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Periodic Solutions of Neutral Differential Equations

by

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Abstract

In this paper, we study the existence of periodic solutions of neutral differential equations with infinite delay and delay equations of $n^{th}$ order. Some necessary and sufficient conditions for existence of periodic solutions are obtained.

Keywords: Neutral differential equations, Delay, Periodic solutions.
1 Introduction

The existence of periodic solution of differential delay equations has been studied by many authors (see, e.g., Banks 1988, Burton 1985 and Hale 1977). In many cases the conditions for existence of periodic solutions are not particularly easy to check. Here we shall develop a simple technique for neutral differential equations with infinite delay and delay equations of \( n^{th} \) order. This will lead to easily testable conditions for the existence of periodic solutions.

In section 2 we prove a simple lemma on Fourier series and apply it in section 3 to equations with infinite delay. Delay equations of \( n^{th} \) order will be considered in section 4 and finally in section 5 some examples will be given.

2 Fourier Exponential Series

Let \( f(t) : R \rightarrow R^n \) be a continuous periodic function with period \( 2\pi \). Then, its Fourier exponential series is given by (see Kufner and Kadlec, 1971)

\[
f(t) = \sum_{n=-\infty}^{\infty} C_n e^{int}
\]

where

\[
C_n = \frac{1}{2\pi} \int_0^{2\pi} f(t)e^{-int} dt
\]

\( (n = 0, \pm 1, \pm 2, \ldots) \), whereas for a continuous periodic function with period \( 2l \), we have

\[
f(t) = \sum_{n=-\infty}^{\infty} C_n e^{i(n\pi t/l)}
\]
3 Infinite Delay Equations

Consider the equation

$$\frac{d}{dt} \left[ x(t) - \sum_{k=0}^{\infty} C_k x(t - \tau_k) \right] = \sum_{k=0}^{\infty} B_k x(t - \tau_k) + f(t)$$  \hspace{1cm} (5)$$

where \( x \in \mathbb{R}^n; C_k, B_k \in \mathbb{R}^{n \times n}; \tau_k \geq 0 \) and \( \tau_k \to +\infty \) as \( k \to +\infty \); and \( f: \mathbb{R} \to \mathbb{R}^n \) is a continuously differentiable periodic function with period \( 2\pi \), that is \( f(t + 2\pi) = f(t) \). Then, we may write

$$f(t) = \sum_{n=-\infty}^{\infty} f_n \cdot e^{int}$$

where

$$f_n = \frac{1}{2\pi} \int_{0}^{2\pi} f(t) e^{-int} dt$$  \hspace{1cm} (6)$n = 0, \pm 1, \pm 2, \ldots \)

**Theorem 3.1.** If

$$\sum_{k=0}^{\infty} || C_k || < 1, \quad \sum_{k=0}^{\infty} || B_k || < +\infty$$

Then, the equation (5) has a continuously differentiable periodic solution with period \( 2\pi \) if and only if for each \( n \), where \( 0 \leq n \leq [\sum_{k=0}^{\infty} || B_k || + 1]/(1 - \sum_{k=0}^{\infty} || C_k ||) \) (where \([\star]\) denotes the largest integral part of \( \star \)), a solution of the equation

$$\left( inI - in \sum_{k=0}^{\infty} C_k e^{-ir_k n} - \sum_{k=0}^{\infty} B_k e^{-ir_k n} \right) x_n = f_n$$  \hspace{1cm} (7)$$

exist.
Proof: ‘Only If’. Let \( x(t) \) be a continuously differentiable periodic solution of (5) with period \( 2\pi \). Then, \( x(t) \) can be written as

\[
x(t) = \sum_{n=-\infty}^{\infty} x_n e^{int}
\]

where

\[
x_n = \frac{1}{2\pi} \int_{0}^{2\pi} x(t) e^{-int} dt
\]

Then,

\[
x(t - \tau_k) = \sum_{n=-\infty}^{\infty} x_n e^{-ir_k n} e^{int}
\]

putting (8) and (9) into (5), and comparing the coefficients of both sides, we get

\[
\left( inI - \sum_{k=0}^{\infty} C_k e^{-ir_k n} - \sum_{k=0}^{\infty} B_k e^{-ir_k n} \right) x_n = f_n
\]

for \( n = 0, \pm 1, \pm 2, \ldots \) this shows that the solution of (5) exists for all \( n \) and the proof for ‘only if’ is complete.

‘If’. When \( |n| \geq \left[ (\sum_{k=0}^{\infty} \| B_k \| + 1)/(1 - \sum_{k=0}^{\infty} \| C_k \|) \right] \triangleq N \), we have

\[
\| \sum_{k=0}^{\infty} C_k e^{-ir_k n} + \frac{1}{in} \sum_{k=0}^{\infty} B_k e^{-ir_k n} \|
\]

\[
\leq \sum_{k=0}^{\infty} \| C_k \| + \frac{1}{n} \sum_{k=0}^{\infty} \| B_k \|
\]

\[
\leq 1
\]

Hence, for \( |n| \geq N \), the matrix

\[
\left( inI - \sum_{k=0}^{\infty} C_k e^{-ir_k n} - \sum_{k=0}^{\infty} B_k e^{-ir_k n} \right)
\]

\[
= in \left( I - \sum_{k=0}^{\infty} C_k e^{-ir_k n} - \frac{1}{in} \sum_{k=0}^{\infty} B_k e^{-ir_k n} \right)
\]
is invertible. It follows that equation (5) can be solved for $|n| \geq N$. Hence, the solution of (5) exists for all $n = 0, \pm 1, \pm 2, \ldots$.

We shall prove that the series

$$
\sum_{n=-\infty}^{\infty} x_n e^{int}
$$

(12)

converges absolutely and uniformly.

From (7), we have

$$
\left[ |n| \left( 1 - \sum_{k=0}^{\infty} \| C_k \| \right) - \sum_{k=0}^{\infty} \| B_k \| \right] \| x_n \| \leq \| f_n \|
$$

Since $\sum_{k=0}^{\infty} \| C_k \| < 1$ and $\sum_{k=0}^{\infty} \| B_k \| < +\infty$, there existe $N_1 > 0$ such that $|n| \geq N_1$ implies that

$$
|n| \left( 1 - \sum_{k=0}^{\infty} \| C_k \| \right) - \sum_{k=0}^{\infty} \| B_k \| \geq \frac{|n|}{2} \left( 1 - \sum_{k=0}^{\infty} \| C_k \| \right)
$$

Then, for $|n| \geq N_1$, we have

$$
\| x_n \| \leq \frac{2 \| f_n \|}{(1 - \sum_{k=0}^{\infty} \| C_k \|) |n|}
$$

Using lemma 2.1, we get

$$
\| x_n \| \leq \frac{2K}{1 - \sum_{k=0}^{\infty} \| C_k \|} \cdot \frac{1}{|n|^2}
$$

for $|n| \geq N_1$, where

$$
K = \frac{1}{2\pi} \left( 2 \| f(0) \| + \int_{0}^{2\pi} \| f'(t) \| dt \right).
$$

Then,

$$
\sum_{n=-\infty}^{\infty} \| x_n \| \leq \frac{2K}{1 - \sum_{k=0}^{\infty} \| C_k \|} \sum_{|n| = N_1}^{\infty} \frac{1}{|n|^2} + \sum_{n=-N_1}^{N_1} \| x_n \| < +\infty
$$
Hence, the series (12) converges absolutely and uniformly.

Let

$$x(t) = \sum_{n=-\infty}^{\infty} x_n e^{int}$$

it is easy to check that $x(t)$ is a continuously differentiable periodic solution of (5) with period $2\pi$, this completes the proof.

**Theorem 3.2.** If

$$\sum_{k=0}^{\infty} \| C_k \| < 1, \quad \sum_{k=0}^{\infty} \| B_k \| < +\infty.$$  

Then, equation (5) has a unique continuously differentiable periodic solution with period $2\pi$ if and only if

$$\det \left( inI - in \sum_{k=0}^{\infty} C_k e^{-ir_k n} - \sum_{k=0}^{\infty} B_k e^{-ir_k n} \right) \neq 0$$

for each $n$, where $0 \leq n \leq [(\sum_{k=0}^{\infty} \| B_k \| + 1)/(1 - \sum_{k=0}^{\infty} \| C_k \|)].$

**Proof:** Note that under the conditions, the equation (7) have unique solution for all $n = 0, \pm 1, \pm 2, \ldots$. Then, the result follows.

**Corollary 3.1.** If

$$\sum_{k=0}^{\infty} (\| C_k \| + \| B_k \|) < 1, \quad \det (\sum_{k=0}^{\infty} B_k) \neq 0$$

Then, the equation (5) has a unique continuously differentiable periodic solution with period $2\pi$.

The proof is simple and we omit it here.
4 Delay Equations of $n^{th}$ order

Consider the delay equation of $n^{th}$ order

$$\sum_{k=0}^{n} \left[ b_k x^{(n-k)}(t) + c_k x^{(n-k)}(t - \tau_k) \right] = f(t)$$

(13)

where $b_0 \neq 0, b_k, c_k, \tau_k (\geq 0)$ are constants, $f : R \rightarrow R$ is a continuously differentiable periodic function with period $2\pi$. Then, we may write $f(t)$ as

$$f(t) = \sum_{m=-\infty}^{\infty} f_m \cdot e^{int}$$

Theorem 4.1 Assume that $|b_0| > |c_0|$, then the equation (13) has a continuously differentiable periodic solution with period $2\pi$ if and only if a solution of the equation

$$\left[ \sum_{k=0}^{n} (im)^{(n-k)}(b_k + c_k e^{-im\tau_k}) \right] x_m = f_m$$

(14)

exists for each $m$, where $0 \leq m \leq [max(\lambda) + 1]$, $\lambda$ is a solution of

$$\lambda^n (|b_0| - |c_0|) - \sum_{k=1}^{n} \lambda^{(n-k)}( |b_k| + |c_k|) = 0$$

(15)

Proof: 'Only If'. Let $x(t)$ be a continuously differentiable periodic solution of (13) with period $2\pi$. Then, $x(t)$ can be written as

$$x(t) = \sum_{m=-\infty}^{\infty} x_m e^{int}$$

where

$$x_m = \frac{1}{2\pi} \int_{0}^{2\pi} x(t)e^{-int} dt$$

Then,

$$x^{(n-k)}(t) = \sum_{m=-\infty}^{\infty} x(m) \cdot (im)^{(n-k)} \cdot e^{int}$$
and
\[ x^{(n-k)}(t - \tau_k) = \sum_{m=-\infty}^{\infty} x(m) \cdot (im)^{(n-k)} \cdot e^{-im\tau_k} \cdot e^{imt} \]

Substituting these into (13), we get
\[ \sum_{k=0}^{n} (im)^{(n-k)}(b_k + c_k e^{-im\tau_k})x_m = f_m \]
for \( m = 0, \pm 1, \pm 2, \ldots \). This shows that the equation (13) have solution and this completes the proof for 'only if'.

'If'. First, we shall prove that (14) can be solved for \( |m| \leq [\max(\lambda) + 1] \) \( \leq M \). Since
\[ | \sum_{k=0}^{n} (im)^{(n-k)}(b_k + c_k e^{-im\tau_k}) | \]
\[ \geq |m|^n (|b_0| - |c_0|) - \sum_{k=1}^{n} |m|^{(n-k)} (|b_k| + |c_k|) \]  \( (16) \)
and
\[ |m|^n (|b_0| - |c_0|) - \sum_{k=1}^{n} |m|^{(n-k)} (|b_k| + |c_k|) \rightarrow +\infty \]
as \( |m| \rightarrow +\infty \). Then, for \( |m| \geq M \),
\[ |m|^n (|b_0| - |c_0|) - \sum_{k=1}^{n} |m|^{(n-k)} (|b_k| + |c_k|) > 0 \]
Hence, the equation (14) can be solved for \( |m| \geq M \). Then, equation (14) has a solution for all \( m = 0, \pm 1, \pm 2, \ldots \).

We shall prove that the series
\[ \sum_{m=-\infty}^{\infty} x_m \cdot (im)^{(s)} \cdot e^{imt} \]  \( (17) \)
s = 0, 1, 2, \ldots, n - 1, converges absolutely and uniformly.
From (14), we have
\[
\sum_{k=0}^{\infty} (im)^{(n-k-s)}(b_k + c_k e^{-im\tau_k}) x_m \cdot (im)^{(s)} = f_m
\]

Then,
\[
\left| \sum_{k=1}^{n} m \left| (n-k-s) \left( | b_k | + | c_k | \right) \right| x_m \cdot (im)^{(s)} \right| \leq |f_m|
\]

(18)

Since \( |b_0| > |c_0| \), and \( s \leq n-1 \), there must exists a \( M_1 > 0 \) such that \( m \geq M_1 \) implies that
\[
\left| m \right|(n-s) \left( | b_0 | - | c_0 | \right) - \sum_{k=1}^{n} m \left| (n-k-s) \left( | b_k | + | c_k | \right) \right| \geq \frac{|m|}{2} (|b_0| - |c_0|)
\]

(19)

Hence, when \( m \geq M_1 \), we get
\[
| x_m \cdot (im)^{(s)} | \leq \frac{2 |f_m|}{|b_0| - |c_0|} \cdot \frac{1}{|m|}
\]

Using Lemma 2.1, it follows that
\[
| x_m \cdot (im)^{(s)} | \leq \frac{2K}{|b_0| - |c_0|} \cdot \frac{1}{|m|^2}
\]

where
\[
K = \frac{1}{2\pi} \left( 2 |f(0)| + \int_0^{2\pi} |f'(t)| \, dt \right)
\]

Then,
\[
\sum_{m=-\infty}^{\infty} | x_m \cdot (im)^{(s)} | \leq \frac{2K}{|b_0| - |c_0|} \sum_{|m| = M_1}^{\infty} \frac{1}{|m|^2} + \sum_{m = -M_1}^{M_1} | x_m \cdot (im)^{(s)} | < +\infty
\]

(20)
This shows that the series (17) converges absolutely and uniformly. Let
\[ x(t) = \sum_{m=-\infty}^{\infty} x_m e^{imt} \]
then
\[ x^{(s)}(t) = \sum_{m=-\infty}^{\infty} x_m \cdot (im)^{(s)} \cdot e^{imt} \]
for \( s = 1, 2, \ldots, n - 1 \). It is easy to check that \( x(t) \) is a continuously differentiable periodic solution of (13) with period \( 2\pi \). This completes the proof.

**Theorem 4.2.** Assume that \( |b_0| > |c_0| \), then equation (13) has a unique continuously differentiable periodic solution with period \( 2\pi \) if and only if
\[ \sum_{k=0}^{n} (im)^{(n-k)}(b_k + c_k e^{-im\tau_k}) \neq 0 \]
for each \( m \), where \( 0 \leq m \leq \lceil \max(\lambda) + 1 \rceil \), \( \lambda \) is a solution of equation (15).

The proof is similar with the proof of theorem 3.2 and we omit it here.

**Corollary 4.1.** If
\[ b_n + c_n \neq 0, \quad |b_0| - |c_0| - \sum_{k=1}^{n} (|b_k| + |c_k|) > 0 \]
Then, the equation (13) has a unique continuously differentiable periodic solution with period \( 2\pi \).

## 5 Examples

**Example 5.1.** Consider the equation
\[ \frac{d}{dt} \left[ x(t) - \frac{1}{6} \sum_{k=1}^{\infty} \frac{1}{k!} x(t - k) \right] = \frac{1}{6} \sum_{k=1}^{\infty} \frac{1}{k!} x(t - k) + \sin t \quad (21) \]
Since
\[
\frac{1}{6} \sum_{k=1}^{\infty} \frac{1}{k!} + \frac{1}{6} \sum_{k=1}^{\infty} \frac{1}{k!} = \frac{e}{3} < 1
\]

Then, using corollary 3.1 (21) has a unique continuously differentiable periodic solution with period $2\pi$.

**Example 5.2.** Consider the equation
\[
2x^{(n)}(t) - x^{(n)}(t - \tau) + \frac{1}{4} x(t) - \frac{1}{4} x(t - \tau) = \cos t
\] (22)

Using corollary 4.1, it easy to check that equation (22) has a unique continuously differentiable periodic solution with period $2\pi$.

## 6 Conclusions

In this paper, we have investigated the periodic solution of neutral differential equations with infinite delay and neutral differential equations of $n^{th}$ order via the theory of Fourier exponential series. The results seem very simple and the conditions are very mild. Throughout this paper, we have considered periodic solutions with period $2\pi$. By a simple modification we can also consider periodic solutions with a general period $2l$. 

12
References


