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Global Linear Representations of Nonlinear Systems and the A^{d}joint Map

by

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Abstract

In this paper we shall study the global linearization of nonlinear systems on a manifold by two methods . The first consists of an expansion of the vector field in terms of a basis of the space of square integrable vector fields . In the second method we use the adjoint representation of the Lie algebra of vector fields to obtain an infinite-dimensional matrix representation of the system . A connection between the two approaches will be developed.

Keywords : Global Linearization , Adjoint Representation , Exponential Map.

1 Introduction

The exact representation of nonlinear systems by infinite-dimensional linear systems has been studied by many authors (see [1],[3]). These methods assume that the vector field defining the system is analytic and results essentially in Taylor series representations of the solutions . In this paper we shall study two methods which require only C^∞ vector fields ; indeed in the first case one can apply the method to any L^2 vector field . (We shall say precisely what this means shortly.)

In the first method , we expand the nonlinear system in terms of a basis of the (completion of the) space of 'square integrable' vector fields on a manifold. We shall then demonstrate that for the linear operator A obtained by this method we may evaluate e^{At} directly from the series by introducing a sequence of Banach spaces . Section 3 will deal with the local theory , while section 4 will consider the global result in detail .

In section 5 , we shall introduce an entirely different linear representation of a nonlinear system by using the adjoint representation of the (infinite- dimensional) Lie algebra of vector fields on a manifold . This will represent the Lie algebra of vector fields by a Lie algebra of infinite-dimensional matrices . Finally , a connection between this representation and that of sections 3,4 will be established .

2 Notation

$$
\begin{aligned}
L^2_w(R^n) &= \{f : R^n \longrightarrow R^n : \int_{R^n} \| f(x) \|^2 w(x)dx < \infty\} \\
\ell^2 &= \{(x_1, x_2, \ldots) : \sum_{i=1}^{\infty} | x_i |^2 < \infty\} \\
i &: X \subseteq Y, i \textit{ is a continuous inclusion of Banach Spaces} \\
\mathcal{L}(X,Y) &= \textit{space of continuous linear operators from } X \textit{ to } Y \\
T(M) &= \textit{tangent bundle of the differentiable manifold } M \\
T_p &= \textit{tangent space at } p \\
<.,.> &= \textit{inner product on space of vector fields} \\
E(M,T(M)) &= \textit{smooth sections of } T(M) \\
L^2(M,T(M)) &= \textit{completion of } E(M,M)) \textit{ under } <.,.> \\
\gamma^X_t(m) &= \textit{flow of vector field } X \textit{ with initial point } m \in M \\
ad, Ad &: \textit{adjoint maps}
\end{aligned}
$$

3 Local Theory of Linearization

In this section we shall consider a differential equation of the form

$$\dot{x} = f(x), x \in R^n \tag{3.1}$$

The extension of the theory to vector fields on manifolds will be discussed later . Let $L^2_w(R^n)$ be a weighted Hilbert space of square-integrable functions with respect to w such that the identity function I $(I(x) = x)$ belongs to $L^2_w(R^n)$.

Moreover , let $\{e_i\}_{0\le i<\infty}$ be an orthonormal basis for $L_w^2(R^n)$, each element of which is differentiable . Thus we have

$$x = \sum_{i=0}^{\infty} x^i e_i(x) \tag{3.2}$$

where $\{x^i\} \in \ell^2$.

Differentiating $e_n(x)$ along the trajectories of (3.1) , we have

$$\frac{d}{dt} e_n(x) = Je_n(x)\dot{x} = Je_n(x)f(x) \tag{3.3}$$

where

$$Je_n(x) = \left(\frac{\partial}{\partial x_j} e_{ni}(x)\right)$$

is the Jacobian matrix of e_n . Now make the assumption

$$Je_n(x)f(x) \in L_w^2(R^n), n \ge 0 \tag{3.4}$$

Then we can write

$$Je_n(x)f(x) = \sum_{m=0}^{\infty} a_{nm} e_m(x)$$

for some scalars a_{nm} , and so by (3.3) we have

$$\frac{d}{dt} e_n(x) = \sum_{m=0}^{\infty} a_{nm} e_m(x)$$

or

$$\frac{dE}{dt} = AE \tag{3.5}$$

where

$$E = (e_0, e_1, e_2, \ldots)^T$$

and

$$A = (a_{nm})$$

In order to investigate the operator A in more detail , we shall first examine a more abstract situation . Thus , let

$$X_0 \subseteq X_1 \subseteq X_2 \subseteq \dots \tag{3.6}$$

be a sequence of Banach spaces such that

(i) $i_k : X_k \subseteq X_{k+1}$ is a continuous injection for $k \geq 0$, with

$$\| x \|_{X_{k+1}} \leq \| x \|_{X_k} \; for \; all \; x \in X_k .$$

(ii) X_k is dense in X_{k+1} (in the topology of X_{k+1}) .

Then $\{X_k, i_{k,k+1}\}$ is a directed system of Banach spaces and we let X_∞ denote the direct limit of this system . Thus ,

$$X_\infty \doteq \cup_{k=0}^{\infty} X_k$$

and X_∞ has the finest topology for which all the injections

$$i_k : X_k \subseteq X_\infty$$

are continuous .

Next let A be an operator in the space $\mathcal{L}(X_0, X_1)$ and suppose that A has extensions to each X_k , $k \geq 1$ such that $A \in \mathcal{L}(X_k, X_{k+1})$. (We shall use the same letter A for each of these extensions.) Since $i_{k,k+1}$ is a continuous injection , so is $i_{k,k+\ell} \doteq i_{k+\ell-1,k+\ell-2} \circ i_{k+\ell-2,k+\ell-1} \circ \dots \circ i_{k,k+1}$ for $\ell \geq 2$ for $\ell \geq 2$.

Thus $i_{k+1,k+\ell} \circ A \in \mathcal{L}(X_k, X_{k+\ell})$, and by abuse of notation we shall denote this operator by A , it being clear from the context which spaces form the domain and range of A . We write

$$\| A \|_{k,k+\ell} = sup_{x \in X_k, x \neq 0} \frac{\| Ax \|_{X_{k+\ell}}}{\| x \|_{X_k}} \tag{3.7}$$

and we assume that

$$\| A \| \doteq \| A \|_{k,k+1}$$

is independent of k , $k \geq 0$. The notation $\| A \|$ is therefore unambiguously defined . Let $A^\ell = A \circ \ldots \circ A : X_k \longrightarrow X_{k+\ell}$ for each $\ell \geq 1$ (ℓ factors). Then we have

Lemma 3.1

$$\| A^\ell \|_{k,k+\ell} \leq \| A \|^\ell, k \geq 0, \ell \geq 1$$

Proof

If $x \in X_k$, then

$$\begin{aligned} \| A^\ell x \|_{X_{k+\ell}} &\leq \| A \|_{k+\ell-1,k+\ell} \| A^{\ell-1} x \|_{X_{k+\ell-1}} \\ &\leq \| A \|_{k+\ell-1,k+\ell} \| A \|_{k+\ell-2,k+\ell-1} \| A^{\ell-2} x \|_{X_{k+\ell-2}} \\ &\leq \ldots \\ &\leq \| A \|^\ell \| x \|_{X_k} \end{aligned} \tag{3.8}$$

and the result follows .

X_∞ is a Hausdorff topolgical vector space which is not complete . Let $\hat{X}_\infty$ denote the completion of X_∞ .

Theorem 3.2

For any $x \in X_\infty$ the series $\sum_{i=0}^{\infty}\left(\frac{A^i}{i!}t^i\right) x$ converges in $\hat{X}_\infty$.

Proof

Suppose first that $x \in X_0$ and for any integer $N \geq 0$ let $m > n \geq N$. Then consider the series $\sum_{i=n}^{m}\left(\frac{A^i}{i!}t^i\right) x$. We have , for any $k \geq m$,

$$\begin{aligned}
\| \sum_{i=n}^{m}\left(\frac{A^i}{i!}t^i\right) x \|_{X_k} &\leq \sum_{i=n}^{m}\frac{t^i}{i!} \| A^i x \|_{X_k} \\
&\leq \sum_{i=n}^{m}\frac{t^i}{i!} \| A \|^i \| x \|_{X_{k-i}} \qquad (\textit{by lemma } 3.1) \\
&\leq \sum_{i=n}^{m}\frac{t^i}{i!} \| A \|^i \| x \|_{X_0} \qquad (\textit{by } (i)) \qquad (3.9)
\end{aligned}$$

Since $i_k : X_k \subseteq X_\infty$ is continuous , it follows that the sequence $\sum_{i=0}^{\ell}\left(\frac{A^i}{i!}t^i\right) x$ is a Cauchy sequence in X_∞ and so it converges in $\hat{X}_\infty$. Similar reasoning now holds for any $x \in X_\infty$ and the result follows .

Corollary

Define the operator $e^{At} : X_\infty \longrightarrow \hat{X}_\infty$ by

$$e^{At}x = \sum_{i=0}^{\infty}\left(\frac{A^i}{i!}t^i\right) x, x \in X_\infty$$

Then e^{At} is continuous and hence has an extension

$$e^{At} : \hat{X}_\infty \longrightarrow \hat{X}_\infty$$

which is bounded .

Proof

This follows directly from (3.8) and the definition of direct limits .

Example

In order to bring out the significance of the various spaces introduced above we shall now present a trivial example in the form of a one-dimensional system

$$\dot{x} = cos\pi x. \tag{3.10}$$

To simplify the computation we shall consider the system on the finite domain [-1,1] and so the expansions will only be valid on this domain . Since the example is only for illustrative purposes , this is not really a restriction. Indeed , in some situations it may be that we know that the solutions of a system $\dot{x} = f(x)$ belong to some bounded region $\Omega \subseteq R^n$ and so we can then consider the space $L^2(\Omega)$.

Thus consider the o.n. basis $\frac{1}{\sqrt{2}}, cos\pi x, sin\pi x, \ldots, cosn\pi x, sinn\pi x, \ldots$ of $L^2(-1,1)$. This basis can be written in the form $e_1, e_2, e_3, \ldots$ where

$$\begin{aligned} e_1 &= \frac{1}{\sqrt{2}} \\ e_n &= cos\frac{n}{2}\pi x, \quad n \quad even \\ e_n &= sin\frac{n-1}{2}\pi x, \quad n \quad odd \end{aligned}$$

Note first that

$$x = \sum_{n=1}^{\infty} x_n e_n,$$

where

$$x_1 = 0$$

$$
\begin{aligned}
x_n &= 0 \quad , \quad n even \\
x_n &= (-1)^{(n+1)/2}\frac{2}{(n-1)\pi}
\end{aligned}
$$

To evaluate the matrix A in (3.5) note that

$$
\begin{aligned}
\dot{e}_1 &= 0 \\
\dot{e}_2 &= -(\pi/2)e_5 \\
\dot{e}_3 &= \pi/2(1+e_4) \\
\dot{e}_n &= -\frac{n\pi}{4}(e_{n-1}+e_{n+3}), n\ even, n \geq 4 \\
\dot{e}_n &= \frac{n-1}{4}\pi(e_{n-3}+e_{n+1}), n\ odd, n \geq 5
\end{aligned}
$$

This gives rise to an A matrix of the form

$$
A = \begin{bmatrix}
0 & 0 & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & 0 & 0 & -\frac{\pi}{2} & 0 & 0 & \dots & \dots & \dots & \dots \\
\frac{\pi}{\sqrt{2}} & 0 & 0 & \frac{\pi}{2} & 0 & 0 & 0 & \dots & \dots & \dots & \dots \\
0 & 0 & -\pi & 0 & 0 & 0 & -\pi & 0 & \dots & \dots & \dots \\
0 & \pi & 0 & 0 & 0 & \pi & 0 & 0 & \dots & \dots & \dots \\
0 & 0 & 0 & 0 & -\frac{3\pi}{2} & 0 & 0 & 0 & -\frac{3\pi}{2} & 0 & \dots \\
0 & 0 & 0 & \frac{3\pi}{2} & 0 & 0 & 0 & \frac{3\pi}{2} & 0\dots\dots & & \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots
\end{bmatrix} \tag{3.11}
$$

If $x(0) = x_0 \in (-1,1)$ is the initial value , then

$$
e_1(0) = \frac{1}{\sqrt{2}}
$$

$$\begin{aligned} e_n(0) &= \cos\frac{n}{2}\pi x_0, n \; even \\ e_n(0) &= \sin\frac{n-1}{2}\pi x_0, n \; odd \end{aligned}$$

and so , if $E(0) = (e_1(0), e_2(0), \ldots)^T$, then the solution of (3.9) is given by

$$x(t) = \sum_{n=1}^{\infty} x_n(exp(At)E(0))_n$$

For the spaces X_k we may take

$$\begin{aligned} X_0 &= \ell^2 \\ X_k &= \{s = (s_1, s_2, \ldots) : (s_1, \frac{s_2}{2}, \frac{s_3}{3}, \ldots) \in X_{k-1}\} \end{aligned}$$

4 A Global Theory

In this section we shall globalize the results of section 3 and hence obtain a solution to the problem of linearizing equations on manifolds . Thus , let M be a compact differentiable manifold and let $T(M)$ denote the tangent bundle of M with tangent space T_p at $p \in M$. Let $\langle,\rangle$ be a metric on $T(M)$, i.e. $\langle,\rangle_p$ is an inner product on T_p for each $p \in M$ and the function

$$\langle \xi, \eta \rangle : U \to R \quad (\langle \xi, \eta \rangle(p) = \langle \xi(p), \eta(p) \rangle_p)$$

is C^∞ , where ξ, η are local sections of $T(M)$ on $U \subseteq M$. Let μ be a strictly positive smooth measure on M and define an inner product on the space $E(M,T(M))$ of smooth sections of $T(M)$ (i.e. smooth vector fields) by

$$\langle \xi, \eta \rangle = \int_X \langle \xi(p), \eta(p) \rangle_p d\mu$$

for any $\xi, \eta \in E(M,T(M))$ (see [7]).

Let $L^2(M,T(M))$ be the completion of $E(M,T(M))$ in the norm $\langle , \rangle$. Then $L^2(M,T(M))$ is a Hilbert space and as such has a basis $\{v_i\}$ which we may assume belongs to $E(M,T(M))$ i.e. the elements of the basis are smooth . Then we have

$$[v_i, v_j] \in E(M,T(M)) \subseteq L^2(M,T(M)) \tag{4.1}$$

for each i, j , where $[,]$ is the standard Lie bracket . Note that $E(M,T(M))$ is an infinite dimensional Lie algebra , but $L^2(M,T(M))$ is not , since if we define

$$[f,g] = \sum \sum f_i g_j [v_i, v_j]$$

for any $f, g \in L^2(M,T(M))$ where $f = \sum f_i v_i, g = \sum g_j v_j$, then $[f,g] \notin L^2(M,T(M))$, in general .

In order to obtain a global theory of linearization we first consider the local vector field interpretation of the method of section 3 . Assume , temporarily , that f is analytic and consider the differential equation

$$\dot{x} = f(x), x(0) = x \tag{4.2}$$

defined on R^n . Then , as is well known (see , for example , [6]) if we interpret $f(x)$ as the vector field $\sum_{i=1}^n f_i \frac{\partial}{\partial x_i}$, we can write the solution of (4.2) in the form of the Lie series :

$$\begin{aligned} x(t) &= exp\left(t\sum_{i=1}^{n} f_i(x)\frac{\partial}{\partial x_i}\right)x \\ &\doteq \sum_{j=0}^{\infty}\frac{t^j}{j!}\left(\sum_{i=1}^{n} f_i(x)\frac{\partial}{\partial x_i}\right)^j x \end{aligned} \tag{4.3}$$

Assume that the sum in (3.2) is essentially finite , i.e.

$$x = \sum_{k=0}^{\ell} x^k e_k(x)$$

for some ℓ (indeed , we could arrange for $I(x)/ \parallel I \parallel$ to be a member of the basis $\{e_n\}$) . Then we can differentiate to obtain

$$\frac{\partial x}{\partial x_i} = \sum_{k=0}^{\ell} x^k \frac{\partial e_k}{\partial x_i}(x)$$

Consider the operator $\sum f_i(x)\frac{\partial}{\partial x_i}$ which appears in (4.3) . Then

$$\begin{aligned} \sum_{i=1}^{n} f_i(x)\frac{\partial}{\partial x_i}\left(\sum_{j=0}^{\infty} x^j e_j(x)\right) &= \sum_{j=0}^{\infty} x^j \sum_{i=1}^{n} f_i(x)\frac{\partial}{\partial x_i}e_j(x) \\ &= \sum_{j=0}^{\infty} x^j \sum_{m=0}^{\infty} a_{jm}e_m(x) \end{aligned} \tag{4.4}$$

Hence , the solution of the system (4.2) given by the Lie series (4.3) corresponds to the solution of the system (3.5) . Hence , the matrix A in (3.5) can be regarded as a matrix representation of the operator appearing in the Lie series . Note , however , that the analyticity requirement for the existence of the Lie series is no longer necessary in the L^2 theory so that the system (3.5) can be thought of as a direct generalization of the Lie series .

Now let f be a vector field on M which we shall assume for the moment to be differentiable (i.e. $f \in E(M,(M))$) . If $p \in M$ then choose a coordinate system $\phi : U \rightarrow R^n$ for some open set U with $p \in U \subseteq M$. Assume that $\phi(p) = 0$ and denote the coordinates as usual by $x_i = \phi_i(q), q \in U$. Then we can write

$$f = \sum_{i=1}^{n} f_i(x) \frac{\partial}{\partial x_i}, x = \phi(q), q \in U.$$

We have chosen a basis $\{v_i\} \subseteq E(M, T(M))$ of $L^2(M, T(M))$ and again , we can write , locally

$$v_i = \sum_{j=1}^{n} v_{ij}(x) \frac{\partial}{\partial x_j}, x = \phi(q), q \in U.$$

Let $e_i(x)$ denote the vector of local functions $(v_{i1}(x), \ldots, v_{in}(x))$. We must next choose the analogue of x in (3.2) as a combination of the functions $e_i(x)$. To do this define the vector field on $\phi(U) \subseteq R^n$ by

$$X = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} \in T(\phi(U))$$

and let $\phi^*(X) \in T(U)$ be the pullback of X to M . Then we can extend $\phi^*(X)$ to a vector field on M which we denote by $\overline{X}$. Since $\overline{X} \in E(M, T(M))$ we can write

$$\overline{X} = \sum_{i=0}^{\infty} \overline{x}^i v_i$$

Returning to the local expressions on $\phi(U)$ we can therefore write

$$x = \sum_{i=0}^{\infty} \overline{x} e_i(x)$$

Now define the local vector fields E_k on $\phi(U)$ by

$$\begin{aligned} E_k &= \sum_{i=1}^{n} \sum_{\ell=1}^{n} f_\ell(x) \frac{\partial}{\partial x_\ell} v_{ki}(x) \frac{\partial}{\partial x_i} \\ &= \sum_i (J e_k(x) f(x))_i \frac{\partial}{\partial x_i} \end{aligned}$$

Again we can extend each $\phi^*(E_k)$ to M (nonuniquely!) to obtain vector fields $\overline{E}_k$. Thus , for each k , we have

$$\overline{E}_k = \sum_{j=0}^{\infty} \xi_{kj} v_j$$

and , locally ,

$$E_k = \sum_{j=0}^{\infty} \xi_{kj} e_j \tag{4.5}$$

We have therefore proved

Theorem 4.1

Given a vector field X on M we can represent the differential equation defined by X locally in the form

$$\dot{E} = \Xi E$$

where

$$\Xi = (\xi_{kj})$$

is given by (4.5) and $E = (e_0, e_1, e_2, \ldots)^T$ is the local representation of the basis vector fields $v_0, v_1, \ldots$ of $L^2(M, T(M))$.

Moreover , if Ξ satisfies the conditions of theorem 3.2 on the spaces $X_0 \subseteq X_1 \subseteq X_2 \subseteq \ldots$, we can write

$$E(t) = exp(\Xi t)E(0)$$

and so

$$x(t) = \sum_{i=0}^{\infty} \overline{x}^i \left(exp(\Xi t)E(0)\right)_i$$

5 The Adjoint Representation

In this section we shall consider a different approach to the global linear representation of vector fields on a manifold , by applying the adjoint representation to the Lie algebra of vector fields on the manifold . Thus , let $L = A(M, T(M))$ denote the Lie algebra of analytic vector fields on M and suppose that each vector field is complete . If $X \in L$ the flow γ_t^X generated by X defines an element of a Lie transformation group on M given by $m \to \gamma_t^X(m), m \in M$ for each fixed t . Let G denote the (infinite-dimensional) Lie group of diffeomorphisms of M . We can define the exponential map $exp(tX)$ by

$$exp(tX) = \gamma_t^X$$

and by (4.3) it is given locally by the Lie series ; i.e. if

$$X = \sum_{i=1}^{n} f_i(x) \frac{\partial}{\partial x_i}$$

in the local coordinate x , then we have

$$exp(tX)x = \gamma_t^X(x) = exp\left(t \sum_{i=1}^{n} f_i(x) \frac{\partial}{\partial x_i}\right) x$$

(where we have identified points on M with their coordinates).

Now consider the adjoint representation of L defined by

$$ad : L \to \mathcal{L}(L)$$

where

$$(adX)Y = [X, Y], X, Y \in L.$$

(Here $\mathcal{L}(L)$ is the vector space of linear maps from L into L.) As before , let $\{e_i\}$ be a basis of $L^2(M, T(M))$. Moreover , let P_m denote the projection operator

$$P_m : L \rightarrow L_m$$

where L_m is the subspace generated by $\{e_1, \dots, e_m\}$ and define

$$L_f = \cup_{m=1}^{\infty} P_m L$$

i.e. L_f is the linear space of elements of L which have a representation as a finite linear combination of elements of the basis $\{e_i\}$. Then , for any $X \in L_f$ we have

$$\begin{aligned} \langle (adX)e_i, e_j \rangle &= \langle [\sum_{k=1}^{m} X_k e_k, e_i], e_j \rangle \\ &= \sum_{k=1}^{m} X_k \langle [e_k, e_i], e_j \rangle \\ &= \sum_{k=1}^{m} X_k . c_{ki}^{j} \end{aligned}$$

where c_{ki}^{j} are the structure constants of L given by

$$[e_k, e_i] = \sum_{\ell=1}^{\infty} c_{ki}^{\ell} e_\ell$$

Hence , in the adjoint representation the vector field X has the matrix representation

$$A = \sum_{k=1}^{m} X_k . c_{ki}^{j} \tag{5.1}$$

Example

Consider the system in (3.10) again , i.e.

$$\dot{x} = cos\pi x$$

The vector fields $e_1 = \frac{1}{\sqrt{2}}\frac{\partial}{\partial x}, e_2 = cos\pi x\frac{\partial}{\partial x}, e_3 = sin\pi x\frac{\partial}{\partial x}, \ldots, e_{2n} = cosn\pi x\frac{\partial}{\partial x}, e_{2n+1} =$ $sinn\pi x\frac{\partial}{\partial x}, \ldots$ for an o.n. basis of $L^2((-1,1), T(-1,1))$. We have

$$X = cos\pi x\frac{\partial}{\partial x}$$

and so

$$X = e_2.$$

Hence

$$\langle (adX)e_i, e_j\rangle = \langle [e_2, e_i], e_j\rangle.$$

Now

$$\begin{aligned}
[e_2, e_1] &= [cos\pi x\frac{\partial}{\partial x}, \frac{1}{\sqrt{2}}\frac{\partial}{\partial x}] \\
&= \frac{1}{\sqrt{2}}\pi e_3 \\
[e_2, e_2] &= 0 \\
[e_2, e_3] &= \sqrt{2}\pi e_1 \\
[e_2, e_i] &= [cos\pi x\frac{\partial}{\partial x}, cos\ell\pi x\frac{\partial}{\partial x}]\ if\ i = 2\ell\ \ (even) \\
&= -\frac{1}{2}(\ell+1)\pi e_{i-1} - \frac{1}{2}(\ell-1)\pi e_{i+3} \\
[e_2, e_i] &= [cos\pi x\frac{\partial}{\partial x}, sin\ell\pi x\frac{\partial}{\partial x}]\ if\ i = 2\ell+1\ \ (odd) \\
&= \frac{1}{2}(\ell+1)e_{i-3} + \frac{1}{2}(\ell-1)\pi e_{i+1}, i \geq 5
\end{aligned}$$

This gives the matrix representation

$$A = \begin{bmatrix} 0 & 0 & \frac{1}{\sqrt{2}}\pi & 0 & 0 & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & \dots & \dots & \dots \\ \sqrt{2}\pi & 0 & 0 & 0 & 0 & 0 & 0 & \dots & \dots & \dots & \dots \\ 0 & 0 & -\frac{3}{2}\pi & 0 & 0 & 0 & -\frac{1}{2}\pi & 0 & 0 & \dots & \dots \\ 0 & \frac{3}{2}\pi & 0 & 0 & 0 & \frac{1}{2}\pi & 0 & 0 & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & -2\pi & 0 & 0 & 0 & 2\pi & 0 & \dots \\ 0 & 0 & 0 & 2\pi & 0 & 0 & 0 & 2\pi & 0\dots\dots & & \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix} \tag{5.2}$$

We can provide a connection between the two representations given in sections 4 and 5 by noting the following well-known result on adjoint representations . (See [2]) :

Theorem 5.1

Let G be a (finite-dimensional) Lie group and for each $g \in G$ let $c(g) : G \to G$ be the map

$$c(g)(x) = gxg^{-1}.$$

Then , if LG is the Lie algebra of G , $ad : LG \to End(G)$ is the map which makes the diagram

$$\begin{tikzcd}
L(G) \arrow[r, "exp"] \arrow[d, "ad"'] & G \arrow[d, "Ad"] \\
End(LG) \arrow[r, "exp"] & Aut(LG)
\end{tikzcd}$$

commutative , where $Ad : G \rightarrow Aut(LG)$ is the map

$$g \mapsto d(c(g))$$

Moreover ,

$$ad = d(Ad)$$

(d is the differential operator) .

If M is a compact manifold , then the space of diffeomorphisms of M is an infinite-dimensional Lie group (see [5] and [4]) with Lie algebra $E(M, T(M))$. Let $\mathcal{M}$ denote the set of all real infinite matrices and let $I_1 : LG \rightarrow \mathcal{M}, I_2 : LG \rightarrow \mathcal{M}$ be the maps associate with a vector field X the matrices obtained , respectively , by the representations of sections 4 and 5 . Also let $\mathcal{M}_1 = RangeI_1, \mathcal{M}_2 = RangeI_2$. Then $I_1 : LG \rightarrow \mathcal{M}_1$ and $I_2 : LG \rightarrow \mathcal{M}_2$ are injective and so we can define the maps

$$J_1 : \mathcal{M}_1 \rightarrow LG, J_2 : \mathcal{M}_2 \rightarrow LG$$

where $J_1 = I_1^{-1}, J_2 = I_2^{-1}$. Then we have

Theorem 5.2

Let M be a compact differentiable manifold and let (U, ϕ) be a coordinate system around $m \in M$ where $\phi(m) = 0$, and $\phi(U) = R^n$. Let $\{e_i\}$ be a basis of $L^2(R^n)$ so that $\sum_{j=1}^n e_{ij} \frac{\partial}{\partial x_j}$ is a basis of $L^2(R^n, T(R^n))$. If X is a vector field on M , let

$$X = \sum_{i=1}^n X_i(x) \frac{\partial}{\partial x_i}$$

be the local representation of X in U . Suppose that A_1 is the representation of X (in U) (i.e. $A_1 = I_1(X)$) obtained as in section 4 and that A_2 is the matrix representation of X (i.e. $A_2 = I_2(X)$) obtained from the adjoint map as above . Then

$$exp(A_2 t)\mathcal{Y} = \{\langle J_1(exp(A_1 t)Y_1 exp(-A_1 t)), e_i\rangle\}_{1 \le i < \infty}$$

where , for any vector field Y on M , Y_1 is the matrix representation obtained in the same way as A_1 is obtained from X and $\mathcal{Y}$ is the column vector

$$\mathcal{Y} = \{\langle Y, e_i \rangle\}$$

Proof

This follows from the above definitions and the fact that $Ad(\gamma_t^X)Y$ is given by

$$\partial / \partial s \mid_0 a(t,s)$$

where

$$a(t,s) = exp\left(t \sum X_i \frac{\partial}{\partial x_i}\right) exp\left(s \sum Y_i \frac{\partial}{\partial x_i}\right) exp\left(-s \sum X_i \frac{\partial}{\partial x_i}\right)$$

and I_1 gives a matrix representation of the Lie series .

6 Conclusions

We have presented two new methods of linearizing nonlinear dynamical systems , which do not require analyticity of the vector field . The first is an essentially local procedure , but the adjoint representation provides a globally valid matrix representation of the vector field . It also has the advantage of being a Lie algebra homomorphism and so it will enable systems of vector fields which generate a subalgebra of $E(M, T(M))$ to be studied by matrix Lie algebra techniques - a point which will be investigated further in future papers .

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