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ON IMPROVING THE ROBUSTNESS OF THE SINGLE SCAN POLYNOMIAL TRACKING

by

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On Improving the Robustness of the Single Scan Polynomial Tracking

1. Abstract

This report presents some recent work accomplished for the improvement in the robustness of polynomial tracking algorithms. The single scan problem is stated and solved in linear algebra paradigm and the resulting solution has been found to be identical to the one developed in a previous report using the finite differences and Taylor's series approach. To improve the robustness of the single scan algorithm an averaging version has been derived. The improvement is achieved by sampling more frequently than the order of the field generating function.

2. Introduction

Earlier work [1, 2] on polynomial tracking revealed that the improvement in the robustness is possible only by increasing the sampling interval. As, the maximum sampling interval is limited by the entire width of the window available for the scanning, it is desirable to investigate other approaches for possible improvement in the robustness of the algorithm.

Consider a field generated by an nth order polynomial bearing coefficients $a_1, a_2, \ldots, a_n$. Given n samples, H_i, $i=1,2\ldots,n$, captured in a particular scan at the sampling interval Δy, the single scan algorithm determines the position y_1, of the first sample H_1, using the relation [1]

$$y_1 = \frac{1}{n!\, a_n} \frac{\sum_{i=1}^{n} (-1)^{n+i} H_i\, {}_{i-1}^{n-1}C}{\Delta y^{n-1}} - \frac{a_{n-1}}{n a_n} - \frac{n-1}{2} \Delta y \qquad (1)$$

Equation (1) shows that for the determination of y_1,

- the number of samples required are equal to the order of the field generating polynomial. More samples in a given sampling window could not be used if available. This follows as the relation was generically derived for the deterministic case.
- only two coefficients a_{n-1} and a_n, of the field generating polynomial (and not $a_1, a_2, \ldots, a_{n-2}$) are used.

In this report the single scan tracking algorithm is reformulated and re-derived in an attempt to improve the robustness of the method by using more polynomial parameters (parameter rich) or more sampled values (data heavy).

In section 3 the single scan problem is reformulated and solved using linear algebra paradigm. It has been found that the solution obtained is identical to the one obtained using the finite difference and Taylor series approach [1]. This tends to imply that the information carried by the lower order coefficients is redundant. Instead, it is possible that the robustness of the tracking algorithm could be improved by using more sampled values. Following this approach, a variant of the single scan algorithm is developed in section 4 that uses more sampled values instead of just n. Noise tolerance of the averaging algorithm and that of the generic single scan algorithm are compared in section 5. Finally, conclusions are drawn in Section 6.

3. A New Formulation of the Single Scan Solution of the Polynomial Tracking Problem

Let H be a measure of the physical property to be scanned for subsequent tracking. Assuming a polynomial representation we write

$$H(y) = a_1 y + a_2 y^2 + \ldots + a_n y^n \tag{2}$$

of order n, with appropriate choice of constant parameters values $a_1, a_2, \ldots, a_n$, to represent all the features of interest within the field. Consider n samples of $H(y)$, equispaced at Δy, are taken. The object of the sampling is to find y_1, the position of first sampling point given the polynomial parameters and the sampling interval Δy.

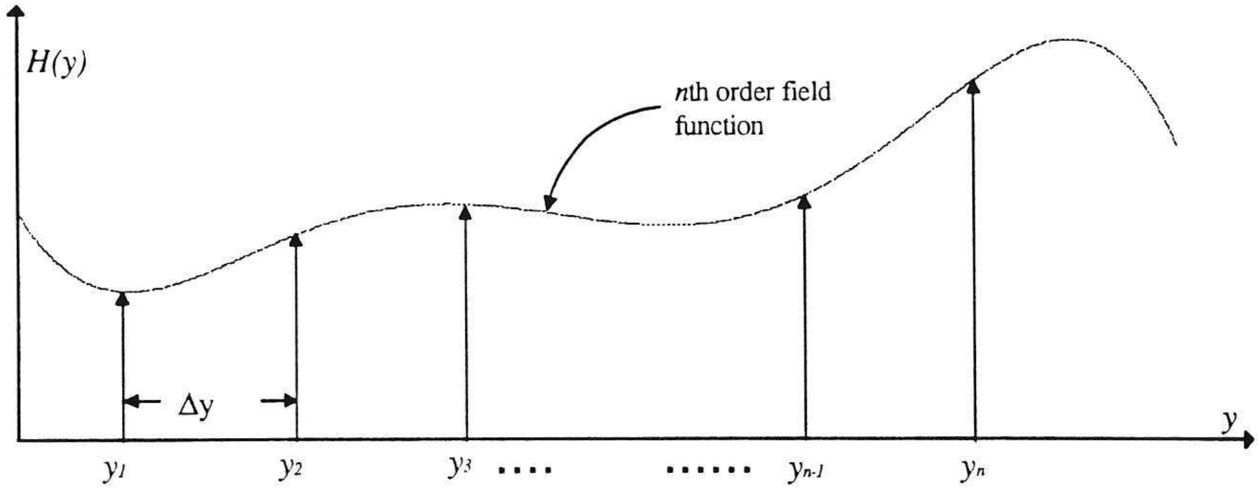

Fig. 1 A polynomial generating function sampled at the interval Δy.

Fig. 1 above shows an nth order field function scanned at n points $y_1, y_2, \ldots, y_n$ furnishing $H(y_i)=H_i$, $i=1,2,\ldots,n$. For these n sampling points we can write the following set of n equations.

$$\begin{array}{ccccc} a_1 y_1 + & a_2 y_1^2 + & \cdots & +a_n y_1^n = & H_1 \\ a_1 y_2 + & a_2 y_2^2 + & \cdots & +a_n y_2^n = & H_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1 y_n + & a_2 y_n^2 + & \cdots & +a_n y_n^n = & H_n \end{array} \tag{3}$$

Fig. 1 clearly shows that

$$y_i = y_1 + (i-1)\Delta y \qquad for \quad i = 1,2,\ldots,n \tag{4}$$

Putting this in set (3) of the equations above we get

$$\begin{matrix} a_1(y_1)+ & a_2(y_1)^2+ & \cdots & +a_n(y_1)^n & =H_1 \\ a_1(y_1+\Delta y)+ & a_2(y_1+\Delta y)^2+ & \cdots & +a_n(y_1+\Delta y)^n & =H_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1(y_1+(n-1)\Delta y)+ & a_2(y_1+(n-1)\Delta y)^2+ & \cdots & +a_n(y_1+(n-1)\Delta y)^n & =H_n \end{matrix} \tag{5}$$

Rearranging we get

$$\begin{matrix} c_{11}y_1+ & c_{12}y_1^2+ & \cdots & +c_{1n}y_1^n+d_1 & =H_1 \\ c_{21}y_1+ & c_{22}y_1^2+ & \cdots & +c_{2n}y_1^n+d_2 & =H_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{n1}y_1+ & c_{n2}y_1^2+ & \cdots & +c_{nn}y_1^n+d_n & =H_n \end{matrix} \tag{6}$$

where

$$c_{ij}=f_c(i,j,\Delta y,a_1,a_2,..,a_n) \qquad for\ \ i,j=1,2,\ldots,n \tag{7}$$

$$d_i=f_d(i,\Delta y,a_1,a_2,..,a_n) \qquad for\ \ i=1,2,\ldots,n \tag{8}$$

where general expressions for f_c and f_d are yet to be developed. Now setting

$$y_1^i=x_i \qquad for\ \ i=1,2,..,n \tag{9}$$

and rewriting set (6) of equations in the matrix form yields

$$\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}+\begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}=\begin{bmatrix} H_1 \\ H_2 \\ \vdots \\ H_n \end{bmatrix} \tag{10}$$

This is a set of n linear equations with n unknowns. Clearly, we are interested in finding x_1 $(=y_1)$ only and not the whole vector **x**. Writing equation (10) in a compact form

$$\mathbf{Cx}+\mathbf{d}=\mathbf{H} \tag{11}$$

$$\therefore \quad \mathbf{Cx}=\mathbf{H}-\mathbf{d}=\mathbf{r} \tag{12}$$

$$\therefore \quad \mathbf{x}=\mathbf{C}^{-1}\mathbf{r} \tag{13}$$

Hence

$$y_1 = x_1 = R1(\mathbf{C}^{-1})\mathbf{r} \tag{14}$$

where $R1(\mathbf{C}^{-1})$ is the first row of the matrix $\mathbf{C}^{-1}$. In order to devise an analytical expression for y_1 it is essential to determine first a general expression for $\mathbf{C} \equiv c_{ij},\ i,j = 1,2,\ldots,n$ and a general expression for the first row of $\mathbf{C}^{-1}$. In addition to this an expression for $\mathbf{d} \equiv d_i,\ i = 1,2,\ldots,n$ is required as well in order to evaluate $\mathbf{r}$. It has been found that the tensor c_{ij} can be written as

$$c_{ij} = a_j + \sum_{k=j+1}^{n} a_k \Delta y^{k-j} (i-1)^{k-j} \overset{k}{\underset{j}{C}} \qquad i = 1,2,..,n;\ j = 1,2,..,n \tag{15}$$

$$\textit{where} \qquad \overset{k}{\underset{j}{C}} = \frac{k!}{j!(k-j)!} \tag{16}$$

The tensor d_i follows the relation

$$d_i = \sum_{k=1}^{n} a_k \Delta y^k (i-1)^k \qquad \textit{for} \quad i = 1,2,\ldots,n \tag{17}$$

Now the main task left is to derive a general expression for the first row of inverse of $\mathbf{C}$, i.e. $R1(\mathbf{C}^{-1})$. Let

$$R1(\mathbf{C}^{-1}) = \mathbf{b} \equiv b_i \qquad \textit{for} \quad i = 1,2,\ldots,n \tag{18}$$

It has been found that the first row of $\mathbf{C}^{-1}$ can be written as

$$b_i = \frac{(-1)^{n+i} \overset{n-1}{\underset{i-1}{C}}}{n!\,a_n \Delta y^{n-1}} \qquad \textit{for} \quad i = 1,2,\ldots,n \tag{19}$$

Now it is straightforward to evaluate y_1 using the equations (14), (17) and (19)

$$y_1 = \sum_{i=1}^{n} \frac{(-1)^{n+i} \overset{n-1}{\underset{i-1}{C}}}{n!\,a_n \Delta y^{n-1}} \left[H_i - \sum_{k=1}^{n} \Delta y^k a_k (i-1)^k \right] \tag{20}$$

so

$$y_1 = \frac{\sum_{i=1}^{n} (-1)^{n+i} \overset{n-1}{\underset{i-1}{C}} H_i}{n!\,a_n \Delta y^{n-1}} - \sum_{i=1}^{n} \sum_{k=1}^{n} \frac{(-1)^{n+i} \overset{n-1}{\underset{i-1}{C}} \Delta y^k a_k (i-1)^k}{n!\,a_n \Delta y^{n-1}} \tag{21}$$

Equation (21) shows that the formula consist of two components. The first one is the stochastic component that depends upon H_i, $i=1,2,..,n$. and, therefore, will be effected by the observation process noise. The second part is deterministic and is fully characterised only by the sampling interval and polynomial parameters. It has been found that the deterministic part in equation (21) reduces to $\left(\frac{a_{n-1}}{na_n}+\frac{n-1}{2}\Delta y\right)$.

Therefore (21) implies

$$y_1=\frac{\sum_{i=1}^{n}(-1)^{n+i}\overset{n-1}{\underset{i-1}{C}}H_i}{n!\,a_n\Delta y^{n-1}}-\left(\frac{a_{n-1}}{na_n}+\frac{n-1}{2}\Delta y\right) \tag{22}$$

This is exactly the result as obtained in [1] using Taylor series and the finite differences treatment. It is to be noted that using two independent methods, the same formula evolves and that the evaluation of y_1 depends on only two highest order coefficients a_n and a_{n-1} (and not on $a_1, a_2, \ldots, a_{n-2}$) of the field generating polynomial. This tends to suggest that for this type of positional calculation the lower order polynomial coefficients carry redundant information and as such the algorithm could not be made parameter rich to a further degree.

4. A Successive Averaging Variant of the Single Scan Algorithm

Consider an nth order polynomial. Position of the first sampling point (sampling being done at the interval Δy) can be determined using the single scan algorithm. Instead of determining only this first location we repeat the process for positions y_0, y_0+T,..,$y_0+(m-1)T$ as shown in Fig. 2 considering sufficient sampling points are available.

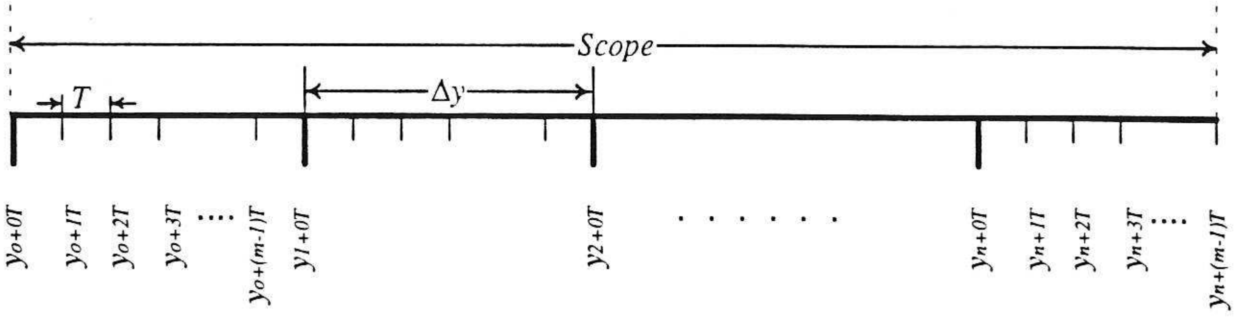

Fig. 2. Sampling Scheme for the Averaging Algorithm

Here $mT=\Delta y$. Now,

$$y_0=y_0+0T=\frac{d_0}{\Delta y^{n-1}n!\,a_n}-\frac{a_{n-1}}{na_n}-\frac{n-1}{2}\Delta y \tag{23}$$

$$y_1=y_0+T=\frac{d_1}{\Delta y^{n-1}n!\,a_n}-\frac{a_{n-1}}{na_n}-\frac{n-1}{2}\Delta y \tag{24}$$

•

•

$$\mathrm{y}_k = y_0 + kT = \frac{d_k}{\Delta y^{n-1} n!\ a_n} - \frac{a_{n-1}}{na_n} - \frac{n-1}{2}\Delta y \tag{25}$$

•

•

$$\mathrm{y}_{m-1} = y_0 + (m-1)T = \frac{d_{m-1}}{\Delta y^{n-1} n!\ a_n} - \frac{a_{n-1}}{na_n} - \frac{n-1}{2}\Delta y \tag{26}$$

where

$$d_k = \sum_{i=1}^{n} (-1)^{n+i} H_{k+1+m(i-1)} \overset{n-1}{\underset{i-1}{C}} \tag{27}$$

Now adding equations (23) to (26) gives

$$my_0 + T\sum_{k=0}^{m-1} k = \frac{1}{\Delta y^{n-1} n!\ a_n}\sum_{k=0}^{m-1} d_k \ - \ \frac{ma_{n-1}}{na_n} \ - \ \frac{m(n-1)}{2}\Delta y \tag{28}$$

Solving for y_0 and setting $\sum_{k=0}^{m-1} d_k = d$ we get

$$y_0 = \frac{d}{\Delta y^{n-1} m\ n!\ a_n} - \frac{a_{n-1}}{na_n} - \frac{n-1}{2}\Delta y - \frac{m-1}{2}T \tag{29}$$

Setting $T = \Delta y/m$.

$$y_0 = \frac{d}{\Delta y^{n-1} m\ n!\ a_n} - \frac{a_{n-1}}{na_n} - \frac{\Delta y}{2}\left(n - \frac{1}{m}\right)$$

Now d can be written

$$\begin{aligned} d &= \sum_{k=0}^{m-1} d_k = \sum_{k=0}^{m-1}\sum_{i=1}^{n} (-1)^{n+i} H_{k+1+m(i-1)} \overset{n-1}{\underset{i-1}{C}} \\ &= \sum_{i=1}^{n} (-1)^{n+i} \overset{n-1}{\underset{i-1}{C}} \left[\sum_{k=0}^{m-1} H_{k+1+m(i-1)}\right] \end{aligned} \tag{30}$$

We now see how this process of averaging does affect the robustness of the tracking. Following a similar approach as in [1], the mean and the variance of y_0 can be written as

$$E\{y_0\} = y_0 \tag{31}$$

and

$$Var\{y_0\} = \frac{\sum_{i=1}^{n}\left[\overset{n-1}{\underset{i-1}{C}}\right]^2 \sigma^2}{m(\Delta y^{n-1}\ n!a_n)^2} \tag{32}$$

and hence ϕ, the noise amplification factor [2] can be written as

$$\phi = \frac{\sqrt{\sum_{i=1}^{n}\left[\overset{n-1}{\underset{i-1}{C}}\right]^2}}{m^{\frac{1}{2}}\Delta y^{n-1}\ n!a_n}$$

Apparently, as m increases, the noise amplification factor reduces. Yet, for a given fixed sampling window (called scope) increasing m means a smaller sampling interval Δy. The next section considers under what conditions this averaging version outperforms the generic version.

5. Comparison of Methods with a Constant Scope

In this section the robustness of the generic version (GV) and the successive averaging version (AV) developed in the preceding section will be compared for a fixed scope. Rewriting the noise amplification factors for both of the methods.

Generic Version: $$\phi_{GV} = \frac{1}{\Delta y^{n-1} n!a_n}\left(\sum_{i=1}^{n}\left[\overset{n-1}{\underset{i-1}{C}}\right]^2\right)^{1/2} \tag{33}$$

Averaging Version: $$\phi_{AV} = \frac{1}{m^{\frac{1}{2}}\Delta y^{n-1}\ n!a_n}\left(\sum_{i=1}^{n}\left[\overset{n-1}{\underset{i-1}{C}}\right]^2\right)^{1/2} \tag{34}$$

Consider a fixed scope l. The sampling intervals Δy for both versions can be determined as

Generic Version:

$$l = \Delta y(n-1) \tag{35}$$

$$\therefore\quad \Delta y = \frac{l}{n-1}$$

Averaging Version:

Fig. 2. shows that

$$l = \Delta y(n-1) + (m-1)T$$

and

$$\because \quad T = \frac{\Delta y}{m}$$

$$\therefore \quad \Delta y = \frac{l}{n - \frac{1}{m}} \tag{36}$$

So the noise amplification factor for the generic version can be written as

$$\phi_{GV}(n) = \frac{(n-1)^{n-1}\left(\sum_{i=1}^{n}\left[\overset{n-1}{\underset{i-1}{C}}\right]^2\right)^{1/2}}{l^{n-1}a_n\, n!} \tag{37}$$

While for the averaging version this is

$$\phi_{AV}(n) = \frac{(n-\frac{1}{m})^{n-1}\left(\sum_{i=1}^{n}\left[\overset{n-1}{\underset{i-1}{C}}\right]^2\right)^{1/2}}{l^{n-1}a_n\, m^{\frac{1}{2}}\, n!} \tag{38}$$

Clearly, for those values of m and n when the ratio

$$r_\phi = \frac{\phi_{AV}}{\phi_{GV}} \tag{39}$$

is less than 1, the averaging version would outperforms the generic version. Smaller the ratio more is the improvement. Equations (37) and (38) implies that

$$r_\phi = \left(\frac{n - 1/m}{n-1}\right)^{n-1}\frac{1}{m^{\frac{1}{2}}} \tag{40}$$

Table 1 enumerates the ratio r_ϕ for different values of m and n. The shaded region signifies the area where the averaging algorithm degenerates. Evidently, with increasing m the ratio increases to a maximum value before decreasing. This trend can be explained as follows. For a given scope, when m is increased, the sampling interval in the case of the averaging algorithm decreases resulting a performance degradation. When m is further increased, the reduction in the performance is offset by the improvement due to the averaging process. When m is increased beyond this threshold, the improvement appears to follow the law of diminishing returns.

Please note that a larger value of m is required to achieve any improvement for a polynomial field of higher degree. As an example, for a field generating function of order 8, 10% reduction in the noise amplification factor would require 6 times as many samples required for the generic version. For a polynomial field of order 3, on the other hand, 6 times as much sampling renders a 20% reduction in noise amplification factor.

Table 1. Variation of the r_ϕ with m and n

m	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$
1	1.0	1.0	1.0	1.0	1.0	1.0	1.0
2	1.06066	1.10485	1.12286	1.13265	1.1388	1.14303	1.14611
3	0.96225	1.0264	1.05412	1.06961	1.07951	1.08639	1.09144
4	0.875	0.945312	0.976562	0.99427	1.00568	1.01364	1.01952
5	0.804984	0.876539	0.908871	0.927342	0.939301	0.947678	0.953874
6	0.748455	0.819332	0.851707	0.870305	0.882385	0.890866	0.897149
7	0.701934	0.771356	0.803312	0.82174	0.833739	0.842176	0.848432
8	0.662913	0.730585	0.761915	0.780034	0.791853	0.800173	0.806348
9	0.62963	0.695473	0.726092	0.74384	0.755432	0.763601	0.769668
10	0.600833	0.664869	0.694752	0.712106	0.723453	0.731454	0.7374
11	0.575613	0.637908	0.667064	0.684019	0.695116	0.702945	0.708767
12	0.553294	0.613936	0.642385	0.65895	0.6698	0.677459	0.683156
13	0.533366	0.592446	0.620219	0.636407	0.647016	0.654509	0.660084
14	0.515432	0.573044	0.600173	0.615999	0.626378	0.63371	0.639167
15	0.499185	0.555415	0.581933	0.597415	0.607572	0.614751	0.620095

6. Conclusions and Discussion

The single scan tracking problem has been solved in the linear algebra paradigm. It has been found that the resulting solution is identical to the one obtained using the finite difference and Taylor series approach as described in an earlier report [1]. It is noteworthy that the algorithm utilises only two (most significant) of the coefficients of the field generating polynomial to determine the position of the first sampling point. The evolution of an identical solution (of the problem) from two independent methods tends to suggest that the information carried by the lower order coefficients is redundant for this type of estimation problems. Hence, as such, a parameter-richer algorithm appears to be unobtainable.

The data-heavy approach, on the other hand has worked. A variant of the single scan algorithm is developed that uses more data points than the order of the field generating function by sampling more frequently. For a constant scope, the performance of the generic version and the derived averaging version has been compared. The conditions for the derived version to outperform the generic version has been evaluated. The improvement is found to be very significant.

7. References

[1] J. B. Edwards and R. Iqbal, "Tracking the features of a spatially distributed continuous field- The idealised 2D, deterministic case" , AC & SE research report No. 492, University of Sheffield, Sheffield, UK, 1993.

[2] R. Iqbal and J. B. Edwards, "Noise analysis in tracking the features of a spatially distributed field" , AC & SE research report No. 501, University of Sheffield, Sheffield, UK, 1994.