Finite Horizon Optimal Stopping of Time-Discontinuous Functionals with Applications to Impulse Control with Delay*

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Abstract

We study finite horizon optimal stopping problems for continuous time Feller-Markov processes. The functional depends on time, state and external parameters, and may exhibit discontinuities with respect to the time-variable. Both left and right-hand discontinuities are considered. We investigate the dependence of the value function on the parameters, initial state of the process and on the stopping horizon. We construct ε-optimal stopping times and provide conditions under which an optimal stopping time exists. We demonstrate how to approximate this optimal stopping time by solutions to discrete-time problems. Our results are applied to the study of impulse control problems with finite time horizon, decision lag and execution delay.

Keywords: optimal stopping, Feller Markov process, discontinuous functional, impulse control, decision lag, execution delay

1. Introduction

The interest in optimal stopping and impulse control of continuous-time Markov processes has been continually fuelled by applications to such areas as finance, resource management or production scheduling. The theory of optimal stopping has undergone intense development for almost three decades. The mathematical framework was built in seminal papers by Bismut and Skalli [6], El Karoui [11], El Karoui et al. [12], Fakeev [14], and Mertens [18] with extensions in El Karoui et al. [13]. A topic sparking a lot of interest was the regularity of the value function. Bismut [7]

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applied methods of convex analysis. The time-discretization technique, used in the present paper, was explored in Mackevicius [16]. The penalty method, introduced by Stettner and Zabczyk [27], was further extended in [28]. A survey of various results and approaches to optimal stopping for standard Markov processes can be found in [29].

Another strand of literature was devoted to stopping of diffusion processes in which the differential structure of their generators was used to form suitable variational inequalities. A predominant solution technique was pioneered in the classical monograph by Bensoussan and Lions [5], who studied the stopping of non-degenerate diffusions where the cost/reward was described by a continuous function. Generalizations covered degeneracy of the diffusion (Menaldi [17]), removal of the discounting factor and relaxation of many assumptions regarding the functional and the coefficients of the diffusion (see Fleming, Soner [10]). Recently, Lamberton [15] obtained continuity and variational characterization of the value function for stopping of one-dimensional diffusions with bounded and Borel-measurable reward function. Bassan and Ceci studied semi-continuous reward functions and for diffusions and certain jump-diffusions ([1, 2]). They proved that the optimal stopping with a lower/upper semi-continuous reward function yields a lower/upper semi-continuous value function. Under further conditions the existence of optimal stopping times was also shown but without an explicit contruction.

Our paper is rooted in probabilistic methods developed in Mackevicius [16]. At the heart of our interest is the optimal stopping problem

$$v(x, T_1, T_2, b) = \sup_{T_1 \le \tau \le T_2} \mathbb{E}^x \big\{ F(\tau, X(\tau), b) \big\},$$

where $(X(t))$ is a Feller-Markov process and b is a parameter. If F is continuous and bounded it is well known that the value function v is continuous and the optimal stopping time is characterized by the first hitting time of the set on which the value function coincides with F (see [29] and the references therein). This paper studies optimal stopping problems with a time-discontinuous function F which appears naturally in the study of impulse control with decision lag (see Section 5). We demonstrate that certain kinds of discontinuities prevent existence of optimal stopping times, while others, even though the value function is discontinuous, have solutions in a standard Markovian form. We show how the discontinuities in F are transferred to the value function v. The results when F is right-continuous with respect to the time variable are summarized in Theorems 3.1 and 3.10. The left-continuous case can be found in Theorem 4.2 and Corollary 4.3.

Our research complements the papers on variational techniques in two dimensions. Firstly, the (possibly piecewise) continuity of the value function is proved for the class of weakly Feller-Markov processes (this is a wide class of processes comprising, inter alia, Levy processes and diffusions) on locally compact separable spaces therefore providing a universal basis for the search of further smoothness results in far more technical realm of variational inequalities. Secondly, we provide explicit formulas for ε-optimal and optimal stopping times for discontinuous functionals. These results also benefit numerical methods for solution of stopping problems by variational methods by providing detailed estimates on the magnitude of discontinuities, their exact positions and the relation between the value function and optimal stopping rules.

The results of the present paper rely on an approximation of the continuous-time stopping problem with appropriately constructed discrete-time counterparts (see Theorem 3.1). This approach provides an alternative method for numerical computation of the value function. We prove that the optimal stopping time can be approximated by appropriately modified optimal-stopping times for the related discrete-time stopping problems, see the proof of Lemma 3.4. This proof is of its own interest as, to the best of our knowledge, it offers a new method to prove the existence and form of ε-optimal and optimal stopping times even in the well-studied case of continuous and bounded F.

The properties of weak Feller processes that enable our approach are collected in Section 2. We would like to turn reader's attention to Proposition 2.1, which states that the study of weak Feller processes can be limited, with high probability, to compact sets. We also show that the assumptions in the definition of weak Feller processes cannot be relaxed without surrendering properties of the value function and its relation to the optimal stopping time. An example provided at the end of Subsection 3.1 demonstrates that if the semigroup only maps the space of continuous bounded functions into itself, the value function of the stopping problem with a continuous bounded F may not be continuous and the optimal stopping time is not determined by the coincidence of F with the value function.

Main application, as well as the motivation for the research presented in this paper, is the problem of finite-horizon impulse control in the presence of execution delay and decision lag, with many applications in finance and decision-making processes (regulatory delays, delayed data availability, liquidity risk, real-options, see [3, 8]). It appears that the discontinuities of the kind studied in this paper are natural when there is either delay in execution of impulses or decision lag. A simple version of the control problem when the execution delay is equal to the decision lag and the underlying process is a jump-diffusion is solved by Øksendal and Sulem [21]. They transform the problem into a sequence of no-delay optimal stopping problems using variational techniques. Bruder and Pham [8] consider more general controls (the execution delay is a multiplicity of the decision lag) and a diffusion as the underlying. They prove, using variational approach, that there exists a solution and provide a sketch of a numerical algorithm. Different techniques are employed by Bayraktar and Egami [3] who give explicit formulas for optimal strategies if there is no decision lag (the execution of impulses might be delayed) and the set of admissible control strategies is restricted to threshold strategies. Our results, presented in Section 5, are closest in their spirit to [8]. However, in our setting the underlying process is weakly Feller on a locally compact separable state space and no relation between the length of decision lag and execution delay is imposed. We rephrase the problem as a finite system of optimal stopping problems which can be solved explicitly. We prove the existence and the form of an optimal control as well as point out the discontinuities in the value functions of the auxiliary optimal stopping problems. In our opinion, our method has several advantages compared to those used in the aforementioned papers. Firstly, our results hold for general weak Feller processes. Theorem 5.1 can be viewed as a universal tool to assess basic smoothness properties of the value function as well as the existence of optimal strategies. Secondly, our proofs address only the inherent difficulties of the control problem leaving aside the technicalities of the variational approach. This enables us to provide a detailed construction of a strategy and a proof of its optimality. Finally, our system of auxiliary optimal stopping problems can suit as a basis for numerical solution: it can be split into separate stopping problems which, after smoothing (see Theorem 3.5), have representations in the form of variational inequalities as in [8].

The paper is organized as follows. Section 2 collects properties of weakly Feller processes.

They are used to study, in Section 3, stopping problems for functionals with right-continuous dependence on time. Left-continuous functionals are dealt with in Section 4. Impulse control problem is formulated and solved in Section 5.

2. Properties of weak Feller processes

Consider a standard Markov right continuous process $\big(X(t)\big)$ defined on a locally compact separable space E endowed with a metric ρ with respect to which every closed ball is compact. The Borel σ-algebra on E is denoted by $\mathcal{E}$. Let $\mathcal{C}$ be the space of continuous bounded functions $E \to \mathbb{R}$ with the supremum norm. Denote by $\mathcal{C}_0$ its linear subspace comprising functions vanishing at infinity, i.e., functions $f : E \to \mathbb{R}$ such that $\lim_{\|x\| \to \infty} f(x) = 0$.

It is assumed throughout this paper that $\big(X(t)\big)$ satisfies the weak Feller property:

$$P_t\, \mathcal{C}_0 \subseteq \mathcal{C}_0$$

where P_t is the transition semigroup of the process $\big(X(t)\big)$, i.e., $P_t h(x) = \mathbb{E}^{\,x} \big\{h\big(X(t)\big)\big\}$ for any bounded measurable $h : E \to \mathbb{R}$. Right continuity of $\big(X(t)\big)$ and Theorem T1, Chapter XIII in [19] implies that the semigroup P_t satisfies the following uniform continuity property:

$$\forall\, f \in \mathcal{C}_0 \qquad \lim_{t \to 0+} P_t f = f \;\text{ in } \mathcal{C}_0. \tag{1}$$

A class of weak Feller processes consists of numerous stochastic processes commonly used in mathematical practice, as general as non-exploding diffusions, jump-diffusions and Levy processes.

Due to the weak Feller property the study of many optimal stopping problems can be restricted to compact state spaces. Indeed, the following proposition states that the process does not leave a compact ball around its initial point with arbitrarily large probability over a finite time. Let

$$\gamma_T(x, R) = \mathbb{P}^x \left\{\exists\; s \in [0, T] \quad \rho\big(x, X(s)\big) \geq R\right\}. \tag{2}$$

PROPOSITION 2.1 *For any compact set $K \subseteq E$*

$$\sup_{x \in K} \gamma_T(x, R) \to 0 \tag{3}$$

as $R \to \infty$.

Proof. The proof exploits ideas of Proposition 1 and Lemma 1 of [27]. Fix a compact set $K \subseteq E$. The proof consists of two steps:

Step 1. For each $\varepsilon > 0$ there are compact sets $L_1, L_2 \subseteq E$ such that $K \subseteq L_i$, $i = 1, 2$,

$$\inf_{x \in K} P_T 1_{\{L_1\}}(x) \geq 1 - \varepsilon \tag{4}$$

and

$$\sup_{x \notin L_2} \sup_{t \in [0, T]} P_t 1_{\{L_1\}}(x) \leq \varepsilon. \tag{5}$$

To prove this, consider a family of continuous functions g_K^n such that $\|g_K^n\| \leq 1$ ($\|\cdot\|$ stands for the supremum norm), $g_K^n(y) = 1$, for $y \in B(K,n)$, and $g_K^n(y) = 0$, for $y \notin B(K,n+1)$, where $B(K,n) := \{z \in E : \rho(z,K) \leq n\}$. These functions are in $\mathcal{C}_0$ and $g_K^n(x)$ converge pointwise to a constant function equal to 1 as $n \to \infty$. Due to the dominated convergence theorem $P_T g_K^n(x)$ also converges to 1. The sequence $P_T g_K^n$ is non-descreasing hence by Dini's theorem (see [25, Thm 7.13]) $P_T g_K^n$ converges uniformly on compact sets to 1. This implies that there exists n^* such that $P_T g_K^{n^*}(x) \geq 1 - \varepsilon$ for all $x \in K$. This completes the proof of (4) with $L_1 = B(K, n^* + 1)$ since $1_{\{L_1\}}(x) \geq g_K^{n^*}(x)$.

By (1) and the weak Feller property the mapping $(t,x) \mapsto P_t f(x)$ is continuous for any $f \in \mathcal{C}_0$. This implies that $h(x) = \sup_{t \in [0,T]} P_t g_K^{n^*+1}(x)$ is continuous. The proof that $h \in \mathcal{C}_0$ is performed by contradiction. Assume that there exists a sequence (x_n) converging to infinity such that $h(x_i) \geq \delta > 0$ (h is non-negative by definition). Let $(t_i) \subset [0,T]$ be such that $h(x_i) = P_{t_i} g_K^{n^*+1}(x_i)$, $i = 1, 2, \ldots$. Consider a subsequence t_{i_j} converging to some t^*. For large j the following inequality holds:

$$\left| h(x_{i_j}) - P_{t^*} g_K^{n^*+1}(x_{i_j}) \right| = \left| P_{t_{i_j}} g_K^{n^*+1}(x_{i_j}) - P_{t^*} g_K^{n^*+1}(x_{i_j}) \right| \leq \frac{\delta}{2}.$$

On the other hand, $P_{t^*} g_K^{n^*+1}$ is in $\mathcal{C}_0$ (by the definition of weak Feller property), which implies that $\lim_{j \to \infty} P_{t^*} g_K^{n^*+1}(x_{i_j}) = 0$. This is a contradiction of the assumption $h(x_i) \geq \delta$.

Since $h \in \mathcal{C}_0$ there exists $r > 0$ such that $h(x) \leq \varepsilon$ for $x \notin B(K,r)$. This implies that $L_2 = B(K,r)$ satisfies (5) because $g_K^{n^*+1}(x) \geq 1_{\{L_1\}}(x)$.

Step 2. Let $\tau = \inf\big\{s \geq 0 : \rho\big(K, X(s)\big) \geq R\big\}$, where R is such that $L_2 \subset B(K,R)$. For $x \in K$, using (4) and (5) we have

$$\begin{aligned}
1 - \varepsilon &\leq \mathbb{P}^x \{X(T) \in L_1\} \\
&= \mathbb{P}^x \{X(T) \in L_1, \tau \leq T\} + \mathbb{P}^x \{X(T) \in L_1, \tau > T\} \\
&\leq \mathbb{E}^x \left\{ 1_{\tau \leq T} P_{X(\tau)} \{X(T - \tau) \in L_1\} \right\} + \mathbb{P}^x \{\tau > T\} \\
&\leq \varepsilon \mathbb{P}^x \{\tau \leq T\} + \mathbb{P}^x \{\tau > T\} = 1 - \mathbb{P}^x \{\tau \leq T\} (1 - \varepsilon)
\end{aligned}$$

and therefore

$$\mathbb{P}^x \{\tau \leq T\} \leq \frac{\varepsilon}{1 - \varepsilon},$$

which completes the proof. ■

COROLLARY 2.2

i) $P_t \mathcal{C} \subset \mathcal{C}$ *(the Feller property).*

ii) $\lim_{t \to 0} P_t f(x) = f(x)$ *uniformly on compact subsets of* E *for* $f \in \mathcal{C}$.

Proof. Let $f \in \mathcal{C}$ and $K \subseteq E$ be a compact set. By Proposition 2.1 there exists a sequence $r_n \to \infty$ such that

$$\sup_{x \in K} \gamma_t(x, r_n) \leq 2^{-n}.$$

Define continuous functions $g_n : E \to \mathbb{R}$ satisfying the following properties: $0 \le g_n(x) \le 1$, $g_n(x) = 1$ for $x \in B(K, r_n)$, and $g_n(x) = 0$ for $x \notin B(K, r_n + 1)$. Functions $f_n(x) = f(x)g_n(x)$ are in $\mathcal{C}_0$. By the weak Feller property $P_t f_n(x)$ are continuous. The construction of r_n yields

$$\sup_{x \in K} \big| P_t f(x) - P_t f_n(x) \big| \le \frac{\|f\|}{2^n}.$$

Therefore, $P_t f_n$ converges uniformly on K to $P_t f$, which implies that $P_t f$ is continuous on K. Arbitrariness of K yields that $P_t f \in \mathcal{C}$.

To prove (ii) notice that

$$|P_t f(x) - f(x)| \le |P_t f(x) - P_t f_n(x)| + |P_t f_n(x) - f_n(x)| + |f_n(x) - f(x)|,$$

where f_n is defined above. Therefore,

$$\sup_{x \in K} \big| P_t f(x) - f(x) \big| \le \frac{\|f\|}{2^n} + \|P_t f_n - f_n\|.$$

and letting $t \to 0$ and then $n \to \infty$ we complete the proof.

■

In what follows we shall denote by E^B a locally compact space of parameters endowed with the metric ρ_B.

LEMMA 2.3 *Let $u : E \times E^B \to \mathbb{R}$ be a continuous bounded function. Then the mapping*

$$E \times E^B \times [0, \infty) \ni (x, b, d) \mapsto P_d u(x, b),$$

where $P_d u(x, b) = \mathbb{E}^x \{u(X(d), b)\}$, is continuous.

Proof. Take a sequence $(x_k, b_k, d_k) \subseteq E \times E^B \times [0, \infty)$ converging to (x, b, d). Let $K = \{x, x_1, x_2, \ldots\}$, $B = \{b, b_1, b_2, \ldots\}$. By Proposition 2.1 for any $\varepsilon > 0$ there exists a compact set $L \subset E$ such that

$$\sup_{x \in K} \mathbb{P}^x \Big(\exists s \in [0, d+1] \quad X(s) \notin L \Big) < \varepsilon.$$

Define a continuous function $g : E \to [0, 1]$ such that $g(x) = 1$ for $x \in L$, $g(x) = 0$ for $x \notin B(L, 1)$. The function $\bar{u}(x, b) = g(x)u(x, b)$ has a compact support for any fixed $b \in E^B$ and $|P_t u(x, b) - P_t \bar{u}(x, b)| \le \varepsilon \|u\|$ for $(x, b) \in K \times B$. Therefore

$$\begin{aligned} \big| P_d u(x, b) - P_{d_k} u(x_k, b_k) \big| &\le 2\varepsilon \|u\| + \big| P_d \bar{u}(x, b) - P_{d_k} \bar{u}(x_k, b_k) \big| \\ &\le 2\varepsilon \|u\| + \big| P_d \bar{u}(x, b) - P_{d_k} \bar{u}(x_k, b) \big| + \big| P_{d_k} \bar{u}(x_k, b_k) - P_{d_k} \bar{u}(x_k, b) \big|. \end{aligned}$$

The second term converges to 0 by the Feller property (see Corollary 2.2). The third term converges to 0 by uniform continuity of $\bar{u}$ on $E \times B$. ■

The following lemma explores another aspect of continuity of weak Feller processes.

LEMMA 2.4 *([9, Theorem 3.7]) For any compact set $K \subseteq E$ and any $\varepsilon, \delta > 0$ there is $h_0 > 0$ such that*

$$\sup_{0 \le h \le h_0} \sup_{x \in K} \mathbb{P}^x\{X(h) \notin B(x,\delta)\} < \varepsilon.$$

3. Optimal stopping of right-continuous functionals

This section studies optimal stopping problems with the reward function that is right-continuous with respect to time. Notice that this type of discontinuity complies with the right-continuity of weakly Feller processes. The properties of value function are explored and existence of ε-optimal and optimal (if exists) stopping time is proved.

3.1. Optimal stopping of a simple discontinuous functional

Fix $T^* \ge 0$ and let $f, g \in \mathcal{C}([0,T^*] \times E \times E^B)$. Define the functional

$$J(s,T,x,b,\tau) = \mathbb{E}^x\Big\{1_{\{\tau < T-s\}}\, f(s+\tau, X(\tau), b) + 1_{\{\tau \ge T-s\}}\, g(T, X(T-s), b)\Big\}, \tag{6}$$

where $T \in [0,T^*]$, $s \in [0,T]$, $x \in E$, $b \in E^B$ and $\tau \ge 0$. The goal is to maximize the functional over all stopping times τ. Denote by w the corresponding value function:

$$w(s,T,x,b) = \sup_{\tau} J(s,T,x,b,\tau). \tag{7}$$

In the following theorem we study the continuity of w and characterize optimal (if exists) and ε-optimal stopping times. If the functions f and g do not coincide at the time $T-s$ the functional is discontinuous and an optimal stopping time may not exist.

THEOREM 3.1

i) The function w is continuous and bounded on $\big\{(s,T,x,b) \in [0,T^*] \times [0,T^*] \times E \times E^B : s < T\big\}$ (there might be a discontinuity at $s = T$), and

$$\lim_{s \to T-} w(s,T,x,b) = \max\big(f(T,x,b), g(T,x,b)\big)$$

uniformly[1] in $(T,x,b) \in [0,T^*] \times K \times B$, for any compact $K \subseteq E$ and $B \subseteq E^B$.

ii) For each $\varepsilon > 0$ and $s \in [0,T]$ the stopping time

$$\tau_s^\varepsilon = \inf\{t \ge 0 : w(t+s,T,X(t),b) \le F(t+s,X(t),b) + \varepsilon\}, \tag{8}$$

[1] The uniformity of convergence is understood as

$$\lim_{\delta \to 0+} \sup_{x \in K} \sup_{b \in B} \sup_{T \in [\delta,T^*]} \big|w(T-\delta,T,x,b) - \max(f(T,x,b), g(T,x,b))\big| = 0.$$

where

$$F(u,x,b)=\begin{cases} f(u,x,b), & u<T,\\ g(T,x,b), & u=T,\end{cases} \tag{9}$$

is ε-optimal, i.e. $J(s,T,x,b,\tau_s^\varepsilon)\ge w(s,T,x,b)-\varepsilon$,

iii) If $g\ge f$ then the function w is continuous on $\big\{(s,T,x,b)\in[0,T^*]\times[0,T^*]\times E\times E^B:\ s\le T\big\}$ (there is no discontinuity at $s=T$) and the stopping time

$$\tau_s=\inf\{t\ge 0: w(t+s,T,X(t),b)\le F(t+s,X(t),b)\} \tag{10}$$

is optimal for the functional $J(s,T,x,b,\cdot)$. Moreover,

$$\lim_{\varepsilon\to 0^+}\tau_s^\varepsilon=\tau_s.$$

The proof of the above theorem consists of several lemmas. Let $\Delta_n(s,T)=\frac{T-s}{n}$ for $T\in[0,T^*]$ and $s\le T$. Consider the following discretized stopping problem

$$w^n(s,T,x,b)=\sup_{\tau\in\mathcal{T}_{\Delta_n(s,T)}}\mathbb{E}^x\Big\{1_{\tau<T-s}f(\tau+s,X(\tau),b) \\ +1_{\tau\ge T-s}g(T,X(T-s),b)\Big\}, \tag{11}$$

where $\mathcal{T}_{\Delta_n(s,T)}$ is the class of stopping times taking values in the set $\mathcal{H}^n(s,T):=\{0,\Delta_n(s,T),\dots,n\Delta_n(s,T)\}$. The family of stopping problems (w^n) can be decomposed into a sequence of simple maximization problems:

$$w^1(s,T,x,b)=\max\Big(f(s,x,b),P_{T-s}\,g(T,x,b)\Big),$$
$$w^{n+1}(s,T,x,b)=\max\Big(f(s,x,b),P_{\Delta_{n+1}(s,T)}w^n(s+\Delta_{n+1}(s,T),T,x,b)\Big),\quad n=1,2,\dots,$$

where $P_t w(s,T,x,b)=\mathbb{E}^x\big\{w\big(s,T,X(t),b\big)\big\}$. Indeed, $w^n(s+\Delta_{n+1}(s,T),T,x,b)$ is a value function for the problem in which stopping is allowed in the moments

$$\big\{0,\Delta_n\big(s+\Delta_{n+1}(s,T),T\big),\dots,n\Delta_n\big(s+\Delta_{n+1}(s,T),T\big)\big\},$$

which simplifies to $\{0,\Delta_{n+1}(s,T),\dots,n\Delta_{n+1}(s,T)\}$.

Let $\mathcal{D}=\big\{(s,T,x,b)\in[0,T^*]\times[0,T^*]\times E\times E^B:\ s<T\big\}$. Notice that the difference $\overline{\mathcal{D}}\setminus\mathcal{D}$ consists of the points of the form (T,T,x,b). The following lemma explores continuity properties of the value functions w^n and their extensions to $\overline{\mathcal{D}}$. Notice that w^n may be discontinuous at $\overline{\mathcal{D}}\setminus\mathcal{D}$ (take, e.g., $f=1$ and $g=0$).

LEMMA 3.2 *Functions w^n are continuous and bounded on $\mathcal{D}$. Their restrictions to $\mathcal{D}$ have unique continuous extensions $\bar w^n$ to functions on $\overline{\mathcal{D}}$ that satisfy*

$$\bar w^n(T,T,x,b)=\max\big(f(T,x,b),g(T,x,b)\big).$$

Proof. Boundedness of w^n follows directly from the boundedness of the functional. Continuity on $\mathcal{D}$ is proved via induction. The function w^1 is continuous as a maximum of continuous functions. For w^n, $n > 1$, it suffices to show the continuity of $P_{\Delta_{n+1}(s,T)} w^n$ under the assumption that w^n is continuous and bounded on $\mathcal{D}$. This follows from Lemma 2.3.

Consider the following auxiliary maximization problem: for $T \in (0, T^*]$ and $n = 1, 2, \ldots$

$$v^n(s,T,x,b) = \max\big(f(s,x,b), P_{\Delta_n(s,T)} h\big(s+\Delta_n(s,T),T,x,b\big)\big), \quad (s,x,b) \in [0,T]\times E\times E^B, \tag{12}$$

where $h : [0,T^*] \times [0,T^*] \times E \times E^B \to \mathbb{R}$ is a bounded continuous function. Lemma 2.3 implies that $P_{\Delta_n(s,T)} h(s + \Delta_n(s,T), T, x, b)$ converges to $h(T,T,x,b)$ as $s \to T$ uniformly in $n = 1, 2, \ldots$ and $(T,x,b) \in [0,T^*] \times K \times B$ for compact $K \subseteq E$, $B \subseteq E^B$. Uniform convergence of $f(s,x,b)$ to $f(T,x,b)$ for $(T,x,b) \in [0,T^*] \times K \times B$ follows from the uniform continuity of f on compact sets. Finally, we have

$$\lim_{s\to T-} v^n(s,T,x,b) = \max\big(f(T,x,b), h(T,T,x,b)\big) \tag{13}$$

uniformly in $T \in [0,T^*]$, $x \in K$, $b \in B$ and $n = 1, 2, \ldots$.

Existence of the continuous extension $\bar{w}^n$ follows from the following result: for any compact subset $K \subseteq E$, $B \subseteq E^B$

$$\lim_{s\to T-} w^n(s,T,x,b) = \max\big(f(T,x,b), g(T,x,b)\big) \tag{14}$$

uniformly on $(T,x,b) \in [0,T^*] \times K \times B$. The proof of the limit (14) is performed by induction. The value function $w^1(s,T,x,b)$ can be written as the maximization problem (12) with $h(s,T,x,b) = g(T,x,b)$. Hence, $\lim_{s\to T-} w^1(s,T,x,b) = \max\big(f(T,x,b), g(T,x,b)\big)$ uniformly in $(T,x,b) \in [0,T^*] \times K \times B$. Next, assume that the convergence (14) holds for w^n. The value function w^{n+1} on $\mathcal{D}$ has the form (12) with

$$h(s,T,x,b) = \begin{cases} w^n(s,T,x,b), & s < T, \\ f \vee g(T,x,b), & s \geq T. \end{cases}$$

Since w^n satisfies (14), h is continuous. By (13) the limit property (14) is satisfied by w_{n+1}. ■

The following lemma provides an estimate of the approximation error of w by w^n on the set $\mathcal{D}$. The estimate is one-sided as $w^n \leq w$ by construction: it represents the optimization of the same functional but on a restricted set of stopping times. The value functions w and w^n are identical on the set $\overline{\mathcal{D}} \setminus \mathcal{D}$: $w(T,T,x,b) = g(T,x,b) = w^n(T,T,x,b)$.

LEMMA 3.3 *For every compact set $K \subseteq E$, $B \subseteq E^B$ and $\varepsilon > 0$ there exists n_0 such that for $n \geq n_0$*

$$\sup_{x\in K} \sup_{b\in B} \sup_{T\in[0,T^*]} \sup_{s\in[0,T)} \big(w(s,T,x,b) - w^n(s,T,x,b)\big) \leq \varepsilon\big(4 + 11\|f\| + 3\|g\|\big).$$

Proof. By Proposition 2.1 there exists a compact set $L \subseteq E$ such that

$$\sup_{x \in K} \mathbb{P}^x\{X(s) \notin L \text{ for some } s \in [0, T^*]\} < \varepsilon.$$

Functions f, g are uniformly continuous on $[0, T^*] \times L \times B$, so there exists $\delta > 0$ such that

$$\sup_{b \in B} \sup_{x \in L} \sup_{y \in B(x,\delta)} \sup_{t,s \in [0,T^*],\ |t-s| \le \delta} |f(s,x,b) - f(t,y,b)| + |g(s,x,b) - g(t,y,b)| < \varepsilon.$$

By Lemma 2.4 there is $h_0 > 0$ such that

$$\sup_{0 \le h \le h_0} \sup_{x \in L} \mathbb{P}^x\{X(h) \notin B(x,\delta)\} < \varepsilon.$$

Set $n_0 = T^*/(h_0 \wedge \delta)$ so that for $n \ge n_0$ we have $\Delta_n(s,T) \le h_0 \wedge \delta$, which enables us to use the estimates formulated above.

Fix $(T, x, b) \in [0, T^*] \times K \times B$ and $s \in [0, T)$. We have

$$\begin{aligned} &w(s,T,x,b) - \tilde{w}^n(s,T,x,b) \\ &\le \sup_{0 \le \tau \le T-s} \mathbb{E}^x\Big\{1_{\{\tau < T-s\}} f(s+\tau, X(\tau), b) + 1_{\{\tau \ge T-s\}} g(T, X(T-s), b) \\ &\qquad\qquad - 1_{\{\hat\tau < T-s\}} f(s+\hat\tau, X(\hat\tau), b) - 1_{\{\hat\tau \ge T-s\}} f \vee g(T, X(T-s), b)\Big\}, \end{aligned}$$

where $\hat\tau$ is a stopping time derived from τ in the following way:

$$\hat\tau = \inf\{t \in \{0, \Delta_n(s,T), \ldots, n\Delta_n(s,T)\} : t \ge \tau\},$$

and $\tilde{w}^n$ is the value function of an auxiliary discrete optimal stopping problem

$$\tilde{w}^n(s,T,x,b) = \sup_{\tau \in \mathcal{T}_{\Delta_n(s,T)}} \mathbb{E}^x\Big\{1_{\tau < T-s} f(\tau+s, X(\tau), b) + 1_{\tau \ge T-s} f \vee g(T, X(T-s), b)\Big\}.$$

The difference between w and $\tilde{w}^n$ can be bounded in the following way:

$$\begin{aligned} &w(s,T,x,b) - \tilde{w}^n(s,T,x,b) \\ \le\ &\sup_{0 \le \tau \le T-s} \mathbb{E}^x\Big\{1_{\{\tau \le T-s-\Delta_n(s,T)\}} \big(f(s+\tau, X(\tau), b) - f(s+\hat\tau, X(\hat\tau), b)\big)\Big\} \\ &+ \sup_{0 \le \tau \le T-s} \mathbb{E}^x\Big\{1_{\{T-s-\Delta_n(s,T) < \tau < T-s\}} \big(f(s+\tau, X(\tau), b) - f \vee g(T, X(T-s), b)\big)\Big\} \\ &+ \sup_{0 \le \tau \le T-s} \mathbb{E}^x\Big\{1_{\{\tau \ge T-s\}} \big(g(T, X(T-s), b) - f \vee g(T, X(T-s), b)\big)\Big\}. \end{aligned}$$

Assume $n \ge n_0$. Consider the first term. By the strong Markov property of $X(t)$ and the results summarized at the beginning of the proof we have

$$\begin{aligned} &\mathbb{E}^x\Big\{1_{\{\tau \le T-s-\Delta_n(s,T)\}} \big(f(s+\tau, X(\tau), b) - f(s+\hat\tau, X(\hat\tau), b)\big)\Big\} \\ &\quad = \mathbb{E}^x\Big\{1_{\{\tau \le T-s-\Delta_n(s,T)\}} \mathbb{E}^{X(\tau)}\big\{f(s+\tau, X(0), b) - f(s+\hat\tau, X(\hat\tau-\tau), b)\big|\mathcal{F}_\tau\big\}\Big\} \\ &\quad \le 2\|f\|\, \mathbb{P}^x\Big\{\big\{X(\tau) \notin L\big\} \text{ or } \big\{X(\tau) \in L,\ X(\hat\tau) \notin B(X(\tau), \delta)\big\}\Big\} + \varepsilon \le 4\varepsilon\|f\| + \varepsilon. \end{aligned}$$

The second term is dominated by

$$\sup_{0\le\tau\le T-s} \mathbb{E}^x\Big\{1_{\{T-s-\Delta_n(s,T)<\tau<T-s\}}\big(f(s+\tau,X(\tau),b)-f(T,X(T-s),b)\big)\Big\}$$

since $-(f\vee g)\le -f$ and the estimation as above can be used. The third term is non-positive. Consequently $w(s,T,x,b)-\tilde{w}^n(s,T,x,b)\le 2\varepsilon(1+4\|f\|)$.

The next step of the proof is to show the relation between $\tilde{w}^n$ and w^n. Obviously, $\tilde{w}^n$ dominates w^n. The results summarized at the beginning of the proof imply for $y\in L$ and $n\ge n_0$ the following inequalities:

$$\begin{aligned}
&\mathbb{E}^y\big\{f\vee g\big(T,X(\Delta_n(s,T)),b\big)\big\}\le f\vee g(T-\Delta_n(s,T),y,b)+\varepsilon(1+\|f\vee g\|),\\
&\mathbb{E}^y\big\{g\big(T,X(\Delta_n(s,T)),b\big)\big\}\ge g(T-\Delta_n(s,T),y,b)-\varepsilon(1+\|g\|).
\end{aligned}$$

These inequalities drive the following estimates for the value functions $\tilde{w}^1$ and w^1 on $y\in L$:

$$\begin{aligned}
&\tilde{w}^1\big(T-\Delta_n(s,T),T,y,b\big)\le f\vee g\big(T-\Delta_n(s,T),y,b\big)+\varepsilon(1+\|f\vee g\|),\\
&w^1\big(T-\Delta_n(s,T),T,y,b\big)\ge f\vee g\big(T-\Delta_n(s,T),y,b\big)-\varepsilon(1+\|g\|).
\end{aligned}$$

Hence, the difference $(\tilde{w}^1-w^1)\big(T-\Delta_n(s,T),y,b\big)$ is bounded by $2\varepsilon+\varepsilon\|f\|+2\varepsilon\|g\|$ for $y\in L$.

Now we reduce the task of bounding $\tilde{w}^n-w^n$ to the estimation of the difference $\tilde{w}^1-w^1$:

$$\begin{aligned}
&\tilde{w}^n(s,T,x,b)-w^n(s,T,x,b)\\
&\le \sup_{\tau\in\mathcal{T}_{\Delta_n(s,T)}}\mathbb{E}^x\Big\{1_{\tau<T-s-\Delta_n(s,T)}f(\tau+s,X(\tau),b)\\
&\qquad\qquad +1_{\tau\ge T-s-\Delta_n(s,T)}\tilde{w}^1\big(T-\Delta_n(s,T),T,X(T-s-\Delta_n(s,T)),b\big)\\
&\qquad\qquad -1_{\tau<T-s-\Delta_n(s,T)}f(\tau+s,X(\tau),b)\\
&\qquad\qquad -1_{\tau\ge T-s-\Delta_n(s,T)}w^1\big(T-\Delta_n(s,T),T,X(T-s-\Delta_n(s,T)),b\big)\Big\}\\
&\le \mathbb{E}^x\Big\{(\tilde{w}^1-w^1)\big(T-\Delta_n(s,T),T,X(T-s-\Delta_n(s,T)),b\big)\Big\}.
\end{aligned}$$

Inserting the bound for $\tilde{w}^1-w^1$ we obtain

$$\begin{aligned}
&\tilde{w}^n(s,T,x,b)-w^n(s,T,x,b)\\
&\le \|\tilde{w}^1-w^1\|\,\mathbb{P}^x\big\{X(T-s-\Delta_n(s,T))\notin L\big\}\\
&\quad +(2\varepsilon+\varepsilon\|f\|+2\varepsilon\|g\|)\,\mathbb{P}^x\big\{X(T-s-\Delta_n(s,T))\in L\big\}\\
&\le \varepsilon(2\|f\|+\|g\|)+(2\varepsilon+\varepsilon\|f\|+2\varepsilon\|g\|)\le 2\varepsilon+3\varepsilon\|f\|+3\varepsilon\|g\|.
\end{aligned}$$

To complete the proof combine this estimate with the bound for the difference $w-\tilde{w}^n$. ■

Lemma 3.3 implies that w is continuous on $\mathcal{D}$. Since the approximation is uniform in $(s,T,x,b)\in\mathcal{D}\cap[0,T^*]\times[0,T^*]\times K\times B$ for any compact set $K\subseteq E$, $B\subseteq E^B$ we have

$$\lim_{s\to T-} w(s,T,x,b)=\max\big(f(T,x,b),g(T,x,b)\big)$$

uniformly in $x \in K$, $b \in B$ and $T \in [0, T^*]$, which completes the proof of Theorem 3.1(i).

The form of an ε-optimal stopping time is obtained in Lemma 3.4. A general theory cannot be applied because of the discontinuity of the functional. To the best of our knowledge the proof of the optimality of the stopping time presented below is original even in the standard case of continuous functionals.

LEMMA 3.4 *For each $\varepsilon > 0$, $s \in [0, T]$ the stopping time*

$$\tau_s^\varepsilon = \inf\{t \geq 0 : w(s+t, T, X(t), b) \leq F(s+t, X(t), b) + \varepsilon\}$$

is ε-optimal, i.e., $J(s, T, x, \tau_s^\varepsilon, b) \geq w(s, T, x, b) - \varepsilon$.

Proof. Fix $b \in E^B$ and $T \in [0, T^*]$. Consider the discretization (11). Functions w^n satisfy the following supermartingale property:

$$\mathbb{E}^x\big\{w^n(s+t', T, X(t'), b)\big|\mathcal{F}_t\big\} \leq w^n(s+t, T, X(t), b), \qquad t, t' \in \mathcal{H}^n(s, T),\ t \leq t'.$$

Take arbitrary $0 \leq t \leq t' \leq T - s$ and two non-increasing sequences $(t_n), (t'_n)$ converging to t, t' such that $t_n \leq t'_n$ and $t_n, t'_n \in \mathcal{H}^n(s, T)$. The supermartingale property of w^n implies that

$$\mathbb{E}^x\big\{w^n(s+t'_n, T, X(t'_n), b)\big|\mathcal{F}_{t_n}\big\} \leq w^n(s+t_n, T, X(t_n), b).$$

Due to the right-continuity of $t \mapsto X(t)$ and the convergence of w^n to w (see Lemma 3.3) we have $\lim_{n\to\infty} w^n(s+t_n, T, X(t_n), b) = w(s+t, T, X(t), b)$. The right-continuity of the filtration $(\mathcal{F}_t)$ and the dominated convergence theorem imply that

$$\lim_{n\to\infty} \mathbb{E}^x\big\{w^n(s+t'_n, T, X(t'_n), b)\big|\mathcal{F}_{t_n}\big\} = \mathbb{E}^x\big\{w(s+t', T, X(t'), b)\big|\mathcal{F}_t\big\}.$$

Hence $t \mapsto w(s+t, T, X(t), b)$ is a right-continuous supermartingale.

By Proposition 2.1 and Lemma 3.3 for any k there exist a compact set $L^k \subseteq E$ and a positive integer n_k such that

$$\mathbb{P}^x\big(\forall\, t \in [0, T] \quad X(t) \in L^k\big) \geq 1 - \frac{1}{k}, \tag{15}$$

$$w(s, T, y, b) \leq w^{n_k}(s, T, y, b) + \frac{1}{k}, \qquad y \in L^k, \quad s \in [0, T]. \tag{16}$$

The optimal stopping time for w^{n_k} is given by

$$\tau_s^k = \inf\big\{t \in \mathcal{H}^{n_k}(s, T) : \quad w^{n_k}(s+t, T, X(t), b) \leq F(s+t, X(t), b)\big\},$$

where F is defined in (9). Clearly

$$\mathbb{E}^x w^{n_k}(s + \tau_s^k, T, X(\tau_s^k), b) = w^{n_k}(s, T, x, b).$$

Furthermore, we have

$$\begin{aligned}
\mathbb{E}^x\big\{w(s+\tau_s^\varepsilon, T, X(\tau_s^\varepsilon), b)\big\} \\
&= \mathbb{E}^x\Big\{w^{n_k}(s+\tau_s^k, T, X(\tau_s^k), b)\Big\} \\
&\quad + \mathbb{E}^x\Big\{1_{\{\tau_s^\varepsilon \le \tau_s^k\}}\Big(w(s+\tau_s^\varepsilon, T, X(\tau_s^\varepsilon), b) - w^{n_k}(s+\tau_s^k, T, X(\tau_s^k), b)\Big)\Big\} \\
&\quad + \mathbb{E}^x\Big\{1_{\{\tau_s^\varepsilon > \tau_s^k\}}\Big(w(s+\tau_s^\varepsilon, T, X(\tau_s^\varepsilon), b) - w^{n_k}(s+\tau_s^k, T, X(\tau_s^k), b)\Big)\Big\}.
\end{aligned}$$

The first term is equal to $w^{n_k}(s,T,x,b)$. The second term is non-negative since by the supermartingale property of w and the domination of w^{n_k} by w we have

$$\mathbb{E}^x\Big\{1_{\{\tau_s^\varepsilon \le \tau_s^k\}}\, w(s+\tau_s^\varepsilon, T, X(\tau_s^\varepsilon), b)\Big\} \ge \mathbb{E}^x\big\{1_{\{\tau_s^\varepsilon \le \tau_s^k\}}\, w^{n_k}(s+\tau_s^k, T, X(\tau_s^k), b)\big\}.$$

The third term is bounded from below by $-2\|F\|\mathbb{P}^x(\tau_s^\varepsilon > \tau_s^k)$. Inequalities (15) and (16) imply $\mathbb{P}^x(\tau_s^\varepsilon > \tau_s^k) \le 1/k$ for $k \ge 1/\varepsilon$. Consequently,

$$\mathbb{E}^x\big\{w(s+\tau_s^\varepsilon, T, X(\tau_s^\varepsilon), b)\big\} \ge w^{n_k}(s,T,x,b) - \frac{2\|F\|}{k} \ge w(s,T,x,b) - \frac{2\|F\|+1}{k}.$$

Letting $k \to \infty$ we obtain $\mathbb{E}^x\big\{w(s+\tau_s^\varepsilon, T, X(\tau_s^\varepsilon), b)\big\} \ge w(s,T,x,b)$. The converse inequality follows directly from the supermartingale property of w. Therefore

$$\mathbb{E}^x\big\{w(s+\tau_s^\varepsilon, T, X(\tau_s^\varepsilon), b)\big\} = w(s,T,x,b). \tag{17}$$

By the right-continuity of the process $X(t)$, the continuity of $(t,x) \mapsto F(t,x,b)$ for $(t,x) \in [0,T-s) \times E$ and the fact that $w(T,T,x,b) = F(T,x,b)$ we have that

$$w(s+\tau_s^\varepsilon, T, X(\tau_s^\varepsilon), b) \le F(s+\tau_s^\varepsilon, X(\tau_s^\varepsilon), b) + \varepsilon. \tag{18}$$

Taking the expectation of both sides of (18) and using (17) we finally obtain

$$w(s,T,x,b) \le \mathbb{E}^x\big\{F(s+\tau_s^\varepsilon, X(\tau_s^\varepsilon), b)\big\} + \varepsilon. \tag{19}$$

■

Assume now that $g \ge f$. The value function w is continuous on its whole domain $\overline{\mathcal{D}}$ accordingly to the statement (i) of the present theorem. The stopping time τ_s is well-defined. We prove its optimality by showing that it can be approximated by τ_s^ε as $\varepsilon \to 0^+$. First notice that $\tau_s^\varepsilon \le \tau_s$. As the sequence $(\tau_s^\varepsilon)_{\varepsilon>0}$ is non-decreasing as ε decreases to 0 there exists $\tau_s^0 = \lim_{\varepsilon\to 0^+} \tau_s^\varepsilon$ with the property $\tau_s^0 \le \tau_s$. By Theorem 3.13 of [9] the process $X(t)$ is quasi-left continuous, i.e., $X(\tau_s^\varepsilon) \to X(\tau_s^0)$ a.s. Continuity of w and upper semicontinuity of F ($F(u,x,b)$ may have an upward jump as u tends to T) yields, almost surely,

$$\begin{aligned}
&\lim_{\varepsilon\to 0+} w(s+\tau_s^\varepsilon, T, X(\tau_s^\varepsilon), b) = w(s+\tau_s^0, T, X(\tau_s^0), b), \\
&\lim_{\varepsilon\to 0+} F(s+\tau_s^\varepsilon, X(\tau_s^\varepsilon), b) \le F(s+\tau_s^0, X(\tau_s^0), b).
\end{aligned}$$

Therefore, by (18)

$$w(s+\tau_s^0, T, X(\tau_s^0), b) \le F(s+\tau_s^0, X(\tau_s^0), b).$$

Combining this result with the trivially satisfied opposite inequality we obtain

$$w(s+\tau_s^0, T, X(\tau_s^0), b) = F(s+\tau_s^0, X(\tau_s^0), b).$$

Consequently $\tau_s^0 = \tau_s$ a.s. By the dominated convergence theorem applied to (19) we have $w(s,T,x,b) \le \mathbb{E}^x F(s+\tau_s, X(\tau_s), b)$, which proves optimality of τ_s. The proof of Theorem 3.1 is complete.

The proof of Lemma 3.4 offers a numerical approach for computation of ε-optimal stopping times. It shows that an ε-optimal stopping time can be obtained from a solution to an appropriate discrete-time stopping problem.

Under the assumptions of Theorem 3.1(iii) the optimal stopping problem with the discontinuous functional (6) can be transformed into a stopping problem with a continuous functional. Define

$$r(s,T,x,b) = \mathbb{E}^x g\big(T, X(T-s), b\big), \qquad (s,T,x,b) \in [0,T^*] \times [0,T^*] \times E \times E^B, \quad s \le T.$$

Function r is continuous by the Feller property (see Corollary 2.2).

THEOREM 3.5 *Assuming that $g \ge f$, the value function w has the following representation:*

$$w(s,T,x,b) = \sup_{\tau \le T-s} \mathbb{E}^x \Big\{ f\big(s+\tau, X(\tau), b\big) \vee r\big(s+\tau, T, X(\tau), b\big) \Big\}.$$

The optimal stopping time for the above functional,

$$\tau_s^* = \inf\{t \in [0, T-s] : w\big(s+t, T, X(t), b\big) = f\big(s+t, X(t), b\big) \vee r\big(s+t, T, X(t), b\big)\},$$

defines an optimal stopping for the functional (6) by

$$\tau_s' = \begin{cases} \tau_s^*, & w\big(s+\tau_s^*, T, X(\tau_s^*), b\big) = f\big(s+\tau_s^*, X(\tau_s^*), b\big), \\ T-s, & w\big(s+\tau_s^*, T, X(\tau_s^*), b\big) > f\big(s+\tau_s^*, X(\tau_s^*), b\big). \end{cases}$$

Proof. Consider a discrete stopping problem

$$v^n(s,T,x,b) = \sup_{\tau \in \mathcal{T}_{\Delta_n(s,T)}} \mathbb{E}^x \big\{ f(s+\tau, X(\tau), b) \vee r(s+\tau, T, X(\tau), b) \big\}. \tag{20}$$

We shall prove by induction that v^n is identical to w^n introduced in the proof of Theorem 3.1. Noting $g(T,x,b) = r(T,T,x,b) \ge f(T,x,b)$ we have

$$\begin{aligned} v^1(s,T,x,b) &= \max\Big(f(s,x,b) \vee r(s,T,x,b), P_{T-s}(f \vee r(\cdot, T, \cdot, \cdot))(T,x,b) \Big) \\ &= \max\Big(f(s,x,b), r(s,T,x,b), P_{T-s} g(T,x,b) \Big) \\ &= \max\Big(f(s,x,b), P_{T-s} g(T,x,b) \Big) = w^1(s,T,x,b). \end{aligned}$$

Given the inductive assumption $v^n = w^n$ we have:

$$\begin{aligned}
v^{n+1}(s,T,x,b) &= \max\Big(f(s,x,b) \vee r(s,T,x,b), P_{\Delta_{n+1}(s,T)} v^n(s+\Delta_{n+1}(s,T),T,x,b)\Big)\\
&= \max\Big(f(s,x,b), P_{\Delta_{n+1}(s,T)} r(s+\Delta_{n+1}(s,T),T,x,b),\\
&\qquad\qquad P_{\Delta_{n+1}(s,T)} v^n(s+\Delta_{n+1}(s,T),T,x,b)\Big)\\
&= \max\Big(f(s,x,b), P_{\Delta_{n+1}(s,T)} v^n(s+\Delta_{n+1}(s,T),T,x,b)\Big)\\
&= \max\Big(f(s,x,b), P_{\Delta_{n+1}(s,T)} w^n(s+\Delta_{n+1}(s,T),T,x,b)\Big)\\
&= w^{n+1}(s,T,x,b).
\end{aligned}$$

The third equality results from the observation $v^n(s,T,x,b) \geq r(s,T,x,b)$. Lemma 3.3 implies that v^n converges to the value function of the problem

$$(s,T,x,b) \mapsto \sup_{\tau \leq T-s} \mathbb{E}^x \Big\{ f\big(s+\tau, X(\tau), b\big) \vee r\big(s+\tau, T, X(\tau), b\big) \Big\}.$$

Due to $v^n = w^n$, this value function is equal to w.

The optimality of the stopping time τ_s^* follows from Theorem 3.1. Its relation to the optimal stopping time for the functional (6) is evident by the following identity:

$$\mathbb{E}^x \Big\{ f\big(s+\tau_s^*, X(\tau_s^*), b\big) \vee r\big(s+\tau_s^*, T, X(\tau_s^*), b\big) \Big\} = \mathbb{E}^x \Big\{ F\big(s+\tau_s', X(\tau_s'), b\big) \Big\},$$

where F is defined as in the statement of Theorem 3.1. ■

The stopping time τ_s' constructed in Theorem 3.5 might not coincide with τ_s defined in (10). Indeed, take any weak Feller process $X(t)$ and define $f(s,x,b) = 2s \wedge 1$ and $g(s,x,b) = 1$ for any $s \geq 0$. Fix a stopping horizon $T = 1$. Simple computations show that $w(s,1,x,b) = 1$ and $r(s,1,x,b) = 1$. Hence $\tau_0^* = 0$ and $\tau_0' = 1$ whereas $\tau_0 = 0.5$.

Theorem 3.5 may appear at first sight as a shortcut to the proof of Theorem 3.1. The transformation of a discontinuous stopping problem into a continuous one is valid *only* under the assumption that $g \geq f$. If this assumption is not satisfied the relation between ε-optimal stopping times for these two problems is unclear. The transformed problem has an optimal solution while the original one can only be approximated by ε-optimal times. It can be however shown that the value function of the transformed problem is identical on the set $\mathcal{D}$ to the value function of the original one.

Theorem 3.1 implies the following standard result. The methods of proof are different from usually used, especially in the case of ε-optimal strategies.

COROLLARY 3.6 *The value function of a standard optimal stopping problem*

$$w(s,T,x) = \sup_{\tau \leq T-s} \mathbb{E}^x \Big\{ f\big(s+\tau, X(\tau)\big) \Big\}$$

is continuous and bounded for a continuous bounded f. Optimal stopping time is given by $\tau = \inf\{t \geq 0 : w(s+t,T,X(t)) \leq f(s+t,X(t))\}$.

The following example (which is a slight modification of an example from [27]) shows that the assumption $P_t\,\mathcal{C}_0 \subseteq \mathcal{C}_0$ can not be replaced by $P_t\,\mathcal{C} \subseteq \mathcal{C}$.

Example. Let $E = E_0 \cup E_1$, with $E_0 = \big\{(0,1),(0,\frac{1}{2}),\ldots,(0,\frac{1}{n}),\ldots,(0,0)\big\}$, $E_1 = \{(1,0),(2,0),\ldots(n,0),\ldots\}$ with the topology induced by $\mathbb{R}^2$. Define a Markov process in the following fashion. The state $(0,0)$ is absorbing. The process starting from $(0,\frac{1}{n})$, after an independent exponentially distributed time with parameter 1, is shifted to the state $(n,0)$ and then after an independent exponentially distributed time with parameter n^2 is shifted to $(0,\frac{1}{n+1})$. One can check that such a process is Markov with a transition operator P_t satisfying $P_t\,\mathcal{C} \subseteq \mathcal{C}$. Let $f(s,x) = 0$ for $x \in E_0$ and $f(s,x) = 1$ for $x \in E_1$. Then $w(s,T,(0,\frac{1}{n})) = 1 - e^{-(T-s)}$ and $w(s,T,(0,0)) = 0$, which means that the value function is discontinuous in $(0,0)$.

3.2. Constrained optimal stopping of a simple discontinuous functional

Consider the following optimal stopping problem: for $0 \le T_1 \le T_2 \le T^*$

$$\tilde{w}(T_1,T_2,x,b) = \sup_{\tau \ge T_1} \mathbb{E}^{\,x}\Big\{1_{\{\tau<T_2\}}\, f(\tau,X(\tau),b) + 1_{\{\tau \ge T_2\}}\; g(T_2,X(T_2),b)\Big\}. \tag{21}$$

The difference between this problem and the problem studied in Subsection 3.1 lies only in the set of stopping times over which the optimization is performed. In (21) they are bounded from below by T_1 whereas in (7) they are unrestricted. One can expect some similarities in the optimal control strategies and in the properties of the value functions of these stopping problems. This issue is explored in the following proposition:

PROPOSITION 3.7

i) The value function $\tilde{w}$ has the following representation

$$\tilde{w}(T_1,T_2,x,b) = \mathbb{E}^{\,x} w\big(T_1,T_2,X(T_1),b\big),$$

where w is defined in (7)*.*

ii) The function $\tilde{w}$ is continuous and bounded on $\mathcal{D}_1 = \{(T_1,T_2,x,b) \in [0,T^*] \times [0,T^*] \times E \times E^B : T_1 < T_2\}$ and

$$\lim_{T_1 \to T_2-} \tilde{w}(T_1,T_2,x,b) = f \vee g(T_2,x,b)$$

uniformly in $T_2 \in [0,T^*]$ and (x,b) in compact subsets of $E \times E^B$.

iii) An ε-optimal stopping time is given by

$$\tau^\varepsilon = \inf\left\{t \ge T_1 : w(t,T_2,X(t),b) \le F(t,X(t),b) + \varepsilon\right\}, \tag{22}$$

where $F(u,x,b) = f(u,x,b)$ for $u < T_2$ and $F(T_2,x,b) = g(T_2,x)$.

iv) If $g \geq f$ then the function $\tilde{w}$ is continuous on its domain $\overline{\mathcal{D}}_1$ and an optimal stopping time is given by

$$\tau = \inf\{t \geq T_1 : w(t, T_2, X(t), b) \leq F(t, X(t), b)\} \tag{23}$$

with

$$\lim_{\varepsilon \to 0^+} \tau^\varepsilon = \tau.$$

Proof. Fix $T_1 \leq T_2 \leq T^*$. Let $\Delta_n = \frac{T_2 - T_1}{n}$. Consider a discretized stopping problem:

$$\tilde{w}^n(T_1, T_2, x, b) = \sup_{\tau \in \tilde{\mathcal{T}}_{\Delta_n(T_1, T_2)}} \mathbb{E}^x \left\{ 1_{\tau < T_2} f(\tau, X(\tau), b) + 1_{\tau \geq T_2} f \vee g(T_2, X(T_2), b) \right\},$$

where $\tilde{\mathcal{T}}_{\Delta_n(T_1,T_2)}$ denotes the set of all stopping times with values in $\{T_1 + k\Delta_n : k = 0, 1, \ldots, n\}$. The above supremum can be written as

$$\tilde{w}^n(T_1, T_2, x, b) = \mathbb{E}^x w^n\big(T_1, T_2, X(T_1), b\big),$$

where $w^n(s, T, x, b)$ is defined in (11). An argument analogous to the one in the proof of Lemma 3.3 extends this relation to the value functions $\tilde{w}$ and w. Corollary 2.2 implies (ii) and the first part of assertion (iv). The form of optimal stopping times and convergence of τ^ε to τ can be proved identically as in Theorem 3.1. ∎

3.3. Optimal stopping of a functional with multiple discontinuities

The purpose of this subsection is to extend the results of previous sections to functionals with multiple discontinuities. Consider the following parametrized optimal stopping problem

$$w(s, T, x, b) = \sup_{\tau \leq T - s} \mathbb{E}^x F\big(s + \tau, X(\tau), b\big), \qquad (s, T, x, b) \in \Delta \times E \times E^B, \tag{24}$$

where $F : [0, T^*] \times E \times E^B \to \mathbb{R}$ is a bounded function and $\Delta = \{(s, T) \in [0, T^*] \times [0, T^*] : s \leq T\}$. Notice that the role of T is different than in (6): it only limits the set of stopping times over which the optimization is performed and does not affect the functional.

THEOREM 3.8 *Assume for a compact set $B \subseteq E^B$ the function F has the following decomposition:*

$$F(t, x, b) = \sum_{i=0}^{N^*} 1_{\{t \in [t_i(b), t_{i+1}(b))\}} f_i(t, x, b) + 1_{\{t = T^*\}} f_{N^*+1}(T^*, x, b),$$
$$(t, x, b) \in [0, T^*] \times E \times B, \tag{25}$$

where

- *$N^* \geq 0$ is a number depending on B,*

- $t_0, t_1, \dots, t_{N^*+1} : B \to [0, T^*]$ *is a sequence of continuous functions such that* $t_0(b) \le \dots \le t_{N^*+1}(b)$, $t_0 \equiv 0$ *and* $t_{N^*+1} \equiv T^*$,
- $f_0, f_1, \dots, f_{N^*+1} : [0, T^*] \times E \times B \to \mathbb{R}$ *is a sequence of continuous bounded functions such that*
$$f_i(t_{i+1}(b), x, b) \le f_{i+1}(t_{i+1}(b), x, b), \qquad i = 0, \dots, N^*.$$

The value function w *has the following decomposition:*

$$w(s,T,x,b) = \sum_{i=0}^{N^*} 1_{\{T \in [t_i(b), t_{i+1}(b))\}}\, w_i(s,T,x,b) + 1_{\{T = T^*\}}\, w_{N^*+1}(s, T^*, x, b),$$
$$(s,T,x,b) \in \Delta \times E \times B, \quad (26)$$

where $w_0, \dots, w_{N^*+1} : \Delta \times E \times B \to \mathbb{R}$ *are continuous bounded functions. Moreover,*

$$0 \le w(s, t_i(b), x, b) - w(s, t_i(b)-, x, b) \le \sup_{y \in E} \big(F(t_i(b), y, b) - F(t_i(b)-, y, b) \big),$$
$$(x,b) \in E \times B, \quad s < t_i(b), \quad i = 1, \dots, N^* + 1. \quad (27)$$

The assertion (27) can be rewritten in terms of the functions $(f_i)_{i=0,\dots,N^*}$ as follows:

$$0 \le w_{i+1}(s, t_{i+1}(b), x, b) - w_i(s, t_{i+1}(b), x, b) \le \sup_{y \in E} \big(f_{i+1}(t_{i+1}(b), y, b) - f_i(t_{i+1}(b), y, b) \big),$$
$$(x,b) \in E \times B, \quad s < t_{i+1}(b), \quad i = 0, \dots, N^*.$$

Proof of Theorem 3.8. It is sufficient to prove the theorem for $T < T^*$ since the case $T = T^*$ can be easily reduced to the former one by a suitable extension of the time horizon T^* and the functions f_i.

Fix a compact set $B \subseteq E^B$ and the decomposition of F: an integer N^*, functions $t_0, \dots, t_{N^*+1}$ and $f_0, \dots, f_{N^*+1}$. Let $N \in \{0, \dots, N^*\}$. For $b \in B$ and $T \in [t_N(b), t_{N+1}(b))$ the value function w is defined as

$$w(s,T,x,b) = \sup_{\tau \le T-s} \mathbb{E}^x \Big\{ \sum_{i=0}^{N-1} 1_{\{t_i \le s+\tau < t_{i+1}(b)\}}\, f_i\big(s+\tau, X(\tau), b\big)$$
$$+ 1_{\{t_N(b) \le s+\tau \le T\}}\, f_N\big(s+\tau, X(\tau), b\big) \Big\}.$$

Consider a sequence of value functions:

$$u_N(s,T,x,b) = \sup_{\tau \le T-s} \mathbb{E}^x \big\{ f_N\big(s+\tau, X(\tau), b\big) \big\},$$
$$u_i(s,T,x,b) = \sup_{\tau \le t_{i+1}(b)-s} \mathbb{E}^x \Big\{ 1_{\{\tau < t_{i+1}(b)-s\}}\, f_i\big(s+\tau, X(\tau), b\big)$$
$$+ 1_{\{\tau = t_{i+1}(b)-s\}}\, u_{i+1}\big(t_{i+1}(b), T, X(t_{i+1}(b)-s), b\big) \Big\}$$
$$i = 0, \dots, N-1.$$

Domains of functions u_i, denoted by $\mathcal{D}_i$, are as follows:

$$\mathcal{D}_N = \big\{(s,T,x,b) \in \Delta \times E \times B : T \in [t_N(b), t_{N+1}(b)],\ s \in [t_N(b), T]\big\},$$
$$\mathcal{D}_i = \big\{(s,T,x,b) \in \Delta \times E \times B : T \in [t_N(b), t_{N+1}(b)],\ s \in [t_i(b), t_{i+1}(b)]\big\}, \qquad i = 0, \ldots, N-1.$$

For convenience the domains include $T = t_{N+1}(b)$.

By Theorem 3.1 the function u_N is continuous on $\mathcal{D}_N$. Continuity of u_i, $i = N-1, \ldots, 0$ is proved by a backward induction. Assume u_{i+1} is continuous on $\mathcal{D}_{i+1}$. The definition of u_i already has the form (6): the function f_i is continuous and $f_i(t_{i+1}(b), x, b) \le u_{i+1}(t_{i+1}(b), T, x, b)$ as $f_i(t_{i+1}(b), x, b) \le f_{i+1}(t_{i+1}(b), x, b) \le u_{i+1}(t_{i+1}(b), T, x, b)$. Theorem 3.1 implies that u_i is continuous on $\mathcal{D}_i$. The functions u_i and u_{i+1} are identical on $\mathcal{D}_i \cap \mathcal{D}_{i+1}$, so the function

$$v_N(s,T,x,b) = \sum_{i=0}^{N} 1_{\{s \in [t_i(b), t_{i+1}(b)) \cap [0,T]\}}\ u_i(s,T,x,b) \tag{28}$$

is continuous on $\hat{\mathcal{D}} = \big\{(s,T,x,b) \in \Delta \times E \times B : T \in [t_N(b), t_{N+1}(b)]\big\}$. Due to the Bellman principle (it can be proved by discretization as in Theorem 3.1) the function $u_i(s,T,x,b)$ is the value function of the optimal stopping problem starting at s, i.e., for $(s,T,x,b) \in \mathcal{D}_i$, $T < t_{N+1}(b)$

$$u_i(s,T,x,b) = \sup_{\tau \le T-s} \mathbb{E}^x F\big(s+\tau, X(\tau), b\big).$$

Therefore, v_N coincides with w on the set $\hat{\mathcal{D}} \cap \{(s,T,x,b) : T < t_{N+1}(b)\}$. Since $\hat{\mathcal{D}}$ is closed, v_N can be trivially extended as a continuous bounded function to the domain $\Delta \times E \times B$. This extension satisfies all the conditions of the function w_N in the representation (26).

Let $\eta = \sup_{y \in E} \big(F\big(t_{N+1}(b), y, b\big) - F\big(t_{N+1}(b)-, y, b\big)\big)$. We have

$$\begin{aligned}
&w(s, t_{N+1}(b), x, b) \\
&= \sup_{\tau \le t_{N+1}(b)-s} \mathbb{E}^x \bigg\{ \sum_{i=0}^{N} 1_{\{t_i(b) \le s+\tau < t_{i+1}(b)\}}\, f_i\big(s+\tau, X(\tau), b\big) \\
&\qquad\qquad + 1_{\{s+\tau \ge t_{N+1}(b)\}}\, f_{N+1}\big(t_{N+1}(b), X(t_{N+1}), b\big) \bigg\} \\
&\le \sup_{\tau \le t_{N+1}(b)-s} \mathbb{E}^x \bigg\{ \sum_{i=0}^{N} 1_{\{t_i(b) \le s+\tau < t_{i+1}(b)\}}\, f_i\big(s+\tau, X(\tau), b\big) \\
&\qquad\qquad + 1_{\{s+\tau \ge t_{N+1}(b)\}}\, f_N\big(t_{N+1}(b), X(t_{N+1}(b)), b\big) \bigg\} + \eta \\
&= w\big(s, t_{N+1}(b)-, x, b\big) + \eta.
\end{aligned}$$

The last equality follows from the continuity of w_N and its coincidence with w for $T < t_{N+1}(b)$. This implies (27) for $i = N+1$. ■

COROLLARY 3.9 *An optimal stopping time for the problem (24) is given by the formula*

$$\tau_s = \inf\big\{t \geq 0 : \quad \big(s+t, X(t)\big) \in I(T,b)\big\},$$

where

$$I(T,b) = \big\{(t,x) \in [0,T] \times E : \quad w(t,T,x,b) \leq F(t,x,b)\big\}.$$

Proof. Recalling the argument at the beginning of the proof of Theorem 3.8, we can assume that $T < T^*$. Fix b, T and the decomposition (26) of w. By Theorem 3.1 an optimal stopping time is given by $\tau_s = \inf\big\{t \geq 0 : \quad \big(s+t, X(t)\big) \in \tilde{I}(T,b)\big\}$ with the stopping region

$$\tilde{I}(T,b) = \bigcup_{i=0,\dots,N^*} \big\{(t,x) \in [t_i(b), t_{i+1}(b)) \times E : \quad u_i(t,T,x,b) \leq F(t,x,b)\big\}.$$

Functions $(u_i)_{i=0,\dots,N^*}$ are defined in the proof of Theorem 3.8. Due to (28), the set $\tilde{I}(T,b)$ coincides with $I(T,b)$. ■

3.4. Constrained parametrized optimal stopping with an integral term

The setting of the preceding section is extended to functionals with an integral term. Let $F : [0,T^*] \times E \times E^B \to \mathbb{R}$ be a bounded function and $f : [0,T^*] \times E \times E^B \to \mathbb{R}$ be a continuous bounded function. Consider the following optimal stopping problem:

$$w(T_1,T_2,x,b) = \sup_{T_1 \leq \tau \leq T_2} \mathbb{E}^x \bigg\{ \int_0^\tau f\big(s, X(s), b\big) ds + F\big(\tau, X(\tau), b\big) \bigg\}, \qquad (T_1,T_2,x,b) \in \Delta \times E \times E^B, \quad (29)$$

where Δ is the set of admissible time constraints, i.e $\Delta = \{(T_1,T_2) \in [0,T^*]^2 : T_1 \leq T_2\}$.

THEOREM 3.10 *Assume for a compact set $B \subseteq E^B$ the function F has the decomposition* (25)*. The value function w can be written as*

$$w(T_1,T_2,x,b) = \sum_{i=0}^{N^*} 1_{\{T_2 \in [t_i(b), t_{i+1}(b))\}}\, w_i(T_1,T_2,x,b) + 1_{\{T_2 = T^*\}}\, w_{N^*+1}(T_1,T^*,x,b), \qquad (T_1,T_2,x,b) \in \Delta \times E \times B, \quad (30)$$

where $w_0, \dots, w_{N^*+1} : \Delta \times E \times B \to \mathbb{R}$ are continuous bounded functions. The discontinuities of w are bounded as follows:

$$0 \leq w(T_1, t_i(b), x, b) - w(T_1, t_i(b)-, x, b) \leq \sup_{y \in E} \big(F(t_i(b), y, b) - F(t_i(b)-, y, b)\big), \qquad T_1 < t_i(b), \quad (x,b) \in E \times B,$$

for $i = 1, \dots, N^*+1$. Moreover, there exists an optimal stopping time for every x, T_1, T_2 and b.

Proof. Notice that for a stopping time τ

$$\mathbb{E}^x\Big\{\int_0^\tau f(s,X(s),b)ds\Big\} = H(0,x,b) - \mathbb{E}^x\big\{H(\tau,X(\tau),b)\big\},$$

where

$$H(t,x,b) = \mathbb{E}^x\Big\{\int_0^{T^*-t} f(t+s,X(s),b)ds\Big\}, \qquad t\in[0,T^*], \quad x\in E, \quad b\in E^B.$$

Due to Corollary 2.2 the function H is continuous and bounded.

Above observation drives the following reformulation of the functional (29):

$$w(T_1,T_2,x,b) = H(0,x,b) + \sup_{T_1\le\tau\le T_2}\mathbb{E}^x\Big\{-H\big(\tau,X(\tau),b\big) + F\big(\tau,X(\tau),b\big)\Big\}.$$

Assertions of the present theorem follow from Theorem 3.8.

■

COROLLARY 3.11 *An optimal stopping time is given by*

$$\tau = \inf\big\{t\ge T_1: \quad \tilde{w}(t,T_2,X(t),b)\le \tilde{F}(t,X(t),b)\big\},$$

where

$$\tilde{w}(s,T,x,b) = \sup_{\tau\le T-s}\mathbb{E}^x\tilde{F}(s+\tau,X(\tau),b)$$

with $\tilde{F}(t,x,b) = F(t,x,b) - H(t,x,b)$.

Proof. Analogous to the proof of Corollary 3.9. ■

4. Optimal stopping of left-continuous functionals

This section explores properties of value functions of optimal stopping problems with left-continuous reward functions. The main difficulty arising here stems from the fact that the functional is itself left-continuous whereas the process $X(t)$ is right-continuous. It prevents the application of the most natural discretization technique as in the previous section. The problem, however, can be reformulated in a way that permits the use of the results for right-continuous functionals.

Consider a parametrized optimal stopping problem

$$\begin{aligned} w(s,T,x,b) = \sup_{\tau\le T-s}\mathbb{E}^x\Big\{&1_{\{\tau\le t_1(b)-s\}}\, f_1\big(s+\tau,X(\tau),b\big) \\ &+ 1_{\{\tau>t_1(b)-s\}}\, v\big(t_1(b)\vee s,T,X((t_1(b)-s)\vee 0),b\big)\Big\}, \end{aligned}$$

where $t_1 : E^B \to \mathbb{R}$ is continuous, $f_1 : [0, T^*] \times E \times E^B \to \mathbb{R}$ and $v : \Delta \times E \times E^B \to \mathbb{R}$ are continuous and bounded. Notice the peculiarity of the functional. The function v is evaluated at a fixed time $(t_1(b) - s) \vee 0$ in contrast to the standard policy of the evaluation at τ. This construction is motivated by the presumption that v is the value function of a stopping problem and the evaluation at $(t_1(b) - s) \vee 0$ is optimal.

LEMMA 4.1 *Assume $f_1(t_1(b), x, b) \geq v(t_1(b), t_1(b), x, b)$. The value function has the decomposition*

$$w(s, T, x, b) = 1_{\{s \leq t_1(b)\}}\, w_1\big(s, T, x, b\big) + 1_{\{s > t_1(b)\}}\, v\big(s, T, x, b\big) \tag{31}$$

for a continuous bounded function $w_1 : \Delta \times E \times E^B \to \mathbb{R}$ satisfying

$$0 \leq w_1\big(t_1(b), T, x, b\big) - v\big(t_1(b), T, x, b\big) \leq \big(f_1(t_1(b), x, b) - v(t_1(b), T, x, b)\big) \vee 0. \tag{32}$$

An optimal stopping time is $\tau_s = 0$ for $s > t_1(b)$ and

$$\tau_s = \inf\big\{t \in [0, t_1(b) - s] : w(s + t, T, X(t), b) \leq f_1(s + t, X(t), b)\big\} \wedge (T - s), \tag{33}$$

for $s \leq t_1(b)$ with the convention $\inf \emptyset = \infty$.

Proof. For $s \leq t_1(b)$ define the following auxiliary value function

$$\begin{aligned} \tilde{w}(s, T, x, b) = \sup_{\tau \leq T - s} \mathbb{E}^x \Big\{ & 1_{\{\tau < t_1(b)\}}\, f_1\big(s + \tau, X(\tau), b\big) \\ & + 1_{\{\tau \geq t_1(b)\}}\, f_1\big(t_1(b), X(t_1(b) - s), b\big) \vee v\big(t_1(b), T, X(t_1(b) - s), b\big) \Big\}. \end{aligned}$$

This value function dominates w. Theorem 3.8 implies the value function $\tilde{w}$ has the form

$$\tilde{w}(s, T, x, b) = 1_{\{T < t_1(b)\}}\, \tilde{w}_1(s, T, x, b) + 1_{\{T \geq t_1(b)\}}\, \tilde{w}_2(s, T, x, b),$$

for continuous bounded $\tilde{w}_1, \tilde{w}_2$ and there exists an optimal stopping time $\tilde{\tau}_s$ given by

$$\tilde{\tau}_s = \inf\big\{t \in [0, T - s] : \tilde{w}(s + t, T, X(t), b) \leq \tilde{F}(s + t, T, X(t), b)\big\}$$

for $\tilde{F}(t, T, x, b) = 1_{\{t < t_1(b)\}}\, f_1(t, x, b) + 1_{\{t \geq t_1(b)\}}\, f_1(t_1(b), x, b) \vee v(t_1(b), T, x, b)$ (to be absolutely precise in the application of Theorem 3.8 the variable T has to be doubled: as a terminal time for stopping and as an additional parameter due to its appearance in v). By (27) we have

$$\begin{aligned} 0 \leq \tilde{w}_2(s, t_1(b), x, b) - \tilde{w}_1(s, t_1(b), x, b) & \\ \leq \sup_{y \in E} \big\{ \big(v(t_1(b), t_1(b), y, b) - f_1(t_1(b), x, b)\big)\big\} \vee 0 = 0, & \end{aligned}$$

where the last equality results from the assumption $v(t_1(b), t_1(b), y, b) \leq f_1(t_1(b), x, b)$. This implies $\tilde{w}$ is continuous.

For fixed T, x, b and $s \leq t_1(b)$ define a stopping time

$$\tau_s = \begin{cases} \tilde{\tau}_s, & \tilde{\tau}_s < t_1(b) - s, \\ t_1(b) - s, & \tilde{\tau}_s \geq t_1(b) - s \quad \text{and} \\ & \qquad f_1\big(t_1(b), X(t_1(b)-s), b\big) \geq v\big(t_1(b), T, X(t_1(b)-s), b\big), \\ T - s, & \tilde{\tau}_s \geq t_1(b) - s \quad \text{and} \\ & \qquad f_1\big(t_1(b), X(t_1(b)-s), b\big) < v\big(t_1(b), T, X(t_1(b)-s), b\big). \end{cases}$$

This stopping time is identical to the one defined in (33). It is also optimal for w as it is shown below. First notice

$$w(s,T,x,b) \geq \mathbb{E}^x\Big\{1_{\{\tau_s \leq t_1(b)\}}\, f_1\big(s+\tau_s, X(\tau_s), b\big) + 1_{\{\tau_s > t_1(b)\}}\, v\big(t_1(b), T, X(t_1(b)-s), b\big)\Big\}.$$

For $t_1(b) \neq T$ the right-hand side equals to $\tilde{w}(s,T,x,b)$. The assumption $f_1(t_1(b),x,b) \geq v(t_1(b), t_1(b), x, b)$ extends this result to $T = t_1(b)$. Therefore, $\tilde{w}(s,T,x,b)$ coincides with $w(s,T,x,b)$ for $s \leq t_1(b)$ and we put $w_1 = \tilde{w}$.

Above arguments do not hold for $s > t_1(b)$ as there might be a strict inequality between $\tilde{w}$ and w. However, the stopping problem becomes trivial since $w(s,T,x,b) = v(s,T,x,b)$ on $s > t_1(b)$. An optimal stopping time is $\tau_s = 0$.

Inequalities (32) follow from the following indentity:

$$w_1\big(t_1(b), T, x, b\big) = \tilde{w}_1\big(t_1(b), T, x, b\big) = f_1\big(t_1(b), x, b\big) \vee v\big(t_1(b), T, x, b\big).$$

■

Notice that the optimal stopping time for w cannot be written in the standard form

$$\inf\{t \in [0, T-s] : \quad w\big(s+t, T, X(t), b\big) \leq F\big(s+t, T, X(t), b\big)\},$$

where

$$F(t,T,x,b) = 1_{\{t \leq t_1(b)\}}\, f_1\big(t,x,b\big) + 1_{\{t > t_1(b)\}}\, v\big(t,T,x,b\big),$$

because the process $t \mapsto F(s+t, T, X(t), b)$ might not be right-continuous.

Now we turn our attention towards a parametrized optimal stopping problem with multiple discontinuities

$$w(s,T,x,b) = \sup_{\tau \leq T-s} \mathbb{E}^x F\big(s+\tau, X(\tau), b\big), \qquad (s,T,x,b) \in \Delta \times E \times B, \tag{34}$$

where $F : [0, T^*] \times E \times E^B \to \mathbb{R}$ is a bounded function.

THEOREM 4.2 *Assume that for a compact set $B \subseteq E^B$ the function F has the following decomposition:*

$$F(t,x,b) = 1_{\{t=0\}}\, f_0(0,x,b) + \sum_{i=1}^{N^*+1} 1_{\{t \in (t_{i-1}(b), t_i(b)]\}}\, f_i(t,x,b), \qquad (t,x,b) \in [0,T^*] \times E \times B, \tag{35}$$

where

- $N^* \geq 0$ *is a number depending on* B,
- $t_0, t_1, \dots, t_{N^*+1} : B \to [0, T^*]$ *is a sequence of continuous functions such that* $t_0(b) \leq \dots \leq t_{N^*+1}(b)$, $t_0 \equiv 0$ *and* $t_{N^*+1} \equiv T^*$,
- $f_0, f_1, \dots, f_{N^*+1} : [0, T^*] \times E \times B \to \mathbb{R}$ *is a sequence of continuous bounded functions such that*
$$f_i(t_i(b), x, b) \geq f_{i+1}(t_i(b), x, b), \qquad i = 0, \dots, N^*. \tag{36}$$

The value function w *has the following decomposition:*

$$w(s, T, x, b) = 1_{\{s=0\}}\, w_0(0, T, x, b) + \sum_{i=1}^{N^*+1} 1_{\{s \in (t_{i-1}(b), t_i(b)]\}}\, w_i(s, T, x, b),$$
$$(s, T, x, b) \in \Delta \times E \times B,$$

where $w_0, \dots, w_{N^*+1} : \Delta \times E \times B \to \mathbb{R}$ *are continuous bounded functions. Moreover,*

$$0 \leq w_i(t_i(b), T, x, b) - w_{i+1}(t_i(b), T, x, b) \leq f_i(t_i(b), x, b) - f_{i+1}(t_i(b), x, b),$$
$$T \geq t_i(b), \quad (x, b) \in E \times B, \quad i = 0, \dots, N^*. \tag{37}$$

An optimal stopping time is given by the formula

$$\tau_s = \inf\{t \in [0, T - s] : \quad w\big(s + t, T, X(t), b\big) \leq F\big(s + t, X(t), b\big)\}. \tag{38}$$

The last assertion of the theorem provides a relation between jumps of the functions w and F. It can be rewritten as

$$0 \leq w(t_i(b), T, x, b) - w(t_i(b)+, T, x, b) \leq F(t_i(b), y, b) - F(t_i(b)+, y, b),$$
$$i = 0, \dots, N^*, \quad (x, b) \in E \times B.$$

This implies that the stopping time τ_s defined in the theorem is well-defined and the infimum is attained.
Proof. Fix a compact set $B \subseteq E^B$ and a decomposition (35). Define the value function

$$v^{N^*+1}(s, T, x, b) = \sup_{\tau \leq T - s} \mathbb{E}^x\Big\{f_{N^*+1}(s + \tau, X(\tau), b)\Big\},$$

and, for $i = N^*, \dots, 0$,

$$v^i(s, T, x, b) = \sup_{\tau \leq T - s} \mathbb{E}^x\Big\{1_{\{s+\tau \leq t_i(b)\}}\, f_i(s + \tau, X(\tau), b)$$
$$+ 1_{\{s+\tau > t_i(b)\}}\, v^{i+1}(t_i(b) \vee s, T, X((t_i(b) - s) \vee 0), b)\Big\}.$$

By Theorem 3.1 the function v^{N^*+1} is continuous, bounded and coincides with w for $s > t_{N^*}(b)$.
Consider the following inductive hypotheses for v^i, $0 \in \{1, \dots, N^*\}$:

i) v^i coincides with w on the set $\mathcal{D}^i = \{(s,T,x,b) \in \Delta \times E \times B : s > t_{i-1}(b)\}$, where $t_{-1} \equiv -\infty$,

ii) v^i has a decomposition

$$v^i(s,T,x,b) = 1_{\{s \le t_i(b)\}}\, v_1^i(s,T,x,b) + 1_{\{s > t_i(b)\}}\, v^{i+1}(s,T,x,b) \tag{39}$$

for a continuous bounded function $v_1^i : \Delta \times E \times B \to \mathbb{R}$.

First, we show that these hypotheses are satisfied for v^{N^*}. By the definition of v^{N^*+1} and inequalities (36) we have

$$v^{N^*+1}\big(t_{N^*}(b), t_{N^*}(b), x, b\big) = f_{N^*+1}\big(t_{N^*}(b), x, b\big) \le f_{N^*}\big(t_{N^*}(b), x, b\big).$$

The assumptions of Lemma 4.1 are satisfied and v^{N^*} has the following representation:

$$v^{N^*}(s,T,x,b) = 1_{\{s \le t_{N^*}(b)\}}\, v_1^{N^*}(s,T,x,b) + 1_{\{s > t_{N^*}(b)\}}\, v^{N^*+1}(s,T,x,b)$$

for a continuous bounded function $v_1^{N^*} : \Delta \times E \times B \to \mathbb{R}$. Due to the Bellman principle v^{N^*} coincides with w on the set $\{(s,T,x,b) \in \Delta \times E \times B : s > t_{N^*-1}(b)\}$.

Assume the inductive hypotheses for v^{i+1}. Define an auxiliary stopping problem

$$\begin{aligned}\tilde{v}^i(s,T,x,b) = \sup_{\tau \le T-s} \mathbb{E}^x \Big\{ & 1_{\{s+\tau \le t_i(b)\}}\, f_i(s+\tau, X(\tau), b) \\ & + 1_{\{s+\tau > t_i(b)\}}\, v_1^{i+1}(t_i(b) \vee s, T, X((t_i(b)-s) \vee 0), b) \Big\},\end{aligned}$$

where v_1^{i+1} is the function from decomposition (39). We infer from the definition of v^{i+1} and the inequalities (36) that

$$v_1^{i+1}\big(t_i(b), t_i(b), x, b\big) = f_{i+1}\big(t_i(b), x, b\big) \le f_i\big(t_i(b), x, b\big).$$

Lemma 4.1 implies $\tilde{v}^i$ can be written as:

$$\tilde{v}^i(s,T,x,b) = 1_{\{s \le t_i(b)\}}\, v_1^i(s,T,x,b) + 1_{\{s > t_i(b)\}}\, v_1^{i+1}(s,T,x,b)$$

for a continuous bounded function $v_1^i : \Delta \times E \times B \to \mathbb{R}$. The value function $\tilde{v}^i$ coincides with v^i for $s \le t_{i+1}(b)$. By the Bellman principle the function v^i agrees with w on the set $\{(s,T,x,b) \in \Delta \times E \times B : s > t_{i-1}(b)\}$. This completes the proof of the hypotheses (i)-(ii) for v^i.

Put $w_{N^*+1} = v^{N^*+1}$ and $w_i = v_1^i$ for $i = 0, \ldots, N^*$. This definition is justified by conditions (i)-(ii), for $i \le N^*$ and by the construction of v^{N^*+1}.

Inequalities (37) follow from (32) in Lemma 4.1:

$$\begin{aligned} w_i\big(t_i(b), T, x, b\big) - w_{i+1}\big(t_i(b), T, x, b\big) &= v_1^i\big(t_i(b), T, x, b\big) - v_1^{i+1}\big(t_i(b), T, x, b\big) \\ &\le \Big(f_i\big(t_i(b), x, b\big) - v_1^{i+1}\big(t_i(b), T, x, b\big)\Big) \vee 0 \le f_i\big(t_i(b), x, b\big) - f_{i+1}\big(t_i(b), x, b\big). \end{aligned}$$

An optimal stopping time can be extracted from optimal stopping times τ_s^i for the partial value functions $v^0, \ldots, v^{N^*+1}$. Fix s, T, x, b and let i be such that $s \in (t_{i-1}(b), t_i(b)]$, with the convention $t_{-1} \equiv -\infty$. The procedure is as follows. If $\tau_s^i \le t_i(b)$ it is optimal to stop at τ_s^i. Otherwise, the control is handed over to the level $i+1$. Since f_i dominates f_{i+1} at $t = t_i(b)$ the inequality $\tau_s^i > t_i(b)$ implies $\tau_{t_i(b)}^{i+1} > t_i(b)$. Again, it is optimal to stop at $\tau_{t_i(b)}^{i+1}$ if $\tau_{t_i(b)}^{i+1} \le t_{i+1}(b)$, and to continue to the level $i+2$ if $\tau_{t_i(b)}^{i+1} > t_{i+1}(b)$. This routine is repeated until the terminal time T is reached.

Thanks to the representation (33) of the stopping times $\tau_\cdot^k$, $k = i, \ldots, N^*$, the stopping time offered by the above procedure can be written as

$$
\begin{aligned}
&\inf\{t \in [s, t_i(b) \wedge T] : \quad w\big(t, T, X(t-s), b\big) \le f_i\big(t, X(t-s), b\big)\} \\
&\wedge \inf\{t \in (t_i(b) \wedge T, t_{i+1}(b) \wedge T] : \quad w\big(t, T, X(t-s), b\big) \le f_{i+1}\big(t, X(t-s), b\big)\} \\
&\ldots \\
&\wedge \inf\{t \in (t_{N^*}(b) \wedge T, t_{N^*+1}(b) \wedge T] : \quad w\big(t, T, X(t-s), b\big) \le f_{N^*+1}\big(t, X(t-s), b\big)\}
\end{aligned}
$$

with the convention $\inf \emptyset = \infty$. Above expression simplifies to the formula (38) in the statement of the theorem. ■

An analogous argument as in the proof of the Theorem 3.10 extends the above result to functionals with an integral term and a restricted stopping region:

COROLLARY 4.3 *Assume $F : [0, T^*] \times E \times E^B \to \mathbb{R}$ is a bounded function and $f : [0, T^*] \times E \times E^B \to \mathbb{R}$ is a continuous bounded function. Consider an optimal stopping problem:*

$$
w(T_1, T_2, x, b) = \sup_{T_1 \le \tau \le T_2} \mathbb{E}^x \bigg\{ \int_0^\tau f\big(s, X(s), b\big) ds + F\big(\tau, X(\tau), b\big) \bigg\},
\qquad (T_1, T_2, x, b) \in \Delta \times E \times E^B. \quad (40)
$$

If F has the decomposition (35) *for a compact set $B \subseteq E^B$, the value function w can be decomposed as follows:*

$$
w(T_1, T_2, x, b) = 1_{\{T_1 = 0\}}\, w_0(0, T_2, x, b) + \sum_{i=1}^{N^*+1} 1_{\{T_1 \in (t_{i-1}(b), t_i(b)]\}}\, w_i(T_1, T_2, x, b),
\qquad (T_1, T_2, x, b) \in \Delta \times E \times B, \quad (41)
$$

where $w_0, \ldots, w_{N^*+1} : \Delta \times E \times B \to \mathbb{R}$ are continuous bounded functions. The discontinuities of w are bounded as follows:

$$
0 \le w(t_i(b), T_2, x, b) - w(t_i(b)+, T_2, x, b) \le F(t_i(b), x, b) - F(t_i(b)+, x, b),
\qquad T_2 > t_i(b), \quad (x, b) \in E \times B,
$$

for $i = 0, \ldots, N^* + 1$. Moreover, there exists an optimal stopping time for every x, T_1, T_2 and b.

5. Impulse control with decision lag and execution delay

The theory of optimal stopping of left-continuous functionals is nicely illustrated by its application to impulse control problems. As in the previous sections a Markov process $\big(X(t)\big)$ is defined on a locally compact separable metric space $(E, \mathcal{E})$ and satisfies the weak Feller property. Now it is controlled using impulses. Impulse strategy is a sequence of pairs (τ_i, ξ_i), where (τ_i) are stopping times with respect to the history $(\mathcal{F}_t)$ and variables ξ_i are $\mathcal{F}_{\tau_i}$-measurable. The pair (τ_i, ξ_i) is interpreted in the following way: at the moment $\tau_i + \Delta$ the process X_t is shifted to the state given by $\Gamma\big(X_-(\tau_i + \Delta), \xi_i\big)$, where $X_-(\tau_i + \Delta)$ represents the state of the process strictly before the exercise of the impulse (the process does not have to be left-continuous so this value may not coincide with the left-hand limit of the controlled process). A deterministic $\Delta \geq 0$ imposes a delay in the execution of the impulse. We write $\Pi = \big((\tau_1, \xi_1), (\tau_2, \xi_2), \dots\big)$ and denote such controlled process by $\big(X^\Pi(t)\big)$. Notice that the filtration $(\mathcal{F}_t)$ depends on the control and on the initial state of the process $(X^\Pi(t))$.

There are two time points related to an impulse (τ_i, ξ_i). At τ_i, called the *ordering time*, a decision is made upon the action ξ_i. It is then *executed* at time $\tau_i + \Delta$. This naming convention will be used throughout this section.

Let $h \geq 0$ and Θ be a compact set of actions. The set of admissible controls $\mathcal{A}(x)$ consists of impulse strategies $\Pi = \big((\tau_1, \xi_1), (\tau_2, \xi_2), \dots\big)$ such that $\tau_{i+1} \geq \tau_i + h$ and $\xi_i \in \Theta$. Value $h \geq 0$ has the meaning of a decision lag, i.e., it is the minimal time gap separating ordering times. If a new impulse (τ_i, ξ_i) is ordered at the moment when a pending impulse (τ_k, ξ_k) is scheduled to be executed, i.e., when $\tau_i = \tau_k + \Delta$, the decision about ξ_i is made *after* the shift of X^Π determined by ξ_k.

A mathematically precise construction of the probability space on which the controlled process is defined can be found in a seminal paper by Robin [23] and his thesis [24]. Let $\big(X_k^\Pi(t)\big)$ be a sequence of processes defined inductively in the following way:

$$
\begin{aligned}
X_0^\Pi(t) &= X(t), \quad t \geq 0, \\
X_{i+1}^\Pi(t) &= \begin{cases} X_i^\Pi(t), & t \leq \tau_{i+1} + \Delta, \\ X^{\Gamma\left(X_i^\Pi(\tau_{i+1}+\Delta), \xi_{i+1}\right)}(t - \tau_{i+1} - \Delta), & t > \tau_{i+1} + \Delta, \end{cases}
\end{aligned}
$$

where $\big(X^\mu(t)\big)$ denotes a process starting from an initial distribution μ. Intuitive meaning of $X_i^\Pi(t)$ is that of a process controlled by first i impulses, i.e. by $\Pi_i = \big((\tau_1, \pi_1), \dots, (\tau_i, \xi_i)\big)$. The controlled process $X^\Pi(t)$ can be composed of the segments $X_i^\Pi(t)$ in the following way:

$$
X^\Pi(t) = 1_{\{t \leq \tau_1\}} X_0^\Pi(t) + \sum_{i=1}^\infty 1_{\{\tau_i < t \leq \tau_{i+1}\}} X_i^\Pi(t). \tag{42}
$$

Consider an optimal control problem with a finite horizon $T > 0$ and a functional given by

$$J(x, \Pi, T) = \mathbb{E}^x \Big\{ \int_0^T e^{-\alpha s} f\big(t, X^\Pi(t)\big) dt + e^{-\alpha T} g\big(X^\Pi(T)\big) \\ + \sum_{i=1}^{\infty} 1_{\{\tau_i + \Delta \le T\}} e^{-\alpha(\tau_i+\Delta)} c\big(X^\Pi_{i-1}(\tau_i + \Delta), \xi_i\big) \Big\}, \quad (43)$$

where $\alpha \geq 0$ is a discount factor, f measures a running reward (cost), g is a terminal reward (cost) and c is the cost for impulses. Although the probability measure with respect to which the expectation in (43) is computed depends on the control Π we omit this dependence in the notation.

Our goal is to find the value function

$$v(x) = \sup_{\Pi \in \mathcal{A}(x)} J(x, \Pi, T)$$

and an admissible strategy $\Pi^* \in \mathcal{A}(x)$ for which the supremum is attained. Such Π^* is called an *optimal strategy*.

We make the following standing assumptions:

(A1) Functions $c : E \times \Theta \to \mathbb{R}$, $f : [0, T] \times E \to \mathbb{R}$ and $g : E \to \mathbb{R}$ are continuous and bounded.

(A2) The function $\Gamma : E \times \Theta \to E$ is continuous.

The main result of this section is summarized in the theorem below.

THEOREM 5.1 *Assume (A1)-(A2) and $h > 0$. Then the value function v is continuous and bounded and for every $x \in E$ there exists an optimal strategy.*

Theorem 5.1 generalizes and complements several existing results on optimal control with and without delay [1, 2, 8, 20, 21]. Its formulation, suggesting a standard approach in solving optimal control problems, is misleading. The controlled process (X^Π_t) is no longer Markovian due to the accumulation of pending impulses. An approach, suggested in [8], leads via a system of optimal stopping problems of Markovian type. Our solution is influenced by this idea, but differs from [8] in many points. Our setting is much more general as we only assume the underlying process to be defined on a locally compact separable state space and to satisfy the weak Feller property. Our proofs benefit from the discretization techniques which do not rely on a convenient form of the infinitesimal generator of the underlying Markov process. In contrast, existing results employ formulations via partial differential equations and are often limited by technical assumptions arising from the theory of PDEs.

In Subsection 5.1 we develop a system of optimal stopping problems possessing certain Markovian properties. The stopping problems comprising the system have time-discontinuous functionals. These discontinuities come naturally as a result of the decision lag h and delay Δ limiting admissible strategies and their execution. Stopping techniques developed in previous sections enable us to solve these discontinuous stopping problems and prove the existence and form of optimal strategies in full detail. This part of the development is pursued in Subsection 5.2. The proof of Theorem 5.1 is located in Subsection 5.3. It is followed by a discussion of the relation of our findings to the existing results.

5.1. Reduction to optimal stopping problems

As it has been pointed out before, the controlled process (X_t^Π) is no longer Markovian due to the accumulation of pending impulses. Our solution is based on a decomposition of the optimal control problem into an infinite-dimensional system of optimal stopping problems (we will show later that it is sufficient to consider only a finite system of stopping problems). For $n \geq 0$ denote by

$$v_i^n(x, s, d, \pi) : E \times [0, T^*] \times [0, h] \times ([0, \Delta] \times \Theta)^i \to \mathbb{R}, \qquad i = 0, \ldots, n,$$

the value function for the maximization of the functional (43) under the conditions described by the parameters:

- n is the maximum number of impulses that can be ordered, $n \geq 0$,
- the first new impulse can be ordered after at least d units of time, $d \in [0, h]$,
- x is a starting point for the process $(X(t))$, $x \in E$,
- s denotes the time until maturity T, so the optimization horizon is s, $s \in [0, T]$,
- i is the number of pending impulses (stored in π), $i \geq 0$,
- π consists of i pairs $((\delta_1, \xi_1), \ldots, (\delta_i, \xi_i))$, where $\xi_k \in \Theta$ is the action, δ_k is the time until the execution of the action ξ_k and $\delta_1 < \cdots < \delta_i \leq s$.

The role of s in the parameters of (v_i^n) is different than in previous sections: it denotes the time until maturity T. This choice is motivated by two observations. Firstly, it allows us to skip the maturity T in the parameters of v_i^n and reduces the dimension of the problem. Secondly, all the points of discontinuities of the above value functions are naturally expressed relative to the distance to the maturity T (see Theorem 5.3).

To simplify the notation, define an operator

$$Mv_i^n\big(x, s, (\delta_1, \xi_1), \ldots, (\delta_{i-1}, \xi_{i-1})\big) = \sup_{\xi \in \Theta} v_i^n\big(x, s, h, (\delta_1, \xi_1), \ldots, (\delta_{i-1}, \xi_{i-1}), (\Delta, \xi)\big).$$

The following standard result holds:

LEMMA 5.2 *The operator M maps a continuous bounded function into a continuous bounded function.*

We first provide formulas for functions v_i^0, i.e., when no new impulses are allowed:

$$\begin{aligned}
&v_i^0\big(x, s, d, (\delta_1, \xi_1), \dots, (\delta_i, \xi_i)\big) \\
&= \mathbb{E}^x \bigg\{ \int_0^{\delta_1} e^{-\alpha u} f(T - s + u, X(u)) du + e^{-\alpha\delta_1} c(X(\delta_1), \xi_1) \\
&\qquad + e^{-\alpha\delta_1} v_{i-1}^0\big(\Gamma(X(\delta_1), \xi_1), s - \delta_1, (d - \delta_1) \vee 0, (\delta_2 - \delta_1, \xi_2), \dots, (\delta_i - \delta_1, \xi_i)\big) \bigg\},
\end{aligned} \tag{44}$$

and

$$v_0^0(x, s, d) = \mathbb{E}^x \bigg\{ \int_0^s e^{-\alpha u} f(T - s + u, X(u)) du + e^{-\alpha s} g(X(s)) \bigg\}. \tag{45}$$

If $n > 0$ and $i > 0$, the value function v_i^n is separately defined on three subsets of the parameter space:

i) $s - \Delta < d$: no impulse can be ordered because the time between possible decision about an impulse and the maturity is shorter than the delay of the execution Δ. This is based on the assumption that all pending impulses are executed before or at the maturity. Impulses ordered after the moment $s - \Delta$ do not affect the value of the functional.

ii) $s - \Delta \geq d$ and $\delta_1 < d$: it is possible to order a new impulse, but a pending impulse (δ_1, ξ_1) is executed before a new one can be ordered,

iii) $s - \Delta \geq d$ and $\delta_1 \geq d$: it is possible to order a new impulse before the execution of a pending impulse (δ_1, ξ_1).

In (i) and (ii) no impulses can be ordered before δ_1. The value functions can be written as

$$\begin{aligned}
v_i^n\big(x, s, d, (\delta_1, \xi_1), \dots, (\delta_i, \xi_i)\big) &= \mathbb{E}^x \bigg\{ \int_0^{\delta_1} e^{-\alpha u} f(T - s + u, X(u)) du + e^{-\alpha\delta_1} c(X(\delta_1), \xi_1) \\
&+ e^{-\alpha\delta_1} v_{i-1}^n\Big(\Gamma(X(\delta_1), \xi_1), s - \delta_1, (d - \delta_1) \vee 0, (\delta_2 - \delta_1, \xi_2), \dots, (\delta_i - \delta_1, \xi_i)\Big) \bigg\}.
\end{aligned} \tag{46}$$

We divide (iii) into three subcases:
a) $\delta_1 \leq s - \Delta$ (by the conditions in (iii) we have $d \leq \delta_1$)

$$\begin{aligned}
v_i^n\big(x, s, d, (\delta_1, \xi_1), \dots, (\delta_i, \xi_i)\big) &= \sup_{d \leq \tau \leq \delta_1} \mathbb{E}^x \bigg\{ \int_0^{\tau} e^{-\alpha u} f(T - s + u, X(u)) du \\
&+ 1_{\{\tau < \delta_1\}}\, e^{-\alpha\tau} M v_{i+1}^{n-1}\big(X(\tau), s - \tau, (\delta_1 - \tau, \xi_1), \dots, (\delta_i - \tau, \xi_i)\big) \\
&+ 1_{\{\tau = \delta_1\}}\, e^{-\alpha\delta_1} \Big(c(X(\delta_1), \xi_1) \\
&\qquad + v_{i-1}^n\Big(\Gamma(X(\delta_1), \xi_1), s - \delta_1, 0, (\delta_2 - \delta_1, \xi_2), \dots, (\delta_i - \delta_1, \xi_i)\Big)\Big) \bigg\},
\end{aligned} \tag{47}$$

b) $\delta_1 > s-\Delta > 0$ (by the conditions in (iii) we have $d \le s-\Delta$)

$$v_i^n\big(x,s,d,(\delta_1,\xi_1),\dots,(\delta_i,\xi_i)\big) = \sup_{d\le\tau\le s-\Delta} \mathbb{E}^x\bigg\{ \int_0^\tau e^{-\alpha u} f(T-s+u,X(u))du$$
$$+ 1_{\{\tau<s-\Delta\}}\ e^{-\alpha\tau} M v_{i+1}^{n-1}\big(X(\tau),s-\tau,(\delta_1-\tau,\xi_1),\dots,(\delta_i-\tau,\xi_i)\big)$$
$$+ 1_{\{\tau=s-\Delta\}}\ e^{-\alpha(s-\Delta)} v_i^n\Big(\big(X(s-\Delta),\Delta,0,(\delta_1-s+\Delta,\xi_1),\dots,(\delta_i-s+\Delta,\xi_i)\big)\Big)\bigg\}, \tag{48}$$

c) $\delta_1 > s-\Delta = 0$ (by the conditions in (iii) we have $d=0$)

$$v_i^n\big(x,\Delta,0,(\delta_1,\xi_1),\dots,(\delta_i,\xi_i)\big)$$
$$= \max\Bigg(M v_{i+1}^{n-1}\big(x,\Delta,(\delta_1,\xi_1),\dots,(\delta_i,\xi_i)\big),$$
$$\mathbb{E}^x\bigg\{ \int_0^{\delta_1} e^{-\alpha u} f(T-s+u,X(u))du + e^{-\alpha\delta_1} c(X(\delta_1),\xi_1)$$
$$+ e^{-\alpha\delta_1} v_{i-1}^n\big(\Gamma(X(\delta_1),\xi_1),\Delta-\delta_1,0,(\delta_2-\delta_1,\xi_2),\dots,(\delta_i-\delta_1,\xi_i)\big)\bigg\}\Bigg). \tag{49}$$

Formula (49) has the following meaning. When time until maturity equals Δ there are only two choices: either to order an impulse immediately (it will be executed at T; no more impulses can be ordered afterwards) or to execute only the pending impulses.

If $n > 0$ and $i = 0$, there are two possibilities:

i) $d > s-\Delta$: no more impulses can be ordered

$$v_0^n\big(x,s,d\big) = v_0^0\big(x,s,d\big), \tag{50}$$

ii) $d \le s-\Delta$: a new impulse can be ordered

$$v_0^n\big(x,s,d\big) = \sup_{d\le\tau\le s-\Delta} \mathbb{E}^x\bigg\{ \int_0^\tau e^{-\alpha u} f(T-s+u,X(u))du$$
$$+ e^{-\alpha\tau}\max\Big(M v_1^{n-1}\big(X(\tau),s-\tau\big), v_0^0\big(X(\tau),s-\tau,0\big)\Big)\bigg\}. \tag{51}$$

The relations developed above are heuristic. In what follows we shall show that there is a unique solution (v_i^n) to the system of equations (44) - (51) and $v_i^n(x,s,d,\pi)$ is the optimal value of the cost functional J with initial condition $(x,T-s)$, a new impulse order allowed after d units of time, i impulses in the memory π and at most n new impulse orders.

5.2. Solution to the system of optimal stopping problems

It is an inherent property of our model that functions v_i^n may not be continuous. They are however piecewise continuous, which is one of the findings of the theorem below. Using results from previous sections we are able to prove that the stopping problems (47), (48) and (51) have optimal solutions. These solutions are the building blocks of the optimal control for the problem (43).

THEOREM 5.3 *Assume (A1)-(A2) and $h > 0$. There is a unique solution (v_i^n) to the system of equations (44)-(51). The functions v_i^n are the value functions for the functional J with i impulses in the memory and at most n new impulse orders allowed. Furthermore:*

i) Functions v_i^0 are bounded and continuous with respect to all arguments.

ii) For $n > 0$ the functions v_i^n have the following decomposition:

$$\begin{aligned} v_i^n\big(x, s, d, (\delta_1, \xi_1), \ldots, (\delta_i, \xi_i)\big) \\ &= 1_{\{s \geq d+\Delta+Nh\}}\, u_{i,N+1}^n\big(x, s, d, (\delta_1, \xi_1), \ldots, (\delta_i, \xi_i)\big) \\ &\quad + \sum_{m=1}^{N} 1_{\{s \in [d+\Delta+(m-1)h, d+\Delta+mh)\}}\, u_{i,m}^n\big(x, s, d, (\delta_1, \xi_1), \ldots, (\delta_i, \xi_i)\big) \\ &\quad + 1_{\{s < d+\Delta\}}\, u_{i,0}^n\big(x, s, d, (\delta_1, \xi_1), \ldots, (\delta_i, \xi_i)\big), \end{aligned} \tag{52}$$

where $N = \max\{m : T - \Delta - mh \geq 0\}$, the functions $u_{i,0}^n, u_{i,1}^n, \ldots, u_{i,N+1}^n : E \times [0, T] \times [0, h] \times ([0, \Delta] \times E^\xi)^i \to \mathbb{R}$ are continuous, bounded and

$$\begin{aligned} u_{i,m}^n\big(x, s, s - \Delta - mh, (\delta_1, \xi_1), \ldots, (\delta_i, \xi_i)\big) \\ &\leq u_{i,m+1}^n\big(x, s, s - \Delta - mh, (\delta_1, \xi_1), \ldots, (\delta_i, \xi_i)\big), \\ &\qquad m = 0, \ldots, N. \end{aligned} \tag{53}$$

iii) All optimal stopping problems used in the construction of (v_i^n) have solutions, i.e., there exists stopping times for which the suprema are attained.

Proof. The functions v_i^0, $i \geq 0$, are uniquely determined by equations (44)-(45). They are the value functions for the functional J with no future orders allowed. Lemma 2.3 implies v_0^0 is continuous and bounded. Further, an inductive argument shows v_i^0, $i \geq 1$, are continuous and bounded. The inductive step follows from Lemma 2.3 or, directly, from Corollary 4.3 with $T_1 = T_2 = \delta_1$.

The rest of the proof relies on the induction with respect to the ordering $\preccurlyeq$ on the set of indices $(n, i) \in \{0, 1, \ldots, N\} \times \{0, 1, \ldots, N\}$ defined as follows:

$$(n', i') \preccurlyeq (n, i) \qquad \text{if} \qquad n' < n, \quad \text{or} \quad (n' = n \quad \text{and} \quad i' \leq i). \tag{54}$$

First we prove that the system of equations (44)-(51) defines functions v_i^n in an explicit way. It is clearly true for v_0^i. Assume $v_{i'}^{n'}$ is defined for all $(n', i') \preccurlyeq (n, i)$ such that $(n', i') \neq (n, i)$.

If $n > 0$ the equation (49) defines $v_i^n(x, \Delta, d, (\delta_1, \xi_1), \dots, (\delta_i, \xi_i))$ for $d = 0$. This is extended to arbitrary $d \in [0, h]$ via (46)-(48). For $n = 0$ equations (50)-(51) provide explicit formulas for v_i^n.

The proof of the continuity of v_i^n follows by induction with respect to the ordering $\preccurlyeq$. Assertion (i) implies conditions (52)-(53) are satisfied for $n = 0$. Assume, as an inductive hypothesis, they are satisfied for all $(n', i') \preccurlyeq (n, i)$ such that $(n', i') \neq (n, i)$.

Preliminary step ($n > 0$, $i = 0$): If $d > s - \Delta$ the function v_0^n coincides with v_0^0, which is continuous by assertion (i). Otherwise, v_0^n is given by (51). It can be written equivalently as

$$v_0^n\big(x, s, d\big) = \sup_{d \le \tau \le s - \Delta} \mathbb{E}^x \bigg\{ \int_0^\tau e^{-\alpha u} f(T - s + u, X(u)) du + F_0^n\big(\tau, X(\tau), s\big) \bigg\},$$

where

$$F_0^n\big(t, x, s\big) = e^{-\alpha t} \max\Big(M v_1^{n-1}(x, s-t), v_0^0\big(x, s-t, 0\big)\Big).$$

By the inductive hypothesis (52) the function F_0^n has the following decomposition:

$$\begin{aligned} F_0^n\big(t, x, s\big) &= 1_{\{t \le s - \Delta - Nh\}}\, f_{0,N+1}^n\big(t, x, s\big) \\ &\quad + \sum_{m=2}^{N} 1_{\{t \in (s - \Delta - mh, s - \Delta - (m-1)h]\}}\, f_{0,m}^n\big(t, x, s\big) \\ &\quad + 1_{\{t > s - \Delta - h\}}\, f_{0,1}^n\big(t, x, s\big), \end{aligned}$$

where

$$f_{0,m}^n\big(t, x, s\big) \;=\; e^{-\alpha t} \max\Big(M u_{1,m-1}^{n-1}(x, s-t), v_0^0(x, s-t, 0)\Big), \qquad m \;=\; 1, \dots, N+1.$$

Lemma 5.2 with the set of parameters $E^B = [0, T]$, $b = (s)$ implies that $(f_{0,m}^n)$ are continuous. We infer from the inductive assumption (53) that

$$f_{0,m}^n\big(s - \Delta - mh, x, s\big) \le f_{0,m+1}^n\big(s - \Delta - mh, x, s\big), \qquad m = 1, \dots, N.$$

By virtue of Corollary 4.3, with the same set of parameters $E^B = [0, T]$, the value function

$$w_0^n(T_1, T_2, x, s) = \sup_{T_1 \le \tau \le T_2} \mathbb{E}^x \bigg\{ \int_0^\tau e^{-\alpha u} f(T - s + u, X(u)) du + F_0^n\big(\tau, X(\tau), s)\big) \bigg\}$$

has the decomposition

$$\begin{aligned} w_0^n\big(T_1, T_2, x, s\big) &= 1_{\{T_1 \le s - \Delta - Nh\}}\, w_{0,N+1}^n\big(T_1, T_2, x, s\big) \\ &\quad + \sum_{m=2}^{N} 1_{\{T_1 \in (s - \Delta - mh, s - \Delta - (m-1)h]\}}\, w_{0,m}^n\big(T_1, T_2, x, s\big) \\ &\quad + 1_{\{T_1 > s - \Delta - h\}}\, w_{0,1}^n\big(T_1, T_2, x, s\big), \end{aligned}$$

with continuous functions $w^n_{0,1}, w^n_{0,2}, \dots, w^n_{0,N+1}$ such that

$$w^n_{0,m}(s-\Delta-mh, T_2, x, s) \le w^n_{0,m+1}(s-\Delta-mh, T_2, x, s), \qquad m = 1, \dots, N.$$

Comparing with (51), we obtain $v^n_0(x,s,d) = w^n_0(d, s-\Delta, x, s)$ for $d \le s-\Delta$.

We summarize the results on v^n_0:

$$v^n_0(x,s,d) = \begin{cases} v^0_0(x,s,d), & d > s-\Delta, \\ w^n_0(d, s-\Delta, x, s), & d \le s-\Delta. \end{cases}$$

Decomposition (52) of v^n_0 is thus given by

$$\begin{aligned} u^n_{0,0}(x,s,d) &= v^0_0(x,s,d), \\ u^n_{0,m}(x,s,d) &= w^n_{0,m}(d, s-\Delta, x, s), \qquad m = 1, \dots, N+1. \end{aligned}$$

Inequalities (53) for $m = 1, \dots, N$ result from those for $w^n_{0,m}$. The relation for $m = 0$, $u^n_{0,0}(x,s,s-\Delta) \le u^n_{0,1}(x,s,s-\Delta)$, follows directly from

$$v^0_0(x,s,s-\Delta) \le w^n_0\big(s-\Delta, s-\Delta, x, s\big).$$

Having proved the assertions of theorem for $i = 0$, i.e. when there are no pending impulses, we turn our attention to the case $n > 0$, $i > 0$. Value functions v^n_i were defined on three disjoint subsets of parameters. We will first consider them separately and merge the results at the end of the proof.

Case (i) and (ii): We infer from the representation (46) and inductive assumption (52) that in case (i), i.e., for $s-\Delta < d$,

$$v^n_i\big(x,s,d,(\delta_1,\xi_1),\dots,(\delta_i,\xi_i)\big) = \hat g^n_{i,0}\big(x,s,d,(\delta_1,\xi_1),\dots,(\delta_i,\xi_i)\big),$$

where

$$\begin{aligned} \hat g^n_{i,0}\big(x,s,d,(\delta_1,\xi_1),\dots,(\delta_i,\xi_i)\big) &= \mathbb{E}^x\bigg\{\int_0^{\delta_1} e^{-\alpha u} f(T-s+u, X(u))du + e^{-\alpha\delta_1} c(X(\delta_1),\xi_1) \\ &+ e^{-\alpha\delta_1} u^n_{i-1,0}\big(\Gamma(X(\delta_1),\xi_1), s-\delta_1, (d-\delta_1)\vee 0, (\delta_2-\delta_1,\xi_2),\dots,(\delta_i-\delta_1,\xi_i)\big)\bigg\}, \end{aligned}$$

and in the case (ii), i.e. for $s-\Delta \ge d > \delta_1$,

$$\begin{aligned} v^n_i\big(x,s,d,(\delta_1,\xi_1),\dots,(\delta_i,\xi_i)\big) &= 1_{\{s \ge d+\Delta+Nh\}}\, \hat g^n_{i,N+1}\big(x,s,d,(\delta_1,\xi_1),\dots,(\delta_i,\xi_i)\big) \\ &+ \sum_{m=1}^N 1_{\{s \in [d+\Delta+(m-1)h, d+\Delta+mh)\}}\, \hat g^n_{i,m}\big(x,s,d,(\delta_1,\xi_1),\dots,(\delta_i,\xi_i)\big), \end{aligned}$$

where, for $m = 1, \dots, N+1$,

$$\begin{aligned} \hat g^n_{i,m}\big(x,s,d,(\delta_1,\xi_1),\dots,(\delta_i,\xi_i)\big) &= \mathbb{E}^x\bigg\{\int_0^{\delta_1} e^{-\alpha u} f(T-s+u, X(u))du + e^{-\alpha\delta_1} c(X(\delta_1),\xi_1) \\ &+ e^{-\alpha\delta_1} u^n_{i-1,m}\big(\Gamma(X(\delta_1),\xi_1), s-\delta_1, d-\delta_1, (\delta_2-\delta_1,\xi_2),\dots,(\delta_i-\delta_1,\xi_i)\big)\bigg\}. \end{aligned}$$

Lemma 2.3 implies the continuity of $\hat{g}^n_{i,m}$, $m = 0, \ldots, N+1$ (the set of parameters is $E^B = [0,T] \times [0,h] \times ([0,\Delta] \times \Theta)^i$, $b = (s, d, (\delta_1, \xi_1), \ldots, (\delta_i, \xi_i))$). The semigroup of the process $X(t)$ is monotonous, i.e., maps non-negative functions into non-negative ones. This, together with the assumption (53), proves that, for $\delta_1 < s - \Delta$,

$$\hat{g}^n_{i,m}\big(x, s, s-\Delta-mh, (\delta_1,\xi_1), \ldots, (\delta_i,\xi_i)\big) \le \hat{g}^n_{i,m+1}\big(x, s, s-\Delta-mh, (\delta_1,\xi_1), \ldots, (\delta_i,\xi_i)\big), \quad m = 0, \ldots N. \tag{55}$$

Above results can also be obtained via Corollary 4.3.

Case (iii): We will use a shorthand notation $D = (s - \Delta) \wedge \delta_1$. Formulas (47)-(49) can be equivalently written as

$$v^n_i\big(x, s, d, (\delta_1,\xi_1), \ldots, (\delta_i,\xi_i)\big) = \sup_{d \le \tau \le D} \mathbb{E}^x \bigg\{ \int_0^\tau e^{-\alpha u} f(T - s + u, X(u)) du \tag{56}$$
$$+ F^n_i\big(\tau, X(\tau), s, (\delta_1,\xi_1), \ldots, (\delta_i,\xi_i)\big) \bigg\},$$

where

$$\begin{aligned}
&F^n_i\big(t, x, s, (\delta_1,\xi_1), \ldots, (\delta_i,\xi_i)\big) \\
&\quad = 1_{\{t<D\}}\, e^{-\alpha t} M v^{n-1}_{i+1}\big(x, s-t, (\delta_1 - t, \xi_1), \ldots, (\delta_i - t, \xi_i)\big) \\
&\qquad + 1_{\{t=D\}}\, e^{-\alpha D} \max\bigg(M v^{n-1}_{i+1}\big(x, s-D, (\delta_1 - D, \xi_1), \ldots, (\delta_i - D, \xi_i)\big), \\
&\qquad\qquad h^n_i\big(x, s-D, (\delta_1 - D, \xi_1), \ldots, (\delta_i - D, \xi_i)\big) \bigg),
\end{aligned}$$

and

$$\begin{aligned}
&h^n_i\big(x, s, (\delta_1,\xi_1), \ldots, (\delta_i,\xi_i)\big) \\
&\quad = 1_{\{\delta_1 \le s-\Delta\}} \Big(c\big(x, \xi_1\big) + v^n_{i-1}\big(\Gamma(x,\xi_1), s - \delta_1, 0, (\delta_2 - \delta_1, \xi_2), \ldots, (\delta_i - \delta_1, \xi_i)\big) \Big) \\
&\qquad + 1_{\{\delta_1 > s-\Delta\}}\, e^{-\alpha \delta_1}\, \mathbb{E}^x \Big\{ c\big(X(\delta_1), \xi_1\big) \\
&\qquad\qquad + v^n_{i-1}\big(\Gamma(X(\delta_1), \xi_1), s - \delta_1, 0, (\delta_2 - \delta_1, \xi_2), \ldots, (\delta_i - \delta_1, \xi_i)\big) \Big\}.
\end{aligned}$$

Inductive hypotheses, monotonicity of the operator M and Lemma 5.2 imply that $M v^{n-1}_{i+1}$ has a decomposition of the type (52)-(53) with the functions $M u^{n-1}_{i+1,m}$, $m = 0, \ldots, N+1$. The functional F is therefore left-continuous for $t < D$ (in the notation of Section 4). Left-continuity clearly fails at $t = D$. Due to the decomposition of v_{i-1}^n we have

$$\begin{aligned} &h_i^n\big(x, s, (\delta_1, \xi_1), \dots, (\delta_i, \xi_i)\big) \\ &\quad = 1_{\{\delta_1 \le s - \Delta - Nh\}}\ h_{i,N+1}^n\big(x, s, (\delta_1, \xi_1), \dots, (\delta_i, \xi_i)\big) \\ &\qquad + \sum_{m=1}^{N-1} 1_{\{\delta_1 \in (s-\Delta-mh, s-\Delta-(m-1)h]\}}\ h_{i,m}^n\big(x, s, (\delta_1, \xi_1), \dots, (\delta_i, \xi_i)\big) \\ &\qquad + 1_{\{\delta_1 > s-\Delta\}}\ h_{i,0}^n\big(x, s, (\delta_1, \xi_1), \dots, (\delta_i, \xi_i)\big), \end{aligned}$$

with

$$\begin{aligned} h_{i,0}^n\big(x, s, (\delta_1, \xi_1), \dots, (\delta_i, \xi_i)\big) &= e^{-\alpha\delta_1}\ \mathbb{E}^x\Big\{c\big(X(\delta_1), \xi_1\big) \\ &\quad + u_{i-1,0}^n\big(\Gamma(X(\delta_1), \xi_1), s - \delta_1, 0, (\delta_2 - \delta_1, \xi_2), \dots, (\delta_i - \delta_1, \xi_i)\big)\Big\}, \\ h_{i,m}^n\big(x, s, (\delta_1, \xi_1), \dots, (\delta_i, \xi_i)\big) &= c(x, \xi_1) \\ &\quad + u_{i-1,m}^n\big(\Gamma(x, \xi_1), s - \delta_1, 0, (\delta_2 - \delta_1, \xi_2), \dots, (\delta_i - \delta_1, \xi_i))\big), \quad m = 1, \dots, N+1. \end{aligned}$$

Functions $h_{i,m}^n$, $m = 0, \dots, M+1$, are continuous and bounded by Lemma 2.3.

Thanks to the decomposition of h_i^n the function F_i^n can be written as

$$\begin{aligned} &F_i^n\big(t, x, (s, (\delta_1, \xi_1), \dots, (\delta_i, \xi_i))\big) \\ &\quad = 1_{\{t \le s - \Delta - Nh\}}\ f_{i,N+1}^n\big(t, x, (s, (\delta_1, \xi_1), \dots, (\delta_i, \xi_i))\big) \\ &\qquad + \sum_{m=1}^{N} 1_{\{t \in (s-\Delta-mh, s-\Delta-(m-1)h]\}}\ f_{i,m}^n\big(t, x, (s, (\delta_1, \xi_1), \dots, (\delta_i, \xi_i))\big) \\ &\qquad + 1_{\{t > s-\Delta\}}\ f_{i,0}^n\big(t, x, (s, (\delta_1, \xi_1), \dots, (\delta_i, \xi_i))\big), \end{aligned}$$

with

$$\begin{aligned} &f_{i,m}^n\big(t, x, (s, (\delta_1, \xi_1), \dots, (\delta_i, \xi_i))\big) \\ &\quad = 1_{\{t < \delta_1 \wedge (s-\Delta)\}}\ e^{-\alpha t}\ M u_{i+1,m}^{n-1}\big(x, s - t, (\delta_1 - t, \xi_1), \dots, (\delta_i - t, \xi_i)\big) \\ &\qquad + 1_{\{t \ge \delta_1 \wedge (s-\Delta)\}}\ e^{-\alpha t} \max\Bigg(M u_{i+1,m}^{n-1}\big(x, s - t, (\delta_1 - t, \xi_1), \dots, (\delta_i - t, \xi_i)\big), \\ &\qquad\qquad h_{i,m}^n\big(x, (s - t, (\delta_1 - t, \xi_1), \dots, (\delta_i - t, \xi_i))\big)\Bigg). \end{aligned}$$

Functions $f_{i,m}^n$ are not continuous; they can have an upward jump at $t = (s - \Delta) \wedge \delta_1$. Combination of Corollary 4.3 with Theorem 3.10 (the set of parameters is $E^B = [0, T] \times ([0, \Delta] \times \Theta)^i$, $b = (s, (\delta_1, \xi_1), \dots, (\delta_i, \xi_i)))$ implies that for $T_2 \ge \delta_1 \wedge (s - \Delta)$ the function w_i^n defined as

$$\begin{aligned} w_i^n\big(T_1, T_2, x, s, (\delta_1, \xi_1), \dots, (\delta_i, \xi_i)\big) = \sup_{T_1 \le \tau \le T_2} \mathbb{E}^x\Big\{ &\int_0^\tau e^{-\alpha u} f(T - s + u, X(u)) du \\ &+ F_i^n\big(\tau, X(\tau), s, (\delta_1, \xi_1), \dots, (\delta_i, \xi_i)\big)\Big\}, \end{aligned}$$

has the following decomposition

$$
\begin{aligned}
&w_i^n\big(T_1, T_2, x, s, (\delta_1, \xi_1), \dots, (\delta_i, \xi_i)\big) \\
&\quad = 1_{\{T_1 \le s-\Delta-Nh\}}\ w_{i,N+1}^n\big(T_1, T_2, x, s, (\delta_1, \xi_1), \dots, (\delta_i, \xi_i)\big) \\
&\qquad + \sum_{m=1}^{N} 1_{\{T_1 \in (s-\Delta-mh, s-\Delta-(m-1)h]\}}\ w_{i,m}^n\big(T_1, T_2, x, s, (\delta_1, \xi_1), \dots, (\delta_i, \xi_i)\big) \\
&\qquad + 1_{\{T_1 > s-\Delta\}}\ w_{i,0}^n\big(T_1, T_2, x, s, (\delta_1, \xi_1), \dots, (\delta_i, \xi_i)\big)
\end{aligned}
$$

with continuous bounded functions $w_{i,m}^n : [0,T]^2 \times E \times [0,T] \times [([0,\Delta] \times \Theta)^i \to \mathbb{R}$ satisfying the following set of inequalities for $m = 0, \dots, N$:

$$
\begin{aligned}
&w_{i,m}^n\big(s-\Delta-mh, T_2, x, s, (\delta_1, \xi_1), \dots, (\delta_i, \xi_i)\big) \\
&\qquad\qquad \le w_{i,m+1}^n\big(s-\Delta-mh, T_2, x, s, (\delta_1, \xi_1), \dots, (\delta_i, \xi_i)\big).
\end{aligned} \tag{57}
$$

Notice that $v_i^n\big(x, s, d, (\delta_1, \xi_1), \dots, (\delta_i, \xi_i)\big) = w\big(d, \delta_1 \wedge (s-\Delta), x, s, (\delta_1, \xi_1), \dots, (\delta_i, \xi_i)\big)$ on $d \le \delta_1 \wedge (s-\Delta)$.

Final step: The results derived above are used to obtain (52)-(53) for v_i^n. Findings in case (i), $s - \Delta < d$, imply $u_{i,0}^n = \hat{g}_{i,0}^n$. Functions $u_{i,m}^n$, for $m > 0$, are defined through cases (ii) and (iii). Indeed, on $s - \Delta \ge d$ we have

$$
\begin{aligned}
u_{i,m}^n\big(x, s, d, (\delta_1, \xi_1), \dots, (\delta_i, \xi_i)\big) &= 1_{\{\delta_1 < d\}}\ \hat{g}_{i,m}^n\big(x, s, d, (\delta_1, \xi_1), \dots, (\delta_i, \xi_i)\big) \\
&\quad + 1_{\{\delta_1 \ge d\}}\ w_{i,m}^n\big(d, \delta_1 \wedge (s-\Delta), x, s, (\delta_1, \xi_1), \dots, (\delta_i, \xi_i)\big).
\end{aligned}
$$

Continuity of $u_{i,m}^n$ can only be violated at $d = \delta_1$. It is however not the case because

$$
\hat{g}_{i,m}^n\big(x, s, \delta_1, (\delta_1, \xi_1), \dots, (\delta_i, \xi_i)\big) = w_{i,m}^n\big(\delta_1, \delta_1, x, s, (\delta_1, \xi_1), \dots, (\delta_i, \xi_i)\big)
$$

on $s - \Delta \ge d$. The function $u_{i,m}^n$ can be extended in a continuous way to its whole domain, i.e. $s \ge 0$.

Inequalities (55) and (57) imply (53) for $m = 1, \dots, N$. Since

$$
w_{i,0}^n\big(s-\Delta, s-\Delta, x, s, (\delta_1, \xi_1), \dots, (\delta_i, \xi_i)\big) \ge \hat{g}_{i,0}^n\big(x, s, s-\Delta, (\delta_1, \xi_1), \dots, (\delta_i, \xi_i)\big).
$$

inequalities (55) and (57) justify (53) for $m = 0$ as well. Bellman principle and the existence of solutions to all considered optimal stopping problems imply v_i^n is the value function for the functional J with i impulses in the memory and at most n future impulse orders. ■

REMARK 5.4 *It might be tempting to use the technique pioneered in Theorem 3.5 to remove the discontinuity of F_i^n in equation* (56) *at $t = D$ in the following fashion. Define for $t \in [0, D]$*

$$
\begin{aligned}
&r_i^n\big(t, x, s, (\delta_1, \xi_1), \dots, (\delta_i, \xi_i)\big) \\
&\quad = \mathbb{E}^x\bigg\{e^{-\alpha(D-t)} \max\Big(Mv_{i+1}^{n-1}\big(X(D-t), s-D, (\delta_1 - D, \xi_1), \dots, (\delta_i - D, \xi_i)\big), \\
&\qquad\qquad h_i^n\big(X(D-t), s-D, (\delta_1 - D, \xi_1), \dots, (\delta_i - D, \xi_i)\big)\Big)\bigg\},
\end{aligned}
$$

and

$$\tilde{F}_i^n\big(t,x,s,(\delta_1,\xi_1),\dots,(\delta_i,\xi_i)\big)$$
$$= e^{-\alpha t}\max\Big(Mv_{i+1}^{n-1}\big(X(t),s-t,(\delta_1-t,\xi_1),\dots,(\delta_i-t,\xi_i)\big),$$
$$r_i^n\big(t,X(t),s,(\delta_1,\xi_1),\dots,(\delta_i,\xi_i)\big)\Big).$$

The value function in (56) *can be equivalently written with $\tilde{F}_i^n$ in place of F_i^n. The intuition standing behind this result comes from Theorem 3.5. Formal justification goes via time-discretization and an analogous but more laborious proof than that of Theorem 3.5.*

However promising it looks, the approach proposed above does not benefit our problem. The decomposition of r_i^n depends on D and the points of discontinuity do not coincide with those in (52). *This leads to multiplication of the number of discontinuities and requires further steps to prove the properties of v_i^n.*

5.3. Main theorem and remarks

Proof of Theorem 5.1. Due to a non-zero decision-lag h, the maximum number of impulses on the interval $[0,T]$ is bounded by $N = \lceil T/h \rceil$. Therefore, $v(x) = v_0^N(x,0,0)$, which by Theorem 5.3 is continuous. An optimal strategy can be constructed from the solutions to the stopping problems considered in the proof of Theorem 5.3. These optimal stopping times exist by Corollary 4.3. Actions are determined by maximizers of appropriate suprema. Due to the compactness of Θ and continuity of $u_{i,m}^n$ with respect to ξ there maximizers can be chosen to be measurable. ■

The discontinuities of the value functions solving the system of optimal stopping problems (44)-(51) are due to the delay $\Delta > 0$ and the decision lag $h > 0$. If both quantities coincide, $\Delta = h$, the optimal control problem can be reformulated as a sequence of no-delay optimal stopping problems. Øksendal and Sulem [21] studied such a problem with a jump-diffusion as the underlying process $(X(t))$ and a random time horizon defined as the first exit time from a given open set. The very idea of their approach can be accommodated in our general setting with a finite horizon and yields analogous results.

Bruder and Pham [8] consider controls where the execution delay is a multiplicity of the decision lag, i.e. $\Delta = mh$. This assumption is crucial for their method of solution because it allows to divide the time between the ordering and execution of the impulse into m intervals of the length h on which only one impulse can be ordered. We relax this condition in the present paper. It forces the introduction of parameter d in the functions v_i^n as well as the construction of a new system of optimal stopping problems (see Subsection 5.1).

Our paper can be naturally extended in two directions. The first one is the removal of the decision lag h. It should, intuitively, smooth out the resulting system of optimal stopping problems leaving only one discontinuity at $s = \Delta$. On the other hand, when $h = 0$ it is possible to have strategies leading to an infinite number of pending impulses, which has two consequences: the system of optimal stopping problems is truly infinite and its solution might not result in a valid control policy (the resulting sequence of stopping times can have an accumulation point smaller than the ordering horizon $T - \Delta$).

The second extension of the paper is into infinite horizon functionals. It requires the introduction of discounting and the removal of the final payoff g. A simple example of such problem is studied by Bar-Ilan and Sulem [4] in the realm of inventory models. Our intuition suggests that such infinite horizon models can be solved via an infinite system of optimal stopping problems with continuous functionals.

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