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Investment strategies and compensation of a mean-variance optimizing fund manager

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Abstract

This paper introduces a general continuous-time mathematical framework for solution of dynamic mean-variance control problems. We obtain theoretical results for two classes of functionals: the first one depends on the whole trajectory of the controlled process and the second one is based on its terminal-time value. These results enable the development of numerical methods for mean-variance problems for a pre-determined risk-aversion coefficient. We apply them to study optimal trading strategies pursued by fund managers in response to various types of compensation schemes. In particular, we examine the effects of continuous monitoring and scheme's symmetry on trading behaviour and fund performance.

Keywords: mean-variance, continuous-time stochastic control, viscosity solutions, investment strategy, managerial compensation

1. Introduction

Markowitz's seminal paper [32] introduced the mean-variance criterion into portfolio optimization. Single-period problems, which are mathematically tractable, have enjoyed popularity both in the academia to model investor preferences and behavior (see, e.g., Epstein [17], Ormiston and Schlee [33], Tobin [38]) and among practitioners (see, e.g., Bodie et al. [13], Litterman [31]). An extension of this theory to continuous-time models proved to be difficult due to fundamental problems introduced by the variance term. A natural approach to continuous-time optimization is to use dynamic programming, which relies on markovianity of functionals. The variance is, however, not markovian. There are three main alternatives. The first involves the study of risk-sensitive functionals (see, e.g.,

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Bielecki et al. [11]), whose second order Taylor expansion has the form of a mean-variance functional

$$E(H) - \frac{\gamma}{2} Var(H), \tag{1}$$

where H is a random outcome of the investment and γ is the risk-aversion coefficient. The second alternative to dynamic programming hinges on the use of martingale methods (see, e.g., Bielecki et al. [10]). Although these methods can be used to obtain closed-form solutions for a class of mean-variance problems, they turned out to be unsuitable as a basis for efficient numerical algorithms for general mean-variance problems.

A substantial progress in the theory for mean-variance functionals was due to a third approach, closely related to the one we employ in this paper. This approach, introduced by Li and Ng [29] in a discrete-time setting, embeds the mean-variance problem into a class of auxiliary stochastic control problems that can be solved by dynamic programming methods (see also Leippold et al. [28]). An extension of this method to a continuous-time framework is presented in Zhou and Li [41], and further employed by Fu et al. [19] and Lim [30]. These papers put several constraints on the optimization problem in order to obtain auxiliary control problems in a linear-quadratic form. In particular, the random variable H in the mean-variance functional (1) is assumed to be a linear function of the portfolio wealth process. Wang and Forsyth [40] design numerical schemes for auxiliary linear-quadratic problems formulated in [41] and construct an efficient frontier.

In this paper we present a mathematical framework for the solution of general mean-variance stochastic control problems in continuous time. This framework extends the continuous-time theory of Zhou and Li [41] in two aspects. First, we allow the random variable H in the mean-variance functional (1) to be specified either as a continuous function of the portfolio wealth at a terminal time (in general, the value of the controlled process) or as an integral of a continuous function of the portfolio wealth (in general, the value of the controlled process) over time. A particular case when H depends linearly on the portfolio wealth at terminal time is covered theoretically in Zhou and Li [41] and numerically in Wang and Forsyth [40]. Second, we relax assumptions on the dynamics of the controlled process to cover non-homogeneous degenerate diffusions with Lipschitz coefficients of linear growth.

To the best of our knowledge, mean-variance optimization problems based on the integral of a function of the value of the controlled process over time have not been widely studied. A closely related paper by Aivaliotis and Veretennikov [4] provides theoretical approximation results via regularization; their solution leads to randomized strategies. In our paper, the optimization problem is solved directly using the theory of viscosity solutions to Hamilton-Jacobi-Bellman equations (see Fleming and Soner [18], Pham [36]). In particular, our results enable computation of (non-randomized) optimal strategies in a feedback form. The justification of their optimality – the verification theorem – requires very restrictive assumptions (for the latest results see Gozzi et al. [22]) that our control problem does not satisfy. We, therefore, resort to numerically testing the optimality of strategies extracted from numerical solutions of the HJB equations.

Our theoretical results are used to develop numerical algorithms to maximize functional (1) for a given (pre-determined) risk-aversion coefficient γ. Wang and Forsyth [40] solve auxiliary markovian optimisation problems which are parametrised neither by risk-aversion nor by the expectation of terminal value. Once the optimal strategy is known, they can compute the risk-aversion, the expectation and the variance. This proves to be sufficient if one aims at graphing an efficient frontier. Our approach is different as we endeavour to find an optimal strategy for a pre-determined risk-aversion coefficient. We reformulate the mean-variance problem as a superposition of a static and a dynamic optimization problem, which is equivalent to solving a set of parametrized HJB equations and maximizing the resulting value functions over a compact interval valued parameter. We demonstrate that, for practical applications, our approach leads to an efficient numerical algorithm.

Recently, Basak and Chabakauri [8] and [9] proposed another view on mean-variance optimisation. They introduced a notion of optimality in an intra-personal game theoretic sense. This has the advantage of turning the optimisation problem markovian. It should, however, be noticed that strategies optimal in a game-theoretic sense might not be optimal in a classical sense; and vice versa.

Theoretical results of this paper are applied to a study of a delegated portfolio management problem. It is a common practice in the asset management industry to use mean-variance preferences for choosing portfolios (see Bodie et al. [13] and Littermann [31]). We assume that fund managers apply the same type of preferences to their compensation and tend to follow trading strategies that maximize their satisfaction from compensation. The mathematical framework introduced in this paper allows us to study trading strategies pursued by fund managers in response to various types of compensation (incentive) schemes. We also analyze implications of complex schemes on distributional properties of the fund's wealth process. We consider symmetric (e.g., co-ownership) and asymmetric (with a hurdle rate provision) schemes based on the terminal wealth and on the continuously monitored wealth.

Incentives have been proven to be a significant factor influencing the behavior and performance of fund managers. Agarwal et al. [2] examine, in an empirical study, the influence of incentives and managerial discretion on the performance of hedge funds. They find that managers with performance-related incentives – the inclusion of hurdle rate provisions, or co-ownership – are associated with a better performance. We study numerically the implications of the above incentives on trading decisions of fund managers. Managers with symmetric (co-ownership) compensation schemes show a superior performance over those remunerated by schemes with hurdle-rate provisions: the resulting Sharpe ratio of the terminal wealth is higher.

Incentives also influence the riskiness of trading strategies pursued by fund managers. Elton et al. [16] find that managers with asymmetric incentive contracts tend to follow riskier strategies than those with symmetric compensation schemes.[1] In particular, they observe that asymmetric schemes encourage large variations in the riskiness of portfolios over time: a poor performance at any time triggers a sharp increase in the risk taking. Our numerical results show that such behavior is optimal for a fund manager with mean-variance preferences.

Our numerical study contributes also to the discussion about the frequency of portfolio monitoring (see, e.g., Agarwal et al. [2] and Goetzmann et al. [21]). We analyze trading strategies and portfolio performance when the manager's compensation is based on her performance sampled continuously over the whole investment period. We observe a fall in Sharpe ratios for symmetric and asymmetric schemes. This agrees with the empirical findings of Agarwal et al. [2]. A continuous examination of the fund's wealth diminishes managerial discretion, which, according to [2], impacts on the fund performace. One would, however, expect that the closer scrutiny offered by such compensation schemes lowers the riskiness of investment decisions. We demonstrate that the opposite is true: the variance of excess returns increases.

A classical but more challenging problem is the design of compensation schemes that align preferences of an investor and a fund manager. Existing literature offers results in the case of preferences represented by utility functions (see, e.g., Carpenter [14] and Ou-Yang [34]). Mean-variance optimality criterion has only been used in a static (single-period) framework (Baptista [6] and Carlier et al. [15]).

The outline of the remaining part of the paper is as follows. Section 2 introduces a general mathematical framework for the solution of mean-variance stochastic control problems and prepares the ground for design of efficient numerical schemes. The problem of managerial compensation in a

[1] The fixed fee in the paper by Elton et al. [16] can be represented in our framework as a symmetric compensation contract.

continuous-time market model alongside with various types of compensation schemes and discussion of numerical methods used for computation of optimal investment strategies is presented in Section 3. Analysis of the trading strategies is performed in Section 4. Section 5 concludes. In the Electronic Supplement, Section A introduces numerical schemes and verifies their convergence, Section B collects proofs.

2. Theoretical framework and results

In this section we present a general framework for the solution of mean-variance dynamic optimization problems. Subsection 2.1 studies functionals depending on the value of controlled process at the terminal time. In Subsection 2.2, these ideas are extended to functionals based on the whole trajectory of the controlled process. Our exposition is geared towards numerical computations, necessary for practical applications.

The state is described by a d-dimensional non-homogeneous stochastic differential equation (SDE) driven by a d_1-dimensional Wiener process (W_t)

$$dX_t = b(\alpha_t, t, X_t)\, dt + \sigma(\alpha_t, t, X_t)\, dW_t, \qquad X_{t_0} = x, \tag{2}$$

where $b : A \times [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : A \times [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^{d\times d_1}$. The process $(\alpha_t,\ t_0 \le t \le T)$ is from the class $\mathcal{A}$ of all progressively measurable processes (with respect to the filtration generated by the Wiener process (W_t)) with values in a compact set $A \subset \mathbb{R}^\ell$. Its role is to control the dynamics of the diffusion (X_t). Equation (2) has a pathwise unique (weak) solution if the following conditions are satisfied (see, e.g., Gikhman and Skorokhod [20, Section 6.7] or Fleming and Soner [18, Appendix D]):[2]

(A_0) The functions σ, b are Borel with respect to (a, t, x) and continuous with respect to (a, x) for every t; moreover, there exists a constant K such that

$$\begin{aligned} \|\sigma(a, t_1, x) - \sigma(a, t_2, z)\| &\le K\,(\|x - z\| + |t_1 - t_2|) \\ \|b(a, t_1, x) - b(a, t_2, z)\| &\le K\,(\|x - z\| + |t_1 - t_2|) \end{aligned} \qquad \text{(Lipschitz condition)}$$

and

$$\begin{aligned} \|b(a, t, x)\| &\le K(1 + \|x\|) \\ \|\sigma(a, t, x)\| &\le K(1 + \|x\|) \end{aligned} \qquad \text{(linear growth condition)}$$

[2] These conditions are superficial for the uniqueness of solutions. We will, however, need them later in the study of the value function of the mean-variance optimization problem.

The solution (X_t) depends on the initial condition x, time t_0, and the control (α_t); to simplify notation we indicate this dependence in the expectation $E^{\alpha}_{t_0,x}$ and the variance $Var^{\alpha}_{t_0,x}$. When this is not sufficient we shall write $X_t^{\alpha,t_0,x}$.

2.1. Mean-variance optimization for terminal-time functionals

Consider a (Markowitz-type) *mean-variance control* problem

$$u(t_0, x) := \sup_{\alpha\in\mathcal{A}} \left\{E^{\alpha}_{t_0,x} h(X_T) - \theta \operatorname{Var}^{\alpha}_{t_0,x} h(X_T)\right\}, \tag{3}$$

where $h : \mathbb{R}^d \to \mathbb{R}$ is a continuous function satisfying a polynomial growth condition:

$$|h(x)| \leq K_1(1 + ||x||^m)$$

for some constants $K_1 > 0$ and $m \geq 1$.

A major obstacle in solving this type of control problems is the non-markovianity. The value function u does not satisfy the Bellman principle as a strategy that is optimal for some t_0 does not have to be optimal for investment starting at $t > t_0$ (this property is also called time-inconsistency). We show that the optimization problem (3) can be represented as a superposition of a dynamic markovian control problem and a static optimization problem. Our approach has the advantage of leading to numerical methods solving the original Markowitz problem for a pre-determined risk aversion parameter θ. Related literature focuses mostly on graphing of an efficient frontier.

Using a dual representation $x^2 = \sup_{\psi\in\mathbb{R}}\{-\psi^2 - 2\psi x\}$ (as in Aivaliotis and Veretennikov [4]), we rewrite the variance term:

$$\operatorname{Var}^{\alpha}_{t_0,x} h(X_T) = E^{\alpha}_{t_0,x}(h(X_T))^2 - (E^{\alpha}_{t_0,x} h(X_T))^2 = E^{\alpha}_{t_0,x}(h(X_T))^2 - \sup_{\psi\in\mathbb{R}}\{-\psi^2 - 2\psi E^{\alpha}_{t_0,x} h(X_T)\}.$$

Inserting this into (3) yields the following representation of the value function:

$$u(t_0, x) = \sup_{\psi\in\mathbb{R}}\{U(t_0, x, \psi) - \theta\psi^2\}, \tag{4}$$

where, for a fixed $\psi \in \mathbb{R}$, $U(t_0, x, \psi)$ is the value function of a markovian control problem

$$U(t_0, x, \psi) = \sup_{\alpha\in\mathcal{A}} E^{\alpha}_{t_0,x}\left\{(1 - 2\theta\psi)h(X_T) - \theta\,(h(X_T))^2\right\}.$$

Notice that if ψ^* is the maximizer in (4) and α^* is an optimal strategy for $U(t_0, x, \psi^*)$, then $\psi^* = -E^{\alpha^*}_{t_0,x}\{h(X_T)\}$. This follows from the fact that in the dual representation of the square function, $x^2 = \sup_{\psi}\{-\psi^2 - 2\psi x\}$, the supremum is attained for $\psi = -x$.

The following theorem provides a representation of the value function U as a viscosity solution to an appropriate Hamilton-Jacobi-Bellman (HJB) equation.[3] The proofs of this and the following theorems are in Section B of the Electronic Supplement. Recall that h is assumed to be continuous and of polynomial growth; these conditions will not be reiterated in the coming theorems.

THEOREM 2.1. *Under assumption (A_0) for every $\psi \in \mathbb{R}$ the value function U is the unique continuous polynomially growing viscosity solution to the HJB equation*

$$\begin{cases} U_{t_0} + \sup_{a \in A} \left\{ b(a, t_0, x)^T U_x + \frac{1}{2} tr(\sigma\sigma^T(a, t_0, x) U_{xx}) \right\} = 0, \\ U(T, x, \psi) = (1 - 2\theta\psi) h(x) - \theta (h(x))^2, \end{cases} \tag{5}$$

where U_{t_0} denotes the partial derivative with respect to time t_0, U_x is a gradient vector arising in the differentiation with respect to the state x, and U_{xx} is a matrix of the second derivatives.

The value function u is related to U via the formula (4). This is not a simple quadratic relation with respect to ψ, since the function U depends on ψ in a non-linear way (due to the supremum operator). The following theorem explores the dependence of U on ψ.

THEOREM 2.2.

i) The function U is convex and continuous in ψ.

ii) The value function u is given by

$$u(t_0, x) = \sup_{\psi_{min} \le \psi \le \psi_{max}} \{ U(t_0, x, \psi) - \theta\psi^2 \},$$

where

$$\psi_{min} = - \sup_{\alpha \in \mathcal{A}} E^{\alpha}_{t_0,x} h(X_T) \qquad \textit{and} \qquad \psi_{max} = - \inf_{\alpha \in \mathcal{A}} E^{\alpha}_{t_0,x} h(X_T). \tag{6}$$

The above theorem shows that the supremum in (4) can be computed over a compact interval $[\psi_{min}, \psi_{max}]$. This, together with the continuity of U with respect to ψ, ensures that the optimal ψ exists. In applications to managerial compensation discussed in this paper, this interval is very small and the optimal ψ can be found efficiently (see Subsection 3.3).

The function under supremum in (4) is a difference of two convex functions. Although the numerical maximization of such functions is not as fast as the maximization of concave functions, one has at one's disposal advanced numerical techniques, see, e.g., Horst and Hoang [23, Chapter 10].

[3]Cf. Fleming and Soner [18], Pham [36] and Aivaliotis and Palczewski [3] for the exposition of the theory of viscosity solutions to Hamilton-Jacobi-Bellman equations and their relation to value functions of optimal stochastic control problems.

Let us summarize the findings of this section. The mean-variance problem (3) can be solved using a family of auxiliary HJB equations (5). Formula (4) provides the relation between the value function U, being the unique viscosity solution to the equation (5), and the value function u. For a given (t_0, x) the algorithm of finding the value of u is as follows:

Step 1 Compute the function $g(\psi) = U(t_0, x, \psi) - \theta\psi^2$ for $\psi_{min} \le \psi \le \psi_{max}$.

Step 2 Find the maximum of g on the interval $[\psi_{min}, \psi_{max}]$. Since g is continuous there is ψ^* for which the maximum is attained.

An optimal strategy for $U(t_0, x, \psi^*)$ is also optimal for the original mean-variance problem $u(t_0, x)$. The value function U is characterized as a viscosity solution to an HJB equation; it might not be a classical solution due to a possible degeneracy of the controlled diffusion (X_t). The computation of an optimal strategy from a viscosity solution to the HJB equation – the verification theorem – requires very restrictive assumptions; the latest results can be found in Gozzi et al. [22]. These assumptions are not satisfied by our managerial compensation example based on the Black-Scholes model. We, therefore, propose to rely on a numerical verification of optimality of a strategy extracted from the HJB equation.

2.2. Mean-Variance Optimization for Integral Functionals

In this subsection we extend the above theory to mean-variance functionals with an integral term:

$$v(t_0, x) := \sup_{\alpha \in \mathcal{A}} \left\{ E^{\alpha}_{t_0,x} \left(\int_{t_0}^{T} f(\alpha_s, s, X_s)\, ds \right) - \theta \,\mathrm{Var}^{\alpha}_{t_0,x} \left(\int_{t_0}^{T} f(\alpha_s, s, X_s)\, ds \right) \right\}. \tag{7}$$

Proceeding as in the previous subsection we reformulate the variance term and obtain

$$v(t_0, x) = \sup_{\psi} \{ V(t_0, x, \psi) - \theta\psi^2 \}, \tag{8}$$

where

$$V(t_0, x, \psi) = \sup_{\alpha \in \mathcal{A}} E^{\alpha}_{t_0,x} \left\{ (1 - 2\theta\psi) \int_{t_0}^{T} f(\alpha_s, s, X_s)\, ds - \theta \Big(\int_{t_0}^{T} f(\alpha_s, s, X_s)\, ds \Big)^2 \right\}.$$

The auxiliary optimization problem for V is still not markovian due to the term involving the square of an integral. Following Aivaliotis and Veretennikov [4] we propose an approach to reformulate this quadratic term. Fubini's theorem implies

$$E^{\alpha}_{t_0,x}\Big(\Big(\int_{t_0}^{T} f(\alpha_s, s, X_s)\, ds \Big) \Big(\int_{t_0}^{T} f(\alpha_t, t, X_t)\, dt \Big) \Big)$$
$$= 2E^{\alpha}_{t_0,x}\Big(\int_{t_0}^{T} f(\alpha_t, t, X_t) \Big(\int_{t_0}^{t} f(\alpha_s, s, X_s)\, ds \Big) dt \Big).$$

Define a new state process (X_t, Y_t) by the following stochastic differential equation

$$\begin{aligned} dX_t &= b(\alpha_t, t, X_t)\,dt + \sigma(\alpha_t, t, X_t)\,dW_t, \\ dY_t &= f(\alpha_t, t, X_t)\,dt. \end{aligned} \tag{9}$$

If we assume a polynomial growth of f, then under (A_0) this equation has a pathwise unique (weak) solution for any initial value $(t_0, x_0, y_0) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}$ (see, as before, Gikhman and Skorokhod [20, Section 6.7] or Fleming and Soner [18, Appendix D]). Using this extended state process we have $V(t_0, x, \psi) = \hat{V}(t_0, x, 0, \psi)$, where

$$\begin{aligned} \hat{V}(t_0, x, y, \psi) &= \sup_{\alpha\in\mathcal{A}} E^{\alpha}_{t_0,x,y}\Big\{(1-2\theta\psi)\int_{t_0}^{T} f(\alpha_t, t, X_t)\,dt - 2\theta\int_{t_0}^{T} f(\alpha_t, t, X_t)Y_t\,dt\Big\} \\ &= \sup_{\alpha\in\mathcal{A}} E^{\alpha}_{t_0,x,y}\Big\{\int_{t_0}^{T} f(\alpha_t, t, X_t)[1-2\theta\psi-2\theta Y_t]\,dt\Big\}. \end{aligned}$$

The functional defining $\hat{V}$ is of markovian type. Since the function f steers the dynamics of the extended state process, we impose on it analogous assumptions as on b and σ:

(A_1) The function f is Borel with respect to (a, t, x) and continuous with respect to (a, x); moreover, there exists a constant K_2 such that

$$\begin{aligned} &|f(a, t_1, x) - f(a, t_2, z)| \le K_2\,(\|x - z\| + |t_1 - t_2|)\,, \\ &|f(a, t, x)| \le K_2(1 + \|x\|). \end{aligned}$$

The following theorem states the HJB equation for the value function $\hat{V}$. This result is used in the design of a numerical scheme approximating $\hat{V}$.

THEOREM 2.3. *Under assumptions (A_0)-(A_1), for every $\psi \in \mathbb{R}$ the value function $\hat{V}(\cdot, \psi)$ is the unique continuous polynomially growing viscosity solution to the following HJB equation:*

$$\begin{cases} \hat{V}_{t_0}(t_0, x, y, \psi) \\ + \sup_{a\in A}\Big\{b(a, t_0, x)^T\hat{V}_x + \frac{1}{2}tr(\sigma\sigma^T(a, t_0, x)\hat{V}_{xx}) + (1-2\theta\psi-2\theta y+\hat{V}_y)f(a, t_0, x)\Big\} = 0, \\ \hat{V}(T, x, y, \psi) = 0. \end{cases} \tag{10}$$

Note that even if we had assumed uniform non-degeneracy of the process (X_t) (which we did not do), equation (10) would be degenerate as Y_t is degenerate. Standard methods are, therefore, insufficient to prove existence and uniqueness of classical or weak solutions. In Aivaliotis and Veretennikov [4], this obstacle is overcome by a regularization of the process Y_t; an independent diffusion part is added with a small constant diffusion coefficient. Subsequently, it is shown that the regularized value function converges to the actual one uniformly when this artificial diffusion coefficient vanishes.

In Theorem 2.3 we apply methods of viscosity solutions to show that equation (10) has a unique solution which corresponds to the value function $\hat{V}$. The reason for this development is three-fold. First, an optimal markovian strategy can be extracted from (10) (see the discussion of the verification theorem and optimal strategies at the end of Subsection 2.1). In contrast, solutions to regularized problems offer only a possibility of extracting a randomized markovian ε-optimal strategy for the original problem. Second, the representation of the value function as a solution to an HJB equation allows for the construction of an approximating numerical scheme. Third, we do not have to assume the uniform non-degeneracy of σ. Yet another reason is the elegance of the theory which can embrace both the original and the regularized equation in the same framework.

Another feature that sets apart this paper from the earlier work by Aivaliotis and Veretennikov [4] is the unboundedness of the diffusion coefficients and of the function f. This is a crucial feature that enables the study of the compensation schemes described in Section 3.

The extension of the state space (9) allows us to consider the stochastic control problem (7) as a special case (3) with $h(x, y) = y$. This, however, leads to an inefficient numerical scheme. Our formulation of the value function $\hat{V}$ with an integral-type functional and the integrand depending on X_t as well as on Y_t smooths out errors induced by numerical approximations of the process Y_t.

The following theorem explores the dependence of $\hat{V}$ on ψ.

THEOREM 2.4.

i) The function $\hat{V}$ is continuous and convex in ψ. If f is non-negative, $\hat{V}$ is decreasing in ψ.

ii) There exists a constant C such that

$$|\hat{V}(t_0, x, y, \psi) - \hat{V}(t_0, x, y, \psi')| \leq C\,(1 + \|x\|)\,|\psi - \psi'|.$$

iii) The value function v is given by

$$v(t_0, x) = \sup_{\psi_{min} \leq \psi \leq \psi_{max}} \{\hat{V}(t_0, x, 0, \psi) - \theta\psi^2\},$$

where

$$\psi_{min} = -\sup_{\alpha \in \mathcal{A}} E^{\alpha}_{t_0,x} \int_{t_0}^{T} f(\alpha_s, s, X_s)\,ds, \qquad \textit{and} \qquad \psi_{max} = -\inf_{\alpha \in \mathcal{A}} E^{\alpha}_{t_0,x} \int_{t_0}^{T} f(\alpha_s, s, X_s)\,ds.$$

Thanks to the properties stated in the above theorem the computation of $v(t_0, x)$ can be performed by a similar algorithm to the one presented at the end of Subsection 2.1.

3. The manager's compensation problem

In this section we introduce a delegated portfolio management problem in a framework of a financial market with one risk-free asset with a continuously compounded return r and a stock (risky asset) whose price follows a geometric diffusion with a constant drift μ and a constant volatility σ:

$$dS_t = \mu S_t dt + \sigma S_t dW_t.$$

The manager invests clients' money in the two available assets. Her strategy is described by a progressively measurable process (π_t) which represents the proportion of the total wealth at time t invested in the stock. The dynamics of the total wealth $\hat{X}_t$ are, therefore, given by the following equation:

$$d\hat{X}_t = \hat{X}_t(r\,dt + \pi_t(\mu - r)\,dt + \pi_t\sigma\,dW_t). \tag{11}$$

We constrain the stock investment π_t to a bounded interval which represents restrictions on short-selling and borrowing; this puts limits on leverage levels. The bound on the leverage is imposed by regulators on many financial institutions (for example, the FSA in the UK requires that firms set their leverage limits according to the risk-management regulatory requirements [1]). In the simplest case, when no borrowing and short-selling is allowed, $\pi_t \in [0, 1]$.

The manager is remunerated according to the performance of the fund under her supervision. The amount of compensation depends on the market evolution, the strategy (π_t) she implements and the compensation contract (hereafter also called the compensation scheme) agreed between investors and the manager. We assume that the fund manager applies mean-variance preferences to her compensation and follows trading strategies that are optimal for her, i.e., that maximize her satisfaction from compensation. This choice of preferences is in line with the common practice in asset management industry (Bodie et al. [13] and Littermann [31]).

Given a compensation contract the manager endeavors to find a strategy (π_t) that maximizes the mean-variance criterion

$$E(H^\pi) - \frac{\gamma}{2}Var(H^\pi), \tag{12}$$

where $\gamma > 0$ is her risk-aversion coefficient and H^π is a random variable representing the amount of compensation the manager receives if she follows a trading strategy (π_t).

In this paper we focus on two approaches to constructing managers' compensation schemes. The first one, which is based on continuous (over the whole investment period) performance monitoring, requires novel results presented in the previous section. The second one depends on a well-documented terminal time (e.g., end-of-year) performance measurement. Our theoretical findings facilitate numerical solution of the resulting optimization problems for both approaches.

3.1. Compensation schemes based on continuously monitored performance

Empirical evidence suggests that frequent monitoring has a negative effect on the fund performance (see, e.g., Agarwal et al [2] and Hunton et al. [25]). We will show that this phenomenon can arise in a fully rational setting, i.e., without psychological/behavioral factors that might influence decision-making in a real-life environment. In the modelling framework of this section, we compare optimal responses to contracts based on a terminal-time and continuous monitoring of funds wealth. Continuous monitoring can be regarded as a limit case, when inter-observation intervals become infinitesimally small.

Another aspect of compensation schemes explored in this paper is their symmetry. Investment Company Amendments Act of 1970 ruled that all performance based compensation schemes employed by US investment companies (such as mutual or pension funds) have to be symmetric. A symmetric compensation contract pays out premiums to the manager when she outperforms a target, but imposes a penalty for under-performance. This can be viewed as a means of risk sharing or co-ownership by the manager.

In a symmetric scheme with continuous monitoring, the manager's compensation is proportional to the cumulative future value of the difference between the portfolio return and the benchmark return:

$$H^{\pi} = C \int_0^T \Big(\frac{\hat{X}_t^{\pi}}{\hat{X}_0} - e^{\beta t} \Big) e^{r(T-t)} dt, \tag{13}$$

where β is the benchmark growth rate, $\hat{X}_t^{\pi}$ denotes the value of the portfolio π at the time t and $C > 0$ is the factor translating accumulated returns into manager's compensation. Notice that the coefficient $e^{r(T-t)}$ ensures that the performance measure takes into account the time-value of money.

An intuition would suggest that the compensation scheme (13) encourages close tracking of the benchmark. This is however not true as the manager's optimization problem can be written in the following equivalent form: [4]

$$E\Big(\int_0^T \big(\frac{e^{r(T-t)}\hat{X}_t^\pi}{\hat{X}_0} - 1\big)dt\Big) - \theta Var\Big(\int_0^T \big(\frac{e^{r(T-t)}\hat{X}_t^\pi}{\hat{X}_0} - 1\big)dt\Big), \tag{14}$$

where $\theta = \frac{C\gamma}{2}$. This separates the portfolio choice from the benchmark rate β and relates it only to the future value of the cumulative returns in a clear analogy to the standard Markowitz problem.

Although the regulation banning the use of asymmetric incentive plans in mutual or pension funds holds, asymmetric schemes are still popular with hedge funds which are characterized by strong performance incentives to their managers. Typically, there is a small annual management fee and a performance-based bonus payment. The latter, for most funds, is paid if the return exceeds a hurdle rate (the benchmark rate) or a high-water mark (the previous maximum). The manager's incentive payment with the hardle-rate provision is given by

$$H^\pi = C\int_0^T \big(\frac{\hat{X}_t^\pi}{\hat{X}_0} - e^{\beta t}\big)^+ e^{r(T-t)}dt, \tag{15}$$

where β is the hurdle rate and $(x)^+ = \max(x, 0)$ is the positive part of x. Since compensation schemes with high-water mark provision can be placed, in terms of the strength of incentives, between the symmetric and hurdle-rate schemes, they will not be part of the analysis presented in this paper. However, they can be accomondated by our mathematical framework and numerical methods.

The dynamics of portfolio wealth (11) is multiplicative, i.e., for any trading strategy (π_t) the return $\hat{X}_t^\pi/\hat{X}_0$ is independent of $\hat{X}_0$. The initial wealth, $\hat{X}_0$, can, therefore, be fixed to 1. Under the assumption that short-selling is not allowed, the compensation schemes (13) and (15) can be written in a unified way:

$$H^\pi = C\int_0^T \big(\hat{X}_t^\pi - Ke^{\beta t}\big)^+ e^{r(T-t)}dt. \tag{16}$$

For a symmetric scheme, one takes $K = 0$. The hurdle rate payoff is obtained when $K = 1$.

The corresponding manager's optimization problem takes the form

$$E\Big(\int_0^T \big(\hat{X}_t^\pi - Ke^{\beta t}\big)^+ e^{r(T-t)}dt\Big) - \theta Var\Big(\int_0^T \big(\hat{X}_t^\pi - Ke^{\beta t}\big)^+ e^{r(T-t)}dt\Big) \to \max, \tag{17}$$

[4]We call optimization problems equivalent if their optimal strategies are identical and their functionals are related in a deterministic way. Manager's optimization problem takes the following form

$$CE\Big(\int_0^T \big(\frac{\hat{X}_t^\pi}{\hat{X}_0} - e^{\beta t}\big)e^{r(T-t)}dt\Big) - \frac{\gamma C^2}{2}Var\Big(\int_0^T \big(\frac{\hat{X}_t^\pi}{\hat{X}_0} - e^{\beta t}\big)e^{r(T-t)}dt\Big).$$

We divide this functional by C and notice that $Var\Big(\int_0^T (\frac{\hat{X}_t^\pi}{\hat{X}_0} - e^{\beta t})e^{r(T-t)}dt\Big) = Var\Big(\int_0^T (\frac{e^{r(T-t)}\hat{X}_t^\pi}{\hat{X}_0} - 1)dt\Big)$. Changing $E\Big(\int_0^T (\frac{\hat{X}_t^\pi}{\hat{X}_0} - e^{\beta t})e^{r(T-t)}dt\Big)$ into $E\Big(\int_0^T (\frac{e^{r(T-t)}\hat{X}_t^\pi}{\hat{X}_0} - 1)\Big)$ shifts the value of the functional by a constant but does not affect the optimal strategy. This proves equivalence of (14) and the original manager's optimization problem (12) with the compensation scheme (13).

where $\theta = \frac{C\gamma}{2}$.

3.2. Compensation schemes based on terminal-time performance

Compensation schemes introduced in the previous subsection have their counterparts based on the fund's wealth at the terminal time. In a symmetric scheme, the manager's compensation is proportional to the difference between the portfolio return and the benchmark return:

$$H^\pi = C\Big(\frac{\hat{X}_T^\pi}{\hat{X}_0} - e^{\beta T}\Big). \tag{18}$$

As in the case of the continuous monitoring, the trading strategy maximizing manager's satisfaction does not depend on the benchmark rate. Taking $\beta = 0$ simplifies manager's optimization problem to

$$E\Big(\frac{\hat{X}_T^\pi}{\hat{X}_0} - 1\Big) - \theta Var\Big(\frac{\hat{X}_T^\pi}{\hat{X}_0} - 1\Big) \tag{19}$$

with $\theta = \frac{C\gamma}{2}$. Since the quantity $\frac{\hat{X}_T^\pi}{\hat{X}_0} - 1$ is the return of the portfolio, the optimization problem (19) is consistent with a standard Markowitz portfolio optimization approach. Hence, by an appropriate selection of C a mean-variance investor can align their risk aversion with the risk aversion of the fund manager (see Starks [37] for similar results). This also implies that the symmetric compensation scheme (18) is equivalent with manager's co-ownership of the fund.

A remuneration scheme with a hurdle-rate provision results in the payoff

$$H^\pi = C\Big(\frac{\hat{X}_T^\pi}{\hat{X}_0} - e^{\beta T}\Big)^+,$$

where β is the hurdle rate.

Similar to the case of continuous monitoring, above two schemes can be represented in a unified form:

$$H^\pi = C\Big(\hat{X}_T^\pi - Ke^{\beta T}\Big)^+, \tag{20}$$

where the choice $K = 0$ leads to a symmetric scheme whereas $K = 1$ corresponds to a scheme with the hurdle-rate provision. An optimization problem faced by a manager with mean-variance preferences and the risk aversion coefficient γ is

$$E\Big(\Big(\hat{X}_T^\pi - Ke^{\beta T}\Big)^+\Big) - \theta Var\Big(\Big(\hat{X}_T^\pi - Ke^{\beta T}\Big)^+\Big) \to \max. \tag{21}$$

3.3. Numerical approach

Optimization problems (17) and (21) are solved numerically. We apply explicit-implicit scheme to the discretized Hamilton-Jacobi-Bellman equations (10) and (5) for the auxiliary optimal control problems. Maximization with respect to ψ is constrained to a closed interval (see Theorems 2.2 and 2.4) which is found by solving appropriate optimal control problems (see Column 3 in Table 1). In the case with continuous monitoring, this has to be done numerically. When compensation scheme is based on the terminal value of the fund, the interval for ψ is given analytically:

LEMMA 3.1. *Assume that* $\pi_t \in [0, \pi_{\max}]$ *for some* $\pi_{\max} > 0$ *and* $\mu > r$*. For any* $t_0 < T$*, we have*

$$\psi_{max} = -(e^{rT}x - e^{\beta T}K)^+,$$
$$\psi_{min} = -xe^{\pi_{max}(\mu-r)T+rT}\Phi(d_1) + e^{\beta T}K\Phi(d_2),$$

where Φ *is the cumulative distribution function of the standard normal distribution and*

$$d_1 = \frac{\log(x/K) + \pi_{max}(\mu - r)(T - t_0) + \frac{1}{2}\pi_{max}^2\sigma^2(T - t_0) - (\beta - r)T}{\pi_{max}\sigma\sqrt{T - t_0}},$$
$$d_2 = d_1 - \pi_{max}\sigma\sqrt{T - t_0}.$$

The proof of the above lemma (see the Electronic Supplement) is surprisingly difficult. First, the optimization problems (6) with the payoff (20) are of a non-standard form: the objective function is not concave which prevents the use of the well-developed theory of continuous-time utility optimization. Another difficulty stems from the fact that the objective function is not differentiable.

Numerical solution of HJB equations requires trimming of the state space (to a bounded region: interval or rectangle) as well as discretizing it. The trimming of the state space and the choice of discretization grid in time and space is usually guided by experience. Existing mathematical results explore the speed of convergence as the bounded region expands and the number of discretization points tend to infinity; such results do not allow to assess the precision of a given computation. Moreover, the verification theorem for viscosity solutions to HJB equations (see [22]) is not applicable in our setting. Hence, it is not known whether the strategy extracted from the discretized HJB approximates the optimal strategy. We propose an approach that fills these gaps. We store the strategy obtained while solving the discretized HJB for the optimal ψ^* and use it to run a Monte Carlo simulation. In each Monte Carlo run we generate a trajectory of the stock prices and invest according to the computed optimal strategy. The collection of samples of manager's compensation is stored and used to approximate the value of their mean-variance functional. Table 1 displays results for a selection

T	K	$[\psi_{min}, \psi_{max}]$	ψ^*	PDE value	Monte Carlo
		Terminal-time performance			
5	1	[-0.441503, -0]	-0.222016	0.156465	(0.154375, 0.157366)
5	0	[-1.65037, -1.28531]	-1.34597	1.31546	(1.31516, 1.31706)
		Continuous monitoring			
5	1	[-1.2517742, -0.0000004]	-0.8463702	0.6296674	(0.590237, 0.636558)
5	0	[-7.3013320, -6.4201271]	-6.6405067	6.5496516	(6.53762, 6.56716)

Table 1: Numerical approximations of the value function u (equation (3)) and the value function v (equation (7)) at the initial time $t_0 = 0$ and the initial wealth $x = 1$. Column 3 displays the interval in which the search of ψ takes place. Column 4 shows the optimum. An approximation of $u(0,1)$, computed using this optimal ψ, is given in Column 5. Last column displays a 98% confidence interval for the value of the functional (3) computed by a Monte Carlo simulation with 100000 runs using the strategy extracted from the discretization of the Hamilton-Jacobi-Bellman equations (5) and (10). In the computation we use $r = 0.05$, $\mu = 0.1$, $\sigma = 0.2$, $C = 1$ and $\beta = 0.06$. The risk aversion coefficient $\gamma = 6$. Trading strategies are constrained to the interval $[0, 1]$ ($\pi_{\max} = 1$). For terminal-time performance: time $[0, T]$ is divided into 300 steps; the state space grid (in terms of $\log(x)$) contains 2000 points spread evenly over $[-2, 3]$. For continuous monitoring: time is divided into 5000 steps for $T = 5$; the state space grid is two-dimensional; one dimension (in terms of $\log(x)$) contains 200 points spread evenly over $[-2, 3]$; the other (in terms of y) has 500 points located uniformly on $[-1, 10]$.

of compensation schemes and model parameters. They hint that the approach presented in this paper with an auxiliary variable ψ performs well. It is also efficient; in the case of terminal value functional, all the computation of the value function , i.e., the maximization over ψ and solution of auxiliary optimization problems required around 5 seconds.[5]. In comparison, each Monte Carlo computation took 90 seconds. Understandably, computational intensity of numerical scheme solving the problem with the continuous monitoring increases dramatically due to the introduction of another state variable y. In addition, the functional is based on an integral over time. Its computation requires a dense time grid, which has a further impact on the computation time. It is still, however, practical taking a few minutes with most of the time used for the determination of ψ^*. This can be sped up by doing pre-runs with rougher grids.

Interested reader is referred to Section A of the Electronic Supplement for a more detailed presentation of numerical methods and proofs of their convergence.

4. Trading strategies

In this section we analyze optimal trading strategies for the various managerial compensation schemes presented in Section 3. Specifically, we look at

(SC) symmetric compensation scheme ($K = 0$),

[5] All computations were performed on a Dell Lattitude E6400 laptop with Intel Core2 Duo 2.54GHz processor.

	scheme	K	β	C	utility
Terminal time	SC	0	0	1	1.314
	HR	1	0.06	41	1.303
Continuous monitoring	SC	0	0	0.2	1.309
	HR	1	0.06	15	1.385

Table 2: Results for the calibration of four models: two for terminal-time performance and two for continuous-monitoring of performance. Time horizon is $T = 5$. The last column displays the values of the mean-variance objective function for calibrated proportionality factors C. Due to computational complexity (fine discretization of the state space is required for large values of θ) the utility value in the last row is not as well fitted as for other functionals.

(HR) asymmetric compensation scheme with a hard hurdle rate provision ($K = 1, \beta = 0.6$),

based on the continuously monitored performance (see (16)) and on the terminal value (see (20)). We fix the market parameters at: $\mu = 0.1, \sigma = 0.2$, and $r = 0.05$. The manager's risk aversion is $\gamma = 6$.[6] We assume that borrowing is not allowed, i.e., the maximum investment in the stock is 100% of the wealth ($\pi_{max} = 1$). Relaxation of this assumption is discussed in Subsection 4.4. We constrain the analysis to the investment horizon $T = 5$.[7]

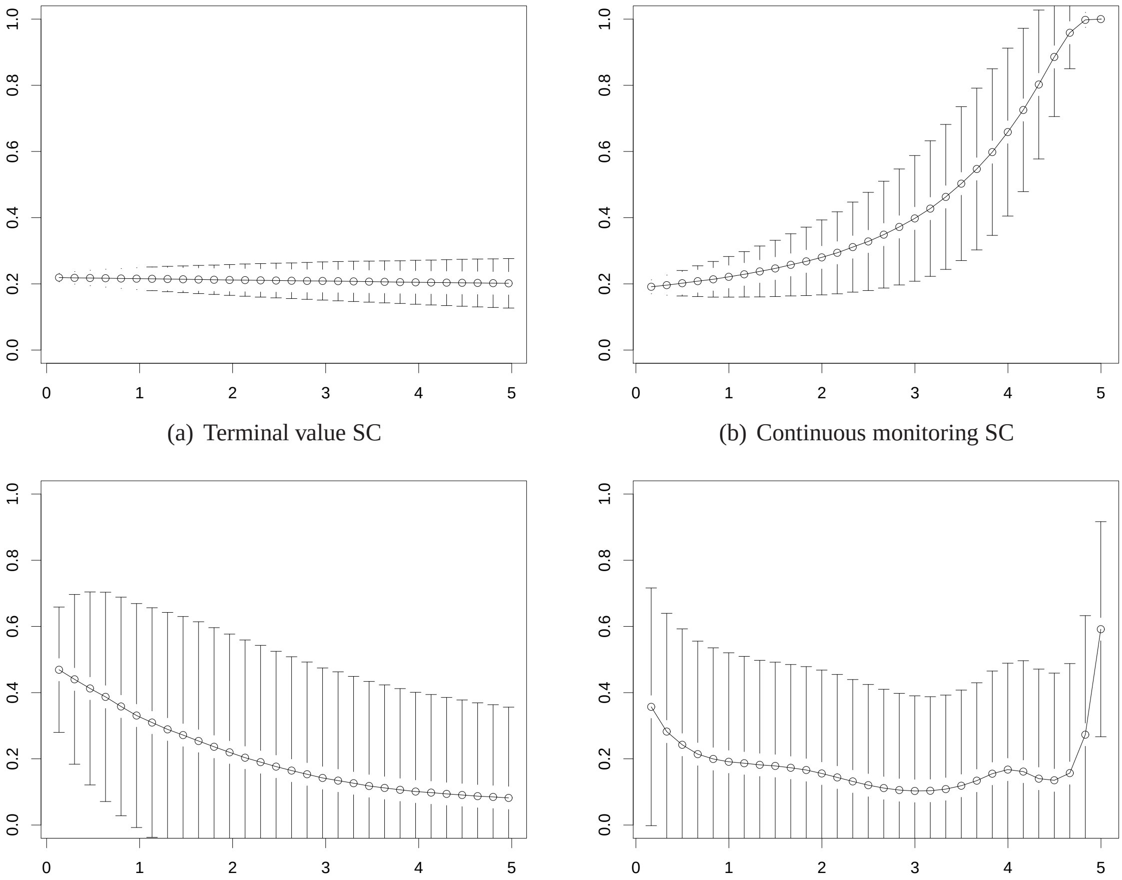
4.1. Consistent comparison of strategies for different functionals

The risk-adjusted expected payoff to the manager heavily depends on the choice of compensation scheme (see Table 1). Indeed, for terminal-wealth based schemes and $C = 1$, the symmetric risk-adjusted payoff is 1.315 whereas the asymmetric one with the hurdle rate provision is 0.156. To ensure the level playing field, we propose to choose the compensation scaling factor C for each scheme in such a way that the manager is indifferent to the choice of his remuneration scheme, i.e., such that the value

$$E(H^\pi) - \frac{\gamma}{2} Var(H^\pi)$$

is independent of the choice of the scheme. This sets our approach apart from the majority of the literature analyzing relations between investment strategies and fund manager's compensation contracts; these studies rarely take into account that different incentives imply different levels of payoff to managers, see, e.g., Carpenter [14], Kouwenberg and Ziemba [26], Panageas and Westerfield [35].

Table 2 shows the results of the calibration of four compensation schemes to be studied in this section.



(a) Terminal value SC

(b) Continuous monitoring SC

(c) Terminal value HR

(d) Continuous monitoring HR

Figure 1: Compensation schemes based on terminal-time performance and continuously monitored performance. Plots of realized portfolio positions along simulated trajectories of stock price. The horizontal axis displays time in years. The vertial axis is the portfolio position – the proportion of the wealth invested in the stock. The circles are the averages of portfolio positions at any given time. The length of the bar above/below the mean represents the standard deviation of the portfolio positions. Trading strategies were constrained to the interval $[0, 1]$ ($\pi_{max} = 1$). Model parameters (C, K and β) are collected in Table 2. The manager's risk aversion is $\gamma = 6$. The market parameters are $\mu = 0.1, \sigma = 0.2$, and $r = 0.05$. These plots are obtained from a Monte Carlo simulation with 10000 realizations of stock prices.

4.2. Trading strategies for terminal value based compensation

Figure 1 displays realized portfolio positions along simulated trajectories of the stock price. Circles represent the average stock position at each time. The length of the bar above/below the mean shows the standard deviation. Optimal strategy for the asymmetric scheme (Panel (c)) is more volatile and tends to invest more in the risky asset than the one for the symmetric scheme (Panel (a)). This gives support to the findings of Elton et al. [16] who demonstrate that stronger incentives encour-

[6]Risk-aversion between 4 and 6 is commonly shown to be typical for the stock market investments, see, e.g., [12, 31].

[7]Results for other investment horizons are similar. We choose the investment horizon longer than a year because differences between compensation schemes are more pronounced for longer horizons and graphs are easier to analyze.

	Terminal time		Continuous monitoring	
Compensation scheme	SC	HR	SC	HR
Sharpe ratio	0.98	0.63	0.57	0.52
Expected annualized excess return	1.91%	1.89%	3.50%	1.72%
Standard deviation of annualized excess return	1.95%	3.00%	6.15%	3.35%

Table 3: Sharpe ratios for returns over 5 years obtained by following optimal strategies. These numbers were computed via Monte Carlo simulation with 10000 realizations of stock prices.

age higher riskiness of trading strategies (higher investment in the risky asset); in our setting the HR scheme offers stronger incentives. The explanation of this phenomenon is however more complicated. Managers remunerated with any of the schemes increase the risky investment after a period of poor performace in order to recoup the losses. However, compared to the symmetric scheme, the optimal strategy for the asymmetric scheme prescribes higher stock investment for small values of wealth and smaller stock investment for large values. This is rather unintuitive; one would expect managers paid according to the asymmetric schemes to hold larger stock positions in all cases. It seems that the primary goal of managers with HR schemes is to beat the benchmark. Their optimal strategy shows strong reactions to the values of the wealth below the benchmark with this behavior becoming even more pronounced towards the end of the investment period. This agrees with results obtained by other authors in various frameworks, see Carpenter [14], and Kouwenberg and Ziemba [26].

Above results suggest that asymmetric schemes induce benchmark tracking behavior. Recall that symmetric schemes cannot enforce benchmark tracking. Investors willing to include the benchmark into manager's compensation should therefore consider including an asymmetric scheme as part of the compensation contract.

Arya and Mittendorf [5] associate stronger incentives with higher abilities of portfolio managers. In our setting, the asymmetric scheme induces strategies that react quickly to market changes resulting in a much larger variation compared to symmetric schemes (see Figure 1(a) and (c)). Therefore, fund managers with asymmetric compensation schemes have to make quicker and more precise trading decisions, which requires more skills than strategies prescribed by the symmetric scheme.

Distributional properties of the annualized fund returns obtained by following optimal strategies are presented in Table 3. Agarwal et al. [2] and Arya and Mittendorf [5] suggest that stronger incentives encourage better performance. Our findings demonstrate that it is not necessarily true. The Sharpe ratio as well as the expected return of a fund whose manager is remunerated by a scheme with a hurdle-rate provision is lower than those for the fund with a symmetric compensation. It is not the strength but an appropriate choice of incentives that leads to a better performance, see Kouwenberg and Ziemba [26] for detailed analysis of the interplay between the symmetric (co-ownership) and asymmetric incentives in the fund management.

4.3. Continuously monitored performance

Figure 1, Panels (b) and (d), shows realized portfolio positions obtained via a Monte Carlo simulation. A common feature of both strategies is the increase of the stock investment when time approaches the investment horizon T.[8] Such behavior was also observed by Panageas and Westerfield [35] in a different modeling framework in which they studied the impact of high-water marks and investment horizon on portfolio positions. Their conclusion was that it is not the asymmetry of the compensation scheme but the finiteness of the investment horizon that encourages fund managers to opt for large stakes in the stock towards the end of the investment period. Our findings for terminal-value based compensation schemes contradict their conclusions: trading strategies for all types of compensation do not show any tendencies of increasing stock positions when time approaches T. In our opinion the integral form of the functional, the feature that is shared by Panageas and Westerfield's model and our compensation schemes based on continuously monitored performance, contributes mostly to this behavior. It results from the interaction between the risk aversion and the ability to accumulate performance over time. High proportion of risky investment at the beginning of the investment period increases the probability of a substancial decrease in the portfolio value. Such drop means that no compensation is accumulated until the loss is recouped increasing the riskiness of the compensation earned by the manager. On the other hand, a big loss closer to the terminal time has disproportionately smaller consequences.

Compensation schemes based on continuous monitoring of portfolio value lead to inferior properties of the terminal time wealth distributions compared to the terminal-value based schemes (see Table 3). This is in agreement with the findings of Agarwal et al. [2] who notice that the discretion of fund managers impacts the returns: the larger the flexibility that managers enjoy the better their investment results. Our results show that this phenomenon is not caused by psychological reaction to close scrutiny but is rather an optimal behavior. Continuous monitoring of portfolio wealth prevents managers from implementing strategies with long-term goals (and possible short-term losses) since

[8]This behavior persists when changing model parameters, investment horizon T and the maximal stock holding π_{max}.

Compensation scheme	SC	HR
Sharpe ratio	0.98	0.57
Expected annualized excess return	1.89%	2.13%
Standard deviation of annualized excess return	1.94%	3.72%

Table 4: Sharpe ratios for returns over 5 years obtained by following optimal strategies for terminal-wealth based functionals and the upper bound on the stock position $\pi_{max} = 2$. These numbers were computed via Monte Carlo simulation with 10000 realizations of stock prices.

the short-term performance impacts their compensation in the same way as the long-term does. This not only lowers the Sharpe ratio compared to the terminal-wealth based schemes, but also boosts the riskiness of the excess returns. This comes as a surprise since continuous monitoring (closer scrutiny of managers' decisions) should, intuitively, encourage better management.

4.4. *Relaxation of the borrowing constraint*

In this subsection we relax the borrowing constraint; we allow the leverage of up to 2 ($\pi_{max} = 2$). We restrict our attention to terminal-wealth based schemes for two reasons as we have already shown the superiority of terminal-wealth based schemes over the continuously monitored schemes. This behavior is further amplified by the relaxation of the borrowing constraint.

Table 4 displays properties of the annualized excess returns. The relaxation of the borrowing constraint results in the decrease of the Sharpe ratio for the asymmetric scheme; bigger investment flexibility benefits the manager (via higher risk-adjusted expected payoff) but worsens the Sharpe ratio for the fund. The returns implied by an optimal strategy for the symmetric scheme are unchanged. Indeed, this strategy is unaffected by the relaxation of the borrowing constraint. This finding lends support to the claims of supremacy of symmetric schemes over the asymmetric ones.

The relaxation of the restriction on the borrowing yields an increase of the expected excess return for the HR scheme; the corresponding quantity for the symmetric scheme remains unchanged. A manager with an asymmetric scheme exploits an increased investment flexibility available on the market to boost the returns; the expected excess return for the HR scheme are substantially higher than for the SC scheme. This supports the view, which we share with Arya and Mittendorf [5], that asymmetric compensation schemes should only be awarded to highly skilled managers.

5. Conclusions

In this paper, we presented theoretical and numerical results for the optimization of a mean-variance functional based on the terminal-time value or on the whole trajectory of the underlying process. An optimization problem of this form cannot be studied directly by employing dynamic programming methods. We reformulated it as a superposition of a static and a dynamic optimization problem, where the latter is feasible for dynamic programming methods. We characterized its value function as the unique continuous, polynomially growing, viscosity solution to an appropriate degenerate Hamilton-Jacobi-Bellman equation. Our reformulation of the mean-variance problem allowed us to numerically calculate the value function and an optimal strategy for a pre-determined risk-aversion coefficient.

We applied this theory to the delegated portfolio management problem: given a compensation contract, a mean-variance optimizing fund manager seeks a trading strategy that maximizes her (risk-adjusted) compensation. Our mathematical and numerical framework can accomodate optimization problems induced by complex (real-world) contracts. In particular, we were able to study the relatively unexplored relations between strategies pursued by managers remunerated according to schemes based on the terminal wealth and on the continuously monitored wealth. Surprisingly, terminal-wealth based schemes turned out to induce more prudent investment behavior and superior performance than the schemes that rely on the continuously monitored wealth.

Goetzmann et al. [21] show, in a discrete-time setting, an advantage of frequent monitoring of portfolio value. Our continuous-time results contradict this finding.[9] Future research will aim at explaining this paradox. We will also try to design compensation schemes that benefit from the continuous monitoring of the portfolio value and lead to trading strategies outperforming those induced by terminal-time based schemes.

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[9]The model of the financial market studied in Section 4 is complete. Therefore, many options and other derivative contracts on the stock price, in particular many of those considered in Goetzmann et al. [21], are included in our framework through the replicating strategies. The limitation in the use of options stems from the requirement that the total position in the stock must be non-negative.

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Electronic supplement

A. Numerical schemes for manager's compensation problems

In this section we design numerical schemes for the solution of the manager's optimization problems introduced in Section 3. In the first part we discuss numerical schemes for functionals depending on the terminal value of the wealth process. The second part is devoted to functionals based on the continuously monitored performance. We follow the same order of presentation as in Section 2 to introduce our numerical methodology in the simpler terminal-wealth case.

To simplify the notation, we introduce a discounted wealth process $X_t = e^{-rt}\hat{X}_t$.[10] This process satisfies the following stochastic differential equation:

$$dX_t = \pi_t X_t((\mu - r)\,dt + \sigma\,dW_t). \tag{A.1}$$

Notice that (X_t) stays positive provided that the initial value is positive.

In this section we assume that short-selling is not allowed and the leverage is bounded, i.e.,

(B$_1$) $\pi_t \in A := [0, \pi_{max}]$.

This assumption allows us to simplify numerical schemes and shorten proofs. Results of this section can be extended, with some effort, to strategies with a bounded leverage ($\pi_t \in [\pi_{min}, \pi_{max}]$ with arbitrary $\pi_{min} \in \mathbb{R}$). The bound on the leverage is imposed by regulators on many financial institutions (see, e.g., [1]).

We also assume (rather naturally) that the rate of return of the risky stock is not smaller that the riskless interest rate:

(B$_2$) $\mu \geq r$.

Finally, we assume that

(B$_3$) $\pi_{max} \leq \frac{2(\mu-r)}{\sigma^2}$.

The above assumptions can also be relaxed at the cost of a more complicated numerical scheme.[11]

[10] We suppress the dependence of both processes on the control (π_t) in order to simplify the notation. This dependence is signalled by the expectation and variance operators.

[11] Under (B$_2$)-(B$_3$) we can always approximate the first derivative with respect to x with a forward difference. Relaxation of these assumptions requires an adaptive numerical scheme that ensures the use of forward and backward differences as appropriate, see [39].

A.1. Numerical solution for the case of terminal wealth-based compensation

In the notation of Subsection 2.1 and using the discounted wealth process, the value function corresponding to the functional (21) has the following form: for $t_0 \leq T$

$$u(t_0, x) = \sup_{\pi \in \mathcal{A}} \left\{ E^{\pi}_{t_0,x} h(X_T) - \theta \operatorname{Var}^{\pi}_{t_0,x} h(X_T) \right\}, \tag{A.2}$$

where $\mathcal{A}$ is the set of all progressively measurable processes with values in $[0, \pi_{max}]$ and

$$h(x) = (xe^{rT} - Ke^{\beta T})^{+}. \tag{A.3}$$

Recall that $v(0, 1)$ corresponds to the optimal utility for a fund manager. The resulting strategies are analyzed in Section 4.

A.1.1. Markovian reparametrization

As (A.2) is not suitable for markovian optimization methods, we rewrite it in the following way (see Section 2):

$$u(t_0, x) = \sup_{\psi \in \mathbb{R}} \{U(t_0, x, \psi) - \theta \psi^2\},$$

where U is the value function corresponding to an auxiliary optimization problem that fits the markovian optimization framework:

$$U(t_0, x, \psi) = \sup_{\pi \in \mathcal{A}} E^{\pi}_{t_0,x} \left\{ (1 - 2\theta\psi) h(X_T) - \theta \left(h(X_T)\right)^2 \right\}.$$

For fixed (t_0, x) the above supremum can be restricted to a compact interval $\psi \in [\psi_{min}, \psi_{max}]$ (see Theorem 2.2), where

$$\psi_{min} = -\sup_{\pi \in \mathcal{A}} E^{\pi}_{t_0,x} h(X_T), \qquad \psi_{max} = -\inf_{\pi \in \mathcal{A}} E^{\pi}_{t_0,x} h(X_T). \tag{A.4}$$

Since h is non-negative, both numbers are non-positive. We will use this property in a number of places later in this subsection. Lemma 3.1 provides analytical expressions for ψ_{min} and ψ_{max}.

A.1.2. Localization and change of variables

By virtue of Theorem 2.1 for any fixed ψ the function $U(\cdot, \psi)$ is the unique continuous polynomially growing viscosity solution to the following HJB equation:

$$\begin{cases} U_{t_0} + \sup_{a \in [0, \pi_{\max}]} \left\{ a(\mu - r) x U_x + \frac{1}{2} a^2 x^2 \sigma^2 U_{xx} \right\} = 0, \\ U(T, x, \psi) = (1 - 2\theta\psi) h(x) - \theta (h(x))^2. \end{cases} \tag{A.5}$$

The state space of this equation is infinite: $x \in \mathbb{R}$. A common approach used for numerical solution of such problems is to localize the equation, i.e., to restrict the state space to a compact interval

$[x_{min}, x_{max}]$ and set appropriate Dirichlet boundary conditions in x_{min} and x_{max} (see Barles, Daher and Romano [7], Wang and Forsyth [39]). We can only state approximate boundary conditions since we do not know the solution U, but this is not a serious limitation. In [7] authors justify that the error incurred by these approximate boundary conditions is small for x far from the boundaries. They study in detail the case when the boundary conditions are equal to the terminal value

$$U(t_0, x_{min}, \psi) = U(T, x_{min}, \psi), \qquad U(t_0, x_{max}, \psi) = U(T, x_{max}, \psi), \qquad t_0 \in [0, T).$$

Using their approach, a localized version of (A.5) reads as follows: for $x \in [x_{min}, x_{max}]$

$$\begin{cases} U_{t_0} + \sup_{a \in [0, \pi_{\max}]} \left\{ a(\mu - r) x U_x + \frac{1}{2} a^2 x^2 \sigma^2 U_{xx} \right\} = 0, \\ U(t_0, x_{min}, \psi) = (1 - 2\theta\psi) h(x_{min}) - \theta (h(x_{min}))^2, \\ U(t_0, x_{max}, \psi) = (1 - 2\theta\psi) h(x_{max}) - \theta (h(x_{max}))^2, \\ U(T, x, \psi) = (1 - 2\theta\psi) h(x) - \theta (h(x))^2. \end{cases} \tag{A.6}$$

Notice that $U(t_0, x, \psi)$ equals 0 for $x \in (-\infty, 0]$. It suffices to restrict our attention to the remaining part of the state space and consider $x_{min} > 0$. This is further justified by the fact that the set $(0, \infty)$ is invariant for the dynamics of (X_t)[12].

The process (X_t) is a controlled geometric Brownian motion and its increments are proportional to the value of the process. Equidistant discretization of the state space of (A.5) would, therefore, be inappropriate. A common solution is a change of variables: $z = \log(x)$. With a slight abuse of notation a new value function with the changed variable is still denoted by U. Equation (A.6) takes the following form:

$$\begin{cases} U_{t_0} + \sup_{a \in [0, \pi_{\max}]} \left\{ [a(\mu - r) - \frac{1}{2} a^2 \sigma^2] U_z + \frac{1}{2} a^2 \sigma^2 U_{zz} \right\} = 0, \\ U(t_0, z_{min}, \psi) = (1 - 2\theta\psi) h(e^{z_{min}}) - \theta (h(e^{z_{min}}))^2, \\ U(t_0, z_{max}, \psi) = (1 - 2\theta\psi) h(e^{z_{max}}) - \theta (h(e^{z_{max}}))^2, \\ U(T, z, \psi) = (1 - 2\theta\psi) h(e^z) - \theta (h(e^z))^2, \end{cases} \tag{A.7}$$

for $z \in [z_{min}, z_{max}]$, where $z_{min} = \log(x_{min})$ and $z_{max} = \log(x_{max})$.

A.1.3. Implicit numerical scheme

Fix $\psi \leq 0$. Equation (A.7) satisfies the strong comparison property for viscosity solutions and, hence, has a unique continuous viscosity solution U (see, e.g., Fleming and Soner [18, Remark V.8.1] or Wang and Forsyth [39, Remark 2.1]). Define an equidistant space grid $z_0, \ldots, z_M$, where $z_0 = z_{min}$

[12] The process (X_t) starting from a positive initial value stays positive.

and $z_M = z_{max}$, and an equidistant time grid $\tau_0, \ldots, \tau_N$, where $\tau_0 = 0$ and $\tau_N = T$. Let $\delta_z = z_1 - z_0$ be the space discretization step and $\delta_\tau = \tau_1 - \tau_0$ be the time discretization step. Denote by U_i^n a discrete approximation to $U(t_n, z_i, \psi)$ and put $U^n = (U_1^n, \ldots, U_M^n)^T$.

Under (B_1)-(B_3) the coefficient by U_z in (A.7) is non-negative. To ensure monotonicity[13] of a numerical scheme for equation (A.7), we approximate U_z with a forward difference, i.e., by $(U(t_0, z + \delta_z, \psi) - U(t_0, z, \psi))/\delta_z$. For a vector of controls $a = (a_1, \ldots, a_M) \in [0, \pi_{max}]^M$ we define a controlled discrete differential operator:

$$
\begin{aligned}
(L^a U^n)_i = {} & \frac{2a_i(\mu - r)\delta_z + (1-\delta_z)(a_i)^2\sigma^2}{2\delta_z^2} U_{i+1}^n + \frac{-2a_i(\mu - r)\delta_z - (2-\delta_z)(a_i)^2\sigma^2}{\delta_z^2} U_i^n \\
& + \frac{(a_i)^2\sigma^2}{2\delta_z^2} U_{i-1}^n.
\end{aligned}
$$

Fully implicit discretization of (A.7) takes the following form:

$$
\begin{cases}
\frac{U_i^{n+1} - U_i^n}{\delta_\tau} + \sup_{a \in [0, \pi_{max}]^M} (L^a U^n) = 0, & i = 1, \ldots, M-1, \quad n = 0, \ldots, N-1 \\
U_0^n = (1 - 2\theta\psi) h(e^{z_0}) - \theta\big(h(e^{z_0})\big)^2, & n = 0, \ldots, N-1 \\
U_M^n = (1 - 2\theta\psi) h(e^{z_M}) - \theta\big(h(e^{z_M})\big)^2, & n = 0, \ldots, N-1 \\
U_i^N = (1 - 2\theta\psi) h(e^{z_i}) - \theta\big(h(e^{z_i})\big)^2, & i = 0, \ldots, M.
\end{cases}
\tag{A.8}
$$

The following theorem ensures the convergence of the above numerical approximation.

THEOREM A.1. *Under assumptions (B_1)-(B_3) the solution of the above discrete Hamilton-Jacobi-Bellman equation converges uniformly to the solution of* (A.7) *as $N, M \to \infty$.*

A.1.4. Algorithm

We can now present an algorithm for the solution of the original mean-variance problem (A.2). Without loss of generality we assume that the initial time equals 0. Fix an initial capital $x > 0$.

1. Fix a space grid $z_0, \ldots, z_M$ in such a way that $\log(x) \approx (z_0 + z_M)/2$, and the step δ_z is appropriately small. We will comment on the selection of the grid later in this section.
2. Fix a time grid $\tau_0, \ldots, \tau_N$. We will comment on the selection of the grid later in this section.
3. Compute ψ_{min} and ψ_{max} using formulas derived in Lemma 3.1.
4. Find a maximum of $g(\psi) = U(0, x, \psi) - \theta\psi^2$ over $\psi \in [\psi_{min}, \psi_{max}]$. For each candidate ψ solve (A.5) as discussed above. The search might be performed either by maximization over a

[13] For the definition of monotonicity see, e.g., [18, Section IX.4].

discrete grid in $[\psi_{min}, \psi_{max}]$ or via DC programming methods.[14] In the examples discussed in this section we use the first approach as in practice the interval $[\psi_{min}, \psi_{max}]$ is fairly small.

5. After finding the optimal ψ^* perform another computation of $U(0, x, \psi^*)$ and store the resulting strategy.

A.1.5. Convergence

Outcomes of the above numerical computations might heavily depend on a few crucial parameters that set the grid on which the discretization of the HJB equation is based: the number of space grid points M and the number of time steps N.

	$K = 0$		$K = 1$	
N	ψ	value	ψ	value
10	-1.0619387	1.0565641	-0.0838146	0.0646984
100	-1.0619612	1.0565812	-0.0882061	0.0693725
200	-1.0619612	1.0565820	-0.0884347	0.0696070
300	-1.0619612	1.0565823	-0.0885697	0.0696854
400	-1.0619612	1.0565824	-0.0886154	0.0697234

Table A.1: Numerical approximations of the value function u at the initial time $t_0 = 0$ and the initial wealth $x = 1$ with varying number of time steps N. The bounds for ψ are $[-1.1051709, -1.0512711]$ for $K = 0$ and $[-0.1096888, -0]$ when $K = 1$. In the computation we use $T = 1$, $r = 0.05$, $\mu = 0.1$, $\sigma = 0.2$, $\beta = 0.06$ and $\theta = 3$. Trading strategies are constrained to the interval $[0, 1]$ ($\pi_{max} = 1$). The state space grid is $z_0 = -2$, $z_M = 3$ with $M = 2000$.

Table A.1 summarizes our experiments with varying number of time steps. For the two cases, the symmetric one ($K = 0$) and the asymmetric one ($K = 1$), we report the optimal value of ψ and the computed approximation to the value function for the initial value $x = 1$. The results for $K = 0$ are surprisingly good: it suffices to take $N = 10$ time steps to obtain a very good precision (the relative difference between the first and the last row is of the order 10^{-5}). Due to this quality of results the optimal ψ is identical for all choices of N (with a small difference for $N = 10$). This property is not shared by the outcomes of simulations for the asymmetric case, where reliable results are obtained for N greater or equal to 100. This difference, in our opinion, is caused by properties of optimal strategies. As we show later, an optimal strategy for the symmetric case is less sensitive to small changes in wealth and in time to maturity T. Indeed, the asymmetric functional requires an optimal strategy to keep the wealth above the benchmark at the same time reducing the risk. This results in frequent adjustments to the portfolio.

[14] The mapping $g(\psi)$ is a difference of two convex functions. There are efficient numerical methods for maximization of such functions (called the DC programming), see, e.g., Horst and Hoang [23, Chapter 10].

	$K=0$		$K=1$	
M	ψ	value	ψ	value
50	-1.0597145	1.0541610	-0.0927347	0.0657653
100	-1.0610219	1.0553888	-0.0885261	0.0696073
1000	-1.0619387	1.0565197	-0.0886611	0.0697280
2000	-1.0619612	1.0565823	-0.0885697	0.0696854
3000	-1.0619837	1.0566032	-0.0887068	0.0696687

Table A.2: Numerical approximations of the value function u at the initial time $t_0 = 0$ and the initial wealth $x = 1$ with varying number of space steps M. The extreme points of the state space grid are $z_0 = -2$, $z_M = 3$. The bounds for ψ are $[-1.1051709, -1.0512711]$ for $K = 0$ and $[-0.1096888, -0]$ when $K = 1$. In the computation we use $T = 1$, $r = 0.05$, $\mu = 0.1$, $\sigma = 0.2$, $\beta = 0.06$ and $\theta = 3$. Trading strategies are constrained to the interval $[0, 1]$ ($\pi_{max} = 1$). The time interval T is divided into $N = 300$ steps.

The results for varying number of space steps are collected in Table A.2. The speed of convergence is similar for both functionals.

We repeated the above convergence experiments for various model parameters and the results were consistent with those presented here.

A.2. Numerical solution for the case of continuously monitored performance

In this subsection we concentrate our efforts on the numerical solution of the optimization problem (17). In terms of the discounted wealth process, the value function takes the following form: for $t_0 \le T$

$$v(t_0, x) = \sup_{\pi\in\mathcal{A}} \left\{ E^{\pi}_{t_0,x}\Big(\int_0^T f(t, X_t)dt\Big) - \theta\,\mathrm{Var}^{\pi}_{t_0,x}\Big(\int_0^T f(t, X_t)dt\Big) \right\}, \tag{A.9}$$

where $\mathcal{A}$ is the set of all progressively measurable processes with values in $[0, \pi_{max}]$ and

$$f(t, x) = (xe^{rT} - Ke^{\beta t + r(T-t)})^{+}.$$

The presentation of the remaining part of this subsection follows a similar pattern as that of Subsection A.1 but differs in important details due to an additional dimension in the controlled process and the type of functionals involved.

A.2.1. Markovian reparametrization

We rewrite (A.9) as follows

$$v(t_0, x) = \sup_{\psi\in\mathbb{R}} \{V(t_0, x, 0, \psi) - \theta\psi^2\},$$

where V is the value function of an auxiliary markovian optimization problem with an extended state space (X_t, Y_t):

$$V(t_0, x, y, \psi) = \sup_{\pi\in\mathcal{A}} E^{\pi}_{x,y}\Big\{ \int_{t_0}^T f(t, X_t)[1 - 2\theta\psi - 2\theta Y_t]\, dt\Big\}. \tag{A.10}$$

The dynamics of (Y_t) is given by a differential equation

$$dY_t = f(t, X_t)dt, \qquad Y_{t_0} = y.$$

Theorem 2.4 implies that the supremum over ψ can be restricted to a compact interval $[\psi_{min}, \psi_{max}]$ with

$$\psi_{min} = -\sup_{\alpha\in\mathcal{A}} E_x^\alpha \int_t^T f(s, X_s)\, ds, \qquad \psi_{max} = -\inf_{\alpha\in\mathcal{A}} E_x^\alpha \int_t^T f(s, X_s)\, ds.$$

Notice that both of these numbers are non-positive since f is non-negative. As previously, this property will be used in the derivation and convergence of the numerical scheme.

Computation of ψ_{min} and ψ_{max} requires numerical solution of stochastic optimization problems: we use an implicit scheme with policy iteration. We omit details as these problems are of a standard form.

A.2.2. Localization and change of variables

For a fixed ψ the value function $V(\cdot, \psi)$ is the unique continuous polynomially growing viscosity solution to the following HJB equation (see Theorem 2.3):

$$\begin{cases} V_t + \sup_{a\in[0,\pi_{max}]} \Big[a(\mu - r)xV_x + \frac{1}{2}a^2\sigma^2x^2V_{xx} + f(t, x)V_y + f(t, x)(1 - 2\theta\psi - 2\theta y)\Big] = 0, \\ V(T, x, y, \psi) = 0. \end{cases} \tag{A.11}$$

Notice that $V(t_0, x, y, \psi) = 0$ for $x \le 0$ so we restrict numerical computations to $(x, y) \in (0, \infty) \times \mathbb{R}$ (this is also an invariant set for the dynamics of the process (X_t, Y_t)). We localize equation (A.11) by choosing a rectangular region $[x_{min}, x_{max}] \times [y_{min}, y_{max}] \subseteq (0, \infty) \times \mathbb{R}$. As in Subsection A.1, the value of V at the terminal time may be used as a boundary condition at times $t < T$. This seems counterintuitive since V is zero at time T, but it is strictly positive (and growing when t decreases) for $y \ge 0$ and $t < T$. We expected that choosing zero boundary conditions for the localized equation would lead to a smaller value of V than required. We tested this hypothesis in two ways. First we doubled the length of the intervals $[x_{min}, x_{max}]$ and $[y_{min}, y_{max}]$ and noticed that this leads to no change of the computed value function for x, y in the middle of the rectangle. Second, we applied boundary conditions derived from asymptotic behavior of the value function as $x \to 0, \infty$ and $y \to \pm\infty$. Results were again identical.

After a change of variables, $z = \log(x)$, a localized version of (A.11) takes the form:

$$\begin{cases} V_t + f(t,e^z)V_y + f(t,e^z)(1-2\theta\psi-2\theta y) \\ \qquad + \sup_{a\in[0,\pi_{max}]}\Big[[a(\mu-r)-\frac{1}{2}a^2\sigma^2]V_z + \frac{1}{2}a^2\sigma^2 V_{zz}\Big] = 0, \\ V(t_0, z_{min}, y) = V(t_0, z_{max}, y) = 0, \\ V(t_0, z, y_{min}) = V(t_0, z, y_{max}) = 0, \\ V(T, z, y, \psi) = 0. \end{cases} \tag{A.12}$$

A.2.3. Implicit numerical scheme

Fix $\psi \le 0$. Using standard arguments ([18, Remark V.8.1], [39, Remark 2.1]) we show that (A.12) satisfies the strong comparison property for viscosity solutions and, hence, has a unique continuous viscosity solution $V(\cdot,\psi)$. Define uniform space grids $z_0,\dots,z_M$, where $z_0 = z_{min}$, $z_M = z_{max}$, and $y_0,\dots,y_L$, where $y_0 = y_{min}$, $y_L = y_{max}$, and a uniform time grid $\tau_0,\dots,\tau_N$, where $\tau_0 = 0$, $\tau_N = T$. Let $\delta_z = z_1 - z_0$, $\delta_y = y_1 - y_0$ be the space discretization steps and $\delta_\tau = \tau_1 - \tau_0$ be the time discretization step. Denote by $V^n_{i,j}$ a discrete approximation to $V(t_n, z_i, y_j, \psi)$ and put $V^n = (V^n_{i,j})$.

Under (B_1)-(B_3) the coefficients standing by V_z and V_y in (A.12) are non-negative. Derivatives V_z and V_y are, therefore, approximated with a forward difference to ensure monotonicity of a numerical scheme. For a matrix of controls $(a_{i,j}) \in [0,\pi_{max}]^{M\times M}$ define a controlled discrete differential operator

$$\begin{aligned}(L^a V^n)_{i,j} = &\frac{2a_{i,j}(\mu-r)\delta_z + (1-\delta_z)(a_{i,j})^2\sigma^2}{2\delta_z^2}V^n_{i+1,j} \\ &+ \frac{-2a_{i,j}(\mu-r)\delta_z - (2-\delta_z)(a_{i,j})^2\sigma^2}{2\delta_z^2}V^n_{i,j} \\ &+ \frac{(a_{i,j})^2\sigma^2}{2\delta_z^2}V^n_{i-1,j}.\end{aligned}$$

Fully implicit discretization of (A.12) takes the following form:

$$\frac{V^{n+1}_{i,j} - V^n_{i,j}}{\delta_\tau} + f(t_n, e^{z_i})\Big(\frac{V^n_{i,j+1} - V^n_{i,j}}{\delta_y} + (1-2\theta\psi-2\theta y_j)\Big) + \sup_{a\in[0,\pi_{max}]^{M\times M}}(L^a V^n) = 0 \tag{A.13}$$

with boundary conditions

$$V^n_{0,j} = V^n_{M,j} = V^n_{i,0} = V^n_{i,L} = V^N_{i,j} = 0$$

for $i = 1,\dots,M$, $j = 1,\dots,L$, and $n = 1,\dots,N$. In a similar way as in Theorem A.1 we prove convergence of the above scheme.

THEOREM A.2. *Under assumptions (B_1)-(B_3) the solution of the above discrete Hamilton-Jacobi-Bellman equation converges uniformly to the solution of* (A.12) *as $L, N, M \to \infty$.*

	$K=0$			$K=1$		
N	$[\psi_{min}, \psi_{max}]$	ψ	value	$[\psi_{min}, \psi_{max}]$	ψ	value
10	-1.08131, -1.05193	-1.0528690	0.6833594	-0.07332, -0.00263	-0.0608738	0.0488927
50	-1.07877, -1.05129	-1.0580779	0.9883867	-0.06959, -0.00015	-0.0626881	0.0515996
100	-1.07848, -1.05128	-1.0591700	1.0231266	-0.06917, -0.00004	-0.0622946	0.0518549
200	-1.07834, -1.05127	-1.0596893	1.0402415	-0.06896, -0.00001	-0.0621028	0.0519749
500	-1.07826, -1.05127	-1.0599647	1.0504315	-0.06883, -0.00000	-0.0619895	0.0520443
1000	-1.07823, -1.05127	-1.0601576	1.0538152	-0.06879, -0.00000	-0.0619520	0.0520671
2000	-1.07822, -1.05127	-1.0599977	1.0565555	-0.06877, -0.00000	-0.0619333	0.0520784

Table A.3: Numerical approximations of the value function u at the initial time $t_0 = 0$ and the initial wealth $x = 1$ with varying number of time steps N. In the computation we use $T = 1$, $r = 0.05$, $\mu = 0.1$, $\sigma = 0.2$, $\beta = 0.06$ and $\theta = 3$. Trading strategies are constrained to the interval $[0, 1]$ ($\pi_{max} = 1$). The state space grid is $z_0 = -2$, $z_M = 3$ with $M = 200$ and $y_0 = -1$, $y_L = 3$ with $L = 100$.

	$K=0$		$K=1$	
L	ψ	value	ψ	value
50	-1.0580075	0.8039931	-0.0619333	0.0376220
100	-1.0599977	1.0565555	-0.0619333	0.0520784
200	-1.0613756	1.0555008	-0.0619333	0.0519358
400	-1.0613756	1.0549085	-0.0619333	0.0518711
800	-1.0613756	1.0545861	-0.0619333	0.0518389

Table A.4: Numerical approximations of the value function u at the initial time $t_0 = 0$ and the initial wealth $x = 1$ with varying number of space steps L in variable y. The bounds for ψ are independent of L and equal to $[-1.07822, -1.05127]$ for $K = 0$ and $[-0.06877, 0]$ for $K = 1$. In the computation we use $T = 1$, $r = 0.05$, $\mu = 0.1$, $\sigma = 0.2$, $\beta = 0.06$ and $\theta = 3$. Trading strategies are constrained to the interval $[0, 1]$ ($\pi_{max} = 1$). The state space grid is $z_0 = -2$, $z_M = 3$ with $M = 200$ and $y_0 = -1$, $y_L = 3$. The number of time steps is $N = 2000$.

A.2.4. Algorithm and convergence

The algorithm for the solution of the above equation is analogous to that in Subsection A.1.

Our experiments show that numerical results for the mean-variance functional with the integral term (A.9) exhibit large errors for small number of timesteps N, see Table A.3. This is clearly at odds with our findings for the terminal-time based functional – good precision is obtained for the number of time steps N as low as 10. It can be explained by the fact that the functional studied in this subsection is based on the integral over time – a dense time grid is crucial for the precision of its computation. Another observation lends support to this claim. The results are fairly precise for N as low as 50 if $K = 1$ (the asymmetric functional), but the case $K = 0$ requires at least $N = 500$ for acceptable precision. This is contrary to our findings for the terminal-time functionals, where the symmetric case ($K = 0$) requires fewer timesteps. This difference might be caused by the fact that the integrand is smaller for $K = 1$ (it might be even equal to 0 when the wealth of portfolio does not exceed the benchmark) than for $K = 0$ and integration errors are hence less pronounced.

	$K=0$			$K=1$		
M	$[\psi_{min}, \psi_{max}]$	ψ	value	$[\psi_{min}, \psi_{max}]$	ψ	value
50	-1.07886, -1.05127	-1.0595801	1.0558601	-0.06683, -0.00000	-0.0601739	0.0478990
100	-1.07843, -1.05127	-1.0600660	1.0563309	-0.06866, -0.00000	-0.0618301	0.0514420
200	-1.07822, -1.05127	-1.0599977	1.0565555	-0.06877, -0.00000	-0.0619333	0.0520784
400	-1.07811, -1.05127	-1.0599642	1.0566644	-0.06860, -0.00000	-0.0617795	0.0521047
800	-1.07806, -1.05127	-1.0599475	1.0567180	-0.06847, -0.00000	-0.0616580	0.0520548

Table A.5: Numerical approximations of the value function u at the initial time $t_0 = 0$ and the initial wealth $x = 1$ with varying number of space steps M in variable z. In the computation we use $T = 1$, $r = 0.05$, $\mu = 0.1$, $\sigma = 0.2$, $\beta = 0.06$ and $\theta = 3$. Trading strategies are constrained to the interval $[0, 1]$ ($\pi_{max} = 1$). The state space grid is $z_0 = -2$, $z_M = 3$ and $y_0 = -1$, $y_L = 3$ with $L = 100$. The number of time steps is $N = 2000$.

Table A.4 shows that there is little benefit from increasing the number of grid points in the variable y above 100. A similar observation can be made with respect to the number of grid points in the variable z (see Table A.5). Our experiments showed that there is an optimal ratio of the grid step sizes in variables z and y. The decrease of the grid step in one dimension brings little improvement if not accompanied by an adjustment in the other. We discovered that, in the example studied here, the length of y-step should be around twice the length of the z-step.

B. Proofs

Proof of Theorem 2.1. Fix $\psi \in \mathbb{R}$. Define a Hamiltonian corresponding to equation (5) (see [36, Section 3.4] or [3, Section 1.4] for the notation)

$$H^\psi : [0, T) \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^d \to \mathbb{R} \cup \{\infty\},$$

where $\mathbb{S}^d$ is the set of symmetric $d \times d$ matrices, by the formula

$$H^\psi(t, x, p, M) = \sup_{a \in A} \Big\{ b(a, t, x)^T p + \frac{1}{2} tr(\sigma\sigma^T(a, t, x)M) \Big\}.$$

The PDE (5) can now be written as:

$$\begin{cases} U_{t_0}(t_0, x, \psi) + H^\psi(t, x, U_x(t_0, x, \psi), U_{xx}(t_0, x, \psi)) = 0, \\ U(T, x, \psi) = (1 - 2\theta\psi)h(x) - \theta(h(x))^2, \end{cases} \tag{B.14}$$

where U_{t_0} denotes the partial derivative with respect to time t_0 and U_x, U_{xx} denote the first and second derivative with respect to x.

The domain of the Hamiltonian H^ψ is defined as

$$\mathrm{dom}(H^\psi) = \{(t, x, p, M) \in [0, T) \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^d : H^\psi(t, x, p, M) < \infty\}.$$

For any fixed $(t, x, p, M) \in [0, T) \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^d$ and due to the continuity of b and σ with respect to $a \in A$ and the compactness of the set A, we have that $\sup_{a\in A}\left\{b(a, t, x)^T p + \frac{1}{2}tr(\sigma\sigma^T(a, t, x)M)\right\} < \infty$. Therefore the domain of H^ψ is the whole space. By Assumption (A_0), the coefficients b and σ are Lipschitz continuous with respect to x and t uniformly in a. This together with the continuity if b and σ with respect to a and the fact that the set of controls A is compact implies that H^ψ is continuous. By [36, Proposition 4.3.2] or [3, Theorem 3.6] $U(\cdot, \psi)$ is a viscosity subsolution to (5). The value function $U(\cdot, \psi)$ is of polynomial growth under Assumption (A_0) and due to the polynomial growth for the cost function h (see [3, Lemma 3.3]). By [36, Proposition 4.3.1] or [3, Theorem 3.4], $U(\cdot, \psi)$ is a viscosity supersolution to (5). Since $U(\cdot, \psi)$ is both a viscosity subsolution and supersolution then it is a viscosity solution to equation (5).

We shall now prove the uniqueness and continuity of solutions to (5). The value function $U(t_0, x, \psi)$ is continuous at $t_0 = T$ due to the continuity and polynomial growth of h and the estimate (D.4) in [18, Appendix D]. The comparison theorem ([36, Theorem 4.4.5] or [3, Corollary 4.7]) yield the continuity of $U(\cdot, \psi)$ in the whole of its domain $[0, T] \times \mathbb{R}^d$ and assure that $U(\cdot, \psi)$ is a unique polynomially growing continuous viscosity solution to (5). □

Proof of Theorem 2.2. (i): Take $\psi_1, \psi_2 \in \mathbb{R}$. By the definition of U we have

$$
\begin{aligned}
&\frac{1}{2}U(t_0, x, \psi_1) + \frac{1}{2}U(t_0, x, \psi_2) \\
&\geq \sup_{\alpha\in\mathcal{A}} E^\alpha_{t_0,x,y}\Big\{\frac{1}{2}\Big((1 - 2\theta\psi_1)h(X_T) - \theta\,(h(X_T))^2\Big) + \frac{1}{2}\Big((1 - 2\theta\psi_2)h(X_T) - \theta\,(h(X_T))^2\Big)\Big\} \\
&= U(t_0, x, (\psi_1 + \psi_2)/2).
\end{aligned}
$$

This proves convexity, which implies continuity with respect to ψ.

(ii): One can see that U grows at most linearly, which implies that the mapping $g(\psi) = U(t_0, x, \psi)-\theta\psi^2$ attains its maximum in a compact interval. By convexity, U has well-defined directional derivatives. Hence, g also has well-defined directional derivatives and in a point where the maximum is attained the left-hand side derivative is non-negative while the right-hand side derivative is non-positive. We shall show that $\partial^+ g(\psi) > 0$ for $\psi < \psi_{min}$ and $\partial^- g(\psi) < 0$ for $\psi > \psi_{max}$. This will imply that the conditions for maximum can only be satisfied in the interval $[\psi_{min}, \psi_{max}]$.

For the right-hand side differential of U we obtain the following lower bound:

$$\begin{aligned}
\partial_\psi^+ U(t_0, x, \psi) &= \lim_{\delta \downarrow 0} \frac{U(t_0, x, \psi + \delta) - U(t_0, x, \psi)}{\delta} \\
&= -\lim_{\delta \downarrow 0} \frac{U(t_0, x, \psi) - U(t_0, x, \psi + \delta)}{\delta} \\
&\geq -\lim_{\delta \downarrow 0} \frac{\sup_{\alpha \in \mathcal{A}} E^\alpha_{t_0,x}\{(- 2\theta\psi + 2\theta(\psi + \delta))h(X_T)\}}{\delta} \\
&= -\sup_{\alpha \in \mathcal{A}} E^\alpha_{t_0,x}\{2\theta h(X_T)\} \\
&= 2\theta\psi_{min}.
\end{aligned}$$

Therefore, $\partial^+ g(\psi) \geq 2\theta\psi_{min} - 2\theta\psi = 2\theta(\psi_{min} - \psi)$, and $\partial^+ g(\psi) > 0$ when $\psi < \psi_{min}$.

The left-hand side differential of U is bounded from above in a similar fashion:

$$\begin{aligned}
\partial_\psi^- U(t_0, x, \psi) &= \lim_{\delta \downarrow 0} \frac{U(t_0, x, \psi) - U(t_0, x, \psi - \delta)}{\delta} \\
&\leq \lim_{\delta \downarrow 0} \frac{\sup_{\alpha \in \mathcal{A}} E^\alpha_{t_0,x}\{(- 2\theta\psi + 2\theta(\psi - \delta))h(X_T)\}}{\delta} \\
&= \sup_{\alpha \in \mathcal{A}} E^\alpha_{t_0,x}\{ - 2\theta h(X_T)\} \\
&= 2\theta\psi_{max}.
\end{aligned}$$

This implies $\partial^- g(\psi) \leq 2\theta\psi_{max} - 2\theta\psi = 2\theta(\psi_{max} - \psi)$ and $\partial^- g(\psi) < 0$ when $\psi > \psi_{max}$. □

Remark to the proof of Theorem 2.2. Formulas for ψ_{min} and ψ_{max} can be obtained directly from the duality relationship (4) provided that for each ψ there exists an optimal strategy maximizing $U(t_0, x, \psi)$. Indeed, if ψ^* maximizes (4) and α^* is an optimal strategy for $U(t_0, x, \psi^*)$ then

$$\psi^* = -E^{\alpha^*}_{t_0,x}\{h(X_T)\} \in \Big[- \sup_{\alpha \in \mathcal{A}} E^\alpha_{t_0,x}\{h(X_T)\}, - \inf_{\alpha \in \mathcal{A}} E^\alpha_{t_0,x}\{h(X_T)\}\Big].$$

The original proof of Theorem 2.2 does not require the existence of optimal strategies. □

Proof of Theorem 2.3. Fix $\psi \in \mathbb{R}$. We rewrite (10) in a canonical form:

$$\begin{cases}
-\hat{V}_{t_0}(t_0, \tilde{x}, \psi) - H^\psi\Big(t, \tilde{x}, \hat{V}_{\tilde{x}}(t_0, \tilde{x}, \psi), \hat{V}_{\tilde{x}\tilde{x}}(t_0, \tilde{x}, \psi)\Big) = 0, \\
\hat{V}(T, \tilde{x}, \psi) = 0,
\end{cases}$$

where $\tilde{x} = (x, y)$, the Hamiltonian H^ψ is given by

$$\begin{aligned}
H^\psi\Big(t, (x, y), \tilde{p}, \tilde{M}\Big) = \sup_{u \in A} \Big[& b(u, t, x)^T p + \frac{1}{2} tr(\sigma\sigma^T(u, t, x) M), \\
&+ (1 - 2\theta\psi - 2\theta y + p_y) f(u, t, x)\Big],
\end{aligned}$$

with $\tilde{p} = (p, p_y) \in \mathbb{R}^d \times \mathbb{R}$ and M is obtained from $\tilde{M}$ by removing the last row and column. The domain of the Hamiltonian is defined as

$$\mathrm{dom}(H^\psi) = \{(t, \tilde{x}, \tilde{p}, \tilde{M}) \in [0, T) \times \mathbb{R}^{d+1} \times \mathbb{R}^{d+1} \times \mathbb{S}^{d+1} : H^\psi(t, \tilde{x}, \tilde{p}, \tilde{M}) < \infty\}.$$

The assumptions of Lipschitz continuity and linear growth for the drift coefficients (b and f) of the extended process (9) and diffusion coefficient σ (Assumptions (A_0) and (A_1)) implies that H^ψ is continuous and its domain is the whole space. [36, Proposition 4.3.2] or [3, Theorem 3.6] imply that $\hat{V}(\cdot, \psi)$ is a viscosity subsolution to (10). By virtue of [3, Lemma 3.3] the value function $\hat{V}(\cdot, \psi)$ is of polynomial growth because of Assumption (A_0) and the linear growth of the running cost function f. By [36, Proposition 4.3.1] or [3, Theorem 3.4] $\hat{V}(\cdot, \psi)$ is a viscosity supersolution to (10). Since $\hat{V}(\cdot, \psi)$ is both a viscosity subsolution and supersolution then it is a viscosity solution to equation (10).

In order to show uniqueness, we need continuity of the value function $\hat{V}(t_0, x, y, \psi)$ at the terminal time $t_0 = T$. Indeed, function $\hat{V}(\cdot, \psi)$ is continuous at $t_0 = T$ due to the Lipschitz continuity in t, x and linear growth in x of f, b and σ. The comparison theorem [36, Theorem 4.4.5] or [3, Corollary 4.7] yield the continuity of $\hat{V}(\cdot, \psi)$ in the whole of its domain $[0, T] \times \mathbb{R}^d \times \mathbb{R}$ and assure that $\hat{V}(\cdot, \psi)$ is a unique polynomially growing continuous viscosity solution to (10).

□

Proof of Theorem 2.4. Stochastic control problem (7) is a special case of the problem (3) with the state space (9) and $h(x, y) = y$. Theorem 2.2 implies assertion (iii) and the first part of assertion (i). Monotonicity of $\hat{V}$ with respect to ψ is immediate.

(ii): We have

$$\begin{aligned}\hat{V}(t_0, x, y, \psi_1) - \hat{V}(t_0, x, y, \psi_2) &\le \sup_{\alpha \in \mathcal{A}} \Big\{ 2\theta(\psi_2 - \psi_1) E^\alpha_{t_0,x,y} \Big\{ \int_{t_0}^T f(\alpha_t, t, X_t)\, dt \Big\} \Big\} \\ &\le 2\theta |\psi_2 - \psi_1| \sup_{\alpha \in \mathcal{A}} E^\alpha_{t_0,x,y} \Big\{ \int_{t_0}^T |f(\alpha_t, t, X_t)| dt \Big\}.\end{aligned}$$

Under assumption (A_0), formula (D.5) in Fleming and Soner [18] implies that there exist a constant C_1 such that for any $\alpha \in \mathcal{A}$

$$E^\alpha_{t_0,x} \{ \sup_{t \in [t_0, T]} \|X_t\|^2 \} \le C_1 (1 + \|x\|^2). \tag{B.15}$$

Linear growth of f allows us to write

$$
\begin{aligned}
E^{\alpha}_{t_0,x,y}\Big\{\int_{t_0}^{T} |f(\alpha_t,t,X_t)|dt\Big\} &\le E^{\alpha}_{t_0,x,y}\Big\{\int_{t_0}^{T} K_2(1+\|X_t\|)dt\Big\}\\
&\le K_2(T-t_0)\, E^{\alpha}_{t_0,x,y}\{1+\sup_{t\in[t_0,T]}\|X_t\|\}\\
&\le K_2(T-t_0)\,(1+(E^{\alpha}_{t_0,x,y}\{\sup_{t\in[t_0,T]}\|X_t\|^2\})^{1/2})\\
&\le K_2(T-t_0)\,(1+(C_1+C_1\|x\|^2)^{1/2})\\
&\le C_2(1+\|x\|)
\end{aligned}
$$

for some constant C_2. In the above derivation, the third inequality follows form the Hölder inequality, and the fourth – from (B.15). Hence,

$$\hat{V}(t_0,x,y,\psi_1)-\hat{V}(t_0,x,y,\psi_2)\le 2\theta|\psi_2-\psi_1|C_2(1+\|x\|).$$

The corresponding lower bound follows similarly. This proves (ii) with $C = 2\theta C_2$.

□

Proof of Lemma 3.1. The values ψ_{min} and ψ_{max} are determined by the following value functions

$$\overline{v}(t_0,x)=\sup_{\pi\in\mathcal{A}} E^{\pi}_{t_0,x}(X_T e^{rT}-Ke^{\beta T})^+,\qquad \underline{v}(t_0,x)=\inf_{\pi\in\mathcal{A}} E^{\pi}_{t_0,x}(X_T e^{rT}-Ke^{\beta T})^+.$$

Standard theory of optimal control (see, e.g., Fleming and Soner [18]) implies that the value function $\underline{v}(t_0,x)$ is a unique continuous viscosity solution with polynomial growth to the following HJB equation:

$$
\begin{cases}
\underline{v}_{t_0}+\inf_{a\in[0,\pi_{\max}]}\left\{a(\mu-r)x\underline{v}_x+\frac{1}{2}a^2x^2\sigma^2\underline{v}_{xx}\right\}=0,\\
\underline{v}(T,x)=(xe^{rT}-Ke^{\beta T})^+.
\end{cases}
$$

A direct verification shows that

$$\underline{v}(t_0,x)=(e^{rT}x-e^{\beta T}K)^+ \tag{B.16}$$

solves this equation (it is then easy to check that an optimal control is $\pi\equiv 0$). Indeed, for $t_0\in[0,T]$, $x>0$ and $x\neq x^*:=e^{(\beta-r)T}K$ the function $\underline{v}$ is of the class $C^{1,2}$. The HJB equation is satisfied then in a classical sense and, hence, in the viscosity sense. The verification of the viscosity solution property at (t_0,x^*) is more involved. The function $\underline{v}$ is a viscosity subsolution at (t_0,x^*) because the set of test functions is empty at this point (see [36, Definition 4.2.1] or [3, Definition 2.1]). For the proof that $\underline{v}$ is a viscosity supersolution take a test function $\phi\in C^{1,2}$ such that $\phi(t_0,x^*)=\underline{v}(t_0,x^*)$ and $\phi\le\underline{v}$. We have to show that

$$\phi_{t_0}(t_0,x^*)+\inf_{a\in[0,\pi_{\max}]}\Big\{a(\mu-r)x\phi_x(t_0,x^*)+\frac{1}{2}a^2x^2\sigma^2\phi_{xx}(t_0,x^*)\Big\}\le 0.$$

The mapping $t \mapsto \phi(t, x^*)$ has a local maximum at t_0 since the function $\underline{v}$ is independent of the first argument, t_0 (see (B.16)). This implies that $\phi_{t_0}(t_0, x^*) = 0$. The second term is non-positive because the expression under infimum equals 0 for $a = 0$. This completes the proof of the supersolution property of $\underline{v}$.

Determination of the optimal strategy for $\overline{v}$ is more complicated. Let $h^\varepsilon : \mathbb{R} \to \mathbb{R}$, $\varepsilon > 0$, be a family of twice continuously differentiable non-decreasing convex functions approximating h in the supremum norm, i.e., $\|h - h^\varepsilon\|_\infty < \varepsilon$.[15] Define

$$\hat{u}^\varepsilon(t, x) = E^{\pi_{\max}}_{t_0,x}\{h^\varepsilon(X_T)\}, \qquad x > 0, \quad t \in [0, T],$$

where by $\pi_{\max}$ we denote the control equal identically to $\pi_{\max}$. Equivalently,

$$\hat{u}^\varepsilon(t, x) = E\{h^\varepsilon(x\xi_{T-t_0})\},$$

where (ξ_t) satisfies the following stochastic differential equation

$$d\xi_t = \xi_t\, \pi_{\max}((\mu - r)\, dt + \sigma\, dW_t), \qquad \xi_0 = 1.$$

This reformulation implies that $\hat{u}^\varepsilon$ is non-decreasing in x. To prove convexity of $\hat{u}^\varepsilon$ in x, take $x_1, x_2 > 0$:

$$\begin{aligned}\frac{\hat{u}^\varepsilon(t_0, x_1) + \hat{u}^\varepsilon(t_0, x_2)}{2} &= E\Big\{\frac{1}{2}h^\varepsilon(x_1\xi_{T-t_0}) + \frac{1}{2}h^\varepsilon(x_2\xi_{T-t_0})\Big\}\\ &\geq E\Big\{h^\varepsilon\Big(\frac{x_1 + x_2}{2}\xi_{T-t_0}\Big)\Big\} = \hat{u}^\varepsilon\Big(t_0, \frac{x_1 + x_2}{2}\Big),\end{aligned}$$

where the inequality follows from the convexity of h^ε and the positivity of ξ_{T-t_0}.

Theorem 2.9.10 in Krylov [27] implies that $\hat{u}^\varepsilon$ is a $C^{1,2}$ solution with a linear growth of the following PDE problem:

$$\begin{cases}\hat{u}^\varepsilon_{t_0} + \pi_{\max}(\mu - r)x\hat{u}^\varepsilon_x + \frac{1}{2}\pi^2_{\max}x^2\sigma^2\hat{u}^\varepsilon_{xx} = 0,\\ \hat{u}^\varepsilon(T, x) = h^\varepsilon(x).\end{cases} \tag{B.17}$$

We shall prove that

$$\sup_{a\in[0,\pi_{\max}]}\Big\{a(\mu - r)x\hat{u}^\varepsilon_x + \frac{1}{2}a^2x^2\sigma^2\hat{u}^\varepsilon_{xx}\Big\} = \pi_{\max}(\mu - r)x\hat{u}^\varepsilon_x + \frac{1}{2}\pi^2_{\max}x^2\sigma^2\hat{u}^\varepsilon_{xx}.$$

[15] For a continuous function f, the supremum norm $\|f\|_\infty$ is defined in the following way

$$\|f\|_\infty = \sup\{f(x) : \ x \in \mathbb{R}\}.$$

The derivative, with respect to a, of the mapping under supremum equals

$$(\mu - r)x\hat{u}^\varepsilon_x + ax^2\sigma^2\hat{u}^\varepsilon_{xx}.$$

This derivative is non-negative for $a \geq 0$. Indeed, the convexity of $\hat{u}^\varepsilon$ implies $\hat{u}^\varepsilon_{xx} \geq 0$. Since $\hat{u}^\varepsilon$ is non-decreasing in x, we have $\hat{u}^\varepsilon_x \geq 0$. Assumption ($B_2$) gives $\mu - r \geq 0$. Recall also that $x > 0$. Thus, the supremum is realized at $a = \pi_{max}$.[16]

Using above observation we conclude that $\hat{u}^\varepsilon$ satisfies

$$\begin{cases} \hat{u}^\varepsilon_{t_0} + \sup_{a\in[0,\pi_{\max}]}\left\{a(\mu - r)x\hat{u}^\varepsilon_x + \frac{1}{2}a^2x^2\sigma^2\hat{u}^\varepsilon_{xx}\right\} = 0, \\ \hat{u}^\varepsilon(T, x) = h^\varepsilon(x). \end{cases} \tag{B.18}$$

Theorem 3.5.2 in Pham [36] implies that $\hat{u}^\varepsilon$ is the value function corresponding to the following optimal control problem

$$[0, T] \times (0, \infty) \ni (t_0, x) \mapsto \sup_{\pi\in\mathcal{A}} E^\pi_{t_0,x}\, h^\varepsilon(X_T)$$

and the optimal control is $\pi \equiv \pi_{\max}$.

Take a control $\pi \in \mathcal{A}$. For any $\varepsilon > 0$ we have

$$E^\pi_{t_0,x}\, h(X_T) \leq E^\pi_{t_0,x}\, h^\varepsilon(X_T) + \varepsilon \leq \hat{u}^\varepsilon(t_0, x) + \varepsilon = E^{\pi_{\max}}_{t_0,x}\, h^\varepsilon(X_T) + \varepsilon \leq E^{\pi_{\max}}_{t_0,x}\, h(X_T) + 2\varepsilon.$$

The arbitrariness of ε implies that

$$E^\pi_{t_0,x}\, h(X_T) \leq E^{\pi_{\max}}_{t_0,x}\, h(X_T).$$

Hence

$$\overline{v}(t_0, x) = \sup_{\pi\in\mathcal{A}} E^\pi_{t_0,x}\, h(X_T) = E^{\pi_{\max}}_{t_0,x}\, h(X_T).$$

We infer that an optimal strategy is, therefore, constant, $\pi \equiv \pi_{max}$, and

$$\overline{v}(t_0, x) = E\Big(xe^{\pi_{max}(\mu-r)(T-t_0)-\frac{\pi^2_{max}\sigma^2}{2}(T-t_0)+\pi_{max}\sigma(W_T-W_{t_0})}e^{rT} - Ke^{\beta T}\Big)^+.$$

This last expression can be computed in a similar way as the price of a call option in the Black-Scholes model (see, e.g., Appendix to Chapter 13 in Hull [24]). □

[16]If $\hat{u}^\varepsilon_x = \hat{u}^\varepsilon_{xx} = 0$ or $\hat{u}^\varepsilon_{xx} = \mu - r = 0$ then the supremum is attained at any point of the interval $[0, \pi_{\max}]$ so, in particular, at $\pi_{\max}$. Otherwise, the derivative of the mapping under supremum is greater than zero, which implies that the maximum is realized at $a = \pi_{\max}$.

Proof of Theorem A.1. It is easy to verify that assumptions (2.1)-(2.2) of [39] are satisfied. Assumption (3.1) of [39] follows from the monotonicity of the scheme which is ensured by the use of forward differencing and assumptions (B_1)-(B_3). We conclude by applying [39, Theorem 4.1]. ☐

Proof of Theorem A.2. The proof follows similar lines as the proof of Theorem A.1. ☐