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Periodic Solutions of Neutral Differential Equations

by

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Abstract

In this paper, we study the existence of periodic solutions of neutral differential equations with infinite delay and delay equations of n^{th} order. Some necessary and sufficient conditions for existence of periodic solutions are obtained.

Keywords: Neutral differential equations, Delay, Periodic solutions.

1 Introduction

The existence of periodic solution of differential delay equations has been studied by many authors (see,e.g.,Banks 1988, Burton 1985 and Hale 1977). In many cases the conditions for existence of periodic solutions are not particularly easy to check. Here we shall develop a simple technique for neutral differential equations with infinite delay and delay equations of n^{th} order. This will lead to easily testable conditions for the existence of periodic solutions.

In section 2 we prove a simple lemma on Fourier series and apply it in section 3 to equations with infinite delay. Delay equations of n^{th} order will be considered in section 4 and finally in section 5 some examples will be given.

2 Fourier Exponential Series

Let $f(t) : R \to R^n$ be a continuous periodic function with period 2π. Then, its Fourier exponential series is given by (see Kufner and Kadlec, 1971)

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{int}$$

where

$$C_n = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int} dt \tag{1}$$

$(n = 0, \pm 1, \pm 2, \ldots)$,whereas for a continuous periodic function with period $2l$, we have

$$f(t) = \sum_{n=-\infty}^{\infty} C'_n e^{i(n\pi t/l)}$$

3 Infinite Delay Equations

Consider the equation

$$\frac{d}{dt}\left[x(t) - \sum_{k=0}^{\infty} C_k x(t-\tau_k)\right] = \sum_{k=0}^{\infty} B_k x(t-\tau_k) + f(t) \tag{5}$$

where $x \in R^n; C_k, B_k \in R^{n\times n}; \tau_k \geq 0$ and $\tau_k \to +\infty\ ask \to +\infty$; and $f : R \to R^n$ is a continuously differentiable periodic function with period 2π, that is $f(t+2\pi) = f(t)$. Then,we may write

$$f(t) = \sum_{n=-\infty}^{\infty} f_n \cdot e^{int}$$

where

$$f_n = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}dt \tag{6}$$

$n = 0, \pm 1, \pm 2, \ldots$.

Theorem 3.1. If

$$\sum_{k=0}^{\infty} \| C_k \| < 1, \qquad \sum_{k=0}^{\infty} \| B_k \| < +\infty$$

Then, the equation (5) has a continuously differentiable periodic solution with period 2π if and only if for each n, where $0 \leq n \leq [(\sum_{k=0}^{\infty} \| B_k \| + 1)/(1 - \sum_{k=0}^{\infty} \| C_k \|)]$, ($[\bullet]$ denotes the largest integal part of $\bullet$), a solution of the equation

$$\left(inI - in\sum_{k=0}^{\infty} C_k e^{-i\tau_k n} - \sum_{k=0}^{\infty} B_k e^{-i\tau_k n}\right) x_n = f_n \tag{7}$$

exist.

Proof: 'Only If'. Let $x(t)$ be a continuously differentiable periodic solution of (5) with period 2π. Then, $x(t)$ can be written as

$$x(t) = \sum_{n=-\infty}^{\infty} x_n e^{int} \tag{8}$$

where

$$x_n = \frac{1}{2\pi}\int_0^{2\pi} x(t)e^{-int}dt$$

Then,

$$x(t-\tau_k) = \sum_{n=-\infty}^{\infty} x_n e^{-i\tau_k n} e^{int} \tag{9}$$

putting (8) and(9) into (5), and comparing the coeffeicents of both sides, we get

$$\left(inI - in\sum_{k=0}^{\infty} C_k e^{-i\tau_k n} - \sum_{k=0}^{\infty} B_k e^{-\tau_k n}\right) x_n = f_n$$

for $n = 0, \pm1, \pm2, \ldots$. this shows that the solution of (5) existe for all n and the proof for 'only if' is complete.

'If'. When $| n |\geq [(\sum_{k=0}^{\infty} \| B_k \| +1)/(1-\sum_{k=0}^{\infty} \| C_k \|)] \triangleq N$, we have

$$\begin{aligned} & \| \sum_{k=0}^{\infty} C_k e^{-i\tau_k n} + \frac{1}{in}\sum_{k=0}^{\infty} B_k e^{-i\tau_k n} \| \\ \leq & \sum_{k=0}^{\infty} \| C_k \| + \frac{1}{| n |}\sum_{k=0}^{\infty} \| B_k \| \\ \leq & \ 1 \end{aligned} \tag{10}$$

Hence, for $| n |\geq N$, the matrix

$$\begin{aligned} & \left(inI - in\sum_{k=0}^{\infty} C_k e^{-i\tau_k n} - \sum_{k=0}^{\infty} B_k e^{-i\tau_k n}\right) \\ = & \ in\left(I - \sum_{k=0}^{\infty} C_k e^{-i\tau_k n} - \frac{1}{in}\sum_{k=0}^{\infty} B_k e^{-i\tau_k n}\right) \end{aligned} \tag{11}$$

is invertiable. It follows that equation (5) can be solved for $| n | \geq N$. Hence, the solution of (5) exists for all $n = 0, \pm 1, \pm 2, \ldots$.

We shall prove that the series

$$\sum_{n=-\infty}^{\infty} x_n e^{int} \tag{12}$$

converges absolutly and uniformly.

From (7), we have

$$\left[| n | (1 - \sum_{k=0}^{\infty} \| C_k \|) - \sum_{k=0}^{\infty} \| B_k \| \right] \| x_n \| \leq \| f_n \|$$

Since $\sum_{k=0}^{\infty} \| C_k \| < 1$ and $\sum_{k=0}^{\infty} \| B_k \| < +\infty$, there existe $N_1 > 0$ such that $| n | \geq N_1$ implies that

$$| n | (1 - \sum_{k=0}^{\infty} \| C_k \|) - \sum_{k=0}^{\infty} \| B_k \| \geq \frac{| n |}{2}(1 - \sum_{k=0}^{\infty} \| C_k \|)$$

Then, for $| n | \geq N_1$, we have

$$\| x_n \| \leq \frac{2 \| f_n \|}{(1 - \sum_{k=0}^{\infty} \| C_k \|) | n |}$$

Using lemma 2.1, we get

$$\| x_n \| \leq \frac{2K}{1 - \sum_{k=0}^{\infty} \| C_k \|} \cdot \frac{1}{| n |^2}$$

for $| n | \geq N_1$, where

$$K = \frac{1}{2\pi} \left(2 \| f(0) \| + \int_0^{2\pi} \| f'(t) \| \, dt \right).$$

Then,

$$\sum_{n=-\infty}^{\infty} \| x_n \| \leq \frac{2K}{1 - \sum_{k=0}^{\infty} \| C_k \|} \sum_{|n|=N_1}^{\infty} \frac{1}{| n |^2} + \sum_{n=-N_1}^{N_1} \| x_n \| < +\infty$$

Hence,the series (12) converges absolutly and uniformly.

Let

$$x(t) = \sum_{n=-\infty}^{\infty} x_n e^{int}$$

it is easy to check that $x(t)$ is a continuously differentiable periodic solution of (5) with period 2π, this completes the proof.

Theorem 3.2. If

$$\sum_{k=0}^{\infty} \| C_k \| < 1, \qquad \sum_{k=0}^{\infty} \| B_k \| < +\infty.$$

Then, equation (5) has a unique continuously differentiable periodic solution with period 2π if and only if

$$det\left(inI - in\sum_{k=0}^{\infty} C_k e^{-i\tau_k n} - \sum_{k=0}^{\infty} B_k e^{-i\tau_k n} \right) \neq 0$$

for each n, where $0 \leq n \leq [(\sum_{k=0}^{\infty} \| B_k \| +1)/(1 - \sum_{k=0}^{\infty} \| C_k \|)]$.

Proof: Note that under the conditions, the equation (7) have unique solution for all $n = 0, \pm 1, \pm 2, \ldots$. Then, the result follows.

Corollary 3.1. If

$$\sum_{k=0}^{\infty}(\| C_k \| + \| B_k \|) < 1, \qquad det(\sum_{k=0}^{\infty} B_k) \neq 0$$

Then, the equation (5) has a unique continuously differentiable periodic solution with period 2π.

The proof is simple and we omit it here.

4 Delay Equations of n^{th} order

Consider the delay equation of n^{th} order

$$\sum_{k=0}^{n}\left[b_k x^{(n-k)}(t) + c_k x^{(n-k)}(t-\tau_k)\right] = f(t) \tag{13}$$

where $b_0 \neq 0, b_k, c_k, \tau_k(\geq 0)$ are constants, $f : R \rightarrow R$ is a continuously differentiable periodic function with period 2π. Then, we may write $f(t)$ as

$$f(t) = \sum_{m=-\infty}^{\infty} f_m \cdot e^{int}$$

Theorem 4.1 Assume that $| b_0 |>| c_0 |$, then the equation (13) has a continuously differentible periodic solution with period 2π if and only if a solution of the equation

$$\left[\sum_{k}^{n}(im)^{(n-k)}(b_k + c_k e^{-im\tau_k})\right] x_m = f_m \tag{14}$$

exists for each m, where $0 \leq m \leq [max(\lambda)+1]$, λ is a solution of

$$\lambda^n(| b_0 | - | c_0 |) - \sum_{k=1}^{n}\lambda^{(n-k)}(| b_k | + | c_k |) = 0 \tag{15}$$

Proof: 'Only If'. Let $x(t)$ be a continuously differentiable periodic solution of (13) with period 2π. Then, $x(t)$ can be written as

$$x(t) = \sum_{m=-\infty}^{\infty} x_m e^{imt}$$

where

$$x_m = \frac{1}{2\pi}\int_0^{2\pi} x(t)e^{-imt}dt$$

Then,

$$x^{(n-k)}(t) = \sum_{m=-\infty}^{\infty} x(m)\cdot(im)^{(n-k)}\cdot e^{imt}$$

and

$$x^{(n-k)}(t-\tau_k)=\sum_{m=-\infty}^{\infty} x(m)\cdot(im)^{(n-k)}\cdot e^{-im\tau_k}\cdot e^{imt}$$

Substituting these into (13), we get

$$\sum_{k=0}^{n}(im)^{(n-k)}(b_k+c_k e^{-im\tau_k})x_m=f_m$$

for $m=0,\pm1,\pm2,\ldots$. This shows that the equation (13) have solution and this completes the proof for 'only if'.

'If'. First, we shall prove that (14) can be solved for $\mid m\mid\leq[max(\lambda)+1]\triangleq M$. Since

$$\begin{aligned}&\mid\sum_{k=0}^{n}(im)^{(n-k)}(b_k+c_k e^{-im\tau_k})\mid\\ \geq\ &\mid m\mid^n(\mid b_0\mid-\mid c_0\mid)-\sum_{k=1}^{n}\mid m\mid^{(n-k)}(\mid b_k\mid+\mid c_k\mid)\end{aligned}\tag{16}$$

and

$$\mid m\mid^n(\mid b_0\mid-\mid c_0\mid)-\sum_{k=1}^{n}\mid m\mid^{(n-k)}(\mid b_k\mid+\mid c_k\mid)\longrightarrow+\infty$$

as $\mid m\mid\longrightarrow+\infty$. Then, for $\mid m\mid\geq M$,

$$\mid m\mid^n(\mid b_0\mid-\mid c_0\mid)-\sum_{k=1}^{n}\mid m\mid^{(n-k)}(\mid b_k\mid+\mid c_k\mid)>0$$

Hence, the equation (14) can be solved for $\mid m\mid\geq M$. Then, equation (14) has a solution for all $m=0,\pm1,\pm2,\ldots$.

We shall prove that the series

$$\sum_{m=-\infty}^{\infty} x_m\cdot(im)^{(s)}\cdot e^{int}\tag{17}$$

$s=0,1,2,\ldots,n-1$,converges absolutly and uniformly.

From (14), we have

$$\sum_{k=0}^{\infty}(im)^{(n-k-s)}(b_k + c_k e^{-im\tau_k})x_m \cdot (im)^{(s)} = f_m$$

Then,

$$\begin{aligned} &\left[\mid m \mid^{(n-s)} (\mid b_0 \mid - \mid c_0 \mid) - \sum_{k=1}^{n} \mid m \mid^{(n-k-s)} (\mid b_k \mid + \mid c_k \mid)\right] \mid x_m \cdot (im)^{(s)} \mid \\ \leq &\ \mid f_m \mid \end{aligned} \tag{18}$$

Since $\mid b_0 \mid > \mid c_0 \mid$, and $s \leq n-1$, there must exists a $M_1 > 0$ such that $\mid m \mid \geq M_1$ implies that

$$\begin{aligned} &\mid m \mid^{(n-s)} (\mid b_0 \mid - \mid c_0 \mid) - \sum_{k=1}^{n} \mid m \mid^{(n-k-s)} (\mid b_k \mid + \mid c_k \mid) \\ \geq &\ \frac{\mid m \mid}{2}(\mid b_0 \mid - \mid c_0 \mid) \end{aligned} \tag{19}$$

Hence, when $\mid m \mid \geq M_1$, we get

$$\mid x_m \cdot (im)^{(s)} \mid \leq \frac{2 \mid f_m \mid}{\mid b_0 \mid - \mid c_0 \mid} \cdot \frac{1}{\mid m \mid}$$

Using Lemma 2.1, it follows that

$$\mid x_m \cdot (im)^{(s)} \mid \leq \frac{2K}{\mid b_0 \mid - \mid c_0 \mid} \cdot \frac{1}{\mid m \mid^2}$$

where

$$K = \frac{1}{2\pi}\left(2 \mid f(0) \mid + \int_0^{2\pi} \mid f'(t) \mid dt\right)$$

Then,

$$\begin{aligned} &\sum_{m=-\infty}^{\infty} \mid x_m \cdot (im)^{(s)} \mid \\ \leq &\ \frac{2K}{\mid b_0 \mid - \mid c_0 \mid} \sum_{\mid m \mid = M_1}^{\infty} \frac{1}{\mid m \mid^2} + \sum_{m=-M_1}^{M_1} \mid x_m \cdot (im)^{(s)} \mid \\ < &\ +\infty \end{aligned} \tag{20}$$

This shows that the series (17) converges absolutly and uniformly. Let

$$x(t) = \sum_{m=-\infty}^{\infty} x_m e^{imt}$$

then

$$x^{(s)}(t) = \sum_{m=-\infty}^{\infty} x_m \cdot (im)^{(s)} \cdot e^{imt}$$

$s = 1, 2, \ldots, n-1$. It is easy to check that $x(t)$ is a continuously differentiable periodic solution of (13) with period 2π. This completes the proof.

Theorem 4.2. Assume that $\mid b_0 \mid > \mid c_0 \mid$, then equation (13) has a unique continuously differentiable periodic solution with period 2π if and only if

$$\sum_{k=0}^{n} (im)^{(n-k)} (b_k + c_k e^{-im\tau_k}) \neq 0$$

for each m, where $0 \leq m \leq [max(\lambda) + 1]$, λ is a solution of equation (15).

The proof is similar with the proof of theorem 3.2 and we omit it here.

Corollary 4.1. If

$$b_n + c_n \neq 0, \quad \mid b_0 \mid - \mid c_0 \mid - \sum_{k=1}^{n} (\mid b_k \mid + \mid c_k \mid) > 0$$

Then, the equation (13) has a unique continuously differntiable periodic solution with period 2π.

5 Examples

Example 5.1. Consider the equation

$$\frac{d}{dt}\left[x(t) - \frac{1}{6}\sum_{k=1}^{\infty} \frac{1}{k!} x(t-k)\right] = \frac{1}{6}\sum_{k=1}^{\infty} \frac{1}{k!} x(t-k) + \sin t \qquad (21)$$

Since

$$\frac{1}{6}\sum_{k=1}^{\infty}\frac{1}{k!}+\frac{1}{6}\sum_{k=1}^{\infty}\frac{1}{k!}=\frac{e}{3}<1$$

Then, using corollary 3.1 (21) has a unique continuously differentiable periodic solution with period 2π.

Example 5.2. Consider the equation

$$2x^{(n)}(t)-x^{(n)}(t-\tau)+\frac{1}{4}x(t)-\frac{1}{4}x(t-\tau)=\cos t \tag{22}$$

Using corollary 4.1, it easy to check that equation (22) has a unique continuously differentiable periodic solution with period 2π.

6 Conclusions

In this paper, we have investegated the periodic solution of neutral differential equations with infnity delay and neutral differentiale equations of n^{th} order via the theory of Fourier exponential series. The results seem very simple and the conditions are very mild. Through out this paper,we have considered periodic solutions with period 2π. By a simple modification we can also consider periodic solutions with a general period $2l$.

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