a permutation \( \sigma \) of \( \{1, \ldots, n\} \).

**Proof Lemma 3.9.** For each \( q_i \), there are ideals \( r_j \) and \( r_k \) such that \( r_j \subseteq q_i \subseteq r_k \), hence \( q_i = r_j = r_k \) since the ideals \( r_j \) and \( r_k \) are incomparable if distinct. This proves that for each ideal \( q_i \), there exists a unique ideal, say \( r_{\sigma(i)} \), such that \( q_i = r_{\sigma(i)} \). By symmetry, for each ideal \( r_j \) there exists a unique ideal, say \( q_{\tau(j)} \), such that \( r_j = q_{\tau(j)} \). Then, \( s = t \) and \( q_1 = r_{\sigma(1)}, \ldots, q_s = r_{\sigma(s)} \) for the permutation \( \sigma \) of \( \{1, \ldots, n\} \). □

**Uniqueness:** Let \( a = q_1 \cdot \cdots q_s = r_1 \cdot \cdots r_t \) be two products of incomparable primes. Each ideal \( q_i \) contains an ideal \( r_j \), and each ideal \( r_k \) contains an ideal \( q_i \). By Lemma 3.9, \( s = t \) and \( q_1 = r_{\sigma(1)}, \ldots, q_s = r_{\sigma(s)} \) for a permutation \( \sigma \) of \( \{1, \ldots, n\} \).

2. **Uniqueness:** Suppose that an ideal \( a \) has two presentations \( a = q_1 \cap \cdots q_s = r_1 \cap \cdots r_t \) of incomparable primes. The sets \( \{q_1, \ldots, q_s\} \) and \( \{r_1, \ldots, r_t\} \) of incomparable primes satisfy the conditions of Lemma 3.9, and so uniqueness follows.

**Existence:** Let \( I \) be the set of all the ideals of \( \mathbb{A}_n \), and \( I' \) be the set of ideals of \( \mathbb{A}_n \) that are intersection of incomparable primes. Then \( I' \subseteq I \). The map

\[
I \to I', \quad q_1 \cdot \cdots q_s \mapsto q_1 \cap \cdots \cap q_s,
\]

is a bijection since \( |I| < \infty \) and by uniqueness of presentations \( q_1 \cdot \cdots q_s \) (statement 1) and \( q_1 \cap \cdots \cap q_s \) (see above) where \( q_1, \ldots, q_s \) are incomparable primes. Then \( I = I' \). This proves that each ideal \( a \) of \( \mathbb{A}_n \) is an intersection of incomparable primes.

3. Let \( a \) be an ideal of \( \mathbb{A}_n \) and \( a = q_1 \cdot \cdots q_s = r_1 \cap \cdots \cap r_t \) where \( S := \{q_1, \ldots, q_s\} \) and \( T := \{r_1, \ldots, r_t\} \) be sets of incomparable primes. The sets \( S \) and \( T \) satisfy the conditions of Lemma 3.9, and so \( s = t \) and \( q_1 = r_{\sigma(1)}, \ldots, q_s = r_{\sigma(s)} \) for a permutation \( \sigma \) of \( \{1, \ldots, n\} \). This means that \( a = q_1 \cdot \cdots q_s = q_1 \cap \cdots \cap q_s \).

4. Let \( a = q_1 \cdots q_s = q_1 \cap \cdots \cap q_s \) be as in statement 3 and let \( \text{Min}(a) = \{r_1, \ldots, r_t\} \) be the set of minimal primes over \( a \). Then \( \text{Min}(a) \subseteq S := \{q_1, \ldots, q_s\} \) (\( a = q_1 \cdots q_s \subseteq r_t \) implies \( q_j \subseteq r_t \) for some \( j \), and so \( q_j = r_t \) by the minimality of \( r_t \)). Up to order, let \( r_{\tau(i)} = q_1, \ldots, r_{\tau(t)} = q_t \). It remains to show that \( t = s \). Suppose that \( t < s \), we seek a contradiction. This means that each prime \( q_i \), \( i = t + 1, \ldots, s \), contains \( a \) and is not minimal over \( a \). Hence, \( q_i \) contains a minimal prime, say \( q_{r(i)} \), a contradiction (the ideal \( q_i \) and \( q_{r(i)} \) are incomparable). □

**Corollary 3.10** Let \( K \) be a field of characteristic zero, \( a \) and \( b \) be ideals of \( \mathbb{A}_n \) distinct from \( \mathbb{A}_n \) in statement 1, 2 and 5. Then
1. \( a = b \) iff \( \text{Min}(a) = \text{Min}(b) \).

2. \( \text{Min}(a \cap b) = \text{Min}(ab) \) is the set of minimal elements (with respect to inclusion) of the set \( \text{Min}(a) \cup \text{Min}(b) \).

3. \( a \cap b = ab \).

4. If \( a \subseteq b \) then \( ab = a \).

5. \( a \subseteq b \) iff \( \text{Min}(a) \subseteq \text{Min}(b) \) (the \( \subseteq \) means that and each \( q \in \text{Min}(b) \) contains some \( p \in \text{Min}(a) \)).

**Proof.**

1. Statement 1 is obvious due to Theorem 3.8.(4).

2. Let \( M \) be the set of minimal elements of the union \( \text{Min}(a) \cup \text{Min}(b) \). The elements of \( M \) are incomparable, and (by Theorem 3.8.(4))

\[
a \cap b = \cap_{p \in \text{Min}(a)} \cap_{q \in \text{Min}(b)} q = \cap_{r \in M} r.
\]

By Theorem 3.8.(2), \( \text{Min}(a \cap b) = M \). By Lemma 3.6.(2),

\[
ab = \prod_{p \in \text{Min}(a)} p \cdot \prod_{q \in \text{Min}(b)} q = \prod_{r \in M} r = a \cap b.
\]

3. The result is obvious if one the ideals is equal to \( A_n \). So, let the ideals are distinct from \( A_n \). By statement 2, \( \text{Min}(a \cap b) = \text{Min}(ab) \), then, by statement 1, \( a \cap b = ab \).

4. If \( a \subseteq b \) then, by statement 3, \( ab = a \cap b = a \).

5. (\( \Rightarrow \)) If \( a \subseteq b \) then \( \text{Min}(a) \subseteq \text{Min}(b) \) since \( a = \prod_{p \in \text{Min}(a)} p \subseteq \prod_{q \in \text{Min}(b)} q = b \).

(\( \Leftarrow \)) Suppose that \( \text{Min}(a) \subseteq \text{Min}(b) \). For each \( q \in \text{Min}(b) \), let \( S(q) \) be the set (necessarily nonempty) of \( p \in \text{Min}(a) \) such that \( p \subseteq q \). Then \( \text{Min}(a) \supseteq S := \bigcup_{q \in \text{Min}(b)} S(q) \) and

\[
a = \cap_{p \in \text{Min}(a)} p \subseteq \cap_{p \in S} p \subseteq \cap_{q \in \text{Min}(a)} q = b. \quad \square
\]

**Theorem 3.11** Let \( K \) be a field of characteristic zero. Then the lattice of ideals of the algebra \( A_n \) is distributive, i.e. \( (a \cap b)c = ac \cap bc \) for all ideals \( a, b, \) and \( c \).

**Proof.** By Corollary 3.10.(3), \( (a \cap b)c = a \cap b \cap c = (a \cap c) \cap (b \cap c) = ac \cap bc \). \( \square \)

**Theorem 3.12** Let \( K \) be a field of characteristic zero, \( a \) be an ideal of \( A_n \), and \( M \) be the minimal elements with respect to inclusion of a set of ideals \( a_1, \ldots, a_k \) of \( A_n \). Then

1. \( a = a_1 \cdots a_k \) iff \( \text{Min}(a) = M \).

2. \( a = a_1 \cap \cdots \cap a_k \) iff \( \text{Min}(a) = M \).
For arbitrary \( c \), all ideals of the algebra \( A_n \) and so \( \text{Min}(\cdot) \), let \( a = a_1 \cdots a_k \) then, by Theorem 3.8.(4) and Corollary 3.10.(4),

\[
a = \prod_{i=1}^{k} \prod_{q_{ij} \in \text{Min}(a_i)} q_{ij} = \prod_{q \in \mathcal{M}} q.
\]

and so \( \text{Min}(a) = \mathcal{M} \), by Theorem 3.8.(4).

(\( \Rightarrow \)) Suppose that \( a = a_1 \cdots a_k \) then, by Corollary 3.10.(4), \( a = a_1 \cdots a_k \).

The involution \( c \). Let \( K \) be a field of characteristic zero and \( \mathcal{I}(A_n) \) be the set of all ideals of the algebra \( A_n \). Consider the map \( \mathcal{C}_n \setminus \{\emptyset\} \to \mathcal{C}_n \setminus \{\emptyset\} \), \( C \mapsto C + 1 \), where \( C = \{f_1, \ldots, f_s\}, C + 1 := \{f_1 + 1, \ldots, f_s + 1\} \). The map is well-defined: \( C \in \mathcal{C}_n \) iff \( \text{supp}(f_1), \ldots, \text{supp}(f_s) \} \in \text{Inc}_n \) iff \( \text{csupp}(f_1), \ldots, \text{csupp}(f_s) \} \in \text{Inc}_n \) where \( \text{csupp}(f_i) := \{1, \ldots, n\} \setminus \text{supp}(f_i) \) iff \( C + 1 \in \text{Inc}_n \). Consider the map

\[
c : \mathcal{I}(A_n) \to \mathcal{I}(A_n), \quad I_C \mapsto I_{C+1}, \quad C \in \mathcal{C}_n,
\]

where \( c(0) := 0 \). Then, for \( C \in \mathcal{C}_n \), \( c(\sum_{I \in C} I_f) = \sum_{I \in C} c(I_f) \). Note that \( c(A_n) = F^\otimes n \), \( c(p_i) = \prod_{I \not\in p_j} p_j \), and \( c(a_n) = \sum_{i=1}^{n} c(p_i) \).

Let \( C, C' \in \mathcal{C}_n \), we write \( C \preceq C' \) if for each \( f \in C \) there exists \( f' \in C' \) such that \( f \leq f' \), and for each \( g' \in C' \) there exists \( g \in C \) such that \( g \leq g' \).

**Lemma 3.13** Let \( K \) be a field of characteristic zero. Then

1. \( c : \mathcal{I}(A_n) \to \mathcal{I}(A_n) \) is an involution (\( c^2 = \text{id} \)) such that \( f \leq g \) implies \( c(I_f) \supseteq c(I_g) \).
2. \( c(a) = a \) iff \( a = I_C \) for some \( C = \{f_1, f_1 + 1, \ldots, f_s, f_s + 1\} \).
3. If \( C, C' \in \mathcal{C}_n \) and \( C \preceq C' \) then \( c(I_C) \supseteq c(I_{C'}) \).
4. \( c(a + b) \subseteq c(a) + c(b) \) for all ideals \( a \) and \( b \) of \( A_n \). If \( a = I_C \) and \( b = I_{C'} \) for some \( C, C' \in \mathcal{C}_n \) such that \( C \cup C' \in \mathcal{C}_n \) then \( c(a + b) = c(a) + c(b) \).
5. \( c(ab) \supseteq c(a) + c(b) \) for all nonzero ideals \( a \) and \( b \) of \( A_n \).

**Proof.**

1. \( c^2 = \text{id} \) since, for all \( \emptyset \neq C \in \mathcal{C}_n \), \( C + 1 + 1 = C \). The rest is obvious.
2. \( c(I_C) = I_C \) iff \( C + 1 = C \) iff \( C = \{f_1, f_1 + 1, \ldots, f_s, f_s + 1\} \).
3. If \( C \preceq C' \) then \( C' + 1 \preceq C + 1 \) and, so \( c(I_C) \supseteq c(I_{C'}) \).
4. The second statement is obvious: If \( C \cup C' \in \mathcal{C}_n \) then

\[
c(a + b) = c(I_{C \cup C'}) = \sum_{f \in C \cup C'} I_{f+1} = \sum_{f \in C} I_{f+1} + \sum_{f \in C'} I_{f'+1} = a + b.
\]

For arbitrary \( a \) and \( b \), let \( a = I_C \) and \( b = I_D \), and \( a + b = I_E \) for some \( C, D, E \in \mathcal{C}_n \). Then, for each \( e \in E \), either \( e \geq f \) for some \( f \in C \) or \( e \geq g \) for some \( g \in D \). Then, either \( c(I_e) \subseteq c(I_f) \) or \( c(I_e) \subseteq c(I_g) \). Hence, \( c(a + b) \subseteq c(a) + c(b) \).
5. Let \( a = I_C, \ b = I_D, \) and \( ab = I_E \) for some \( C, D, E \in \mathcal{C}_n. \) Then \( E \leq C \) and \( E \leq D. \) By statement 3, \( c(ab) \geq c(a) + c(b). \)

The involution \( \tau. \) Let \( K \) be a field of characteristic zero. For each \( p \in \text{Spec}(\mathbb{A}_n), \) there exists a unique prime ideal \( \tau(p) \) such that \( a_n = p \oplus \tau(p). \) In more detail, if \( p = p_{i_1} + \cdots + p_{i_s} \) then \( \tau(p) = p_{j_1} + \cdots + p_{j_t} \) where \( \{j_1, \ldots, j_t\} := \{1, \ldots, n\} \setminus \{i_1, \ldots, i_s\}. \)

The map \( \tau : \text{Spec}(\mathbb{A}_n) \to \text{Spec}(\mathbb{A}_n) \) is an order reversion involution, i.e. \( p \subseteq q \) implies \( \tau(p) \supseteq \tau(q) \) for \( p, q \in \text{Spec}(\mathbb{A}_n); \) and \( \tau^2 = \text{id}. \) In particular, \( \tau \) is an anti-automorphism of the poset \( \text{Spec}(\mathbb{A}_n). \) \( \tau(a_n) = 0 \) and \( \tau(p_i) = \sum_{j \neq i} p_j. \) Let \( \mathcal{I}_n \) be the set of ideals of \( \mathbb{A}_n \) distinct from \( \mathbb{A}_n. \) The map \( \tau \) can be extended to the map

\[ \tau : \mathcal{I}_n \to \mathcal{I}_n, \quad a = \cap_{q \in \text{Min}(a)} \mapsto \tau(a) := \cap_{q \in \text{Min}(a)} \tau(q). \]

Lemma 3.14 Let \( K \) be a field of characteristic zero, and \( a, b \in \mathcal{I}_n. \) Then

1. \( \tau : \mathcal{I}_n \to \mathcal{I}_n \) is the involution \( (\tau^2 = \text{id}). \)

2. \( \tau(a) = a \) iff \( \tau(\text{Min}(a)) = \text{Min}(a). \)

Proof. 1. The elements of the set \( \text{Min}(a) \) are incomparable, then so are the elements of the set \( \tau(\text{Min}(a)), \) hence \( \tau(\text{Min}(a)) = \text{Min}(\tau(a)). \) Then \( \tau^2 = \text{id}. \)

2. By Corollary 3.10.(1), \( \tau(a) = a \) iff \( \tau(\text{Min}(a)) = \text{Min}(a). \)

A prime ideal of a ring \( R \) is called a completely prime if \( R/p \) is a domain.

Corollary 3.15 Let \( K \) be a field of characteristic zero. Then

1. \( a_n \) is the only completely prime ideal of \( \mathbb{A}_n. \)

2. \( a_n \) is the only ideal \( a \) of \( \mathbb{A}_n \) such that the factor ring \( \mathbb{A}_n/a \) is Noetherian (resp. left Noetherian, resp. right Noetherian).

Proof. 1. By Corollary 3.5, any prime ideal \( p \) of \( \mathbb{A}_n \) is a unique sum \( p = p_{i_1} + \cdots + p_{i_s}. \) Then \( \mathbb{A}_n/p \simeq \mathbb{A}_s \otimes \mathbb{A}_{n-s}. \) The ring \( \mathbb{A}_s \otimes \mathbb{A}_{n-s} \) is a domain iff \( s = n, \) that is \( p = a_n. \)

2. The factor ring \( \mathbb{A}_n/a_n \simeq \mathbb{A}_n \) is Noetherian. It remains to show that \( \mathbb{A}_n/a \) is not left and right Noetherian for all ideals \( a \) distinct from \( a_n. \) By Theorem 3.8.(4), if \( a \neq a_n \) then \( a_n \notin \text{Min}(a). \) Choose \( p \in \text{Min}(a). \) Then \( \mathbb{A}_n/p \simeq \mathbb{A}_s \otimes \mathbb{A}_{n-s} \) for some \( s \geq 1. \) The ring \( \mathbb{A}_s \otimes \mathbb{A}_{n-s} \) is not left or right Noetherian. The ring \( \mathbb{A}_s \otimes \mathbb{A}_{n-s} \) is a factor ring of \( \mathbb{A}_n/a, \) and the result follows. \( \square \)

4 The group of units \( \mathbb{A}_n^* \) of \( \mathbb{A}_n \)

In this section, \( K \) be a field of characteristic zero. Let \( \mathbb{A}_n^*, \mathbb{A}_n^* \) and \( K^* \) be the groups of units of the algebras \( \mathbb{A}_n, \mathbb{A}_n, \) and \( K \) respectively. Using the \( \mathbb{Z}^n \)-grading of the skew polynomial algebra \( \mathbb{A}_n \) (see (2)), it follows that

\[ \mathbb{A}_n^* = (S_n^{-1} P_n)^* = \{K^* \prod_{i \in \mathbb{Z}} \prod_{j=1}^n (H_j + i)^{n_{ij}} \mid (n_{ij}) \in (\mathbb{Z}^n)^{\mathbb{Z}}\} \simeq K^* \times (\mathbb{Z}^n)^{\mathbb{Z}} \] (31)

27
where the abelian group \((\mathbb{Z}^n)^{(\mathbb{Z})}\) is the direct sum of \(\mathbb{Z}\) copies of \(\mathbb{Z}^n\).

The group \(K^* \times H_n\). Let, for a moment, \(n = 1\). In this case we usually drop the subscript 1. For each integer \(i \geq 1\) and \(\lambda \in K^*\), the element \((H - i)_\lambda := H - i + \lambda \pi_{i-1}\) is a unit of the algebra \(-\lim\) and its inverse is equal to

\[
(H - i)_\lambda^{-1} := \begin{cases} 
\rho_{i1} + \lambda^{-1} \pi_0 & \text{if } i = 1, \\
\rho_{i1} + \sum_{j=0}^{i-2} \frac{1}{j+1} \pi_j + \lambda^{-1} \pi_{i-1} & \text{if } i \geq 2.
\end{cases}
\]

As a function of the discrete argument \(H\), \((H - i)_\lambda^{-1}\) coincides with \(\frac{1}{H-i}\) but instead of having pole at \(H = i\) it takes the value \(\lambda^{-1}\). Consider the following subgroup of \(D_n^*\),

\[
H := \left\{ \prod_{i \geq 0} (H + i)^{n_i} \cdot \prod_{i \geq 1} (H - i)^{n_i-1} \mid (n_i) \in \mathbb{Z}^{(\mathbb{Z})} \right\} \simeq \mathbb{Z}^{(\mathbb{Z})}. \tag{32}
\]

For an arbitrary \(n \geq 1\), recall that \(A_n = \otimes_{i=1}^n A_1(i) = A_1^{\otimes n}\). For each tensor multiple \(A_1(i) = A_1\), let \(H(i)\) be the corresponding group \(H\). Their product \(H_n := H(1) \cdots H(n)\) is a subgroup of \(D_n^*\) and \(H_n \simeq H_n \simeq (\mathbb{Z}^n)^{(\mathbb{Z})}\). The natural inclusion \(H_n \simeq (H_n + a_n)/a_n \subset A_n/a_n \simeq A_n\) and (31) yield the isomorphism of groups

\[
K^* \times H_n \rightarrow A_n^*, \quad \lambda \mapsto \lambda, \quad H_s + i \mapsto H_s + i, \quad (H_s - j) \mapsto H_s - j, \tag{33}
\]

where \(\lambda \in K^*, 1 \leq s \leq n, i \in \mathbb{N}\) and \(1 \leq j \in \mathbb{N}\). \(K^*H_n\) is the subgroup of \(D_n^*\) such that \(K^*H_n \simeq K^* \times H_n\).

The group \((1 + F^{\otimes n})^*\) of units of the monoid \(1 + F^{\otimes n}\). We are going to find the group \((1 + F^{\otimes n})^*\) of units of the multiplicative (noncommutative) monoid \(1 + F^{\otimes n}\). Let, for a moment, \(n = 1\). The ring \(F = \oplus_{i,j \in \mathbb{N}} KE_{ij}\) is the union \(M_1(K) := \cup_{d \geq 1} M_d(K) = \lim M_d(K)\) of the matrix algebras \(M_d(K) := \oplus_{1 \leq i,j \leq d-1} KE_{ij},\) i.e. \(F = M_1(K)\).

For each \(d \geq 1\), consider the (usual) determinant \(\det_d = \det : 1 + M_d(K) \rightarrow K, u \mapsto \det(u)\). These determinants determine the (global) determinant

\[
\det : 1 + M_\infty(K) = 1 + F \rightarrow K, \ u \mapsto \det(u),
\]

where \(\det(u)\) is the common value of all determinants \(\det_d(u), d \gg 1\). The (global) determinant has usual properties of the determinant. In particular, for all \(u, v \in 1 + M_\infty(K), \ \det(uv) = \det(u) \cdot \det(v)\). It follows from Cramer’s formula that the group \(GL_\infty(K) := (1 + M_\infty(K))^*\) of units of the monoid \(1 + M_\infty(K)\) is equal to

\[
GL_\infty(K) = \{ u \in 1 + M_\infty(K) \mid \det(u) \neq 0 \}. \tag{34}
\]

Therefore,

\[
(1 + F)^* = \{ u \in 1 + F \mid \det(u) \neq 0 \} = GL_\infty(K). \tag{35}
\]

The kernel

\[
SL_\infty(K) := \{ u \in GL_\infty(K) \mid \det(u) = 1 \}
\]

of the group epimorphism \(\det : GL_\infty(K) \rightarrow K^*\) is a normal subgroup of \(GL_\infty(K)\).
For any $n \geq 1$,

$$F^\otimes n = \bigotimes_{i=1}^n F(i) = \bigotimes_{i=1}^n (\cup_{d_i \geq 1} M_{d_i}(K)) = \cup_{d_1, \ldots, d_n \geq 1} \bigotimes_{i=1}^n M_{d_i}(K) = M_\infty(K).$$

Consider the determinant

$$\det : 1 + F^\otimes n = 1 + M_\infty(K) \to K, \quad u \mapsto \det(u),$$

as in the case $n = 1$. Hence,

$$(1 + F^\otimes n)^* = \{ u \in 1 + F^\otimes n \mid \det(u) \neq 0 \} = (1 + M_\infty(K))^* = \text{GL}_\infty(K). \quad (36)$$

For each element $u \in (1 + F^\otimes n)^*$, using Cramer’s formula one can easily find a formula for the inverse $u^{-1}$, it is Cramer’s formula.

**Lemma 4.1** Let $K$ be a field of characteristic zero and $u \in 1 + F^\otimes n$. The following statements are equivalent.

1. $u \in A_n^*$.
2. The element $u$ has left inverse in $A_n$ ($vu = 1$ for some $v \in A_n$).
3. The element $u$ has right inverse in $A_n$ ($uv = 1$ for some $v \in A_n$).
4. $\det(u) \neq 0$.

Proof. Using the $\mathbb{Z}^n$-grading on $A_n$, it is obvious that the first three statements are equivalent to the fourth. □

Since $F^\otimes n$ is an ideal of $A_n$, the subgroup $(1 + F^\otimes n)^*$ of $A_n^*$ is a normal subgroup: For all $a \in A_n^*$, $a(1 + F^\otimes n)a^{-1} = 1 + a F^\otimes n a^{-1} \subseteq 1 + F^\otimes n$, and so $a(1 + F^\otimes n)^* a^{-1} \subseteq (1 + F^\otimes n)^*$.

**The subgroup** $K^* \times (H_n \times (1 + F^\otimes n)^*)$ of $A_n^*$. Let $A_n'$ be the subgroup of the group $A_n^*$ generated by its subgroups $K^*$, $H_n$ and $(1 + F^\otimes n)^*$. Let us prove that

$$A_n' = K^* \times (H_n \times (1 + F^\otimes n)^*). \quad (37)$$

The subgroup $(1 + F^\otimes n)^*$ of $A_n^*$ (and of $A_n'$) is normal and the subgroup $K^*$ belongs to the centre of $A_n^*$, hence $A_n' = K^* H_n(1 + F^\otimes n)^*$, i.e. each element $a$ of $A_n'$ is a product $a = \lambda \alpha u$ for some elements $\lambda \in K^*$, $\alpha \in H_n$ and $u \in (1 + F^\otimes n)^*$. In order to prove (37) it suffices to show uniqueness of the decomposition $a = \lambda \alpha u$. Since $a + a_n = \lambda \alpha + a_n \in (A_n/a_n)^* \simeq A_n^*$, the uniqueness of $\lambda$ and $\alpha$ follows from (31) and (33). Then $u = (\lambda \alpha)^{-1} a$ is unique as well.

This finishes the proof of (37).

**The group** $A_1^*$ and its commutants.

**Theorem 4.2** Let $K$ be a field of characteristic zero. Then
1. $A_1^* = K^* \times (\mathcal{H} \rtimes (1 + F)^*)$, each unit $a$ of $A_1$ is a unique product $a = \lambda \alpha (1 + f)$ for some elements $\lambda \in K^*$, $\alpha \in \mathcal{H}$, and $f \in F$ such that $\det(1 + f) \neq 0$.

2. $A_1^* = K^* \times (\mathcal{H} \rtimes \text{GL}_\infty(K))$.

3. The centre of the group $A_1^*$ is $K^*$.

4. The commutant $A_1^{*(2)} = [A_1^*, A_1^*]$ of the group $A_1^*$ is equal to $\text{SL}_\infty(K) := \{ v \in (1 + F)^* = M_\infty(K) \mid \det(v) = 1 \}$, and $A_1/\langle A_1^*, A_1^* \rangle \simeq K^* \times \mathcal{H} \times K^*$.

5. All the higher commutants $A_1^{*(i)} := [A_1^*, A_1^{*(i-1)}]$, $i \geq 2$, are equal to $A_1^{*(2)}$.

Proof. 1. By (37), $A_1^* = K^* \times (\mathcal{H} \rtimes (1 + F)^*) \subseteq A_1^*$. It suffices to show the reverse inclusion. By (33), there is the exact sequence of groups

$$1 \to (1 + F)^* \to A_1^* \to (A_1/F)^* \simeq A_1^* \to 1$$

which, using again (33), yields the inclusion $A_1^* \subseteq A_1'$. The rest of statement 1 follows from (36).

2. Statement 2 is equivalent to statement 1 since $(1 + F)^* = \text{GL}_\infty(K)$, see (35).

3. Let $Z$ be the centre of the group $A_1^*$. Since $K^* \subseteq Z$, we have

$$Z = Z \cap A_1^* = Z \cap (K^* \mathcal{H}(1 + F)^*) = K^*(Z \cap (1 + F)^*)$$

We have to show that $Z \cap (1 + F)^* = \{1\}$. Let $z = \alpha u \in Z \cap (1 + F)^*$ where $\alpha = \alpha(H) \in \mathcal{H}$ and $u \in (1 + F)^*$. It remains to show that $z = 1$. $\beta z = z \beta$ for all $\beta \in \mathcal{H}$ iff $\beta u = u \beta$ for all $\beta \in \mathcal{H}$ (since $\mathcal{H}$ is an abelian group) iff $[\beta]u = u[\beta]$ (the equality of infinite matrices) for all $\beta = \beta(H) \in \mathcal{H}$ where $[\beta]$ is the diagonal matrix diag$(\beta(1), \beta(2), \ldots)$ iff $u$ is a diagonal matrix of $(1 + F)^*$. The diagonal entries of the matrix $u$, say $u_i$, $i \in \mathbb{N}$, are elements of $K^*$ such that $u_i = 1$ for all $i \geq d$ for some natural number $d = d(u)$. For all distinct $i, j \in \mathbb{N}$, $1 \leq E_{ij} \in (1 + F)^*$. Now, it follows from the equalities

$$z + \alpha(i + 1)u_i E_{ij} = (1 + E_{ij})z = z + E_{ij} \alpha(j + 1)u_j$$

that $\alpha(i + 1)u_i = \alpha(j + 1)u_j$ where $\alpha(i + 1) := \alpha(H)|_{H=1}$ (we have used that $\alpha E_{ij} = \alpha(i + 1)E_{ij}$ and $E_{ij} \alpha = E_{ij} \alpha(i + 1)$). For all distinct natural numbers $i$ and $j$ such that $i, j > d$, we have $\alpha(i + 1) = \alpha(j + 1)$. This means that the function $\alpha(H)$ is a constant, i.e. $\alpha = 1$. This gives $u_i = u_j$ for all $i, j \in \mathbb{N}$ such that $i \neq j$, i.e. $u_i = 1$. Therefore, $z = 1$, as required.

4. The determinant can be extended from the subgroup $(1 + F)^*$ of $A_1^*$ to the whole group by the rule

$$\det : A_1^* \to K^* \times \mathcal{H} \times K^*, \ \lambda \alpha u \mapsto \lambda \alpha \det(u) := (\lambda, \alpha, \det(u)), \quad (39)$$

where $\lambda \in K^*$, $\alpha \in \mathcal{H}$, and $u \in (1 + F)^*$. It turns out that $\det$ is a group epimorphism. By the very definition, $\det$ is a surjection. It remains to show that $\det(aa') = \det(a)\det(a')$ for all $a := \lambda \alpha u$, $a' := \lambda' \alpha' u' \in A_1^*$. This follows from the following equality

$$\det(\alpha^{-1}ua) = \det(u), \ \alpha \in \mathcal{H}, \ u \in (1 + F)^*. \quad (40)$$
Indeed,
\[ \det(aa') = \det(\lambda\lambda'\alpha' \cdot (\alpha')^{-1}ua'u') = \lambda\lambda'\alpha' \cdot \det((\alpha')^{-1}ua') \det(u') \]
\[ = \lambda\lambda'\alpha' \cdot \det(u) \det(u') = \det(a) \det(a'). \]

The proof of (40):
\[ \det(\alpha^{-1}ua) = \det([\alpha^{-1}]u[\alpha]) \quad \text{(where } [\alpha] := \text{diag}(\alpha(1), \alpha(2), \ldots)) \]
\[ = \det([\alpha^{-1}]_d u [\alpha]_d) \quad \text{for all } d \gg 1 \]
\[ = \det(u_d) = \det(u) \quad \text{for all } d \gg 1 \]

where, for an infinite matrix \( X = \sum_{i,j \in \mathbb{N}} x_{ij} E_{ij} \), \( X_d := \sum_{0 \leq i,j \leq d-1} x_{ij} E_{ij} \) is the sub-matrix of \( X \) of size \( d \times d \).

The kernel of the epimorphism \( \det \), (39), is \( \text{SL}_\infty(K) := \{ u \in (1+F)^* = \text{GL}_\infty(K) \mid \det(u) = 1 \} \). In particular, \( \text{SL}_\infty(K) \) is a normal subgroup of \( \mathbb{A}_1^* \) such that the factor group \( \mathbb{A}_1^*/\text{SL}_\infty(K) \simeq K^* \times \mathcal{H} \times K^* \) is abelian. Hence, \( [\mathbb{A}_1^*, \mathbb{A}_1^*] \subseteq \text{SL}_\infty(K) \) and there is the short exact sequence of groups
\[ 1 \to \text{SL}_\infty(K) \to \mathbb{A}_1^* \xrightarrow{\det} K^* \times \mathcal{H} \times K^* \to 1. \] (41)

The group \( \text{SL}_\infty(K) \) is generated by the transvections \( t_{ij}(\lambda) := 1 + \lambda E_{ij}, \lambda \in K^*, i, j \in \mathbb{N}, i \neq j \). Note that
\[ [H, t_{ij}(\lambda)] = t_{ij}(\frac{i-j}{j+1} \lambda) \] (42)

where \( [a, b] := aba^{-1}b^{-1} \) is the commutator of two elements of a group. In more detail,
\[ [H, t_{ij}(\lambda)] = H t_{ij}(\lambda) H^{-1} t_{ij}(\lambda)^{-1} = t_{ij}(\frac{i+1}{j+1} \lambda) t_{ij}(-\lambda) = t_{ij}(\frac{i-j}{j+1} \lambda). \]

Since \( H \in \mathcal{H} \), we have the inclusion \( \text{SL}_\infty(K) \subseteq [\mathbb{A}_1^*, \mathbb{A}_1^*] \), i.e. \( [\mathbb{A}_1^*, \mathbb{A}_1^*] = \text{SL}_\infty(K) \).

5. Statement 5 follows from (42) and the fact that the group \( \text{SL}_\infty(K) \) is generated by the transvections. \( \square \)

**An inversion formula for** \( u \in \mathbb{A}_1^* \). Let \( K \) be a field of characteristic zero. By Theorem 4.2.(1), each element \( u \) of \( \mathbb{A}_1^* \) can written as \( u = \lambda a(1 + f) \). The inverse \((1 + f)^{-1}\) can be found using Cramer’s formula for the inverse of matrix. Then \( u^{-1} = \lambda^{-1}(1 + f)^{-1} a^{-1} \). Let \( f \in K[x] \) be a given polynomial and \( y \in K[x] \) is an unknown. Then the integro-differential equation \( u y = f \) can be solved explicitly: \( y = u^{-1} f \).

In contrast to differential operators on an affine line, in general, the space of solutions for integro-differential operators is infinite-dimensional: Example. \( E_{ij} y = 0 \).

For an ideal \( I \) of \( \mathbb{A}_n \) such that \( I \neq \mathbb{A}_n \), let \((1 + I)^*\) be the group of units of the multiplicative monoid \( 1 + I \).

**Lemma 4.3** Let \( K \) be a commutative \( \mathbb{Q} \)-algebra, \( I \) and \( J \) be ideals of \( \mathbb{A}_n \) which are distinct from \( \mathbb{A}_n \). Then

1. \( \mathbb{A}_n^* \cap (1 + I) = (1 + I)^* \).
2. \((1 + I)^*\) is a normal subgroup of \(A_n^*\).

**Proof.** 1. The inclusion \(A_n^* \cap (1 + I) \supseteq (1 + I)^*\) is obvious. To prove the reverse inclusion, let \(1 + a \in A_n^* \cap (1 + I)\) where \(a \in I\), and let \((1 + a)^{-1} = 1 + b\) for some \(b \in A_n\). The equality \(1 = (1 + a)(1 + b)\) can be written as \(b = -a(1 + b)\) in \(I\), i.e. \(1 + a \in (1 + I)^*\). This proves the reverse inclusion.

2. For all \(a \in A_n^*\), \((1 + I)a^{-1} = 1 + aIa^{-1} = 1 + I\), and so \((1 + I)^*a^{-1} = a(A_n^* \cap (1 + I))a^{-1} = A_n^* \cap (1 + I) = (1 + I)^*\). Therefore, \((1 + I)^*\) is a normal subgroup of \(A_n^*\). □

Let \(K\) be a field of characteristic zero. By (33), the group homomorphism \(A_n^* \to (A_n/a_n)^* \simeq A_n^*\) is an epimorphism. By Lemma 4.3.(1), its kernel is \(A_n^* \cap (1 + a_n) = (1 + a_n)^*\), and we have the short exact sequence of groups

\[
1 \to (1 + a_n)^* \to A_n^* \to A_n^* \to 1
\]

which together with (33) proves the first statement of the next theorem.

**Theorem 4.4** Let \(K\) be a field of characteristic zero. Then

1. \(A_n^* = K^* \times (\mathcal{H}_n \times (1 + a_n)^*)\).

2. The centre of the group \(A_n^*\) is \(K^*\).

**Proof.** 2. Let \(Z\) be the centre of the group \(A_n^*\). Then \(K^* \subseteq Z\) and, by statement 1,

\[
Z = Z \cap A_n^* = Z \cap (K^* \mathcal{H}_n (1 + a_n)^*) = K^*(Z \cap \mathcal{H}_n (1 + a_n)^*).
\]

It remains to show that \(Z' := Z \cap \mathcal{H}_n (1 + a_n)^* = \{1\}\).

Let us show first that \(Z' = Z \cap (1 + a_n)^*\). Let \(z = \varphi u \in Z'\) for some \(\varphi \in \mathcal{H}_n\) and \(u \in (1 + a_n)^*\). It suffices to show that \(\varphi = 1\). Note that, for each element \(a \in a_n\), there exists a natural number \(c = c(a)\) such that \(aE_{\alpha \beta} = E_{\alpha \beta}a = 0\) for all \(\alpha, \beta \in \mathbb{N}^n\) such that all \(\alpha_i, \beta_i \geq c\). For short, we write \(\alpha, \beta \gg 0\). So, \(uE_{\alpha \beta} = E_{\alpha \beta}u = E_{\alpha \beta}\) for all \(\alpha, \beta \gg 0\). Note that \(E_{\alpha \beta}^2 = 0\) for all \(\alpha \neq \beta\), and so \(1 + E_{\alpha \beta} \in A_n^*\). Now, for all \(\alpha, \beta \gg 0\) such that \(\alpha \neq \beta\), \(z(1 + E_{\alpha \beta}) = (1 + E_{\alpha \beta})z \iff z + \varphi(\alpha + 1)E_{\alpha \beta} = z + E_{\alpha \beta}\varphi(\beta + 1) \iff \varphi(\alpha + 1) = \varphi(\beta + 1) \Rightarrow \varphi = 1\), as required, where \(\varphi(\alpha + 1)\) is the value of the function \(\varphi = \varphi(H_1, \ldots, H_n)\) at \(H_1 = \alpha_1 + 1, \ldots, H_n = \alpha_n + 1\). This proves the claim.

So, it remains to show that \(Z' := Z \cap (1 + a_n)^* = \{1\}\). The result is true for \(n = 1\) (Theorem 4.2.(3)). So, we may assume that \(n \geq 2\). Consider the descending chain \(f_1 \supseteq \cdots \supseteq f_i \supseteq \cdots \supseteq f_n \supseteq f_{n+1} := 0\) of ideals

\[
f_i := \sum_{1 \leq j_1 < \cdots < j_i \leq n} p_{j_1}p_{j_2} \cdots p_{j_i}
\]

of \(A_n\). Note that \(f_1 = a_n, f_n = F^{\otimes n}\), and

\[
(p_{j_1} \cdots p_{j_i} + f_{i+1})/f_{i+1} \simeq p_{j_1} \cdots p_{j_i}/p_{j_1} \cdots p_{j_i} \cap f_{i+1} \simeq F^{\otimes i} \otimes A_{n-i} \simeq F \otimes A_{n-i} \simeq M_{\infty}(A_{n-i}).
\]

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In the proof of the above series of ring isomorphisms without 1 we have used the facts that $F = M_\infty(K)$ and $F^\otimes i \simeq F$.

\[
f_i/f_{i+1} \simeq \prod_{1 \leq j_1 < \cdots < j_i \leq n} (p_{j_1} \cdots p_{j_i} + f_{i+1})/f_{i+1} \simeq (F^\otimes i \otimes A_{n-i})^{(\gamma)} \simeq M_\infty(A_{n-i})^{(\gamma)}
\]

are isomorphisms of rings without 1. Consider the isomorphisms of groups

\[
(1 + f_i/f_{i+1})^* \simeq (1 + \prod_{1 \leq j_1 < \cdots < j_i \leq n} (p_{j_1} \cdots p_{j_i} + f_{i+1})/f_{i+1})^*
\]
\[
\simeq \prod_{1 \leq j_1 < \cdots < j_i \leq n} (1 + (p_{j_1} \cdots p_{j_i} + f_{i+1})/f_{i+1})^* \simeq (1 + F^\otimes i \otimes A_{n-i})^{(\gamma)}
\]
\[
\simeq (1 + M_\infty(A_{n-i})))^{(\gamma)} \simeq GL_\infty(A_{i-1})^{(\gamma)}
\]

where $GL_\infty(A_{i-1} := (1 + M_\infty(A_{n-i})))^*$. The descending chain above yields the descending chain of normal subgroups of $A_n^*$:

\[
(1 + f_1)^* = (1 + a_n)^* \supset \cdots \supset (1 + f_i)^* \supset \cdots \supset (1 + f_n)^* = (1 + F^{\otimes n})^* \supset (1 + f_{n+1})^* = \{1\}
\]

For each $i = 1, \ldots, n$, there is the natural homomorphism of groups $\varphi_i : (1 + f_i)^* \to (1 + f_i/f_{i+1})^*$, the kernel of which is $(1 + f_{i+1})^*$. So, $(1 + f_i)/(1 + f_{i+1})^*$ is a subgroup of $(1 + f_i/f_{i+1})^*$. To prove that $Z' = \{1\}$ is equivalent to show that $Z \cap (1 + f_i)^* = \{1\}$ for all $i = 1, \ldots, n$. To prove this we use a downward induction at $i$ starting with $i = n$. In this case, $f_n = F^{\otimes n}$, and the fact that $Z \cap (1 + F^{\otimes n})^* = \{1\}$ follows from the inclusion

\[
Z \cap (1 + F^{\otimes n})^* \subseteq Z((1 + F^{\otimes n})^*) = Z((1 + M_\infty(K))^*) = \{1\}
\]

since $Z(M_\infty(K)) = 0$.

Suppose that $i < n$ and $Z \cap (1 + f_{i+1})^* = \{1\}$. Using a downward induction on $i$ it remains to show that $Z_i := Z \cap (1 + f_i)^* = \{1\}$. Note that in any ring elements $1 + a$ and $1 + b$ commute iff the elements $a$ and $b$ commute. Using this observation we see that, for any ring $R$, the centre $Z(1 + M_\infty(R))$ of the multiplicative monoid $1 + M_\infty(R)$ is 1 since 1 is the only element of $1 + M_\infty(K)$ that commute with all the elements $1 + E_{kl}$, $k \neq l$. All these elements belong to the group $GL_\infty(R) := (1 + M_\infty(R))^*$, and so it has the trivial centre

\[
Z(GL_\infty(R)) = \{1\}. \quad (44)
\]

In particular, $Z(GL_\infty(A_{n-i})^{(\gamma)}) = \{1\}$. For each subset $J = \{j_1, \ldots, j_i\}$ of $\{1, \ldots, n\}$ that contains exactly $i$ elements $j$, it is seen above that

\[
(p_{j_1} \cdots p_{j_i} + f_{i+1})/f_{i+1} \simeq F^{\otimes i} \otimes A_{n-i} \simeq M_\infty(A_{n-i}).
\]

For each matrix unit $E_{\alpha \beta} = E_{\alpha \beta}(j_1) \cdots E_{\alpha \beta}(j_i) \in p_{j_1} \cdots p_{j_i}$, where $\alpha, \beta \in \mathbb{N}$ and $\alpha \neq \beta$, the elements $1 + E_{\alpha \beta}$ belongs to $(1 + f_i)^*$ since $E_{\alpha \beta}^2 = 0$, and its image under the map $\varphi_i$ is equal to the element $1 + E_{\alpha \beta} + f_{i+1}$. This means that an element $\varphi_i(z)$ commutes with all
elements $1 + E_{\alpha,\beta} + f_{i+1}$ for all possible choices of $J$, i.e. $\varphi_i(z) \in Z(GL_\infty(A_{n-J}(i))) = \{1\}$. This means that $z \in Z \cap (1 + f_{i+1})^* = \{1\}$, i.e. $z = 1$, as required. By induction, $\{1\} = Z \cap (1 + f_1)^* = Z \cap (1 + a_n)^*$. This proves that $Z(A_n^e) = K^*$.

The subgroup $(1 + a_n)^*$ of $A_n^e$. Let $K$ be a commutative $\mathbb{Q}$-algebra. For each $i = 1, \ldots, n$,

$$(p_i + f_2)/f_2 \simeq p_i/p_i \cap f_2 \simeq A_{i-1} \otimes F \otimes A_{n-i}/(a_{i-1} \otimes F \otimes A_{n-i} + A_{i-1} \otimes F \otimes a_{n-i})\
\simeq A_{i-1}/a_{i-1} \otimes F \otimes A_{n-i}/a_{n-i} \simeq A_{i-1} \otimes F \otimes A_{n-i} \simeq M_\infty(A_{n-1})$$

is the series of isomorphisms of rings without 1. The factor ring without 1

$$a_n/f_2 \simeq \prod_{i=1}^{n}(p_i + f_2)/f_2 \simeq \prod_{i=1}^{n}A_{i-1} \otimes F \otimes A_{n-i} \simeq M_\infty(A_{n-1})^n$$

is the direct product of its subrings without 1. It is a semi-simple $A_n$-bimodule where $(p_i + f_2)/f_2, 1 \leq i \leq n$, are the simple isotypic components of the $A_n$-bimodule $a_n/f_2$. Fix a section $s: a_n/f_2 \to a_n$ of the $K$-module epimorphism $a_n \to a_n/f_2, a \mapsto a + f_2$. Then $a_n = \text{im}(s) \oplus f_2$ is a direct sum of $K$-submodules. Using the $K$-basis $B_n$ for $A_n$ considered in the proof of Corollary 2.7 one can easily find such a section which even satisfies the additional property that $\text{im}(s)$ is a free $K$-module.

Consider the ring $K \oplus (a_n/f_2)$ with 1 and the subgroup $(1 + (a_n/f_2))^*$ of its group $(K \oplus (a_n/f_2))^*$ of units. There are canonical group isomorphisms

$$(1 + a_n/f_2)^* \simeq (1 + \prod_{i=1}^{n}(p_i + f_2)/f_2)^*$$

$$\simeq \prod_{i=1}^{n}(1 + (p_i + f_2)/f_2)^*, \ 1 + \sum_{i=1}^{n} p_i \mapsto \prod_{i=1}^{n}(1 + p_i),$$

$$\simeq \prod_{i=1}^{n}(1 + M_\infty(A_{n-1}))^* = \prod_{i=1}^{n}GL_\infty(A_{n-1}) = GL_\infty(A_{n-1})^n.$$ 

We have the group monomorphism

$$(1 + a_n)^*/(1 + f_2)^* \to (1 + a_n/f_2)^*, \ (1 + a)(1 + f_2)^* \mapsto 1 + a + f_2.$$

An invertibility criterion. The next theorem is a criterion of when an element of the monoid $1 + a_n$ is invertible.

**Theorem 4.5** Let $K$ be a commutative $\mathbb{Q}$-algebra, $a \in a_n$. Then $1 + a \in (1 + a_n)^*$ iff

1. $1 + a + f_2 \in (1 + a_n/f_2)^* (\simeq GL_\infty(A_{n-1})^n)$, and

2. $a + (1 + a)c \in (1 + f_2)$ and $a + c(1 + a) \in f_2(1 + a)$ where $c := s((1 + a + f_2)^{-1} - 1)$

(the value of the section $s: a_n/f_2 \to a_n$ at the element $(1 + a + f_2)^{-1} - 1 \in a_n/f_2$).
Suppose that conditions 1 and 2 hold and \( a+(1+a)c = (1+a)r \) (resp. \( a+c(1+a) = l(1+a) \)) for some \( r \in f_2 \) (resp. \( l \in f_2 \)) then \( a^{-1} = 1+c-r \) (resp. \( a^{-1} = 1+c-l \)).

Proof. \((\Rightarrow)\) Suppose that \( (1+a) \in (1+a_n)^* \). Due to the group homomorphism \( (1+a_n)^* \rightarrow (1+a_n/f_2)^* \), we have \( 1+a \mapsto 1+a+f_2 \in (1+a_n/f_2)^* \), i.e. the first condition holds. \( (1+a) \in (1+a_n)^* \) \( \ni (1+a)^{-1} = 1+b \) for some element \( b \in a_n \equiv \text{im}(s) \oplus f_2 \) which can be written as \( b = c + d \) where \( c := s((1+a + f_2)^{-1} - 1) \in \text{im}(s) \) and \( d := b - c \in f_2 \). The equalities \((1+a)(1+c + d) = 1 \) and \((1+c+d)(1+a) = 1 \) can be rewritten as follows \( a + (1+a)c = -(1+a)d \in (1+a)\mathfrak{f}_2 \) and \( a + c(1+a) = -d(1+a) \in \mathfrak{f}_2(1+a) \), and so the second statement holds.

\((\Leftarrow)\) Suppose that conditions 1 and 2 hold, we have to show that \( 1+a \in (1+a_n)^* \), i.e. the element \( 1+a \) has a left and a right inverse. Condition 2 can be written as follows \( a + (1+a)c = (1+a)r \) and \( a+c(1+a) = l(1+a) \) for some elements \( r,l \in f_2 \). These two equalities can be rewritten as \((1+a)(1+c-r) = 1 \) and \((1+c-l)(1+a) = 1 \). This means that \( 1+a \in (1+a_n)^* \) and \( a^{-1} = 1+c-r = 1+c-l \). \(\square\)

An inversion formula for \( u \in K^* \times (H_n \ltimes (1 + F\otimes n)^*) \). Let \( K \) be a field of characteristic zero. By Theorem 4.4.(1), each element \( u \in K_n^* \) is a unique product \( u = \lambda h(1+a) \) for some \( \lambda \in K^* \), \( h \in H_n \), and \( a \in a_n \) such that \( 1+a \in (1+a_n)^* \). Clearly, \( u^{-1} = \lambda^{-1}(1+a)^{-1}h^{-1} \). So, to write down explicitly an inversion formula for \( u \) boils down to finding \((1+a)^{-1}\). As a first step, one should know an inversion formula for elements of \( \text{GL}_{\infty}(A_{n-1}) \) which is not obvious how to do, at the moment. It should not be entirely trivial since as a result one would have a formula for solutions of all invertible integro-differential equations (for all \( n \geq 2 \)). Nevertheless, for elements \( u = \lambda h(1+a) \in K^* \times (H_n \ltimes (1 + F\otimes n)^*) \) one can write down the inversion formula exactly in the same manner as in the case \( n = 1 \). Hence, one obtains explicitly solutions to the equation \( uy = f \) where \( f \in P_n \). Elements of the group \( K^* \times (H_n \ltimes (1 + F\otimes n)^*) \) are called minimal integro-differential operators.

References


