Proceedings Paper:

Müller, H and Wilson, S (2013) Characterising subclasses of perfect graphs with respect to partial orders related to edge contraction. In: Cornelissen, K, Hoeksma, R, Hurink, J and Manthey, B, (eds.) Proceedings of the 12th Cologne-Twente Workshop on Graphs and Combinatorial Optimization (CTW 2013). 12th Cologne-Twente Workshop on Graphs & Combinatorial Optimization, 21-23 May 2013, University of Twente, Enschede, The Netherlands. Centre for Telematics and Information Technology, University of Twente , 183 - 186.

Characterising subclasses of perfect graphs with respect to partial orders related to edge contraction

Haiko Müller[1] and Samuel Wilson[1]

[1]School of Computing, University of Leeds, UK

Characterising graph classes by forbidding a set of graphs has been a common feature in graph theory for many years. Up to date many classes have been characterised with respect to the induced subgraph relation and there are a plethora of results regarding graph classes closed under the minor relation. We consider the characterisation of graph classes closed under partial orders including edge contraction as an operation. More concretely we consider contraction minors and a newly defined partial order, under which many of the graph classes considered here can be characterised by a finite minimal forbidden set.

1 Introduction

The characterisation of graph classes by forbidding a set of graphs has been extensively studied providing characterisations for many graph classes, especially surrounding the class of perfect graphs. It is well known that any class closed under a partial order admits a characterisation by forbidding a set of graphs but finding the forbidden set is harder. With respect to the induced subgraph relation many of the classes surrounding perfect graphs have a description of their forbidden set but little effort has gone into considering a description with respect to different partial orders.

The study of partial orders on the set of all graphs has become increasingly popular thanks to the graph minor theorem by Robertson and Seymour. In the series of over 20 papers they have shown a number of results for classes closed under the minor relation, perhaps most interestingly that for any graph class closed under the minor relationship the minimal forbidden set is finite. This result is shown by proving that the relation forms a well-quasi ordering on the set of all graphs and therefore demonstrating that no infinite antichains exists. The graph minor theorem is a celebrated result but many of the subclasses of perfect graphs are not closed under the minor relation so the result does not apply. In addition there is no similar result for other partial orders under which the subclasses of perfect graphs are closed.

This motivates the search for alternative characterisations of graph classes. The contraction minor relation is an interesting partial order as the complexity for recognising if one graph is contained in another with respect to the contraction minor relation is an open problem,

The research presented in this paper is work towards the Ph.D of the second author, who is funded by EPSRC Doctoral Training Grant.

unlike for the minor relation where the graph containment problem is known to belong to the complexity class FPT. We characterise a number of well studied graph classes by forbidding a set of graphs with respect to the contraction minor relation, we then show an interesting connection between the minimal forbidden set with respect to the induced subgraph relation and the minimal forbidden set for a newly defined partial order ($\leqslant_{\mathrm{ftm}}$), for which most classes considered here have a finite minimal forbidden set.

2 Definitions

Here we consider only finite simple undirected graphs. A graph G is a contraction minor of a graph H denoted $G \leqslant_{\mathrm{c}} H$ if there exists a set of edges $U \subseteq E(H)$ such that the contraction of the edges in U results in a graph isomorphic to G. For completeness we include the definition of the induced subgraph relation, a graph G is an induced subgraph of H denoted $G \leqslant_{\mathrm{i}} H$ if there exists a set of vertices $U \subseteq V(H)$ such $H[V(H) \setminus U]$ is isomorphic to G.

Every class $\mathcal{C}$ of graphs closed under a partial order $\leqslant$ is characterised by a set $\mathcal{F} = \{H \mid H \notin \mathcal{C} \wedge \forall G\,(G \leqslant H \Rightarrow G \simeq H \vee G \in \mathcal{C})\}$ of minimal forbidden graphs, where a partial order can not be determined from the context it is explicitly given as a subscript, e.g. $\mathcal{F}_{\mathrm{c}}$ denotes the minimal forbidden set with respect to contraction minors. For a set $\mathcal{F}$ of graphs, the class of $\mathcal{F}$-free graphs is $\{G \mid \forall H \in \mathcal{F}\ (H \nleqslant G)\}$. This class is closed under $\leqslant$ and $\mathcal{F}$ contains all of the minimal forbidden graphs.

We define the following graphs classes adopted from [1];

$$\mathcal{C}_n = \{C_k \mid k \geq n\} \qquad \mathcal{D}_n = \{2K_1 \bowtie kK_1 \mid k \geq n\}$$
$$\mathcal{K}_n = \{kK_1 \mid k \geq n\} \qquad \mathcal{W}_n = \{C_4 \bowtie kK_1 \mid k \geq n\}$$
$$W_k = C_k \bowtie K_1$$

We define a new partial order denoted $\leqslant_{\mathrm{ftm}}$ where $G \leqslant_{\mathrm{ftm}} H$ if H can be transformed into a graph isomorphic to G by a series of edge contractions and inverse false twin operations where the inverse false twin operation is defined as follows; let G be a graph and let $u, v \in V(G)$ then u, v are false twins if the neighbourhood of u and v are equal and $uv \notin E(G)$. The inverse false twin operation allows for the removal of u.

3 Contribution

We show a number of results for characterising graph classes related to perfect graphs with respect to contraction minors and with respect to $\leqslant_{\mathrm{ftm}}$.

Theorem 3.1. *Let G be a connected graph, we show the following conditions are equivalent;*

(i) G is a chordal graph.

(ii) G does not contain $\{C_n \mid n \geq 4\}$ with respect to $\leqslant_{\mathrm{i}}$ [6].

(iii) G does not contain $\mathcal{D}_2 \cup \{W_4, C_4 \bowtie 2K_1\}$ with respect to $\leqslant_{\mathrm{c}}$.

(iv) G does not contain $\{C_4, W_4\}$ with respect to $\leqslant_{\mathrm{ftm}}$.

It is well known that the class of split graphs is the intersection of the classes chordal and co-chordal. With respect to contraction minors and $\leqslant_{\mathrm{ftm}}$ the set of minimal forbidden graphs for the class of co-chordal graphs is infinite since $\{\overline{C_n} \mid n \geq 6\}$ forms an antichain. Each element of this antichain is comparable to W_4.

Theorem 3.2. *Let G be a connected graph, we show the following conditions are equivalent;*

(i) *G is a split graph.*

(ii) *G does not contain $\{2K_2, C_4, C_5\}$ with respect to* $\leqslant_{\mathrm{i}}$ [2].

(iii) *G does not contain $\mathcal{D}_2 \cup \{W_4, C_4 \bowtie 2K_1, P_5, \overline{P}, 2K_2 \bowtie K_1\}$ with respect to* $\leqslant_{\mathrm{c}}$.

(iv) *G does not contain $\{C_4, W_4, P_5, \overline{P}, 2K_2 \bowtie K_1\}$ with respect to* $\leqslant_{\mathrm{ftm}}$ (Figure 1).

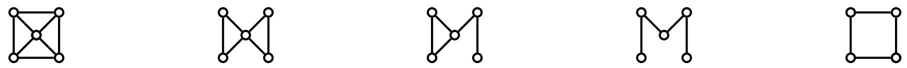

Figure 1: $W_4, 2K_2 \bowtie K_1, \overline{P}, P_5, C_4$: Contraction minimal non-split graphs.

Theorem 3.3. *Let G be a connected graph, we show the following conditions are equivalent;*

(i) *G is a cograph.*

(ii) *G does not contain P_4 with respect to* $\leqslant_{\mathrm{i}}$ [1, Theorem 11.3.3].

(iii) *G does not contain $\{P_4, P_4 \bowtie K_1, \overline{P_5}, C_5, \overline{C_6}\}$ with respect to* $\leqslant_{\mathrm{ftm}}$ (Figure 2).

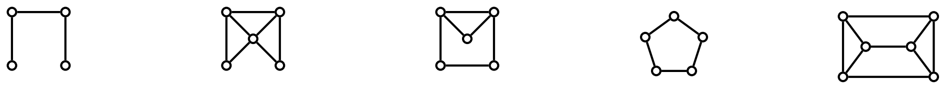

Figure 2: $P_4, P_4 \bowtie K_1, \overline{P_5}, C_5, \overline{C_6}$: Contraction minimal non-cographs

From the above results we distil a general relationship between $\mathcal{F}_{\mathrm{i}}$ and $\mathcal{F}_{\mathrm{ftm}}$ for any class $\mathcal{C}$ closed under vertex deletion and edge contraction.

Theorem 3.4. *Any class $\mathcal{C}$ which has a finite forbidden set with respect to induced subgraphs and is closed under edge contraction has a finite forbidden set with respect to $\leqslant_{\mathrm{ftm}}$ and the order of the largest forbidden graphs is bounded.*

Proof. Let $\mathcal{C}$ be a class closed under vertex deletion and edge contraction and let $\mathcal{F}_{\mathrm{i}}, \mathcal{F}_{\mathrm{ftm}}$ denote the minimal forbidden sets. Let $H \notin \mathcal{C}$ which implies $\exists F \in \mathcal{F}_{\mathrm{i}} \mid F \leqslant_{\mathrm{i}} H$. Contracting the edges $\{uv \mid u, v \in \{V(H) \setminus V(F)\}\}$ leaves an independent set S and a copy of F with some edges between S and $V(F)$. As $\leqslant_{\mathrm{ftm}}$ allows the removal of false twins the number of additional vertices is equal to the number of subsets of vertices in H. $\mathcal{F}_{\mathrm{i}}$ is finite therefore there is a maximum element, let $k = \max\{|F| \mid F \in \mathcal{F}_{\mathrm{i}}\}$. Then the maximum number of vertices of a graph in $\mathcal{F}_{\mathrm{ftm}}$ is $2^k + k$. $\square$

Lemma 3.5. *Let the graph classes $\mathcal{C}, \mathcal{C}'$ be $\mathcal{F}$-free, $\mathcal{F}'$-free respectively, then $\mathcal{C} \cap \mathcal{C}'$ is $(\mathcal{F} \cup \mathcal{F}')$-free.*

- Interval graphs = chordal ∩ co-comparability
- Trivially perfect = chordal ∩ cograph
- co-Trivially perfect = co-chordal ∩ cograph
- Threshold graphs = Trivially perfect ∩ co-Trivially perfect

From these characterisations we obtain the results in Table 1. It is noteworthy that for $\mathcal{C}_1 \cap \mathcal{C}_2$, $\mathcal{F}_1 \cup \mathcal{F}_2 \subseteq \mathcal{G} \setminus (\mathcal{C}_1 \cap \mathcal{C}_2)$ but $\mathcal{F}_1 \cup \mathcal{F}_2$ may contain graphs which are not minimal with respect to the partial order.

Class of ... graphs		Minimal forbidden graphs
co-chordal	$\leqslant_{\mathrm{i}}$	$\overline{C_{n+4}}^*$ for $n \geq 0$
	$\leqslant_{\mathrm{mw}}$	$\overline{C_{n+4}}$ + additional vertices*
trivially perfect	$\leqslant_{\mathrm{i}}$	C_4, P_4 [5]
	$\leqslant_{\mathrm{mw}}$	$P_4, P_4 \bowtie K_1, C_4, W_4$
co-trivially perfect	$\leqslant_{\mathrm{i}}$	$2K_2, P_4$ [1, Theorem 6.6.1]
	$\leqslant_{\mathrm{mw}}$	$2K_2 \bowtie K_1, P_4, P_4 \bowtie K_1, C_5, \overline{C_6}$
threshold	$\leqslant_{\mathrm{i}}$	$2K_2, C_4, P_4$ [1, Theorem 6.6.3]
	$\leqslant_{\mathrm{mw}}$	$P_4, P_4 \bowtie K_1, 2K_2 \bowtie K_1, C_4, W_4$
interval	$\leqslant_{\mathrm{i}}$	$C_{n+4}{}^*, \mathrm{T}_2, \mathrm{X}_{31}, \mathrm{XF}_2{}^{n+1*}, \mathrm{XF}_3{}^{n*}$ for $n \geq 0$ [4]
	$\leqslant_{\mathrm{mw}}$	$\{C_4, W_4, \mathrm{T}_2, \mathrm{X}_{31}, \mathrm{XF}_2{}^1, S_3, \ldots$
co-comparability	$\leqslant_{\mathrm{i}}$	$C_{n+6}, \mathrm{T}_2, \mathrm{X}_2, \mathrm{X}_3, \mathrm{X}_{30}, \mathrm{X}_{31}, \mathrm{X}_{32}, \mathrm{X}_{33}$ $, \mathrm{X}_{34}, \mathrm{X}_{35}, \mathrm{X}_{36}, \mathrm{XF}_2{}^{n+1*}, \mathrm{XF}_3{}^{n*}$ $, \mathrm{XF}_4{}^{n*}, \text{co-}\mathrm{XF}_1{}^{2n+3*}, \text{co-}\mathrm{XF}_5{}^{2n+3*}, \text{co-}\mathrm{XF}_6{}^{2n+2*}, \overline{C_{2n+1}}$ for $n \geq 0$ [3] *
	$\leqslant_{\mathrm{mw}}$	$\{\mathrm{T}_2, \overline{S_3}, S_3, S_3 - e, C_6, D_0, D_1, D_2, \ldots$

Table 1: Forbidden graphs for a collection of graph classes: * denotes an infinite set.

Conclusions

We have presented a set of equivalent definitions for a number of graph classes related to perfect graphs, these graph classes had previously been characterised by forbidding a set of induced subgraphs but the characterisation with respect to other partial order had gone unexplored. This contribution shows that with respect to the contraction minor relation the minimal forbidden set is often infinite, this result is an effect of an infinite series of false twins. We have introduced a new partial order which allows a finite forbidden set characterisation and we have established an upper bound on the size of the minimal forbidden graphs with respect to $\leqslant_{\mathrm{ftm}}$ if the class has a finite minimal forbidden set with respect to induced subgraphs.

References

[1] A. Brandstädt, V.B. Le, and J.P. Spinrad. *Graph classes: a survey.* Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 1999.

[2] S. Foldes and P.L. Hammer. Split graphs. *Congr. Numer*, 19:311–315, 1977.

[3] T. Gallai. Transitiv orientierbare Graphen. *Acta Mathematica Hungarica*, 18:25–66, 1967.

[4] P.C. Gilmore and A.J. Hoffman. A characterization of comparability graphs and of interval graphs. Technical report, DTIC Document, 1962.

[5] M.C. Golumbic. Trivially perfect graphs. *Discrete Mathematics*, 24(1):105 – 107, 1978.

[6] A. Hajnal and J. Surányi. Über die Auflösung von Graphen in vollständige Teilgraphen. *Univ Sci. Budapest, Eötvös Sect. Math. 1*, pages 113 – 121, 1958.