Monograph:
Banks, S.P. (1993) Pseudo-Linear Systems: Periodic Orbits and Stability. Research Report. ACSE Research Report 497 . Department of Automatic Control and Systems Engineering

Pseudo-Linear Systems: Periodic Orbits and Stability

S.P.Banks

Department of Automatic Control and Systems Engineering

University of Sheffield

Mappin Street

SHEFFIELD S1 4DU

Research Report No. 497

Dec 1993

Abstract

In this paper we shall study the existence of periodic orbits and stability of pseudo-linear systems. The Lie algebra generated by the system matrix function will be the main tool.

Keywords:Nonlinear Systems, Periodic Orbits, Stability, Lie algebras.

1 Introduction

In this paper we shall consider systems of the form

$$\dot{x} = A(x)x \tag{1.1}$$

in which $A : \mathbf{R}^n \longrightarrow \mathbf{R}^{n^2}$ is analytic. Note that this system is equivalent to the system

$$\dot{x} = f(x)$$

where

$$f(0) = 0$$

and f is analytic. The main reason for studying the general nonlinear system in the form (1.1) is to apply linear-like techniques. This has already been demonstrated in [1],[2], where the ideas of micro- local optimal control and the use of Lie algebras in stability have been discussed. By the analyticity of $A(x)$ we may write

$$A(x) = \sum_{|\mathbf{i}|=0}^{\infty} A_{\mathbf{i}} x^{\mathbf{i}} \tag{1.2}$$

where $\mathbf{i} = (i_1, \cdots, i_n)$ is a multi-index and

$$x^{\mathbf{i}} = x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \quad , \quad \sum_{|\mathbf{i}|=0}^{\infty} = \sum_{i_1=0}^{\infty} \cdots \sum_{i_n=0}^{\infty} .$$

Of course, $x^{\mathbf{i}}$ is a scalar function, i.e. $x^{\mathbf{i}} : \mathbf{R}^n \longrightarrow \mathbf{R}$ and $A_{\mathbf{i}} \in \mathbf{R}^{n^2}$ for each multi-index $\mathbf{i}$. The dynamics of (1.1) are therefore completely characterized by the infinite set of matrices $A_{\mathbf{i}}$, $\mathbf{i} \geq \mathbf{0}$ (i.e. $i_1 \geq 0, \cdots, i_n \geq 0$).

We shall consider two aspects of the system (1.1)–periodic orbits and stability. In section 2 we shall consider the general case where $\{A(x)\}$ generates a semisimple Lie algebra and in section 3 we shall study Hamiltonian systems and apply results of Conley and Zehnder [4],[5] to obtain the existence of periodic orbits. Finally in section 4 we shall study the stability of Hamiltonian systems. Stability in the more general case where $\{A(x)\}$ generates a semisimple Lie algebra is considered in [3]. For the theory of Lie algebras see [9] or [6].

2 Spectral Theory and Lie Algebras

In this section we shall consider the system (1.1) with respect to the spectral and commutativity properties of the matrices $A_{\mathbf{i}}$. Consider first the simplest and most restrictive case in which all the matrices $A_{\mathbf{i}}$, $\mathbf{i} \geq \mathbf{0}$ are diagonalizable and commute. Then the matrices $A_{\mathbf{i}}$ are simultaneously diagonalizable and we can change coordinates and write equation (1.1) in the form

$$\dot{y}_i = \lambda_i(y)y_i \ , \ 1 \leq i \leq n \ , \ y \in \mathbf{C}^n \tag{2.1}$$

for some functions λ_i. These equations can be integrated to give

$$y_i(t) = e^{\int_0^t \lambda_i(y(t))dt} y_i(0) \ , \tag{2.2}$$

provided the integral exists. Consider the problem of finding periodic orbits of (2.2) (and hence of (1.1)). From (2.2) we require that

$$e^{\int_0^T \lambda_i(y(t))dt} = 1 \ , \ 1 \leq j \leq n$$

for some T, i.e.

$$\int_0^T \lambda_i(y(t))dt = 2k_j\pi i \ , \ 1 \le j \le n \tag{2.3}$$

for some integers k_j. Suppose that the eigenvalues $\lambda_j(y(t))$ are pure imaginary and constant ($\neq 0$) on the torus

$$\mathcal{T}_c : \{y \in \mathbf{C} : |y_1| = c_1, |y_2| = c_2, \cdots, |y_n| = c_n\} \tag{2.4}$$

for some constant $c_i > 0$. Then, if $|y_i(0)| = c_i \ , \ 1 \le i \le n$, it follows from (2.2) that the solution $y(t)$ lies on the torus $\mathcal{T}_c$. By the assumption on the eigenvalues, we may write

$$\lambda_j(y(t)) = i\mu_j \tag{2.5}$$

for some real numbers μ_j. Hence the condition (2.3) becomes

$$T = \frac{2k_j\pi}{\mu_j}. \tag{2.6}$$

We shall call the values $\mu_j \ , \ 1 \le j \le n$,**commensurate with respect to** T if there exist integers $k_1, \cdots, k_n$ such that

$$\mu_j = \frac{T}{2\pi k_j} \ , \ 1 \le j \le n. \tag{2.7}$$

We have therefore proved

Theorem 2.1 Suppose that the matrices A_i defined with respect to the system (1.1) are mutually commutative and diagonalizable. Let P ($\in \mathbf{C}^{n^2}$) be such that

$$PA(x)P^{-1} = \Lambda(x)$$

where $\Lambda(x)$ is diagonal with the eigenvalues $\lambda_i(x)$ of $A(x)$ on the diagonal and suppose that

$$\lambda_i(x) = i\mu_j \ , \ |\sum p_{k\ell}x_\ell| = c_k \tag{2.8}$$

for some constants μ_j and c_k. Then if the numbers $\{\mu_j\}$ are commensurate with respect to T, the system (1.1) has a periodic orbit. □

This result is, of course, highly restrictive because of the condition (2.8). However, by considering almost periodic solutions we can allow much more general conditions on the spectrum of $A(x)$. In fact we shall make the following assumption on the transformed eigenvalues $\lambda_j(y(t))$:

Assumption A: (i) The eigenvalues $\lambda_j(y(t))$ are pure imaginary and continuous.

(ii) $0 < m_j < |\lambda_j(y(t))| < M_j < \infty$ for some constants m_j, M_j.

Consider the following equations corresponding to (2.3):

$$\nu_j(t) \triangleq \int_0^t \mu_j(y(\tau))d\tau = \pm 2\pi \ , \ 1 \leq j \leq n$$

where $\mu_j = i\lambda_j$. By assumption A, it is clear that there exist times $T_1, \cdots, T_n$ such that

$$\nu_j(T_j) = \pm 2\pi \ , \ 1 \leq j \leq n \tag{2.9}$$

and that, moreover,

$$\frac{2\pi}{M_j} \leq T_j \leq \frac{2\pi}{m_j} \ , \ 1 \leq j \leq n.$$

Lemma 1 Given any $\epsilon > 0$ there are integers k_j $(1 \le j \le n)$ such that

$$|k_p T_p - k_q T_q| < \epsilon \ , \ 1 \le p, q \le n.$$

Proof First note that if all the T_i's are rational, say $T_i = p_i/q_i$, then the result is true since we can take

$$k_i = q_i \prod_{\substack{j=1 \\ j \neq i}}^{n} p_j$$

giving

$$k_p T_p = k_q T_q \ , \ 1 \le p, q \le n.$$

Thus we may suppose that at least one T_i (say T_1) is irrational. We may also assume that T_1 and the remaining T_i's are 'mutually irrational' in the sense that T_1/T_i is irrational for $i > 1$. (Otherwise if $T_1/T_2 = v/w$, say, we may consider the reduced set $wT_1 (= vT_2), T_3, \cdots, T_n$.) By dividing by the maximum T_i we may clearly assume that $T_i < 1$ for $1 \le i \le n$. Consider the set

$$S = \{ \sum_{1 \le p, q \le n} |k_p T_p - k_q T_q| \ : \ 1 \le k_p, k_q \le N \}$$

for some N. Each element of S is less than ${}^nC_2 \cdot N$ and since T_1/T_i is irrational for $i \ge 2$ there must be at least N^2 distinct elements in S. Suppose the minimum of the lengths of elements in S is δ. Then we must have

$$(N^2 - 1)\delta < \ {}^nC_2 \cdot N.$$

Hence, choosing N large enough gives the result. $\square$

Theorem 2.2 Suppose that the eigenvalues of $A(x)$ are pure imaginary and satisfy assumption A. Then any solution of (1.1) is almost periodic.

Proof Let N be any positive integer and $\epsilon > 0$. Let $T_1, \cdots, T_n$ be the times defined as in (2.9). Then by lemma 1 there exist integers $k_1, \cdots, k_n$ such that

$$|k_p T_p - k_q T_q| < \epsilon/N \ , \ 1 \leq p, q \leq n.$$

Let $\tau = k_1 T_1$. Then, for the diagonalized system (2.1), the components $y_i(t)$ return to $y_i(0)$ within ϵ/N of $\tau, 2\tau, \cdots, N\tau$. By continuity of the solutions, $x(\tau), x(2\tau), \cdots, x(N\tau)$ will therefore be 'close' to $x(0)$. Since N and ϵ are arbitrary, the result follows. □

In order to generalize the above results to the case where the matrices $A_{\mathbf{i}}$ are not mutually commutative, we shall suppose that the Lie algebra generated by the matrices $A_{\mathbf{i}}$, $\mathbf{i} \geq 0$ is semisimple. Let $\mathcal{L}_S$ denote the Lie algebra generated by the set $S \subseteq gl(\mathbf{C}^n) = \mathbf{C}^{n^2}$. First note the following:

Lemma 2 The Lie algebras generated by the sets $\{A_{\mathbf{i}} \ , \ \mathbf{i} \geq 0\}$ and $\{A(x) : x \in \mathbf{R}^n\}$ are the same; i.e.

$$\mathcal{L}_{\{A_{\mathbf{i}} \ , \ \mathbf{i} \geq 0\}} = \mathcal{L}_{\{A(x) : x \in \mathbf{R}^n\}}.$$

Proof It is clearly sufficient to show that

$$\text{(a)} \quad A_{\mathbf{i}} \in \mathcal{L}_{\{A(x) : x \in \mathbf{R}^n\}}.$$

and that

$$\text{(b)} \quad A(x) \in \mathcal{L}_{\{A_{\mathbf{i}} \ , \ \mathbf{i} \geq 0\}} \quad \text{for all } x \in \mathbf{R}^n$$

To prove (a) note that since $\mathcal{L}_{\{A(x):x\in\mathbf{R}^n\}}$ is a finite-dimensional subspace of $gl(\mathbf{C}^n)$ it is closed, hence $\lim_{\delta\to 0}\frac{A(x)-A(x-\delta e_i)}{\delta} \in \mathcal{L}_{\{A(x):x\in\mathbf{R}^n\}}$ where e_i is the i^{th} unit basis vector of $\mathbf{R}^n$. Hence the partial derivatives $\frac{\partial}{\partial x_i}(x) \in \mathcal{L}_{\{A(x):x\in\mathbf{R}^n\}}$. Similarly, the partial derivatives of all orders of A are in $\mathcal{L}_{\{A(x):x\in\mathbf{R}^n\}}$ and (a) follows. To prove (b), simply note that, for any $x \in \mathbf{R}^n$, the finite sum $\sum_{\mathbf{i}=0}^{\mathbf{K}} A_{\mathbf{i}} x^{\mathbf{i}} \in \mathcal{L}_{\{A_{\mathbf{i}}:\mathbf{i}\geq 0\}}$, by linearity and so $A(x) \in \mathcal{L}_{\{A_{\mathbf{i}}:\mathbf{i}\geq 0\}}$ by the closedness of $\mathcal{L}_{\{A_{\mathbf{i}}:\mathbf{i}\geq 0\}}$. □

Assuming that $g \triangleq \mathcal{L}_{\{A_{\mathbf{i}}:\mathbf{i}\geq 0\}}$ is semisimple we can write the usual Cartan decomposition in the form

$$g = h + \sum_{\alpha\in\Gamma} g_\alpha$$

where h is a Cartan subalgebra and each root space g_α is one dimensional. (Here, Γ is the set of nonzero roots.) It follows that equation (1.1) can be written in the form

$$\dot{x} = H(x)x + \sum_{\alpha\in\Gamma} v_\alpha(x)E_\alpha x \tag{2.10}$$

where the matrices $H(x)$, $x \in \mathbf{R}^n$ are mutually commutative and $E_\alpha \in g_\alpha$ for $\alpha \in \Gamma$.

Since the matrices $H(x)$ are mutually commutative, they are simultaneously diagonalizable and so there exists an invertible matrix P (independent of x) such that $PH(x)P^{-1} = diag\ (\lambda_1(x), \cdots, \lambda_n(x)) \triangleq \Lambda(x)$. Put $y = Px$ in (2.10) and we obtain

$$\dot{y} = \overline{\Lambda}(y)y + \sum_{\alpha\in\Gamma} \overline{v}_\alpha(y)\overline{E}_\alpha y \tag{2.11}$$

where

$$
\begin{aligned}
\overline{\Lambda}(y) &= \Lambda(x) = diag\ (\lambda_1(P^{-1}y), \cdots, \lambda_n(P^{-1}y)), \\
\overline{v}_\alpha(y) &= v_\alpha(P^{-1}y)
\end{aligned}
$$

and

$$\overline{E}_\alpha = PE_\alpha P^{-1}.$$

Consider the j^{th} equation in (2.11):

$$\dot{y}_j = \lambda_j(y)y_j + b_j \tag{2.12}$$

where

$$b_j = \left(\sum_{\alpha\in\Gamma} \overline{v}_\alpha(y)\overline{E}_\alpha y\right)_j .$$

Then

$$y_j(t) = e^{\int_0^t \lambda_j(y)d\tau} y_j(0) + \int_0^t e^{\int_s^t \lambda_j(y)d\tau} b_j(s)ds.$$

Define the diagonal matrix-valued function

$$\mathcal{E}(t,s;y(\cdot)) = diag\left(e^{\int_s^t \lambda_1(y)d\tau}, \cdots, e^{\int_s^t \lambda_n(y)d\tau}\right).$$

Then we may rewrite (2.11) in the form

$$y(t) - \int_0^t \mathcal{E}(t,s;y) \sum_{\alpha\in\Gamma} \overline{v}_\alpha(y(s))\bar{(E)}_\alpha y(s)ds = \mathcal{E}(t,0;y)y(0). \tag{2.13}$$

From (2.12) and (2.13) it follows that

$$\frac{d}{dt}|y_j| \geq (Re\ \lambda_j(y))|y_j| - |b_j|$$

and

$$\frac{d}{dt}|y_j| \leq (Re\ \lambda_j(y))|y_j| + |b_j|$$

i.e.

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}||y||^2 &\geq (Re\ \underline{\lambda}(y)||y||^2 - |b|\ |y| \\ \frac{1}{2}\frac{d}{dt}||y||^2 &\leq (Re\ \overline{\lambda}(y)||y||^2 + |b|\ |y| \end{aligned} \tag{2.14}$$

where $|b| = (|b_1|, \cdots, |b_n|)$, $|y| = (|y_1|, \cdots, |y_n|)$, $\underline{\lambda}(y) = \min\ Re\ \lambda_j(y)$, $\overline{\lambda}(y) = \max\ Re\ \lambda_j(y)$. Now,

$$\begin{aligned} |b|\ |y| &= (|b_1|, \cdots, |b_n|)(|y_1|, \cdots, |y_n|)^T \\ &\leq \sum_{\alpha \in \Gamma} |\overline{v}_\alpha(y)|\ ||\overline{E}_\alpha||\ |y|^2. \end{aligned}$$

Since $|\overline{v}_\alpha(y)|$ is bounded on any sphere $S_a = \{y : ||y|| = a\}$, we may define

$$\xi(a) = \max_{y \in S_a} \sum_{\alpha \in \Gamma} |\overline{v}_\alpha(y)|\ ||\overline{E}_\alpha||.$$

Next we suppose that there exist spheres S_m, S_M with $0 < m < M$ such that each $Re\ \lambda_i(y) > 0$ for $y \in S_m$ and $Re\ \lambda_i(y) < 0$ for $y \in S_M$ and that

Assumption B: $\min_{y \in S_m} \underline{\lambda}(y) > \xi(m)$, $\left|\max_{y \in S_M} \overline{\lambda}(y)\right| > \xi(M)$.

The next lemma then follows from (2.14) and (2.15):

Lemma 3 Under assumption B the annulus

$$\mathcal{A} = \{y : m \leq y \leq M\}$$

is an invariant set for the flow of (2.11). $\square$

The compactness of $\mathcal{A}$ and the continuity of solutions now guarantee the existence of an 'almost cycle' in the sense that for any $\epsilon > 0$ there is a T such that

$$||y(0) - y(T)|| < \epsilon.$$

(Simply consider the points $y(0, y(1), y(2), \cdots$ in the flow of (2.12). This set must have a cluster point.) However, this 'cycle' may not circulate around $\mathcal{A}$. To find conditions under which this holds, consider the 'index'

$$ind_T y_j = \frac{1}{2\pi i} \int_0^T \mathcal{I}m \left(\frac{\dot{y}_j(t)}{y_j(t)} \right) dt.$$

We have

Theorem 2.3 Suppose that

$$(a) \quad \mathcal{I}m\ \lambda_j(y) > \max\ \{\xi(m), \xi(M)\}$$

or

$$(b) \quad \mathcal{I}m\ \lambda_j(y) < -\max\ \{\xi(m), \xi(M)\}$$

for all $y \in \mathcal{A}$. Then there exist times $\tau_1, \cdots, \tau_n$ such that

$$\mathrm{ind}_{\tau_i} y_i = \begin{cases} +1 & \text{(case (a))} \\ -1 & \text{(case (b))} \end{cases}$$

Proof We prove case (a), case (b) being similar. By (2.12) we have

$$\mathcal{I}m\ \frac{\dot{y}_j}{y_j} = \mathcal{I}m\ \lambda_j(y) + \mathcal{I}m\ \frac{e_j}{y_j}$$

so that

$$\mathrm{ind}_\tau y_j = \frac{1}{2\pi}\int_0^\tau \left(\mathcal{I}m\ \lambda_j(y) + \mathcal{I}m\ \frac{e_j}{y_j}\right) dt.$$

Since the integrand is positive for all y the result follows. $\square$

From lemma 1 we can find integers $k_1, \cdots, k_n$ such that

$$|k_p\tau_p - k_q\tau_q| < \epsilon$$

for any $\epsilon > 0$. A cluster point to the set

$$\{y(\ell k_1\tau_1), y(\ell k_2\tau_2), \cdots, y(\ell k_n\tau_n)\}\ ,\ \ell \geq 1$$

will give an 'almost' cycle which circulates around $\mathcal{A}$.

3 Hamiltonian Systems and Periodic Orbits

In this section we shall consider Hamiltonian systems of the form (1.1). Recall that a general Hamiltonian system is given by

$$\dot{x} = J\nabla h(t, x) \tag{3.1}$$

for some Hamiltonian function $h(t, x)$, $x \in \mathbf{R}^{2n}$ and

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \in \mathbf{R}^{(2n)^2}.$$

We shall be concerned only with the time-invariant case, but more general nonautonomous systems can be treated similarly. For linear Hamiltonian systems we have

$$\dot{x} = JSx = Ax$$

where S is symmetric and A is Hamiltonian, i.e.

$$A^T J + JA = 0.$$

The Lie algebra of all such matrices is the classical symplectic Lie algebra $sp(n, \mathbf{R})$.

In order to state a condition under which system (1.1) is Hamiltonian, we introduce the following notation. Let $\mathbf{1}_k$ denote the multi-index which has zeros everywhere except for a one in the k^{th} place and if $\mathbf{i}, \mathbf{j}$ are multi-indices we define

$$\mathbf{i} + \mathbf{j} = (i_1 + j_1, \cdots, i_n + j_n).$$

Let $\{S_{\mathbf{i}}\}_{|\mathbf{i}| \geq 0}$ be a set of symmetric matrices indexed by the multi-indices $\mathbf{i} = (i_1, \cdots, i_n)$ and let the ij^{th} component of any $S_{\mathbf{i}}$ be denoted by $s_{ij}^{\mathbf{i}}$. We shall say that the set $\{S_{\mathbf{i}}\}$ satisfies **condition P** if

$$\text{P:} \quad \sum_j s_{ij}^{\bar{\mathbf{i}} + \mathbf{1}_k - \mathbf{1}_j} = \sum_j s_{kj}^{\bar{\mathbf{i}} + \mathbf{1}_i - \mathbf{1}_j}$$

for any multi-index $\bar{\mathbf{i}}$ and for any $i, k \in \{1, \cdots, n\}$.

Lemma 4 The system (1.1) is Hamiltonian if each $A_{\mathbf{i}}$ is a Hamiltonian matrix and the set $\{S_{\mathbf{i}}\} = \{-JA_{\mathbf{i}}\}$ satisfies condition P.

Proof The system (1.1) is Hamiltonian if and only if it can be written in the form

$$\dot{x} = J \nabla h(x)$$

for some function h and so it suffices to show that $S(x)x = -JA(x)x$ can be written in the form $\nabla h(x)$. By Poincare's lemma, this holds if and only if

$$\frac{\partial}{\partial x_k}\sum_j s_{ij}(x)x_j = \frac{\partial}{\partial x_i}\sum_j s_{kj}(x)x_j$$

for each k, i. Thus, we must have

$$\sum_j \left(\frac{\partial}{\partial x_k}s_{ij}(x)\right)x_j + \sum_j s_{ij}(x)\delta_{jk} = \sum_j \left(\frac{\partial}{\partial x_i}s_{kj}(x)\right)x_j + \sum_j s_{kj}(x)\delta_{ij}.$$

Now,

$$s_{ij}(x) = \sum_{i} s_{ij}^{\mathbf{i}} x^{\mathbf{i}}$$

and since $A_{\mathbf{i}}$ is Hamiltonian, $s_{\mathbf{i}} = -J(s_{ij}^{\mathbf{i}})$ is symmetric. Hence we must have

$$\sum_j \left(\frac{\partial}{\partial x_k}s_{ij}(x)\right)x_j = \sum_j \left(\frac{\partial}{\partial x_i}s_{kj}(x)\right)x_j, \tag{3.2}$$

i.e.

$$\sum_{\mathbf{i}}\sum_j \left(s_{ij}^{\mathbf{i}} x^{\mathbf{i}-\mathbf{1}_k}\right)x_j = \sum_{\mathbf{i}}\sum_j \left(s_{kj}^{\mathbf{i}} x^{\mathbf{i}-\mathbf{1}_k}\right)x_j,$$

or

$$\sum_{\mathbf{i}'}\sum_j s^{\mathbf{i}'+\mathbf{1}_k} x^{\mathbf{i}'} x_j = \sum_{\mathbf{i}'}\sum_j s^{\mathbf{i}'+\mathbf{1}_i} x^{\mathbf{i}'} x_j \quad (\mathbf{i}' = \mathbf{i} - \mathbf{1}_k)$$

or

$$\sum_{\bar{\mathbf{i}}}\sum_j s_{ij}^{\mathbf{i}+\mathbf{1}_k-\mathbf{1}_j} x^{\bar{\mathbf{i}}} = \sum_{\bar{\mathbf{i}}}\sum_j s_{ij}^{\mathbf{i}+\mathbf{1}_i-\mathbf{1}_j} x^{\bar{\mathbf{i}}} \quad (\bar{\mathbf{i}} = \mathbf{i}' + \mathbf{1}_j)$$

The result now follows. □

Now suppose that the matrices $A_{\mathbf{i}}$ in (1.2) are Hamiltonian and satisfy condition P. Then, since $A_{\mathbf{i}} \in sp(n, \mathbf{R}) \subseteq sp(n, \mathbf{C})$ for each $\mathbf{i}$ we can write (1.1) in the form

$$\dot{x} = H(x)x + \sum_{\alpha \in \Gamma} v_\alpha(x) E_\alpha x, \tag{3.3}$$

where H is in a Cartan subalgebra of $sp(n, \mathbf{C})$ and Γ is the system of roots. We may assume (changing variables if necessary) that $H(x)$ is of the form

$$H(x) = \begin{bmatrix} \lambda_1(x) & & & & & \\ & \ddots & & & & \\ & & \lambda_n(x) & & & \\ & & & -\lambda_1(x) & & \\ & & & & \ddots & \\ & & & & & -\lambda_n(x) \end{bmatrix}$$

and the structure formulae for E_α are given by

$$\begin{aligned} E_{\lambda_i - \lambda_k} &= \begin{bmatrix} E_{ik} & 0 \\ 0 & -E_{ki} \end{bmatrix}, \quad E_{-\lambda_i + \lambda_k} = \begin{bmatrix} E_{ki} & 0 \\ 0 & -E_{ik} \end{bmatrix}, \quad i < k, \\ E_{\lambda_i + \lambda_k} &= \begin{bmatrix} 0 & E_{ik} + E_{ki} \\ 0 & 0 \end{bmatrix}, \quad E_{-\lambda_i - \lambda_k} = \begin{bmatrix} 0 & 0 \\ E_{ik} + E_{ki} & 0 \end{bmatrix}, \quad i < k, \\ E_{2\lambda_i} &= \begin{bmatrix} 0 & E_{ii} \\ 0 & 0 \end{bmatrix}, \quad E_{-2\lambda_i} = \begin{bmatrix} 0 & 0 \\ E_{ii} & 0 \end{bmatrix} \end{aligned}$$

where $\pm\lambda_i \pm \lambda_k$ $(i < k)$ and $\pm 2\lambda_i$ are the roots of $sp(n, \mathbf{C})$, and

$$E_{ik} = (\delta_{ik}).$$

Proposition 3.2 If $\{S_i\}$ satisfies condition P then the system (1.2) can be written in the form

$$\dot{x} = J\nabla h(x) \tag{3.4}$$

where

$$h(x) = \int_0^x S(\overline{x})\overline{x}d\overline{x}$$

and the integral is over any path in $\mathbf{R}^{2n}$ from 0 to x.

Proof This follows from Poincare's lemma and the fact that

$$J\nabla h = A(x)x$$

so that

$$\nabla h = -JA(x)x = S(x)x$$

and

$$\dot{h}(x) = \nabla h\dot{x}$$

which gives

$$h(x(t)) = \int_0^t h\dot{x}dt = \int_0^{x(t)} \nabla h(x)dx$$

for any path $t \longrightarrow x(t)$. □

Corollary We may write (3.3) in the form (3.4) where

$$h(x) = -\int_0^x J\left(H(\overline{x})\overline{x} + \sum_{\alpha\in\Gamma} v_\alpha(\overline{x})E_\alpha\overline{x}\right)d\overline{x}. \tag{3.5}$$

□

Our first result follows directly from [4] :

Theorem 3.1 If $H(x) \longrightarrow H = diag(\lambda_1, \cdots, \lambda_n, -\lambda_1, \cdots, -\lambda_n)$, $v_\alpha(x) \longrightarrow v_\alpha$, and

$$||\nabla\lambda_i(x)|| = o(x) \quad , \quad ||\nabla v_\alpha(x)|| = o(x)$$

as $||x|| \longrightarrow \infty$, then the system (3.1) has periodic orbits of all periods, provided the linear system

$$\dot{x} = Hx + \sum_{\alpha\in\Gamma} v_\alpha E_\alpha x$$

is nondegenerate. $\square$

To get more precise results on the number of orbits of period 1, we use the approach of [5] . This is based on Lyapunov-Schmidt reduction to obtain a finite-dimensional approximation to the associated variational problem. Since the details are well known, we shall merely outline the idea in this case. To find periodic solutions of (3.3) i.e. functions $t \longrightarrow x(t) \in \mathbf{R}^{2n}$ such that $x(0) = x(1)$ we introduce the functional

$$I(x) = \int_0^1 \{\frac{1}{2}\langle\dot{x}, Jx\rangle - h(x(t))\}dt$$

on the space of such functions. ($A(x)$ can also be time dependent–this is a simple extension of the following.) Then,

$$\nabla I(x) = -J\dot{x} - \nabla h(x)$$

and so a critical point of I is a periodic solution of (3.3). Let D be the operator on $H^1([0,1],\mathbf{R}^{2n})$ defined by

$$\mathcal{D}(D) = \{u \in H^1([0,1],\mathbf{R}^{2n}) : \; u(0) = u(1) \; \}$$

and

$$Du = -J\dot{u} \quad , \quad u \in \mathcal{D}(D).$$

Also define $F : H \longrightarrow H$ by

$$F(u)(t) = \nabla h(u(t)) \quad , \quad u \in H,$$

where $H = L^2([0,1],\mathbf{R}^{2n})$. Of course,

$$I(u) = \frac{1}{2}\langle Au, u\rangle - \Phi(u),$$

where

$$\Phi(u) = \int_0^1 h(u(t))dt.$$

The operator D is self-adjoint and has compact resolvent. Hence the spectrum of D is a pure point spectrum and $\sigma(D) = 2\pi\mathbf{Z}$. Each eigenvalue has multiplicity $2n$ and eigenspace spanned by the loops

$$t \longrightarrow e^{t\lambda J}e_k = (\cos\ \lambda t)e_k + (\sin\ \lambda t)Je_k,$$

$1 \leq k \leq 2n$ and $\{e_k\}$ is the standard basis of $\mathbf{R}^{2n}$. If the spectral resolution of D is $\{E_\lambda : \lambda \in \mathbf{R}\}$ then we define the projection

$$P(\beta) = \int_{-\beta}^{\beta} dE_\lambda \quad , \quad \beta \notin 2\pi\mathbf{Z}.$$

If $Z = P(H)$ and $Y = (I - P)(H)$, then $H = Z \oplus Y$ and $\dim Z < \infty$. Thus, the equation

$$Du - F(u) = 0 \quad , \quad u \in \mathcal{D}(D) \tag{3.6}$$

is equivalent to the pair

$$DPu - PF(u) = 0 \tag{3.7}$$

$$DP^{\perp}u - P^{\perp}F(u) = 0 \tag{3.8}$$

where $P^{\perp} = I - P$. Put $u = z + y \in Z \oplus Y$.

In order to solve (3.7) for y we note that it is equivalent to the equation

$$y = D_0^{-1}P^{\perp}F(z + y) \tag{3.9}$$

where $D_0 = D|Y$. If we assume that in equation (3.3) we have

$$\begin{gathered} |\lambda_i(x)| \leq c \quad , \quad ||\nabla\lambda_i(x)|| = o(x) \\ |v_\alpha(x)| \leq c \quad , \quad ||\nabla v_\alpha(x)|| = o(x) \end{gathered} \tag{3.10}$$

as $||x|| \longrightarrow \infty$, for some constant c, then it follows that $||h''(x)|| \leq \gamma$ for some constant γ and all x. Thus, if $\beta \geq 2\gamma$, we have

$$||F(u) - F(v)|| \leq \gamma||u - v||$$

so that the right hand side of (3.8) is a contraction operator. Hence we can solve (3.8) for $y = v(z) \in Y$. Clearly v is Lipschitz and if

$$u(z) = z + v(z)$$

we have that (3.8) is equivalent to the equation

$$Du(z) - F(u(z)) = 0.$$

If $g(z) = f(u(z))$, then

$$\nabla g(z) = Dz - PF(u(z)).$$

The idea is then to consider the gradient system

$$\dot{z} = \nabla g(z). \tag{3.11}$$

Note that, as in [5], if $z \in Z$ we define $z = z' + \xi$, where $z' = [z]$ is the mean value of z, so that $z' \in Ker(D)$ and $\xi \in Ker(D)^{\perp} \cap Z$. This, with the assumed periodicity of h allows the conclusion that $(z', \xi) \in T^{2n} \times \mathbf{R}^{2m}$ where $T^{2n} = \mathbf{R}^{2n}/\mathbf{Z}^{2n}$ (the $2n$ dimensional torus). Since we do not assume the periodicity of h (in x) this conclusion does not follow here, and it is necessary to have a nontrivial topology on the space of (z', ξ) to be able to apply index theory. To get around this we assume that the dynamics have an unstable focus at the origin, so that $A(0)$ has positive real eigenvalues. Of course, such a system is not Hamiltonian at 0. We shall assume that the system is Hamiltonian for $||x|| \geq \delta > 0$, where (3.2) is assumed to hold and that this set is invariant. (We clearly do not have analyticity here, so we consider the system in the form (1.1) where $A(x)$ is differentiable.)

Theorem 3.2 Under the above assumptions, if all solutions of (3.1) in $\{x : ||x|| \geq \delta\}$ have mean $[x(\cdot)]$ bounded away from zero, then it has at least $2n$

periodic solutions of period 1.

Proof Since z approximates the solution of (3.1) for large β, we can assume that $z' = [z] \geq \delta'$ for some $\delta' > 0$. Then, as in [5], we see that the system (3.10) splits as

$$\begin{aligned} \frac{dz'}{ds} &= -Q_0 F(u(z)) \\ \frac{d\xi}{ds} &= D\xi - QF(u(z)), \end{aligned}$$

where Q_0 is the orthogonal projection onto $Ker\ D$ and Q projects onto $Z \ominus Ker\ D$. This system is defined on the set $\mathbf{R}^{2n} \backslash \{x : ||x|| < \delta'\} \times \mathbf{R}^{2M}$ for some $M > 0$. Since the first space is homotopic to the $(2n-1)$– sphere it has cup long$=2n-1$ (see [7]). The result now follows as in [5]. □

4 Stability of Nonlinear Hamiltonian Systems

In this section we shall generalize a result of [8] to nonlinear systems. The main results in [8] concern the linear Hamiltonian system

$$\dot{x} = Ax\ ,\ x \in \mathbf{R}^{2n}\ ,\ A \in sp(n, \mathbf{R}). \tag{4.1}$$

The Hamiltonian of this system is

$$V(x) = -\frac{1}{2}\langle JAx, x\rangle.$$

This will be a Lyapunov function for (4.1) if V is a definite form, since then $V > 0$ or $V < 0$ and

$$\dot{V} = -\frac{1}{2}\langle JA\dot{x}, x\rangle - -\frac{1}{2}\langle JAx, \dot{x}\rangle$$

$$
\begin{aligned}
&= -\frac{1}{2}\langle JA \cdot Ax, x\rangle - -\frac{1}{2}\langle JAx, Ax\rangle \\
&= -\frac{1}{2}\langle JA \cdot Ax, x\rangle - -\frac{1}{2}\langle A^T JAx, x\rangle \\
&= 0
\end{aligned}
$$

since $A^T J + JA = 0$. It is also shown that the system (4.1) is strongly stable under small perturbations in A if and only if some linear combination of the n quadratic first integrals $(-1)^k \frac{1}{2}\langle JA^{2k-1}x, x\rangle$, $k = 1, \cdots, n$ is definite.

We shall consider the system

$$
\dot{x} = A(x)x \ , \ x \in \mathbf{R}^{2n} \ , \ A(x) \in sp(n, \mathbf{R})
$$

and we shall assume that $A(x)$ has a representation in the form

$$
A(x) = Hx + \sum_{\alpha \in \Gamma} v_\alpha(x) E_\alpha x \tag{4.2}
$$

where H is indepedent of x. Then we have the following partial generalization of the result in [8].

Theorem 4.1 If some linear combination of the quadratic first integrals $I_k = (-1)^k \frac{1}{2}\langle JH^{2k-1}x, x\rangle$, $k = 1, \cdots, n$ of the linear system

$$
\dot{x} = Hx,
$$

is definite (say $\sum a_k I_k > 0$) and

$$
\sum_{\alpha \in \Gamma} \sum_{k=1}^{n} a_k v_\alpha(x)(-1)^k (JH^{2k-1}E_\alpha + E_\alpha^T JH^{2k-1}) \leq 0
$$

for each x, then the system (4.2) is stable.

Proof Let $V = \sum_{k=1}^{n} a_k(-1)^k \frac{1}{2}\langle JH^{2k-1}x, x\rangle > 0$. Then

$$\dot{V} = \sum_{k=1}^{n} a_k(-1)^k \frac{1}{2}\langle JH^{2k-1}\dot{x}, x\rangle + \sum_{k=1}^{n} a_k(-1)^k \frac{1}{2}\langle JH^{2k-1}x, \dot{x}\rangle.$$

Now,

$$\begin{aligned}\langle JH^{2k-1}Hx, x\rangle &= -\langle H^T JH^{2k-1}x, x\rangle \\ &= -\langle JH^{2k-1}x, Hx\rangle.\end{aligned}$$

Hence,

$$\dot{V} = \sum_{k=1}^{n} a_k(-1)^k \frac{1}{2}\langle JH^{2k-1}E_\alpha x, x\rangle + \sum_{k=1}^{n} a_k(-1)^k \frac{1}{2}\langle JH^{2k-1}x, E_\alpha x\rangle.$$

□

Corollary 4.2 If $-\frac{1}{2}\langle JHx, x\rangle$ is positive definite and

$$\sum_{\alpha\in\Gamma} v_\alpha(x) JHE_\alpha \geq 0$$

then the system (4.2) is stable. □

5 Conclusions

In this paper we have studied pseudo-linear systems from the viewpoint of periodic orbits and stability. The methods have been based on a decomposition of the system in terms of the Lie algebra generated by the coefficient matrix functions of the system. If this Lie algebra is semisimple then significant results can be obtained, particularly in the case of Hamiltonian systems where the Lie algebra has the classical symplectic structure. By diagonalizing the elements in a Cartan subalgebra and regarding the nonzero root matrices as perturbations we have obtained a number of generealizations of some well-known results.

References

[1] S.P.Banks and K.J.Mhana, 'Optimal Control and Stabilization for Nonlinear Systems ', IMA J. Math. Contr. and Inf.,**9**, 179-196,1992.

[2] S.P.Banks and S.Al Jurani, 'Lie Algebras and the Stability of Nonlinear Systems, to appear in Int. J. Control.

[3] S.P.Banks and S.Al Jurani, 'On the Stability of Pseudo-Linear Systems, Preprint.

[4] C.C.Conley and E.Zehnder, 'The Birkhoff-Lewis Fixed Point Theorem and a Conjecture of V.I.Arnold, Inv. Math., **73**, 33-49, 1983.

[5] C.C.Conley and E.Zehnder, 'Morse-Type Index Theory for Flows and Periodic Solutions for Hamiltonian Equations', Comm. Pure App. Maths., **37**, 207-253, 1984.

[6] , J.Humphreys, 'Introduction to Lie Algebras and Representation Theory', Springer-Verlag, New York, 1980.

[7] E.R.Fadell and P.H. Rabinowitz, 'Generalized Cohomology Theories for Lie Group Actions with an Application to Bifurcation Questions for Hamiltonian Systems', Inv. Math., **45**, 139-174, 1978.

[8] M.P.Wojtkowski, 'A Remark on Strong Stability of Linear Hamiltonian Systems', J.Diff. Eqns., **81**, 313-316, 1989.

[9] Z-X Wan, 'Lie Algebras', Pergamon Press, Oxford, 1975.