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An Orthogonal Estimation Algorithm

For

Complex Number Systems

by

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Abstract

An orthogonal estimation algorithm for complex number systems is derived. It is shown that a modified Error Reduction Ratio (ϵRR) test together with the orthogonal estimation algorithm provides an efficient way of identifying both the structure and the unknown parameters of complex number systems. A forward regression procedure is proposed as an optimal search algorithm for this problem and simulated examples which show the application of the method to the parameterisation of linear frequency response functions are included.

1. Introduction

Orthogonal least squares estimation algorithms [Korenberg 1985, Korenberg, Billings and Liu 1988] have been found to be efficient procedures for identifying unknown linear and nonlinear systems. They, do however, fall short in at least one area, the estimation of complex number systems. Complex number systems which are rare in the real world are commonly found in the signal processing field. For example, signals with real and imaginary parts are common in communication systems and in frequency response analysis. It is therefore desirable to have an efficient way of estimating unknown parameters associated with these complex number systems. In the present study it is shown that the original orthogonal least squares estimation algorithm [Korenberg, Billings and Liu 1988] can be reformulated to estimate the unknown parameters associated with complex number systems in such a way that all the properties of the original algorithms are retained. The Error Reduction Ratio (ϵRR) test is also reformulated for the complex number system case to provide an efficient way of identifying the system structure or which terms to include in the model. The parameter estimation algorithm and associated ϵRR test are then combined in a forward regression procedure to provide an optimal search algorithm for this problem. Simulated examples showing the application of the algorithm to the identification of the structure and unknown parameters of linear frequency response functions are included.

2. Orthogonal Least Squares Estimator for Complex Number Systems

Consider a complex number system which can be expressed as

$$Z(\omega) = \sum_{i=1}^{M} \theta_i p_i(\omega) + \xi(\omega) \tag{1}$$

where θ_i, $i=1,\ldots,M$ represent the M real unknown parameters associated with the complex variables $p_i(\omega)$, $i=1,\ldots,M$ and $Z(\omega)$, $\xi(\omega)$ represent the possibly complex dependent variable or output and some modelling error respectively. The objective of the estimation algorithm is to estimate the unknown parameters in eqn.(1). The proposed orthogonal least squares estimation algorithm involves transforming eqn.(1) into an auxiliary equation

$$Z(\omega) = \sum_{i=1}^{M} g_i w_i(\omega) + \xi(\omega) \tag{2}$$

where g_i, $i=1,\ldots,M$ are some constant coefficients and $w_i(\omega)$, $i=1,\ldots,M$ are constructed to be orthogonal over the complex conjugate of the data records such that

$$\overline{w_k^*(\omega)\, w_i(\omega)} = 0 \quad for \quad k \neq i \tag{3}$$

where * denotes complex conjugate and overbar $\overline{}$ denotes time average. A family of orthogonal data records could then be readily constructed by defining

$$
\begin{aligned}
w_1(\omega) &= p_1(\omega) \\
w_i(\omega) &= p_i(\omega) - \sum_{k=1}^{i-1} \alpha_{ki} w_k(\omega) \ , \ k<i
\end{aligned}
\tag{4}
$$

and

$$
\alpha_{ki} = \frac{\overline{w_k^*(\omega)\, p_i(\omega)}}{\overline{w_k^*(\omega)\, w_k(\omega)}} \ , \ k=1,\ldots,i-1
$$

Combining the auxiliary equation, eqn.(2), and the orthogonality of the data records, eqn.(3), gives the parameter estimate

$$
\hat{g}_i = \frac{\overline{Z(\omega)\, w_i^*(\omega)}}{\overline{w_i(\omega)\, w_i^*(\omega)}}
\tag{5}
$$

provided $\overline{w_i(\omega)\, w_i^*(\omega)} \neq 0$. Inspection of eqn.(5) indicates that if $w_i^*(\omega)$ is not related to the output $Z(\omega)$, the estimated orthogonal parameter $\hat{g}_i$ will be small because the average value $\overline{Z(\omega)\, w_i^*(\omega)}$ will not be significant. Hence the orthogonal parameter $\hat{g}_i$ could be used as an indicator of the significance of orthogonal terms $w_i(\omega)$. If the magnitude of the estimated orthogonal parameter $\hat{g}_i$ is less than a certain threshold, say C_g, the associated orthogonal term $w_i(\omega)$ should be regarded as insignificant. That is discard $w_i(\omega)$ from the final estimate if

$$
|\hat{g}_i| < C_g
$$

during the estimation process to avoid numerical problems in eqn.(5).

Once the parameters g_i, $i=1,\ldots,M$ have been estimated using eqn.(5), the original system parameters θ_i, $i=1,\ldots,M$ can be recovered according to the formula

$$
\begin{aligned}
\theta_M &= \hat{g}_M \\
\theta_k &= \hat{g}_k - \sum_{i=k+1}^{M} \alpha_{ki} \theta_i \ , \ k=M-1,\ldots,1
\end{aligned}
\tag{6}
$$

Equations (4), (5) and (6) define the orthogonal least squares estimation algorithm for complex number systems. The algorithm is remarkably simple and easy to implement and retains all the properties of the original orthogonal least squares algorithm. If all the data records and system parameters in eqn.(1) are real, the proposed algorithm reduces to the original orthogonal least squares algorithm [Korenberg, Billings and Liu 1988].

3. Error Reduction Ratio for Complex Number Systems

The error reduction ratio [Korenberg, Billings and Liu 1988], which is a byproduct of the orthogonal least squares estimation, can provide information regarding the significance of variables in the system model. This can act as a selection tool to sort through all possible variables and produce a parsimonious model of the system under investigation. A version for complex number systems can also be derived.

Consider the auxiliary equation

$$Z(\omega) = \sum_{i=1}^{M} g_i w_i(\omega) + \xi(\omega)$$

Multiplying the auxiliary equation with its complex conjugate and taking the time average gives

$$\overline{Z(\omega) Z^*(\omega)} = \sum_{i=1}^{M} g_i g_i^* \overline{w_i(\omega) w_i^*(\omega)} + \overline{\xi(\omega) \xi^*(\omega)} \tag{7}$$

assuming that $\xi(\omega)$ is a zero mean white noise sequence which is independent of the orthogonal data set and the orthogonal property of eqn.(3) holds. The maximum mean square of the magnitude of the prediction errors is achieved when no term is included in the model (M=0) to give

$$| \overline{\xi(\omega) \xi^*(\omega)} |_{M=0} = \overline{Z(\omega) Z^*(\omega)} \tag{8}$$

From eqn.(7), the reduction in the mean square of the magnitude of the prediction errors eqn.(9) as a result of including the term $g_i w_i(\omega)$ can be expressed as a percentage reduction in the total mean square of the magnitude of the errors by defining

$$\epsilon RR_i = \frac{g_i g_i^* \overline{w_i(\omega) w_i^*(\omega)}}{\overline{Z(\omega) Z^*(\omega)}} \times 100 \quad , i=1,\dots,M \tag{10}$$

Notice that while θ_i are always real g_i may be complex.

Hence ϵRR can serve as a selection tool for determining which system variables to include in the final model. A large error reduction ratio will thus indicate that an estimated variable is significant and should be included in the final model. Usually ϵRR_i is tested against a threshold, say C_ϵ, and the i-th term is only included in the model if ϵRR_i exceeds the threshold C_ϵ.

4. Forward Regression Algorithm

The proposed orthogonal estimator coupled with the modified error reduction ratio test provides a powerful estimation algorithm for complex number systems which can be considered as an extension of the forward regression algorithm of Billings et al [1988]. The algorithm can be summarised as follows:

a) Start the estimated model with no terms and define all possible complex variables $p_i(\omega)$, $i=1,\ldots,M$ which might result in the final estimate.

b) Employ the orthogonal least squares estimator and the error reduction ratio, eqns. (4), (5) and (10), to search through all possible variables.

c) Apply the error reduction ratio and orthogonal parameter tests on each variable in turn.
If $|\hat{g}_i| < C_g$ or $\epsilon RR_i < C_\epsilon$, $w_i(\omega)$ will be discarded from the estimation.

d) Test 1: Select the variable which has the maximum error reduction ratio as a candidate to enter into the final model.
Test 2: Alternatively, select the variable which has the maximum $|\hat{g}_i|$ as one of the candidates to enter into the final model.

e) Pass the rest of the variables to the next stage of the estimation.

f) Repeat b), c), d), and e) until the error reduction ratios or the magnitudes of the estimated orthogonal parameters of the remaining variables are all less than the chosen threshold or the variables have been exhausted.

h) Re-construct the actual system parameters using eqn.(6).

The advantage of the forward regression algorithm is that it provides an efficient yet simple method to search through all possible variables which should be included in the model. Notice that throughout it has been assumed that $\xi(\omega)$ is zero mean and white. This assumption will hold in most of the applications as discussed in section 5 below. If the noise does not satisfy these assumptions a noise estimator must be appended to the above algorithm [Korenberg, Billings and Liu 1988] to ensure the estimates are unbiased.

5. Examples

One useful application area for the proposed estimation algorithm is the parameterisation of frequency response functions from a set of frequency response data records. To illustrate the idea, consider a set of N frequency response data $H(j\omega_n)$, $n=1,\ldots,N$ generated from the linear system S_1

$$H(s) = \frac{b_0 s + b_1}{a_0 s^2 + a_1 s + 1} \tag{11}$$

In order to estimate the parameters b_0, b_1, a_0 and a_1, rewrite eqn.(11) as

$$H(j\omega) = -a_0(j\omega)^2 H(j\omega) - ja_1\omega H(j\omega) + jb_0\omega + b_1 \tag{12}$$

which is linear-in-the-parameters.

Comparing eqn.(12) with eqn.(1), we have

$$z(\omega) = \sum_{i=1}^{4} \theta_i p_i(\omega)$$

where

$$Z(\omega) = H(j\omega)$$

$$\theta_1 = a_0 \,, \quad p_1(\omega) = -(j\omega)^2 H(j\omega)$$

$$\theta_2 = a_1 \,, \quad p_2(\omega) = -j\omega H(j\omega)$$

$$\theta_3 = b_0 \,, \quad p_3(\omega) = j\omega$$

$$\theta_4 = b_1 \,, \quad p_4(\omega) = 1$$

The proposed estimation algorithm can then be applied to eqn.(12) to identify the parameters b_0, b_1, a_0 and a_1 respectively. Of course the frequency response data $H(j\omega)$ could have been obtained from several algorithms. Typically $H(j\omega)$ would be the output from a spectrum analyser or the result of digital spectral estimation. In either case it is well known that the estimates of $H(j\omega)$ are unbiased and hence the assumption that $\xi(\omega)$ is zero mean and white in previous sections is clearly justified in these type of applications.

A set of 200 equally spaced frequency response data generated from eqn.(11) in the frequency range 0 to 5 Hz with b_0, b_1, a_0 and a_1 set to 0.5, 1, 0.4 and 2 respectively, was used to estimate the transfer function of the original system. An overparametrised transfer function of the form

$$H(s) = \frac{b_0 s^3 + b_1 s^2 + b_2 s + b_3}{a_0 s^3 + a_1 s^2 + a_2 s + 1}$$

having complex variables 1, $j\omega$, $(j\omega)^2$, $(j\omega)^3$, $-j\omega H(j\omega)$, $-(j\omega)^2 H(j\omega)$ and $-(j\omega)^3 H(j\omega)$ was initially specified for the estimation so that the effectiveness of the proposed algorithm in detecting both the correct model structure and estimating the unknown parameters could be tested. The error reduction ratios for each complex variable, their corresponding orthogonal parameters and magnitudes of the orthogonal parameters for the first five iterations of the estimation are show in Table 1. Complex variables that were selected into the final model at each iteration using test 1 are underlined in Table 1 (e.g. maximum ϵRR or $|\hat{g}|$). At the end of the fourth iteration, the sum of the error reduction ratio for the selected variables is 99.999%. Error reduction ratios for the rest of the complex variables at the fifth iteration all become insignificant. It is therefore reasonable to argue that there are only four parameters in the orthogonal equation and their associated parameters are all underlined in Table 1. The original system parameters which were obtained using eqn.(6) are shown in Table 2. Clearly the actual system parameters can be recovered by the proposed estimation algorithm. However if one of the complex variables, say $j\omega$ is omitted from the estimation, the estimate will be insufficient. This would be clearly revealed by the estimated orthogonal parameters and the actual system parameters.

Since the last estimated orthogonal parameter is complex, the estimated system parameters will also be complex. Table 3 shows the result of the estimation if only three complex variables 1, $-j\omega H(j\omega)$, $-(j\omega)^2 H(j\omega)$ are included in the final estimate. All the estimated system parameters associated with the complex variables are complex. (i.e. the estimate is biased). The estimated system parameters could therefore also serve as an indicating factor of the correctness of the final estimate. If the complex parts of the final estimate are small, the deviation from the original system parameters will also be small. With only three complex variables selected, the sum of the error reduction ratio is 94.438% which clearly indicates that there is a deficiency in the estimates. Another point to note is that in the first iteration of the algorithm, complex variables which are not related to the final estimate are clearly indicated by their error reduction ratio and their associated orthogonal parameters where both of the readings are small.

A further point to note at the first iteration is that the magnitude of the orthogonal parameters corresponding to the complex variables $(j\omega)^2$, $(j\omega)^3$ and $-(j\omega)^3 H(j\omega)$, (i.e. complex variables which should not be included in the system model), are small compared to the terms that should be in the model. The contribution of these terms to the output of the system could be checked using the modified error reduction ratio. Since both of these values were small, it would be reasonable to say that they are insignificant to the final estimate and could be discarded at an early stage of the estimation. This could drastically reduce the computational time as lots of complex variables would be discarded from the estimation at each iteration. In this example, 13 complex variables would pass through the orthogonal estimator instead of 25 in the original case if C_g was chosen correctly.

Table 4 shows the first five iterations for the estimation of the same transfer function, eqn.(11), but with b_0, b_1, a_0 and a_1 set to 0.0002, 0.5, 0.4 and 0.001 respectively (System S_2). The same parameters were specified for the estimation. Tables 5 and 6 respectively show the results of the estimation if 4 and 3 complex variables were included in the final estimate. Even though some of the system parameters b_0 and a_1 are very small, the corresponding magnitudes of the estimated orthogonal parameters are significant, that is magnitudes of the estimated orthogonal parameters can be used to indicate the significance of complex variables regardless of the magnitude of the system parameters. With 4 complex variables included in the final estimate, the original system was fully recovered. However, if only three complex variables were included in the final estimate, the estimated system parameters became complex. Results were very similar to the previous example.

6. Conclusion

An orthogonal least squares algorithm has been derived for complex number systems. The orthogonal property of the algorithm allows each parameter in the auxiliary model to be estimated one at a time by repeated application of a very simple formula. Additional terms can be added to the model without the need to re-estimate all the previous model coefficients and the percentage reduction that each term makes to the mean squared of the magnitude of the output error, the ERR test, provides an extremely simple indication of the significance of each term in the model. Moreover, magnitudes of the estimated orthogonal parameters can be used to indicate whether complex variables are related to the system. Combining the modified error reduction test with orthogonal parameter value checking forms a very powerful sorting tool which can drastically reduce computational time. Other advantages are that the estimate itself provides information on the goodness of fit of the estimate and implementation on a microprocessor should be straightforward.

7. References

Korenberg M.J. [1985]: Orthogonal identification of nonlinear difference equation models; Mid West Symp. on Circuits and Systems, Louisville.

Korenberg M.J., Billings S.A. and Liu Y.P. [1988]: An orthogonal parameter estimation algorithm for nonlinear stochastic systems, Int. J. of Control, 48, 193-210.

Billings S.A., Korenberg M.J. and Chen S. [1988]: Identification of nonlinear output-affine systems using an orthogonal least squares algorithm, Int. J. of Systems Science, 19, 1559-1568.

Iteration	Complex Variables	Error Reduction Ratio	Orthogonal Parameter	
i	p_i	eRR_i	g_i	$\lvert g_i \rvert$
1	1	37.202	0.060939 - j 0.10804	0.12404
	$j\omega$	8.948	-0.003262 - j 0.000761	0.00335
	$(j\omega)^2$	4.398	-1.5447E-5 + j 9.5017E-5	9.6264E-5
	$(j\omega)^3$	2.831	2.8748E-6 + j 3.9268E-7	2.9015E-6
	$-j\omega H(j\omega)$	25.426	8.6349E-20 + j 0.09008	0.09008
	$-(j\omega)^2 H(j\omega)$	8.197	0.002616 + j 2.1415E-21	0.002616
	$-(j\omega)^3 H(j\omega)$	4.272	-5.8897E-23 - j 7.6890E-5	7.6890E-5
2	$j\omega$	25.752	0.007478 + j 0.008503	0.011323
	$(j\omega)^2$	16.016	0.000195 - j 0.000194	0.000275
	$(j\omega)^3$	11.469	-5.6381E-6 - j 5.3604E-6	7.7796E-6
	$-j\omega H(j\omega)$	54.450	0.510078 - j 0.19618	0.546503
	$-(j\omega)^2 H(j\omega)$	25.712	-0.006532 - j 0.005847	0.008767
	$-(j\omega)^3 H(j\omega)$	15.913	-0.000136 + j 0.000173	0.00022
3	$j\omega$	2.652	-0.005896 - j 0.000255	0.005901
	$(j\omega)^2$	1.846	-6.7421E-6 + j 0.000123	0.000123
	$(j\omega)^3$	1.364	3.2450E-6 + j 8.5337E-8	3.2461E-6
	$-(j\omega)^2 H(j\omega)$	2.786	0.004777 + j 0.000199	0.004781
	$-(j\omega)^3 H(j\omega)$	1.872	-4.0609E-6 - j 9.9629E-5	9.9712E-5
4	$j\omega$	5.561	0.5 + j 7.8383E-14	0.5
	$(j\omega)^2$	2.040	0.000294 - j 0.00055	0.000624
	$(j\omega)^3$	1.552	-8.7492E-6 - j 4.7854E-6	9.9724E-6
	$-(j\omega)^3 H(j\omega)$	2.049	-0.000238 + j 0.000439	0.000499
5	$(j\omega)^2$	1.012E-23	-1.1538E-16 - j 1.7417E-15	1.7455E-15
	$(j\omega)^3$	2.708E-24	-1.5355E-17 + j 2.2241E-18	1.5515E-17
	$-(j\omega)^3 H(j\omega)$	1.037E-23	2.0259E-16 + j 1.3988E-15	1.4134E-15

Table 1. First Five Iterations for the Estimation of S_1

Complex Variable	Parameter
$-j\omega H(j\omega)$	2.000
$-(j\omega)^2 H(j\omega)$	0.400
1	1.000
$j\omega$	0.500

Table 2. Estimated S_1 with 4 Variables

Complex Variable	Parameter
$-j\omega H(j\omega)$	0.6055 - j 0.3287
$-(j\omega)^2 H(j\omega)$	0.0048 + j 0.0002
1	0.7767 - j 0.2213

Table 3. Estimated S_1 with 3 Variables

Iteration	Complex Variables	Error Reduction Ratio	Orthogonal Parameter	
i	p_i	ERR_i	g_i	$\lvert g_i \rvert$
1	1	0.5013	0.642381 - j 0.326827	0.720742
	$j\omega$	0.0031	-0.001566 - j 0.00271	0.003130
	$(j\omega)^2$	3.955E-6	-1.8200E-6 + j 4.1912E-6	4.5693E-6
	$(j\omega)^3$	2.115E-6	1.0463E-8 - j 1.2511E-7	1.2554E-7
	$-j\omega H(j\omega)$	99.9882	5.3475E-18 + j 0.6334	0.633400
	$-(j\omega)^2 H(j\omega)$	99.7590	0.400276 - j 8.5205E-18	0.400276
	$-(j\omega)^3 H(j\omega)$	75.4560	-3.0227E-18 - j 0.191722	0.191722
2	1	0.0056	0.076619 + j 0.000268	0.076619
	$j\omega$	0.0013	1.4191E-5 - j 0.002024	0.002024
	$(j\omega)^2$	0.0006	-5.7964E-5 - j 5.0403E-6	5.7966E-5
	$(j\omega)^3$	0.0004	-1.6844E-8 + j 1.7485E-6	1.7486E-6
	$-(j\omega)^2 H(j\omega)$	0.0035	-0.055974 - j 3.1924E-15	0.055974
	$-(j\omega)^3 H(j\omega)$	0.0013	3.3995E-17 + j 0.001619	0.001619
3	$j\omega$	0.0034	3.7353E-6 + j 0.006524	0.006524
	$(j\omega)^2$	0.0021	0.000160 - j 8.5624E-8	0.000160
	$(j\omega)^3$	0.0015	-2.4202E-9 - j 4.5305E-6	4.5305E-6
	$-(j\omega)^2 H(j\omega)$	0.0062	0.399799 - j 0.006669	0.399855
	$-(j\omega)^3 H(j\omega)$	0.0034	-5.7431E-5 - j 0.005218	0.005219
4	$j\omega$	1.44469E-6	0.000200 + j 1.2588E-13	0.000200
	$(j\omega)^2$	1.38380E-6	-2.6510E-8 - j 5.0185E-6	5.0186E-6
	$(j\omega)^3$	1.26819E-6	-1.4976E-7 + j 1.0787E-9	1.4977E-7
	$-(j\omega)^3 H(j\omega)$	1.44468E-6	-0.000160 + j 2.4428E-6	0.000160
5	$(j\omega)^2$	1.3561E-24	-2.4102E-14 - j 2.1621E-15	2.4199E-14
	$(j\omega)^3$	7.9561E-25	-3.0270E-17 + j 3.3802E-16	3.3937E-16
	$-(j\omega)^3 H(j\omega)$	6.9558E-23	7.3220E-12 - j 3.4654E-10	3.4662E-10

Table 4. First Five Iterations for the Estimation of S_2

Complex Variable	Parameter
$-j\omega H(j\omega)$	0.0010
$-(j\omega)^2 H(j\omega)$	0.4000
1	0.5000
$j\omega$	0.0002

Table 5. Estimated S_2 with 4 Variables

Complex Variable	Parameter
$-j\omega H(j\omega)$	-0.0095 + j 0.0003
$-(j\omega)^2 H(j\omega)$	0.3998 - j 0.0067
1	0.4998 - j 0.0039

Table 6. Estimated S_2 with 3 Variables