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# Limit Cycles and Bifurcation in Piecewise-Linear and Piecewise-Analytic Systems:

## I. General Theory.

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Abstract

**The existence of limit cycles and periodic doubling bifurcations in piecewise-linear and piecewise-analytic systems is studied. Some theoretical sufficient conditions are obtained directly in terms of the right hand sides of the system.**

# 1 introduction

In this paper we shall consider some theoretical aspects of limit cycles and bifurcation in piecewise-linear and piecewise-analytic systems. There is of course a vast literature on limit cycles and bifurcation in the general case(see, for instance, Guckenheimer and Holmes 1983, Iooss and Joseph 1980) and some more detailed investigation into piecewise-linear systems(Banks and Khathur 1989; Chua, Komuro and Matsumoto 1986). However, there does not seems to be many attempts at direct sufficient conditions for general systems of this form. In the case of piecewise-linear systems our approach will be to obtain an expression for the solution of the system by using the Campbell-Hausdorff formula. For piecewise-analytic systems we shall use a generalized Lie series for the system solution and obtain analytic series representations. The conditions will then be specified directly in terms of the right hand sides of the equations.

We shall be concerned solely here with the theoretical approach to the problem . In a forthcoming companion paper we shall seek efficient numerical procedures to implement the results presented here.

# 2 Existence of Limit Cycles in Piecewise-Linear Systems

In this section we shall consider the piecewise-linear system defined by the equations

$$
\begin{aligned}
\dot{x} &= A_1 x \quad x \in P_1 \\
\dot{x} &= A_2 x \quad x \in P_2 \\
&\vdots \qquad\quad \vdots \\
\dot{x} &= A_m x \quad x \in P_m
\end{aligned}
\tag{2.1}
$$

where the $P_i$ are polytopes (not necessarily bounded !) of dimension n such that

(*i*). $\overline{P}_1 \cup \ldots \cup \overline{P}_m = \mathbb{R}^n$

(*ii*). $\overline{P}_i \cap \overline{P}_j$ is a polytope of dimension $< n$, if i $\neq$ j

It will be assumed that the system (2.1) has a unique solution for each initial condition $x_0 \in \mathbb{R}^n$ and that each matrix $A_i$ is nonsingular (this will guarantee a single equilibrium point). We shall also assume that no trajectories of the system lie on a boundary $\overline{P}_i \cap \overline{P}_j$.

Let $x_0 \in P_i$ , $1 \leq i \leq m$ and suppose that, on some time interval [0,t] the solution trajectory of (2.1) starting at $x_0$ passes through the sequence of polytopes $P_{i_1}, P_{i_2}, \ldots, P_{i_k}$ where $P_{i_1} = P_i$. Then the solution of (2.1) through $x_0$ is given by

$$x(t) = e^{A_{i_k}\tau} \ldots\ldots e^{A_{i_2}\tau_{i_2}} e^{A_{i_1}\tau_{i_1}} x_0 \tag{2.2}$$

where $\tau_{i_j}$ is the time the trajectory remains in $P_{i_j}$ and

$$\tau = t - \sum_{j=1}^{k-1} \tau_{i_j}.$$

Of course, each $\tau_{i_j}$ depends on $x_0$ and this valid for $\tau \geq 0$ i.e.

$$t \geq \sum_{j=1}^{k-1} \tau_{i_j}$$

Now suppose that a trajectory starts on a boundary, say $x_0 \in \partial P_1$ and passes through a number of polytopes once in some order. For simplicity (by renumbering) we may assume that these are $P_1, P_2, \ldots, P_k$.

If $x_0 \in \overline{P}_1 \cap \overline{P}_k$, then by (2.2) we have

$$x(t) = e^{A_k\tau_k} \ldots e^{A_1\tau_1} x_0 \quad \in \overline{P}_1 \cap \overline{P}_k. \quad \left(t = \sum_{i=1}^{k} \tau_i\right) \tag{2.3}$$

For a limit cycle, we require

$$e^{A_k\tau_k} \ldots e^{A_1\tau_1} x_0 = x_0 \quad ,$$

i.e. the matrix $e^{A_k\tau_k} \ldots e^{A_1\tau_1}$ has an an eigenvalue 1. Before considering the general case it is instructive to determine conditions when the $A_i$'s all commute. In this case we have

$$e^{A_k \tau_k} \ldots e^{A_1 \tau_1} = e^{\sum_{i=1}^{k} A_i \tau_i}$$

and so we can state

**Proposition 2.1** *Given the system (2.1), if a trajectory passes through the polytopes $P_1, P_2, \ldots, P_k, P_1$, then a limit cycle exist if the times $\tau_1, \ldots, \tau_k$ as defined above satisfy the condition:*

$$-2\pi j \quad or \quad 2\pi j \in \sigma(A_1\tau_1 + A_2\tau_2 + \ldots\ldots + A_k\tau_k)$$

*where $\sigma(A)$ is the spectrum of $A$.*

**Proof**. This follows from the spectral mapping theorem which states that

$$exp(\sigma(A)) = \sigma(exp(A)) \quad . \qquad \Box$$

Since the $A_i$'s commute we can find a common diagonalizing transformation matrix T so that

$$T \;\; A_i \;\; T^{-1} \;\; = \begin{pmatrix} \lambda_{i_1} & & 0 \\ & \ddots & \\ 0 & & \lambda_{i_n} \end{pmatrix}$$

where $\lambda_{i_j}$ is the $j^{th}$ eigenvalue of $A_i$. Hence we have

**Corollary 2.1** *Under the condition of proposition 2.1,* $x_0 \in \overline{P}_1 \cap \overline{P}_k$ *lies on a limit cycle if there exist* $1 \leq l \leq n$ *such that*

$$\sum_{i=1}^{k} \lambda_{il} \tau_i = \pm 2\pi j. \qquad \square$$

This result can be interpreted in the following way: for each $l$ we define the map

$$f_l \; : \; \overline{P}_1 \cap \overline{P}_k \longrightarrow C$$

by

$$f_l(x_0) = \sum_{i=1}^{k} \lambda_{il} \tau_i(x_0).$$

Then the requirement is that, for some $l$ ,

$$f_l(x_0) = \pm 2\pi j.$$

The times $\tau_i(x_0)$ are determined as follows. First let $\overline{P}_i \cap \overline{P}_j$ lie in the hyperplane $H_{ij}$ given by

$$< w_{ij}, x > = c_{ij} \tag{2.4}$$

for some constant $c_{ij}$. Then if $x_0 \in H_{12}, \tau_1(x_0)$ is given implicitly by

$$< w_{12}, e^{A_1 \tau_1(x_0)} x_0 > = c_{12} \tag{2.5}$$

Similarly, $\tau_l(x_0)$ is given implicitly by

$$< w_{l,l+1} \; , \; e^{A_l \tau_l(x_0)} \; e^{A_{l-1} \tau_{l-1}(x_0)} \ldots e^{A_1 \tau_1(x_0)} \; x_0 > = c_{l,l+1}. \tag{2.6}$$

and so the $\tau$ 's can be determined recursively. Note, however, that we can only write these implicit relations and so the conditions in corollary 2.1 must be checked numerically.

We shall consider next the case where the $A_i$ matrices do not commute. For this we shall require the Campbell-Hausdorff formula given by the following result(Varadarajan 1974):

**Lemma 2.1** *If $A$, $B \in \mathbb{R}^n$, then we have*

$$exp\ A\ exp\ B \ = \ exp\ C(A:B) \ ,$$

*where*

$$C(A:B) = \sum_{n=1}^{\infty} c_n(A:B) \ ,$$

*and $c_n$ is given recursively by*

$$\begin{aligned} c_0(A:B) &= 0 \\ c_1(A:B) &= A+B \ , \end{aligned}$$

$$\begin{aligned} (m+1)c_{m+1}(A:B) &= \frac{1}{2}[A-B, c_m(A:B)] \\ &+ \sum_{p\geq 1, 2p\leq m} K_{2p} \sum_{\substack{k_1,\ldots,k_{2p}>0 \\ k_1+\ldots+k_{2p}=m}} \\ & \left[c_{k_1}(A:B), \left[\ldots \left[c_{k_{2p}}(A:B), A+B\right]\ldots\right]\right] \end{aligned} \quad (2.7)$$

*Here the coefficients $K_{2p}$ are defined by the power series expansion*

$$\frac{z}{1-e^{-z}} - \frac{1}{2}z = 1 + \sum_{p=1}^{\infty} K_{2p} z^{2p} \qquad \square$$

It will be convenient to write $c_n(A:B)$ in a different form as follows. Note that $c_n(A:B)$ is a linear combination of $n^{th}$ order Lie monomials where the multiplication is Lie bracket multiplication $A.B = [A,B]$. (This multiplication is neither associative nor commutative.) We can therefore write

$$c_n(A:B) = \sum_{i+j=n} p_{n,i,j}(A,B) \quad , \quad n \geq 2 \tag{2.8}$$

where $p_{n,i,j}(A,B)$ is a homogeneous Lie polynomial of order n which has $i$ $A$ factors and $j$ $B$ factors. Of course,

$$p_{n,n,0}(A,B) = p_{n,0,n}(A,B) = 0.$$

Thus, by lemma 2.1 we have, for example,

$$\begin{aligned} c_2(A:B) &= p_{2,2,0}(A,B) + p_{2,1,1}(A,B) + p_{2,0,2}(A,B) \\ &= [A,A] + \frac{1}{2}[A,B] + [B,B] \\ &= \frac{1}{2}[A,B] \quad , \end{aligned}$$

and

$$\begin{aligned} c_3(A:B) &= \frac{1}{12}[A,[A,B]] - \frac{1}{12}[B,[A,B]] \\ &= p_{3,2,1}(A,B) + p_{3,1,2}(A,B). \end{aligned}$$

In general, we have

**Lemma 2.2** *If $A_1, \ldots, A_m \in \mathbb{R}^{n^2}$, then we have*

$$\Pi_{i=1}^{m} (exp\ A_i) = exp\ C(A_1 : A_2 : \ldots : A_m)$$

*where*

$$C(A_1 : A_2 : \ldots : A_m) = \sum_{n=1}^{\infty} c_n(A_1 : \ldots : A_m)$$

*and*

$$c_n(A_1 : \ldots : A_m) = \sum_{i_1+\ldots+i_m=n} p_{n,i_1,\ldots,i_m}(A_1, \ldots, A_m) \qquad \square$$

Here $p_{n,i_1,\ldots,i_m}(A_1, \ldots, A_m)$ is a homogeneous Lie polynomial of order n.

In the general non-commutative case, therefore, the solution(2.3) of our piecewise-linear system becomes

$$\begin{aligned} x(t) &= e^{A_k\tau_k} \ldots e^{A_1\tau_1}\ x_0 \\ &= exp(C(\tau_k A_k : \ldots : \tau_1 A_1))\, x_0 \end{aligned}$$

and we have the following generalization of proposition 2.1

**Theorem 2.1** *A limit cycle will exist in system (2.1) in the sequence of polytopes $P_1, P_2, \ldots, P_k, P_1$ if the times $\tau_1, \ldots, \tau_k$ and the matrices $A_1, \ldots, A_k$ satisfies the condition*

$$-2\pi j \quad or \quad 2\pi j \in \sigma(C(\tau_k A_k : \ldots : \tau_1 A_1)). \qquad \square$$

Note that

$$
\begin{aligned}
C(\tau_k A_k : \ldots : \tau_1 A1) &= \tau_1 A_1 + \ldots + \tau_k A_k \\
&+ \sum_{n=2}^{\infty} \sum_{i_1+\ldots+i_k=n} \tau_1^{i_1} \ldots \tau_k^{i_k} p_{n,i_1,\ldots,i_k}(A_1, \ldots, A_k)
\end{aligned}
$$

since $p_{n,i_1,\ldots,i_k}$ is a homogeneous Lie polynomial.

**Corollary 2.2** *Let $\mathcal{L} = \mathcal{L}(A_1, \ldots, A_k)$ denote the Lie algebra generated by $A_1, \ldots, A_k$. If $\mathcal{L}$ is nilpotent with class of nilpotency $N$ then a limit cycle exists if*

$$
-2\pi j \ \ or \ \ 2\pi j \ \in \sigma \left( \tau_1 A_1 + \ldots + \tau_k A_k + \sum_{n=2}^{N} \sum_{i_1+\ldots+i_k=n} \tau_1^{i_1} \ldots \tau_k^{i_k} p_{n,i_1,\ldots,i_k}(A_1, \ldots, A_k) \right) \quad \square
$$

# 3 Bifurcation conditions

Consider next the system

$$\begin{aligned}
\dot{x} &= A_1(\mu)x \qquad , \quad x \in P_1 \\
&\vdots \\
\dot{x} &= A_m(\mu)x \qquad , \quad x \in P_m
\end{aligned} \tag{3.1}$$

which is similar to that given by (2.1) except that $A_1, \ldots, A_m$ now depend on a parameter $\mu$. We shall consider the problem of bifurcation from a periodic orbit. Again we shall begin by assuming that the matrices $A_i(\mu)$, $1 \leq i \leq m$ all commute in order to see clearly the nature of the bifurcation conditions. Let the solution of (3.1) with initial value $x_0$ be written

$$x(t) = \phi_t(x_0, \mu)$$

Then a periodic point $x_0$ satisfies

$$x_0 = \phi_t(x_0, \mu). \tag{3.2}$$

A bifurcation can occur at a value $\mu$ where the equation (3.2) has multiple solutions. Thus, a bifurcation can occur if

$$det\left[I - \frac{\partial}{\partial x_0}\phi_t(x_0, \mu)\right] = 0. \tag{3.3}$$

If the matrices $A_i(\mu)$, $1 \leq i \leq m$ and the initial point $x_0$ satisfy the conditions of corollary 2.1 then $x_0$ will lie on a periodic orbit. Condition (3.3) for a bifurcation becomes

$$det\left[I - \frac{\partial}{\partial x_0} exp\left[\sum_{i=1}^{k} A_i(\mu)\tau_i\right] x_0\right] = 0$$

Now,

$$\begin{aligned} \frac{\partial}{\partial x_0} exp\left[\sum_{i=1}^{k} A_i(\mu)\tau_i\right] x_0 &= exp\left[\sum_{i=1}^{k} A_i(\mu)\tau_i\right] \\ &+ \sum_{i=1}^{k} exp\left[\sum_{l=1}^{k} A_l(\mu)\tau_l\right] A_i(\mu) x_0 \frac{\partial \tau_i}{\partial x_0} \end{aligned}$$

Hence we have the following result:

**Lemma 3.1** *If $x_0$ is a periodic point for the system (3.1) (with a given parameter $\mu$ ), then it is also a bifurcation point for the existence of new periodic orbits if, for the same parameter $\mu$, we have*

$$0 \in \sigma\left(I - exp\left[\sum_{i=1}^{k} A_i(\mu)\tau_i\right] - \sum_{i=1}^{k} exp\left[\sum_{l=1}^{k} A_l(\mu)\tau_l\right] A_i(\mu) x_0 \frac{\partial \tau_i}{\partial x_0}\right) \quad \square \qquad (3.4)$$

If $x_0$ is a periodic point and we consider the map which follows the periodic trajectory for two orbits then we obtain the solution

$$x_0 = exp\left(2\sum_{i=1}^{k} A_i(\mu)\tau_i\right) x_0$$

(assuming again that the $A_i$'s commute). In order for this 'trivial' doubly periodic orbit to bifurcate into other periodic doubling maps we require, as in lemma 3.1, the condition

$$0 \in \sigma \left( I - exp\left[2\sum_{i=1}^{k} A_i(\mu)\tau_i\right] - 2\sum_{i=1}^{k} exp\left[2\sum_{l=1}^{k} A_l(\mu)\tau_l\right] A_i(\mu)x_0\frac{\partial \tau_i}{\partial x_0}\right) \quad (3.5)$$

This is the condition for a periodic doubling bifurcation. Higher order bifurcation can be determined similarly.

For the noncommutative case we require the well-known formula(Miller 1972)

$$\frac{d}{dt}exp\,(A(t)) = exp\,(A(t))\,f\,(Ad\ A(t))\left(\dot{A}(t)\right) \quad (3.6)$$

where

$$f(z) = \frac{1-e^{-z}}{z} = 1 - \frac{z}{2!} + \frac{z^2}{3!} + \ldots$$

and

$$(Ad\ A)B = [A, B]$$

In section 2 it was shown that the solution of system (2.3) is given by

$$x(t) = exp(C(\tau_k A_k : \ldots : \tau_1 A_1))\ x_0 \quad , \quad t = \sum_{i=1}^{k} \tau_i$$

Put

$$g(\tau_k, \ldots, \tau_1) = C(\tau_k A_k : \ldots : \tau_1 A_1).$$

Then

$$x(t) = exp(g)\ x_0 \quad ,$$

where we shall omit explicit reference to $\tau_k, \ldots, \tau_1$. Thus,

$$\frac{\partial x(t)}{\partial x_{0j}} = (exp(g))_j + exp(g) \left\{ f(Ad\ g) \sum_{l=1}^{k} \frac{\partial g}{\partial \tau_l} \frac{\partial \tau_l}{\partial x_{0j}} \right\} x_0$$

by (3.6) where $(B)_j$ denotes the $j^{th}$ column of the matrix B. Generalizing (3.4) and (3.5) we therefore have

**Theorem 3.1** *If $x_0$ is a periodic point for the system (3.1) for the parameter $\mu$, then it is also a bifurcation point for that $\mu$ if*

$$0 \in \sigma \left( I - exp(g) - \sum_{l=1}^{k} exp(g) \left\{ f(Ad\ g) \frac{\partial g}{\partial \tau_l} \frac{\partial \tau_l}{\partial x_0} \right\} x_0 \right) \tag{3.7}$$

*Similarly, $\mu$ is a periodic doubling bifurcation point if*

$$0 \in \left( I - exp(g_2) - \sum_{l=1}^{k} exp(g_2) \left\{ f(Ad\ g_2) \frac{\partial g_2}{\partial \tau_l} \frac{\partial \tau_l}{\partial x_0} \right\} x_0 \right) \quad , \tag{3.8}$$

*where*

$$g_2(t_{2k}, \ldots, t_{k+1}, t_k, \ldots, t_1) \;=\; C(t_{2k}A_k : \ldots : t_{k+1}A_1 : t_kA_k : \ldots : t_1A_1)$$

*and $g_2$ is evaluated at $(t_{2k}, \ldots, t_{k+1}, t_k, \ldots, t_1) = (\tau_k, \ldots, \tau_1, \tau_k, \ldots, \tau_1)$ in (3.8).* □

Recall that

$$g(\tau_k,\ldots,\tau_1) = \tau_1 A_1 + \ldots + \tau_k A_k + \sum_{n=2}^{\infty} \sum_{i_1+\ldots+i_k=n} \tau_1^{i_1} \ldots \tau_k^{i_k} p_{n,i_1,\ldots,i_k}(A_1,\ldots,A_k)$$

and so

$$\frac{\partial g}{\partial \tau_l} = A_l + \sum_{n=2}^{\infty} \sum_{i_1+\ldots+i_k+n} i_l \tau_1^{i_1} \ldots \tau_l^{i_l-1} \ldots \tau_k^{i_k} p_{n,i_1,\ldots,i_k}(A_1,\ldots,A_k).$$

The derivatives $\partial\tau_l/\partial x_0$ can be found in the following way.

Consider first $\tau_1$, and suppose that $\partial P_1 \cap \partial P_2$ lies in the hyperplane $H_1$ defined by the equation

$$w_{12}\, x \;=\; c_{12}$$

for some row vector $w_{12}$ and a constant $c_{12}$. Then

$$w_{12}\, e^{A_1\tau_1}\, x_0 \;=\; c_{12}$$

by definition of $\tau_1$ and so

$$w_{12} A_1 \frac{\partial \tau_1}{\partial x_{0i}}\, e^{A_1\tau_1}\, x_0 + \left(w_{12}\, e^{A_1\tau_1}\right)_i \;=\; 0$$

Hence,

$$\frac{\partial \tau_1}{\partial x_{0i}} \;=\; -\left(w_{12}\, e^{A_1\tau_1}\right)_i / w_{12} A_1\, e^{A_1\tau_1}\, x_0 \;\; .$$

Note that

$$w_{12} A_1\, e^{A_1\tau_1}\, x_0 = w_{12}\, \dot{x}(\tau_1)$$

and $\dot{x}(\tau_1)$ is not parallel to $H_1$, so that $w_{12} A_1 e^{A_1 \tau_1} x_0 \neq 0$.

Similarly, we have

$$w_{12}\; e^{A_2 \tau_2}\; e^{A_1 \tau_1}\; x_0 = c_{12}$$

where $\partial P_2 \cap \partial P_3$ lies in the hyperplane $H_2$ defined by

$$w_{23}\; x = c_{23}$$

Thus,

$$\frac{\partial \tau_2}{\partial x_{0i}} = \frac{\left\{ -w_{23} e^{A_2 \tau_2} A_1 \frac{\partial \tau_1}{\partial x_{0i}} e^{A_1 \tau_1} x_0 - \left( w_{23} e^{A_2 \tau_2} e^{A_1 \tau_1} \right)_i \right\}}{w_{23} A_2 e^{A_2 \tau_2} e^{A_1 \tau_1} x_0}$$

In general, $\partial \tau_k / \partial x_{0i}$ can be obtained from the equation

$$w_{k,k+1}\; e^{A_k \tau_k} \ldots e^{A_1 \tau_1}\; x_0 \;=\; c_{k,k+1}$$

where $H_k : w_{k,k+1}\; x = c_{k,k+1}$ contains $\partial p_k \cap \partial p_{k+1}$ (where $p_{k+1} = p_1$ if $k = m$).

# 4 Limit Cycles and Bifurcation in Piecewise-Analytic Systems

In this section we shall consider systems defined by piecewise-analytic equations of the form

$$\begin{aligned} \dot{x} &= f_1(x) \quad , \quad x \in D_1 \\ &\vdots \\ \dot{x} &= f_m(x) \quad , \quad x \in D_m \end{aligned} \tag{4.1}$$

where

(*i*) $\overline{D}_1 \cup \ldots \cup \overline{D}_m = \mathbb{R}^n$

(*ii*) $\overline{D}_i \cap \overline{D}_j \subseteq$ *some* $(n-1)$ *dimensional manifold.*

The functions $f_i(x)$ and the boundaries $\partial D_i$ are assumed to be analytic. Consider first the case m=1, so that we have a single equation

$$\dot{x} = f(x) \quad , \quad x(0) = x_0 \tag{4.2}$$

the existence of limit cycles by a global linearization technique has been studied by Banks (1988). Here we shall use a similar technique, based on the Lie series (Banks 1989).

Equation (4.2) has solution given by(see Steinberg 1984)

$$x(t) = exp\left[t\sum_{i=1}^{n} f_i(x_0)\frac{\partial}{\partial x_{0i}}\right] x_0 \tag{4.3}$$

In order to evaluate the solution (4.3) explicitly we must rewrite each term of the form

$$\left(f_1(x_0)\frac{\partial}{\partial x_{01}} + \ldots + f_n(x_0)\frac{\partial}{\partial x_{0n}}\right)^k \tag{4.4}$$

so that the derivations all appear on the right of the of the $f$'s in an expansion of (4.4). We start with

**Lemma 4.1** *Let $X$ be some set and let $R$ be a ring of operators on $X$ such that $X \in R$ and $R$ is generated by $X$ and $\{b_1, \ldots, b_n\}$, where $b_i \neq X$ , $1 \leq i \leq n$. Suppose that for $x \in X$ and $c \in R$ we have*

$$b_i(x\ c) = (b_i\ x)c + x(b_i\ c)\ , \tag{4.5}$$

*where $(b_i x) \in X$.*

*Then any element $r \in R$ of the form*

$$r_k = a_{i_k} b_{i_k} a_{i_{k-1}} b_{i_{k-1}} \ldots a_{i_2} b_{i_2} a_{i_1} b_{i_1}$$

*can be written*

$$r_k = a_{i_k}[b_{i_k}, a_{i_{k-1}}[b_{i_{k-1}}, a_{i_{k-2}}[\ldots[b_{i_2}, a_{i_1}]\ldots]]]b_{i_1} + r'_k$$

*where $r'_k$ consists of terms with at least two $b_{il}$'s on the right, and*

$$[b_k, a_k] = b_k a_k - a_k b_k \in X \qquad \square.$$

**Proof**. By induction. The result is true for $k = 1$. For $k = l$ we have

$$\begin{aligned} r_l &= a_{i_l}[b_{i_l}, a_{i_{l-1}}[b_{i_{l-1}}, a_{i_{l-2}}[\ldots[b_{i_2}, a_{i_1}]\ldots]]]b_{i_1} + r'_l \\ &= a' b_{i_1} + r'_l \quad , \quad \text{say,} \end{aligned}$$

and so, for $k = l+1$, we have

$$\begin{aligned} r_{l+1} &= a_{i_{l+1}} b_{i_{l+1}} r_l \\ &= a_{i_{l+1}} b_{i_{l+1}} (a' b_{i_1}) + a_{i_{l+1}} b_{i_{l+1}} r'_l \\ &= a_{i_{l+1}} \left(b_{i_{l+1}} a'\right) b_{i_1} + a_{i_{l+1}} a' b_{i_{l+1}} b_{i_1} + a_{i_{l+1}} b_{i_{l+1}} r'_l \\ &= a_{i_{l+1}} \left[b_{i_{l+1}}, a'\right] b_{i_1} + r'_{l+1} \quad , \end{aligned}$$

by (4.5). The result now follows. $\square$

Let $X = \mathcal{O}(x)$ be the local ring of analytic functions on $\mathbb{R}^n$ and let $R$ be the ring of operators generated by $\mathcal{O}(x)$ and $\{\partial/\partial x_1, \ldots, \partial/\partial x_n\}$ . Then we can have

**Lemma 4.2** *If $a_1, \ldots, a_n \in \mathcal{O}(x)$, then any element of $R$ of the form*

$$r_k = a_{i_k} \frac{\partial}{\partial x_{i_k}} a_{i_{k-1}} \frac{\partial}{\partial x_{i_{k-1}}} \cdots a_{i_2} \frac{\partial}{\partial x_{i_2}} a_{i_1} \frac{\partial}{\partial x_{i_1}}$$

*can be written in the form*

$$r_k = a_{i_k}\frac{\partial}{\partial x_{i_k}}\left(a_{i_{k-1}}\frac{\partial}{\partial x_{i_{k-1}}}\left(a_{i_{k-2}}\frac{\partial}{\partial x_{i_{k-2}}}\left(\cdots\frac{\partial}{\partial x_{i_2}}a_{i_1}\right)\cdots\right)\right)\frac{\partial}{\partial x_{i_1}} + r'_k$$

*where $r'_k$ contains terms with derivation of order at least 2.*

**Proof**. This follows directly from lemma 4.1 since

$$b_k a_k = [b_k, a_k] + a_k b_k$$

and if $b_k = \partial/\partial x_k$, we have

$$[b_k, a_k] = \frac{\partial a_k}{\partial x_k} \qquad \square$$

**Lemma 4.3** *The solution (4.3) of the system (4.2) is given explicitly by*

$$x(t) = x_0 + \sum_{k=1}^{\infty}\frac{t^k}{k!}\left\{\sum_{i_k=1}^{n}\cdots\sum_{i_1=1}^{n} f_{i_k}\frac{\partial}{\partial x_{i_k}}\left(f_{i_{k-1}}\frac{\partial}{\partial x_{i_{k-1}}}\left(\cdots\frac{\partial f_{i_1}}{\partial x_{i_2}}\right)\cdots\right)(\delta_{i_1 j})\right\}\Big|_{x=x_0} \tag{4.6}$$

*where $(\delta_{i_1 j})$ is the vector* $(0,\ldots,0,\overset{\overset{i_1}{\downarrow}}{1},\ldots,0)$ $\square$

**Remark**. *The terms of the form*

$$\xi_k \triangleq \sum_{i_k=1}^{n}\cdots\sum_{i_1=1}^{n} f_{i_k}\frac{\partial}{\partial x_{i_k}}\left(f_{i_{k-1}}\frac{\partial}{\partial x_{i_{k-1}}}\left(\cdots\frac{\partial f_{i_1}}{\partial x_{i_2}}\right)\cdots\right)(\delta_{i_1 j})\Big|_{x=x_0}$$

*in (4.6) can be obtained recursively by*

$$\begin{aligned}\xi_1 &= \sum_{i_1=1}^{n} f_{i_1}(x_0)(\delta_{i_1 j}) \\ \xi_{k+1} &= \sum_{i_{k+1}=1}^{n} f_{i_{k+1}}(x_0)\frac{\partial}{\partial x_{i_{k+1}}}\xi_k \mid_{x=x_0} \quad ,\end{aligned}$$

*and then*

$$x(t) = x_0 + \sum_{k=1}^{n} \frac{t^k}{k!}\xi_k \qquad \square$$

Define

$$\mathcal{G}_f^l(x_0,t) = \sum_{k=1}^{l} \frac{t^k}{k!}\left\{\sum_{i_k=1}^{n}\cdots\sum_{i_1=1}^{n} f_{i_k}\frac{\partial}{\partial x_{i_k}}\left(f_{i_{k-1}}\frac{\partial}{\partial x_{i_{k-1}}}\left(\cdots\frac{\partial f_{i_1}}{\partial x_{i_2}}\right)\cdots\right)(\delta_{i_1 j})\right\} \mid_{x=x_0}$$

and put

$$\mathcal{G}_f(x_0,t) = \mathcal{G}_f^{\infty}(x_0,t) \ .$$

Then, by (4.6),

$$x(t) = x_0 + \mathcal{G}_f(x_0,t) \ . \tag{4.7}$$

Note that if $f$ is a polynomial function, then $\mathcal{G}_f^l(x_0)$ is a polynomial in $x_0$ and $t$ of order at most $l(m-1)+1$ where $m$ is that maximum of the order of $f_1,\ldots,f_n$ (in $x_1,\ldots .x_n$). From (4.7) we have

**Theorem 4.1** *The equation (4.2) has a limit cycle if there exists a vector $x_0$ and a number $\tau > 0$ such that*

$$\mathcal{G}_f(x_0, \tau) = 0 \qquad \square$$

**Corollary 4.1** *The equation (4.2) has a limit cycle if there exist sequences $x_0(i) \in C^n$, $\tau_i \in \mathbb{R}$ such that*

$$x_0(i) \longrightarrow x_0 \in \mathbb{R}^n \quad , \quad \tau_i \longrightarrow \tau > 0 \qquad \text{and}$$

$$\mathcal{G}^i_f(x_0(i), \tau_i) = 0 \ ,$$

$$\mathcal{G}_f(x_0, \tau) = 0$$

Proof. This follows from the analyticity of the solutions of (4.2). $\square$

Remark. *Corollary 4.1 suggests a numerical procedure for finding limit cycles. For example, if $n = 2$ and we set $x_0(i) = (x_{01}(i), 0)$, then we solve the pair of polynomial equations (in the case where $f$ is polynomial)*

$$\mathcal{G}^i_f((x_{01}(i), 0), \tau_i) = 0$$

*of order at most $i(m-1)+1$. Note that $x_0 + \mathcal{G}_f((x_{01}, 0), \tau)$ is the Poincare' return map for the $x_1$ axis.* $\square$

Once a limit cycle with parameters $x_0$ and $\tau$ has been determined, bifurcation from this limit cycle are determined as before. Thus we have

**Theorem 4.2** *The equation (4.2) is critical for bifurcation to a double period oscillation if it has a limit cycle with parameter $x_0$ and $\tau$ and*

$$det\left(\frac{\partial}{\partial x_0}\mathcal{G}_f(x_0,\tau)\right)=0 \qquad \square \tag{4.8}$$

Note that $\frac{\partial}{\partial x_0}\mathcal{G}_f(x_0,\tau)$ has $(i,j)^{th}$ element

$$\sum_{k=1}^{\infty}\frac{t^k}{k!}\left\{\sum_{i_k=1}^{n}\cdots\sum_{i_1=1}^{n}\frac{\partial}{\partial x_i}\left(f_{i_k}\frac{\partial}{\partial x_{i_k}}\left(f_{i_{k-1}}\frac{\partial}{\partial x_{i_{k-1}}}\left(\cdots\frac{\partial f_{i_1}}{\partial x_{i_2}}\right)\cdots\right)\right)(\delta_{i_1 j}))\right\}|_{x=x_0}$$

Again the condition (4.7) may be replaced by the condition

$$lim_{k\to\infty}\ det\ \left(\frac{\partial}{\partial x_0}\mathcal{G}_f^{\infty}(x_0,\tau)\right)=0.$$

Consider finally the general piecewise-analytic system (4.1). For simplicity we shall suppose $m = 2$, i.e. there are just two regions, the general case is dealt with similarity. We have seen that the solution of (4.1) with $x_0 \in \partial D_1 \cap \partial D_2$ such that the solution remains in $D_1$ for time $\tau_1$ and in $D_2$ for time $\tau_2$ is given by

$$x(t)=x_0+\mathcal{G}_{f_1}(x_0,\tau_1)+\mathcal{G}_{f_2}(x_0+\mathcal{G}_{f_1}(x_0,\tau_1),\tau_2)$$

Hence a limit cycle is given by the condition

$$\mathcal{G}_{f_1}(x_0,\tau_1)+\mathcal{G}_{f_2}(x_0+\mathcal{G}_{f_1}(x_0,\tau_1),\tau_2)=0$$

If this equation has a nontrivial solution $(x_0, \tau_1, \tau_2)$ then the condition for a periodic doubling bifurcation is

$$det\left[\frac{\partial}{\partial x_0}\mathcal{G}_{f_1}(x_0,\tau_1)+\frac{\partial}{\partial x_0}\mathcal{G}_{f_2}(x_0+\mathcal{G}_{f_1}(x_0,\tau_1),\tau_2)\right]=0$$

i.e.

$$\begin{aligned}
&det[\partial_1\mathcal{G}_{f_1}(x_0,\tau_1)+\frac{\partial}{\partial\tau_1}\mathcal{G}_{f_1}(x_0,\tau_1)\frac{\partial\tau_1}{\partial x_0}\\
&\quad+\partial_1\mathcal{G}_{f_2}(x_0+\mathcal{G}_{f_1}(x_0,\tau_1),\tau_2)(I+\partial_1\mathcal{G}_{f_1}(x_0,\tau_1)+\partial_2\mathcal{G}_{f_1}(x_0,\tau_1)\frac{\partial\tau_1}{\partial x_0})\\
&\quad+\partial_2\mathcal{G}_{f_2}(x_0+\mathcal{G}_{f_1}(x_0,\tau_1),\tau_2)\frac{\partial\tau_2}{\partial x_0}]=0
\end{aligned}$$

where $\partial_1, \partial_2$ denote partial derivatives with respect to the first and second variables. This derivatives in this equation can be evaluated explicitly from the expression for $\mathcal{G}_f$ above.

# 5 Conclusions

In this paper we have developed some explicit relations for the existence of limit cycles and periodic doubling bifurcation in piecewise-linear and piecewise-analytic systems. The expressions have been obtained by Lie series methods and result in complex equations which can only be applied numerically. In a companion paper(Banks and Khathur 1989), to appear shortly, we shall examine some efficient computational algorithms for solving these equations.

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