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On the Inversion Problem for Nonlinear Systems

by

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Abstract

A new homotopy technique is applied to the inversion problem for nonlinear systems, both in the input-output and state-space formulations.

Keywords:Inversion problem, Homotopy theory, Lie series.

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1 Introduction

The inversion problem in systems theory has a long history [3,4,6,7,8,9] and has an important position in the field. Regarding a system as an input-output map

$$S : U \longrightarrow Y$$

from an input (Banach) space U into an output (Banach) space Y, the inverse problem is simply to find

$$S^{-1} : \mathcal{R}(S) \subseteq Y \longrightarrow U$$

if it exists. The linear case is well understood and has been completely solved by Silverman [7]. In the nonlinear case, the implicit function theorem [2] is important in the existence proof. In this paper we shall give a new homotopy approach to system inversion and apply it to nonlinear dynamical systems. For this we shall use a global expansion technique based on generalized Lie series developed in [1].

The structure of the paper is as follows. In section 2 we derive the homotopy method, which has been applied in other circumstances [5]. In section 3, the method is applied to systems with a finite-dimensional realization by using Silverman's algorithm and in section 4 we consider the case of a nonlinear analytic dynamical system in state-space form. Finally, a simple example to illustrate the formal technique will be given.

2 General Homotopy Method

In this section we shall consider a general input-output map $S : U \longrightarrow Y$ defined on Banach spaces U, Y. We shall assume that S is Frechet differentiable. If S is (locally or globally) invertible then given $y^* \in Y$ we wish to find

$$u^* = S^{-1}(y^*).$$

Let $(u_0, y_0) \in U \times Y$ be any known input-output pair, i.e.

$$y_0 = S(u_0),$$

and put

$$\overline{S}(u) = S(u) - y^*. \tag{2.1}$$

Define the homotopy H between the functions $\overline{S}$ and the constant function $u \longrightarrow \overline{S}(u_0)$ by

$$H(u,t) = \overline{S}(u) + (t-1)\overline{S}(u_0). \tag{2.2}$$

We shall find a differentiable curve $G : [0,1] \longrightarrow U$ with

$$G(0) = u_0 \quad , \quad G(1) = u^*,$$

i.e. one which connects the known value u_0 to the desired one u^*. Note that

$$H(u_0, 0) = \overline{S}(u_0) - \overline{S}(u_0) = 0$$

and

$$H(u^*, 1) = \overline{S}(u^*) = S(u^*) - y^* = 0.$$

We shall therefore search for $G(t)$ among those functions which satisfy

$$H(G(t),t) = 0. \tag{2.3}$$

Lemma 2.1 G satisfies the first-order differentiable equation

$$\dot{G}(t) = -[\mathcal{F}S(G(t))]^{-1}(y_0 - y^*)\ ,\ G(0) = u_0. \tag{2.4}$$

Proof This follows by differentiating (2.3). Thus,

$$\begin{aligned}
0 &= \frac{d}{dt}H(G(t),t) \\
&= \mathcal{F}_G H(G(t),t)\frac{dG}{dt} + \frac{\partial H}{\partial t}(G(t),t) \\
&= \mathcal{F}_u\ \overline{S}(u)\big|_{u=G}\ \dot{G} + \overline{S}(u_0) \\
&= \mathcal{F}S(G)\dot{G} + S(u_0) - y^*.
\end{aligned}$$

□

If we assume that $\mathcal{F}S(u)$ is continuous and invertible for each u and that $||(\mathcal{F}S(u))^{-1}|| \leq \epsilon$ for all $u \in U$ and some $\epsilon > 0$, then an easy application of the implicit function theorem shows that equation (2.4) has a unique solution $G(t)$. Equation (2.4) can be solved iteratively by Euler's method, i.e.

$$\begin{aligned}
G_{k+1} &= G_k - h(\mathcal{F}S(G_k))^{-1}(y_0 - y^*) \qquad (2.5)\\
G_0 &= u_0
\end{aligned}$$

for $k = 0, 1, \cdots, L-1$. This process will give an approximation G_L to u^*. Using Newton's algorithm we can write

$$\begin{aligned}
G_L^{i+1} &= G_L^i - (\mathcal{F}S(G_L^i))^{-1}(S(G_L^i) - y^*) \qquad (2.6)\\
G_L^0 &= G_L \qquad (2.7)
\end{aligned}$$

for $i = 0, 1, 2, \cdots$. Simple estimates then show that the combined approximations (2.5) and (2.6) converge to a unique solution u^*. (Details can be found in [5]).

3 Application to Systems with s Finite-Dimensional Realization

Suppose that the linear system $\mathcal{F}S(u)$ is shift-invariant and finite- dimensional for each $u \in U$ so it has a state space representation of the form

$$\begin{aligned} \dot{x}(t) &= A_u x(t) + B_u v(t) \\ z(t) &= C_u x(t) + D_u v(t) \end{aligned} \tag{3.1}$$

where $x(t) \in \mathbf{R}^n, v(t) \in \mathbf{R}^p, y(t) \in \mathbf{R}^p$.

In order to solve (2.4) numerically (by, say Euler's method) then we can use Silverman's algorithm [7] to invert the linear system (3.1) at each iteration. (Note that we can obtain (3.1) for the linear system $\mathcal{F}S(u)$ by taking the Laplace transform of the system and realizing it by standard methods in the form (3.1).) This algorithm proceeds by defining a sequence of systems derived from (3.1):

$$\begin{aligned} \dot{x}(t) &= A_u x(t) + B_u v(t) \\ z(k)(t) &= C_u(k) x(t) + D_u(k) v(t) \end{aligned}$$

where $C_u(k)$ and $D_u(k)$ are obtained as follows. If $q_0 = \text{rank } D_u < p$, then

$D_u(0) = \begin{bmatrix} \overline{D}_u(0) \\ 0 \end{bmatrix}$, where $D_u(0) = S_0 D_u$ and S_0 is a nonsingular $p \times p$ matrix. Define $C_u(0) = S_0 C_u$. Then if

$$D_u(k) = \begin{bmatrix} \overline{D}_u(k) \\ 0 \end{bmatrix}$$

where $D_u(k)$ has rank $q_k < p$, then we partition $C_u(k)$ in the same way as

$$C_u(k) = \begin{bmatrix} \overline{C}_u(k) \\ \tilde{C}_u(k) \end{bmatrix}$$

and we find a nonsingular matrix S_{k+1} such that

$$D_u(k+1) \triangleq S_{k+1} \begin{bmatrix} \overline{D}_u(k) \\ \tilde{C}_u(k) B_u \end{bmatrix} = \begin{bmatrix} \overline{D}_u(k+1) \\ 0 \end{bmatrix}$$

where $\overline{D}_u(k+1)$ has q_{k+1} rows and rank q_{k+1}. Then, if $D_u(\alpha)$ has rank p for some α, then the inverse system of (25) is given by

$$\dot{\xi}(t) = (A_u - B_u(D_u(\alpha))^{-1} C_u(\alpha))\xi(t) + B_u(D_u(\alpha))^{-1} z_\alpha(t)$$

where

$$z_\alpha(t) = \left(\prod_{i=0}^{\alpha} S_{\alpha-i} M_{\alpha-i-1} \right) y(t),$$

and M_k is given by

$$M_k = \begin{bmatrix} I_{q_k} & 0 \\ 0 & I_{p-q_k}(d/dt) \end{bmatrix}.$$

From Silverman [7], we have

Theorem 4.1 *The system (3.1) is invertible if and only if there exists a positive integer $\alpha < n$ such that $q_\alpha = p$.* □

Therefore we have

Corollary 4.1 *A nonlinear input-output map $S : U \longrightarrow Y$ is invertible if it is differentiable with a derivative $\mathcal{F}S(u)$ which has a finite-dimensional realization of the form (3.1) such that there exist integers $\alpha(u)$ determined as above for each u satisfying $\alpha(u) < n$.* □

4 Application to Nonlinear Dynamical Systems

Consider the general analytic system

$$\dot{x} = f(x, u) \tag{4.1}$$

$$y = h(x, u). \tag{4.2}$$

(We shall consider the case where dim u = dim y=1, i.e. scalar systems for simplicity. At the expense of more notational complexity, we can easily extend the theory to the general case.) In order to apply the general theory we must first find the input-output map for (4.1). We shall use the global bilinearization technique which uses the Lie series (Banks, [1]). First, augment (4.1) by making u into a state:

$$\begin{aligned} \dot{x} &= f(x, u) \\ \dot{u} &= v \end{aligned} \tag{4.3}$$

$$y = h(x,u)$$

and we obtain the system

$$\begin{aligned} \dot{z} &= \phi(z) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} v \\ y &= h(z) \end{aligned} \qquad (4.4)$$

where $z = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\phi(z) = \begin{pmatrix} f(x,u) \\ 0 \end{pmatrix}$. Now define the functions

$$\begin{aligned} g_1(z) &= h(z) = y \\ g_i(z) &= \begin{cases} \frac{\partial g_{i/2}}{\partial z}(z) \cdot \phi(z) & \text{if } i \text{ even} \\ \frac{\partial g_{(i-1)/2}}{\partial z}(z) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \text{if } i \text{ odd} \end{cases} \end{aligned}$$

Hence, from (4.3) we have

$$\begin{aligned} \dot{g}_i(z) &= \frac{\partial g_i}{\partial z}(z)\dot{z} \\ &= \frac{\partial g_i}{\partial z}(z) \cdot \phi(z) + v(t)\frac{\partial g_i}{\partial z}(z) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= g_{2i}(z) + v(t) g_{2i+1}(z) \end{aligned}$$

and so we obtain the bilinear system

$$\dot{G} = AG + v(t)BG \quad , \quad G(0) = G_0 \qquad (4.5)$$

where

$$G = (g_1, g_2, \cdots)^T \ ,$$

and $A = (a_{ij}), B = (b_{ij})$ are infinite-dimensional (block) matrices defined by

$$\begin{aligned} a_{ij} &= \delta_{2i,j} I_{n\times n} \\ b_{ij} &= \delta_{2i+1,j} I_{n\times n} \ . \end{aligned}$$

The initial state G_0 is given by

$$G_0 = \left(z_0, f(z_0), \begin{pmatrix} 0 \\ 1 \end{pmatrix}, ((\partial f)f)(z_0), \left((\partial f) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)(z_0), \cdots \right)^T$$

where $\partial = \partial/\partial z$. (Note that we are regarding G as an infinite vector of vector-valued functions.) We can evaluate the Volterra series for (4.4) in the usual way:

Let

$$G(t) = \sum_{k=0}^{\infty} \xi_k(t) \ ,$$

where

$$\begin{aligned} \xi_0(t) &= e^{At} G_0 \\ \xi_k(t) &= \int_0^t e^{A(t-s)} B v(s) \xi_{k-1}(s) ds. \end{aligned} \tag{4.6}$$

Hence, iterating (4.5) we obtain

$$\begin{aligned} \xi_k(t) &= \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{k-1}} e^{A(t-\tau_1)} B e^{A(\tau_2-\tau_1)} B \cdots B e^{A(\tau_{k-1}-\tau_k)} B \ \cdot \\ &= \quad e^{A\tau_k} G_0 \cdot v(\tau_1) \cdots v(\tau_k) d\tau_1 \cdots d\tau_k \ , \\ &\triangleq M_k(v, \cdots, v)(t) \end{aligned} \tag{4.7}$$

where M_k is a multilinear operator in v. Let S denote the input-output map

$$S : v \longrightarrow \sum_{k=0}^{\infty} M_k(v, \cdots, v) \quad , \tag{4.8}$$

and let $L_k(v; \cdot)$ denote the Frechet dervative of $M_k(v, \cdots, v)$ at v (i.e. $L_k(v; \cdot)$ is a linear operator $w \longrightarrow L_k(v; w)$). Then

$$L_k(v, \cdot) = \sum_{i=1}^{k} M_k(v, \cdots, \underbrace{\cdot}_{i}, \cdots, v).$$

We therefore have

Lemma 4.1 The Frechet derivative of the Volterra series defined by (4.7) is given by

$$\mathcal{F}S_v(u) = \sum_{k=1}^{\infty} L_k(v; u)$$

where

$$\begin{aligned} L_k(v; u) \quad = \quad & \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{k-1}} e^{A(t-\tau_1)} B e^{A(\tau_2 - \tau_1)} B \cdots B e^{A(\tau_{k-1} - \tau_k)} B \cdot \\ & e^{A\tau_k} G_0 \cdot v(\tau_1) \cdots u(\tau_j) \cdots v(\tau_k) d\tau_1 \cdots d\tau_k \quad , \end{aligned} \tag{4.9}$$

□

Now let P denote the operator $P : G = (g_1, g_2, \cdots)^T \longrightarrow y$ where $g_1 = h(z) = y$. Then we must invert the linear operator L_v defined by

$$L_v(u) = \sum_{k=1}^{\infty} P L_k(v; u)$$

for any v.

Lemma 4.2 Each linear operator $PL_k(v; u)$ can be written as an integral operator of the form

$$y(t) = \int_0^t K_k(t,\tau,v)u(\tau)d\tau$$

for some kernel K_k.

Proof This is proved easily from (4.8) by extending the kernel by zero so that the integrals all range from 0 to t and then using Fubini's theorem to interchange the integrals. $\square$.

The inverse of the locally linearized system obtained by Frechet differentiatingthe input-output map is then found by solving a linear integral equation.

Example Consider the single-input, single-output system

$$\begin{aligned} \dot{x}_1 &= x_1^2 + x_1 u \ , \quad x_1(0) = 1 \ , \ x_2(0) = 0. \\ \dot{x}_2 &= x_1 x_2 \\ y &= x_1 + x_2 \end{aligned}$$

In this case we do not need to augment the state with u as in (4.2) since the system is already linear-analytic. Hence we define

$$\begin{aligned} g_1(x) &= x \\ g_i(x) &= \begin{cases} \frac{\partial g_{i/2}}{\partial x}(x) \cdot \begin{pmatrix} x_1^2 \\ x_1 x_2 \end{pmatrix} & \text{if } i \text{ even} \\ \frac{\partial g_{(i-1)/2}}{\partial x}(x) \cdot \begin{pmatrix} x_1 \\ 0 \end{pmatrix} & \text{if } i \text{ odd} \end{cases} \end{aligned} \qquad (4.10)$$

Then,

$$\dot{g}_i(x) = \frac{\partial g_i}{\partial x}\dot{x}$$

$$
\begin{aligned}
&= \frac{\partial g_i}{\partial x}\begin{pmatrix} x_1^2 \\ \\ x_1x_2 \end{pmatrix} + u\frac{\partial g_i}{\partial x}\begin{pmatrix} x_1 \\ \\ 0 \end{pmatrix} \\
&= g_{2i}(x) + u\partial g_{2i+1}(x).
\end{aligned}
$$

As an example, we shall evaluate the second order term ($k = 2$) in (4.8). Thus,

$$
\begin{aligned}
L_2(v;u) \quad = \quad & \int_0^t \int_0^{\tau_1} e^{A(t-\tau_1)}Be^{A(\tau_1-\tau_2)}Be^{A(\tau_2)}G_0u(\tau_1)v(\tau_2)\tau_1\tau_2 \\
& + \int_0^t \int_0^{\tau_1} e^{A(t-\tau_1)}Be^{A(\tau_1-\tau_2)}Be^{A(\tau_2)}G_0v(\tau_1)u(\tau_2)\tau_1\tau_2
\end{aligned}
$$

Now,

$$
\left(e^{At}\right)_{ij} = \sum_{n=0}^{\infty} \frac{t^n}{n!}\delta_{2^n i,j} I_{2\times 2}
$$

and so

$$
\left(e^{At}B\right)_{ij} = \sum_{n=0}^{\infty} \frac{t^n}{n!}\delta_{2\cdot 2^n i+1,j} I_{2\times 2}
$$

whence

$$
\begin{aligned}
\left(e^{At_1}Be^{At_2}B\right)_{ij} &= \sum_{k=1}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_1=0}^{\infty} \frac{t_1^{n_1}}{n_1!}\delta_{2\cdot 2^{n_1}i+1,k}\frac{t_2^{n_2}}{n_2!}\delta_{2\cdot 2^{n_2}k+1,j} I_{2\times 2} \\
&= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!}\delta_{2\cdot 2^{n_2}(2\cdot 2^{n_1}i+1)+1,j} I_{2\times 2}
\end{aligned}
$$

and finally,

$$
\begin{aligned}
&\left(e^{A(t-\tau_1)}Be^{A(\tau_1-\tau_2)}Be^{A\tau_2}\right)_{ij} \\
&= \sum_{k=1}^{\infty}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty} \frac{(t-\tau_1)^{n_1}}{n_1!}\frac{(\tau_1-\tau_2)^{n_2}}{n_2!}\delta_{2\cdot 2^{n_2}(2\cdot 2^{n_1}i+1)+1,k}\frac{\tau_2^{n_3}}{n_3!}\delta_{2^{n_3}k,j} I_{2\times 2} \\
&= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty} \frac{(t-\tau_1)^{n_1}}{n_1!}\frac{(\tau_1-\tau_2)^{n_2}}{n_2!}\frac{\tau_2^{n_3}}{n_3!}\delta_{2^{n_3}2\cdot 2^{n_2}(2\cdot 2^{n_1}i+1)+1,j} I_{2\times 2}
\end{aligned}
$$

Next, G_0 can be found inductively from the definition (4.9). Thus,

$$
\begin{aligned}
g_1 &= x \\
g_2 &= I \begin{pmatrix} x_1^2 \\ x_1 x_2 \end{pmatrix} \\
g_3 &= I \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \\
g_4 &= \frac{\partial g_2}{\partial x} \begin{pmatrix} x_1^2 \\ x_1 x_2 \end{pmatrix} \\
&= \begin{pmatrix} 2x_1 & 0 \\ x_2 & x_1 \end{pmatrix} \begin{pmatrix} x_1^2 \\ x_1 x_2 \end{pmatrix} \\
&= \begin{pmatrix} 2x_1^3 \\ 2x_1^2 x_2 \end{pmatrix} \\
&\;\;\cdot \\
&\;\;\cdot \\
&\;\;\cdot
\end{aligned}
$$

and

$$
G_0 = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \cdots \right).
$$

Put

$$
(\overline{K})_i(t, \tau_1, \tau_2) =
$$

$$
\sum_{k=1}^{\infty} \left(\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \frac{(t-\tau_1)^{n_1}}{n_1!} \frac{(\tau_1-\tau_2)^{n_2}}{n_2!} \frac{\tau_2^{n_3}}{n_3!} \delta_{2^{n_3} 2 \cdot 2^{n_2}(2 \cdot 2^{n_1}+1)+1,k} I_{2\times 2} \right) G_{0k}
$$

Then L_2 can be written in the form

$$L_2(v;u) = \int_0^t \int_0^{\tau_1} \overline{K}_i(t,\tau_1,\tau_2)u(\tau_1)v(\tau_2)d\tau_1 d\tau_2$$
$$+\int_0^t \int_0^{\tau_1} \overline{K}_i(t,\tau_1,\tau_2)v(\tau_1)u(\tau_2)d\tau_1 d\tau_2$$

Changing variables in the first integral gives the kernel K_2 in lemma 4.2.

5 Conclusion

In this paper we have applied a simple homotopy argument to the inversion problem in systems theory. The method leads to a simple numerical algorithm which requires only the inversion of linear systems along a trajectory in the input space. The numerical aspects of the results will be investigated in a later paper.

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