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RATIONAL S^1-EQUIVARIANT ELLIPTIC COHOMOLOGY. (VERSION 5.2)

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Abstract. We give a functorial construction of a rational S^1-equivariant cohomology theory from an elliptic curve A equipped with suitable coordinate data. The elliptic curve may be recovered from the cohomology theory; indeed, the value of the cohomology theory on the compactification of an S^1-representation is given by the sheaf cohomology of a suitable line bundle on A. This suggests the construction: by considering functions on the elliptic curve with specified poles one may write down the representing S^1-spectrum in the author's algebraic model of rational S^1-spectra [9].

The construction extends to give an equivalence of categories between the homotopy category of module S^1-spectra over the representing spectrum and a derived category of sheaves of modules over the structure sheaf of A.

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1. Introduction.

Two of the most important topological cohomology theories are associated to one dimensional group schemes: ordinary cohomology is associated to the additive group, and K-theory is associated to the multiplicative group. This connection is most transparent in the equivariant context, and because the group schemes are one dimensional it is enough to consider a one dimensional group of equivariance: the circle group $\mathbb{T}$.

Beginning with ordinary cohomology, we use the Borel construction to define an equivariant theory for $\mathbb{T}$-spaces X by $H^*_{\mathbb{T}}(X) = H^*(E\mathbb{T} \times_{\mathbb{T}} X)$. The coefficient ring $H^*_{\mathbb{T}} = H^*(B\mathbb{T}) \cong \mathbb{Z}[x]$ inherits a coproduct from the map $B\mathbb{T} \times B\mathbb{T} \longrightarrow B\mathbb{T}$ classifying tensor product of line bundles, and the resulting Hopf algebra represents the additive group.

This construction works equally well for any complex oriented theory. For instance if we let z denote the natural representation of the circle group on the complex numbers, K-theory of the Borel construction has coefficient ring $K^0(BT) = \mathbb{Z}[[y]]$, with $y = 1-z$, and this represents the multiplicative formal group. However, by working with the correct equivariant theory we may obtain the uncompleted version. Indeed, the coefficient ring $K^0_{\mathbb{T}} = \mathbb{Z}[z, z^{-1}]$ of Atiyah-Segal equivariant K-theory acquires a coproduct from the group multiplication $\mathbb{T} \times \mathbb{T} \longrightarrow \mathbb{T}$, and the resulting Hopf algebra represents the multiplicative group.

Elliptic cohomology was first defined [18, 19] as a non-equivariant complex oriented cohomology theory whose associated formal group is the completion of an elliptic curve around the identity. It is therefore natural to hope for an equivariant cohomology theory giving the associated elliptic curve A without completion. It is the purpose of the present paper to construct such a theory over the rationals and establish its basic properties. The most obvious new feature is that A is not affine, and one of our main tasks is to elucidate the connection between the cohomology theory and the elliptic curve

A programme to extend this work to higher dimensional abelian varieties and higher dimensional tori is underway [12, 13, 14, 11].

In concrete terms, the main purpose of this paper is to construct a rational $\mathbb{T}$-equivariant cohomology theory $EA^*_{\mathbb{T}}(\cdot)$ associated to any elliptic curve A over a $\mathbb{Q}$-algebra. The construction is compatible with base change, and the properties of the cohomology theory when we work over a field may be summarized as follows; we give full details in Section 10 below.

Theorem 1.1. *For any elliptic curve A over a field k of characteristic 0, there is a 2-periodic, multiplicative, rational $\mathbb{T}$-equivariant cohomology theory $EA^*_{\mathbb{T}}(\cdot)$. The value on the one point compactification S^W of a complex representation W of $\mathbb{T}$ with $W^{\mathbb{T}} = 0$ is given as the sheaf cohomology of a line bundle $\mathcal{O}(-D(W))$. To describe this, we write $A[n]$ for the divisor of points of order dividing n in A. If $W = \sum_n a_n z^n$, we consider the divisor $D(W) = \sum_n a_n A[n]$, and the associated line bundle $\mathcal{O}(-D(W))$ on A. The cohomology of S^W is given by*

$$\widetilde{EA}^i_{\mathbb{T}}(S^W) \cong H^i(A; \mathcal{O}(-D(W)) \text{ for } i = 0, 1$$

and the homology by

$$\widetilde{EA}^{\mathbb{T}}_{-i}(S^W) \cong H^i(A; \mathcal{O}(D(W)) \text{ for } i = 0, 1.$$

In particular, the coefficient ring is

$$EA^{\mathbb{T}}_* = k[u, u^{-1}] \otimes \Lambda(\tau)$$

where u (of degree 2) is a generator of $H^0(A; \Omega)$ (i.e., a nowhere zero, regular differential) and τ (of degree -1) is a generator of $H^1(A; \mathcal{O})$.

Remark 1.2. The above properties do not quite determine the cohomology theory. The cohomology theory depends on one auxiliary piece of data: a *coordinate* t_e on A. This is a function vanishing to the first order at the identity, whose zeroes and poles are all at points of finite order. The construction is natural for isomorphisms of the data (A, k, t_e). All three of the inputs, A, k and t_e can be recovered from the cohomology of suitable spaces.

Remark 1.3. For general spaces there is a Hasse long exact sequence describing how to calculate elliptic cohomology. A precise statement is given in 15.3, but the idea is that, just as the arithmetic Hasse square recovers global data from completions at various primes p, compatible in the rationalization, the Hasse sequence recovers elliptic cohomology from Borel cohomology of C-fixed points with coefficients in completions of local rings at points of order $|C|$, compatible with the cohomology of the $\mathbb{T}$-fixed points with coefficients in meromorphic functions.

The first version of $\mathbb{T}$-equivariant elliptic cohomology was constructed by Grojnowksi in 1994 [15]. He was interested in implications for the representation theory of certain elliptic algebras: these implications are the subject of the work of Ginzburg-Kapranov-Vasserot [8] and the context is explained further in [7]. For this purpose it was sufficient to construct a theory on finite complexes taking values in analytic sheaves over the elliptic curve. Later Rosu [21] used this sheaf-valued theory to give a proof of Witten's rigidity theorem for the equivariant Ochanine genus of a spin manifold with non-trivial $\mathbb{T}$-action, and Ando-Basterra do the same for the Witten genus [2]. Ando [1] has related the sheaf valued theory to the representation theory of loop groups.

However, to exploit the theory fully, it is essential to have a theory defined on general $\mathbb{T}$-spaces and $\mathbb{T}$-spectra, and to have a conventional group-valued theory represented by a $\mathbb{T}$-spectrum EA. This allows one to use the full apparatus of equivariant stable homotopy theory. For example, twisted pushforward maps are immediate consequences of Atiyah duality; in more concrete terms, it allows one to calculate the theory on free loop spaces, and to describe algebras of operations. It is also likely to be useful in constructing an integral version of the theory, and we hope it may also prove useful in the continuing search for a geometric definition of elliptic cohomology. The theory we construct has these desirable properties, whilst retaining a very close connection with the geometry of the underlying elliptic curve.

Returning to the geometry, a very appealing feature is that although our theory is group valued, the original curve can still be recovered from the cohomology theory. It is also notable that the earlier sheaf theoretic constructions work over larger rings and certainly require the coefficients to contain roots of unity: the loss of information can be illustrated by comparing the rationalized representation ring $R(C_n) = \mathbb{Q}[z]/(z^n - 1)$ (with components corresponding to *subgroups* of C_n) to the complexified representation ring, isomorphic to the character ring $\mathrm{map}(C_n, \mathbb{C})$ (with components corresponding to the *elements* of C_n).

Finally, the ingredients of the model are very natural invariants of the curve given by sheaves of functions with specified poles at points of finite order: Definition 10.6 simply writes down the representing object in terms of these, and readers already familiar with elliptic curves and the model of [9] may wish to look at this immediately. In fact the algebraic model of [9] gives a generic de Rham model for all $\mathbb{T}$-equivariant theories, and the models of elliptic cohomology theories highlight this geometric structure. These higher de Rham models should allow applications in the same spirit as those made for de Rham models of ordinary cohomology and K-theory [16].

In fact, we are able to go beyond constructing a particular cohomology theory $EA^*_{\mathbb{T}}(\cdot)$ and establish an equivalence between a derived category of sheaves over the elliptic curve and cohomology theories which are modules over $EA^*_{\mathbb{T}}(\cdot)$. Because homotopy theory only sees points of finite order, we use the torsion point topology on the elliptic curve consisting of complements of sets of points of finite order, which is coarser than the Zariski topology, and because a $\mathbb{T}$-equivariant homotopy equivalence is an equivalence in H-equivariant fixed points for all subgroups H, the maps inverted in forming $D_{tp}(\mathcal{O}_A^{\mathrm{tp}}\text{-mod})$ are those inducing isomorphisms of $H^*(A; \mathcal{O}_A(D(W)) \otimes_{\mathcal{O}} (\cdot))$ for all representations W with $W^{\mathbb{T}} = 0$.

Theorem 1.4. *The representing object EA_a in the algebraic category may be taken to be a commutative ring, and there is an equivalence*

$$D_{tp}(\mathcal{O}_A^{\mathrm{tp}}\text{-}mod) \simeq D_{\mathbb{T}}(EA_a\text{-}mod)$$

between derived categories of sheaves of $\mathcal{O}_A$-modules on A and EA_a-modules. These categories both have relative injective dimension 1, so that maps are calculated by a short exact sequence from Hom and Ext groups in an abelian category.

The corresponding $\mathbb{T}$-spectrum EA is a ring up to homotopy, and the above equivalence classifies homotopy EA-module spectra up to equivalence as $\mathcal{O}_A$-modules up to isomorphism. Using the result of [14]*, that EA can be realized as a strictly commutative ring spectrum, the right hand side may be replaced the derived category of EA-module $\mathbb{T}$-spectra, and morphisms of module spectra are thereby also classified.*

This is proved in Section 21. Our construction directly models the representing ring spectrum EA in the author's algebraic model $\mathcal{A}_s$ of rational $\mathbb{T}$-spectra [9]. We describe the abelian category $\mathcal{A}_s$ in detail in Section 4, but it can be viewed as a category of sheaves over the space of closed subgroups of $\mathbb{T}$ [12]. The equivalence is obtained from functors at the level of abelian categories, and (Theorem 22.3) Grojnowski's sheaf $\mathrm{Groj}(X)$ associated to a compact $\mathbb{T}$-manifold X is obtained by applying the functor to the function spectrum EA-module $F(X, EA)$, and then changing to the analytic topology. Thus, for a compact $\mathbb{T}$-manifold X, there is a short exact sequence

$$0 \longrightarrow \Sigma H^1(A; \mathrm{Groj}(X)) \longrightarrow EA^*_{\mathbb{T}}(X) \longrightarrow H^0(A; \mathrm{Groj}(X)) \longrightarrow 0$$

relating the cohomology of Grojnowksi's sheaf to $EA^*_{\mathbb{T}}(X)$.

By way of motivation, we will discuss the way that a $\mathbb{T}$-equivariant cohomology theory is associated to several other geometric objects. Perhaps most familiar is the complete case discussed in Section 2, where the Borel theory for a complex oriented cohomology theory is associated to a formal group. Amongst global groups, the additive and multiplicative ones are the simplest, and in Appendix A we describe how they give rise to ordinary Borel cohomology and equivariant K-theory; the behaviour of the construction on the non-split torus is also notable.

We have divided the paper into six parts. Part 1 explains how equivariant cohomology theories ought to be related to group schemes. Part 2 provides prerequisites on rational $\mathbb{T}$-equivariant cohomology theories. Part 3 provides prerequisites on elliptic curves. Part 4 is extremely short, and just contains the construction. Part 5 describes some properties of the theory. Part 6 builds on the construction to give an equivalence between a derived category of sheaves over A and a derived category of $\mathbb{T}$-spectra. The appendix re-examines equivariant K-theory from the present point of view.

Historical note. Early versions of this paper were under joint authorship with M.J.Hopkins and I.Rosu. This reflected the fundamental influence of their ideas, in the expectation that they would continue to be part of the project. To the disappointment of all of us, circumstances prevented this, and the other authors withdrew.

Rosu's emphasis on the sheaf associated to a sphere [21] was significant. When the author first heard it at the 1997 Glasgow workshop on elliptic cohomology, he believed this would necessitate representing elliptic cohomology by sheaves of spectra. However it led Hopkins towards his vision that a result like the Theorem 1.1 proved here should be true. Work on the present paper began after a breathless conversation between the author and Hopkins in Oberwolfach at the 1998 Homotopietheorie meeting.

The present paper is Version 5.2 of the preprint.

Part 1. Equivariant cohomology theories and group schemes.

In Part 1 we describe how equivariant cohomology theories and group schemes are related in ideal circumstances. We begin with the familiar example of formal groups and complex oriented theories, and then explore how this correspondence should be extended.

2. Formal groups from complex oriented theories.

The purpose of this section is to recall that any complex orientable cohomology theory $E^*(\cdot)$ determines a one dimensional, commutative formal group $\widehat{\mathbb{G}}$ and to explain how the cohomology of various spaces can be described in terms of the geometry of $\widehat{\mathbb{G}}$. This is well known (see especially [3]) but it introduces the geometric language, and motivates our main construction, which *over the rationals* reverses the process by using this geometric data to construct the cohomology theory. Indeed, we will show that the machinery of [9] permits a construction of a 2-periodic rational $\mathbb{T}$-equivariant cohomology theory $E\mathbb{G}^*_{\mathbb{T}}(\cdot)$ from a one dimensional group scheme $\mathbb{G}$ over a $\mathbb{Q}$-algebra, functorial in $\mathbb{G}$ with some additional data. Furthermore, the construction is reversible in the sense that $\mathbb{G}$ can be recovered from $E\mathbb{G}^*_{\mathbb{T}}(\cdot)$. The most interesting case of this is when $\mathbb{G}$ is an elliptic curve, but the affine case is treated in Appendix A.

2.A. **Geometry of formal groups.** Before bringing the cohomology theory into the picture, we introduce the geometric language. When all schemes are affine, the geometric language is equivalent to the ring theoretic language, and all geometric statements can be given meaning by translating them to algebraic ones. It is traditional in topology to stick to algebra, but to prepare for the case of an elliptic curve, we will use the geometric language.

A one dimensional commutative formal group law over a ring k is a commutative and associative coproduct on the complete topological k-algebra $k[[y]]$. Equivalently, it is a complete topological Hopf k-algebra $\mathcal{O}$ together with an element $y \in \mathcal{O}$ so that $\mathcal{O} = k[[y]]$. A topological Hopf k-algebra $\mathcal{O}$ for which such a y exists is the ring of functions on a one dimensional commutative formal group $\widehat{\mathbb{G}}$. The counit $\mathcal{O} \longrightarrow k$, is viewed as evaluation of functions at the identity $e \in \widehat{\mathbb{G}}$, and the augmentation ideal I consists of functions vanishing at e. The element y generates the ideal I, and is known as a *coordinate* at e.

We also need to discuss locally free sheaves $\mathcal{F}$ over $\widehat{\mathbb{G}}$, and in the present affine context these are specified by the $\mathcal{O}$-module $M = \Gamma\mathcal{F}$ of global sections. In particular, line bundles L over $\widehat{\mathbb{G}}$ correspond to modules M which are submodules of the ring of rational functions and free of rank 1. Line bundles can also be described in terms of the zeroes and poles of their generating section: we only need this in special cases made explicit below. The generator f of the $\mathcal{O}$-module M is a section of L, and as such it defines a divisor $D = D_+ - D_-$, where D_+ is the subscheme of $\widehat{\mathbb{G}}$ where f vanishes (with multiplicities), and D_- is the subscheme of $\widehat{\mathbb{G}}$ where f has poles (with multiplicites). This divisor determines L, and we write $L = \mathcal{O}(-D)$. For example,

$$M = I = (y) \text{ corresponds to } \mathcal{O}(-(e)),$$

and

$$M = I^a = (y^a) \text{ corresponds to } \mathcal{O}(-a(e)).$$

Next we may consider the $[n]$-series map $[n] : \mathcal{O} \longrightarrow \mathcal{O}$, which corresponds to the n-fold sum map $n : \widehat{\mathbb{G}} \longrightarrow \widehat{\mathbb{G}}$. We write $\widehat{\mathbb{G}}[n]$ for the kernel of n, and its ring of functions is $\mathcal{O}/([n](y))$. Hence, since $n^*y = [n](y)$ by definition, $M = ([n](y))$ corresponds to $\mathcal{O}(-\widehat{\mathbb{G}}[n])$, and $M = (\ ([n](y))^a\ )$ corresponds to $\mathcal{O}(-a\widehat{\mathbb{G}}[n])$. Finally, if M corresponds to $\mathcal{O}(-D)$ and M' corresponds to $\mathcal{O}(-D')$ then $M^\vee := \mathrm{Hom}(M, \mathcal{O})$ corresponds to $\mathcal{O}(D)$ and $M \otimes M'$ corresponds to $\mathcal{O}(-D - D')$. This gives sense to enough line bundles for our purposes.

2.B. **Complex oriented cohomology theories.** Now suppose that E is a 2-periodic ring valued theory with coefficients E^* concentrated in even degrees. The collapse of the Atiyah-Hirzebruch spectral sequence for $B\mathbb{T}$ shows that E is complex orientable. We may define the $\mathbb{T}$-equivariant Borel cohomology by $E^*_{\mathbb{T}}(X) = E^*(E\mathbb{T} \times_{\mathbb{T}} X)$. We work over the ring $k = E^0_{\mathbb{T}}(\mathbb{T}) = E^0$, and view $E^0_{\mathbb{T}} = E^0(B\mathbb{T})$ as the ring of functions on a formal group $\widehat{\mathbb{G}}$ over

k. The tensor product and duality of line bundles makes $B\mathbb{T}$ into a group object, so $E^0(B\mathbb{T})$ is a topological Hopf algebra and $\widehat{\mathbb{G}}$ is a group. From this point of view, the augmentation ideal $I = \ker(E^0_{\mathbb{T}} \longrightarrow E^0)$ consists of functions vanishing at the identity $e \in \widehat{\mathbb{G}}$. We may also define the module of cotangent vectors at the identity by

$$\omega := I/I^2 = \tilde{E}^0(S^2) = E^{-2} = E_2.$$

This allows us to recover the graded cohomology ring from the ungraded ring since

$$E^{-2n}_{\mathbb{T}}(X) = E^0_{\mathbb{T}}(X) \otimes \omega^n.$$

Now, if W is a complex representation of the circle group $\mathbb{T}$ with $W^{\mathbb{T}} = 0$, we also let W denote the associated bundle over $B\mathbb{T}$ and the Thom isomorphism shows $\tilde{E}^0((B\mathbb{T})^W) = \tilde{E}^0_{\mathbb{T}}(S^W)$ is a rank 1 free module over $E^0_{\mathbb{T}}$, and hence corresponds to a line bundle $\mathbb{L}(W)$ over $\widehat{\mathbb{G}}$, whose global sections are naturally isomorphic to the module

$$\Gamma\mathbb{L}(W) = \tilde{E}^0_{\mathbb{T}}(S^W).$$

From the fact that Thom isomorphisms are transitive we see that $\mathbb{L}(W \oplus W') = \mathbb{L}(W) \otimes \mathbb{L}(W')$. The values of all these line bundles can be deduced from those of powers of z.

Lemma 2.1. (1) $\mathbb{L}(0) = \mathcal{O}$ *is the trivial bundle.*

(2) $\mathbb{L}(z) = \mathcal{O}(-(e))$ *is the sheaf of functions vanishing at e, and its module of sections I is generated by the coordinate y.*

(3) $\mathbb{L}(z^n) = \mathcal{O}(-\widehat{\mathbb{G}}[n])$ *is the sheaf of functions vanishing on $\widehat{\mathbb{G}}[n]$, and its module of sections is generated by the multiple $[n](y)$ of the coordinate y.*

(4) $\mathbb{L}(az^n) = \mathcal{O}(-a\widehat{\mathbb{G}}[n])$ *is the sheaf of functions vanishing on $\widehat{\mathbb{G}}[n]$ with multiplicity a, and its module of sections is generated $([n](y))^a$.*

Proof: The first statement is clear since $\tilde{E}^0_{\mathbb{T}}(S^0) = E^0_{\mathbb{T}}$. For the second we use the equivalence $(B\mathbb{T})^z \simeq (B\mathbb{T})^0/(\mathrm{pt})^0$. The third statement follows from the Gysin sequence since z^k is the pullback of z along the kth power map $B\mathbb{T} \longrightarrow B\mathbb{T}$. The final statement follows from the tensor product property. $\square$

This gives the fundamental connection between the equivariant cohomology of a sphere and sections of a line bundle.

Corollary 2.2. *If $E^*_{\mathbb{T}}(\cdot)$ is a complex oriented 2-periodic cohomology theory with associated formal group $\widehat{\mathbb{G}}$ then for any $a \in \mathbb{Z}$, $n \neq 0$ we have*

$$\tilde{E}^0_{\mathbb{T}}(S^{az^n}) = \mathcal{O}(-a\widehat{\mathbb{G}}[n]). \quad \square$$

3. What to expect when the group is not affine.

This section discusses what happens if we replace the formal group $\widehat{\mathbb{G}}$ (which is affine) in Section 2 by a (one dimensional) group $\mathbb{G}$ with higher cohomology.

3.A. **Odd cohomology.** The main point is that we cannot expect a cohomology theory entirely in even degrees. Now that the group is not affine, $\mathcal{O}$ denotes the structure *sheaf* of $\mathbb{G}$. This is reconciled to the above usage since in the affine case, the structure sheaf is determined by its ring of global sections. In the non-affine case, the cofibre sequence

$$S^{az} \wedge \mathbb{T}_+ \longrightarrow S^{az} \longrightarrow S^{(a+1)z}$$

of based $\mathbb{T}$-spaces forces there to be odd cohomology. Indeed, we expect a corresponding short exact sequence of sheaves

$$\mathcal{O}(-ae)/\mathcal{O}(-(a+1)e) \longleftarrow \mathcal{O}(-ae) \longleftarrow \mathcal{O}(-(a+1)e).$$

Any satisfactory cohomology theory will be functorial, and applying $\tilde{E}^0_{\mathbb{T}}(\cdot)$ will give sections of the associated sheaves. However the global sections functor on sheaves is not usually right exact, and the sequence of sections continues with the sheaf cohomology groups $H^1(\mathbb{G};\cdot)$. It is natural to hope that the long exact cohomology sequence induced by the sequence of spaces should be the long exact cohomology sequence induced by the sequence of sheaves. This gives a natural candidate for the odd cohomology:

$$\tilde{E}^i_{\mathbb{T}}(S^{az}) = H^i(\mathbb{G};\mathcal{O}(-a(e))) \text{ for } i = 0, 1.$$

This explains why it is possible for complex orientable cohomology theories to have coefficient rings in even degrees (formal groups are affine), and how their values on all complex spheres can be the same (formal groups have a regular coordinate). It also explains why we cannot expect either property for a theory associated to an elliptic curve.

3.B. **The definition of type.** We are now ready to formalize the relationship between group schemes and cohomology theories.

Definition 3.1. (i) Given a virtual complex representation W with $W^{\mathbb{T}} = 0$ we define an associated divisor $D(W)$ as follows. We write $W = \sum_n a_n z^n$, and then take $D(W) = \sum_n a_n \mathbb{G}[n]$, where $\mathbb{G}[n] = \ker(n : \mathbb{G} \longrightarrow \mathbb{G})$.
(ii) We say that a 2-periodic $\mathbb{T}$-equivariant cohomology theory $E^*_{\mathbb{T}}(\cdot)$ is of *type* $\mathbb{G}$ if, for any complex representation W,

$$\widetilde{E}^i_{\mathbb{T}}(S^W) \cong H^i(\mathbb{G};\mathcal{O}(-D(W))$$

and

$$\widetilde{E}^{\mathbb{T}}_{-i}(S^W) \cong H^i(\mathbb{G};\mathcal{O}(D(W)).$$

for $i = 0, 1$.

We also require these isomorphisms to be natural for inclusions $j : W \longrightarrow W'$ of representations. To describe this, first note that such a map induces a map $S^W \longrightarrow S^{W'}$ of based $\mathbb{T}$-spaces and hence maps

$$j^* : \widetilde{E}^i_{\mathbb{T}}(S^{W'}) \longrightarrow \widetilde{E}^i_{\mathbb{T}}(S^W)$$

and

$$j_* : \widetilde{E}^{\mathbb{T}}_{-i}(S^W) \longrightarrow \widetilde{E}^{\mathbb{T}}_{-i}(S^{W'}).$$

On the other hand we have inclusion of divisors $D(W) \longrightarrow D(W')$, inducing maps

$$\mathcal{O}(-D(W')) \longrightarrow \mathcal{O}(-D(W))$$

and

$$\mathcal{O}(D(W)) \longrightarrow \mathcal{O}(D(W')).$$

The induced maps in sheaf cohomology are required to be j^* and j_*.

Remark 3.2. The naturality requirement really allows us to identify the homology and cohomology of spheres with spaces of functions or their duals. For example, all the sheaves $\mathcal{O}(-D(V))$ are subsheaves of the constant sheaf

$$\mathcal{K} = \{f \mid f \text{ is a function on } \mathbb{G} \text{ with poles only at points of finite order }\},$$

of meromorphic functions. Thus the naturality requirement shows we may actually identify $\widetilde{E}^0_{\mathbb{T}}(S^{-W})$ with a set of meromorphic functions. In the presence of Serre duality (see Section 11), the first cohomology groups may similarly be identified with duals of spaces of functions.

Remark 3.3. We also need to discuss the appropriate behaviour for for representations W with trivial summands. The convention that

$$E^{\mathbb{T}}_{i+2}(X) = E^T_i(X) \otimes \omega$$

or

$$E_{\mathbb{T}}^{i-2}(X) = E_T^i(X) \otimes \omega$$

leads to an appropriate formula, where ω is the cotangent space at the identity of $\mathbb{G}$. However to obtain a properly natural identification it is better to use sheaves and the fact that

$$H^i(\mathbb{G}; \mathcal{L} \otimes_{\mathcal{O}} \Omega) = H^i(\mathbb{G}; \mathcal{L}) \otimes \omega,$$

where $\otimes$ denotes tensor product over k, and Ω is the sheaf of Kähler differentials on $\mathbb{G}$.

This leads to the requirements

$$\widetilde{E}^i_{\mathbb{T}}(S^W) \cong H^i(\mathbb{G}; \mathcal{O}(-D(W/W^{\mathbb{T}})) \otimes_{\mathcal{O}} \Omega^{\dim_{\mathbb{C}}(W^{\mathbb{T}})})$$

and

$$\widetilde{E}^{\mathbb{T}}_{-i}(S^W) \cong H^i(\mathbb{G}; \mathcal{O}(D(W/W^{\mathbb{T}})) \otimes_{\mathcal{O}} \Omega^{-\dim_{\mathbb{C}}(W^{\mathbb{T}})})$$

for $i = 0, 1$ (here and elsewhere Ω^n denotes the nth tensor power of Ω). The answer for other values of i follows easily.

Remark 3.4. The use of differentials to give suspensions means that a cohomology theory of type $\mathbb{G}$ contains data about Thom isomorphisms. For example, if $S^{\infty W} := \lim\limits_{\to\, U^{\mathbb{T}}=0} S^U$ then $S^z \longrightarrow S^{\infty W}$ induces

$$\begin{array}{ccc} \widetilde{E}^{\mathbb{T}}_2(S^z) & \longrightarrow & \widetilde{E}^{\mathbb{T}}_2(S^{\infty W}) \\ \cong\downarrow & & \downarrow\cong \\ H^0(\mathbb{G}; \mathcal{O}((e)) \otimes_{\mathcal{O}} \Omega) & \longrightarrow & H^0(\mathbb{G}; \mathcal{K} \otimes_{\mathcal{O}} \Omega). \end{array}$$

This picks out a k-subspace of the constant sheaf $\mathcal{K} \otimes_{\mathcal{O}} \Omega = H^0(\mathbb{G}; \mathcal{K} \otimes_{\mathcal{O}} \Omega)$. When $\mathbb{G}$ is an elliptic curve, this is the one dimensional space of invariant differentials.

3.C. **The affine case revisited.** It is worth pointing out that if $\mathbb{G}$ is affine and has a good coordinate, any cohomology theory of type $\mathbb{G}$ is complex orientable and in even degrees (we construct a number of such theories in Appendix A). More precisely, we require that $\mathbb{G}$ has a regular coordinate function y in the sense that the identity $e \in \mathbb{G}$ is defined by the vanishing of y and y is a regular element of the ring $\mathcal{O}$ of functions on $\mathbb{G}$. The multiplication by n map is also required to be flat for $n \geq 1$.

First, since $\mathbb{G}$ is affine, there is no higher cohomology. Thus, the condition that $E^*_{\mathbb{T}}(\cdot)$ is of type $\mathbb{G}$ states that the cohomology of spheres of complex representations is in even degrees, and that if $W^{\mathbb{T}} = 0$,

$$\widetilde{E}^{-2n}_{\mathbb{T}}(S^W) = \mathcal{O}(-D(W)) \otimes \omega^n,$$

where we have identified the sheaf with its space of global sections. It remains to observe that $\mathcal{O}(-D(W))$ is a free module on one generator. Indeed, $\mathbb{G}[n]$ is defined by the vanishing of $n^*(y)$ the pullback of y along the multiplication by n map of $\mathbb{G}$. Since this map is flat, n^*y is a regular element.

Since we have a complex oriented theory we also have Thom classes and Euler classes, and these depend on the coordinate, y. For example, the Thom class of z^n is the chosen generator of $\mathcal{O}(-\mathbb{G}[n])$, and the Euler class is its pullback to $\mathcal{O}$, namely

$$\chi_y(z^n) = [n](y) := n^*(y).$$

Thus we have the idea that the Euler class of z^n is a function whose vanishing defines $\mathbb{G}[n]$.

In characteristic 0 it is elementary to go one step further and decompose the divisor $\mathbb{G}[n]$:

$$\mathbb{G}[n] = \sum_{s|n} \mathbb{G}\langle s\rangle$$

where $\mathbb{G}\langle s\rangle$ is the divisor of points of exact order s. In fact, we define a function $\phi_s(y)$ vanishing to the first order on $\mathbb{G}\langle s\rangle$ recursively by the condition

$$\chi_y(z^n) = \prod_{s|n} \phi_s(y):$$

the formula for $n = 1$ defines $\phi_n(y)$ directly for $n = 1$, and for larger values of n, $\phi_n(y)$ is defined by dividing $\chi_y(z^n)$ by the previously defined $\phi_s(y)$. Each $\phi_n(y)$ is regular by the regularity of y and the flatness requirement.

3.D. **Summary.** We may summarize the correspondence between algebra and topology.

- The suspension $S^{az^n} \wedge E\mathbb{G}$ corresponds to the sheaf $\mathcal{O}(a\mathbb{G}[n])$ and more generally, suspension by z^n corresponds to tensoring with $\mathcal{O}(\mathbb{G}[n])$.
- The subgroup $\mathbb{T}[n]$ of order n (kernel of z^n) corresponds to the subgroup $\mathbb{G}[n]$ of elements of order dividing n (defined by the vanishing of $\chi(z^n)$).
- The inclusion $S^0 \longrightarrow S^{z^n}$ which induces multiplication by the Euler class (in the presence of a Thom isomorphism) corresponds to $\mathcal{O} \longrightarrow \mathcal{O}(\mathbb{G}[n])$.
- We extend the notation, so that

  $$S^{\infty z^n} := \lim_{\rightarrow\, a} S^{az^n} \text{ corresponds to the sheaf } \mathcal{O}(\infty\mathbb{G}[n]) := \lim_{\rightarrow\, a} \mathcal{O}(a\mathbb{G}[n])$$

  and

  $$\tilde{E}\mathcal{F} := \lim_{\rightarrow\, U^{\mathbb{T}}=0} S^U \text{ corresponds to the sheaf } \mathcal{O}(\infty\mathbb{G}[tors]) := \lim_{\rightarrow\, a,n} \mathcal{O}(a\mathbb{G}[n]).$$

- The family $\mathcal{F}$ of finite subgroups corresponds to the set $\mathbb{G}[tors]$ of elements of torsion points.

Part 2. Background on rational $\mathbb{T}$-equivariant cohomology theories.

The method of this paper is only practical because there is a complete algebraic model for rational $\mathbb{T}$-equivariant cohomology theories [9]. In Part 2 we describe this model and explain how to make relevant calculations in it.

4. The model for rational $\mathbb{T}$-spectra.

For most of the paper we work with the representing objects of $\mathbb{T}$-equivariant cohomology theories, namely $\mathbb{T}$-spectra [6]. Thus we prove results about the representing spectra, and deduce consequences about the cohomology theories. More precisely, any suitable $\mathbb{T}$-equivariant cohomology theory $E^*_{\mathbb{T}}(\cdot)$ is represented by a $\mathbb{T}$-spectrum E in the sense that for a based $\mathbb{T}$-space X,

$$\tilde{E}^*_{\mathbb{T}}(X) = [X, E]^*_{\mathbb{T}}.$$

This enables us to define the associated homology theory

$$\widetilde{E}^{\mathbb{T}}_*(X) = [S^0, E \wedge X]^{\mathbb{T}}_*$$

in the usual way. We shall make use of the elementary fact that the Spanier-Whitehead dual of the sphere S^W is S^{-W}, as one sees by embedding S^W as the equator of $S^{W\oplus 1}$. Hence, for example

$$\tilde{E}^0_{\mathbb{T}}(S^W) = [S^W, E]^{\mathbb{T}} = [S^0, S^{-W} \wedge E]^{\mathbb{T}} = \pi^{\mathbb{T}}_0(S^{-W} \wedge E) = \tilde{E}^{\mathbb{T}}_0(S^{-W}).$$

We say that a cohomology theory is *rational* if its values are graded rational vector spaces. A spectrum is rational if the cohomology theory it represents is rational. It suffices to check the values on the homogeneous spaces $\mathbb{T}/H$ for closed subgroups H, since all spaces are built from these up to weak equivalence.

Convention 4.1. Henceforth all spectra and the values of all cohomology theories are rationalized whether or not this is indicated in the notation.

Our results are made possible because there is a complete algebraic model of the category of *rational* $\mathbb{T}$-spectra, and hence of rational $\mathbb{T}$-equivariant cohomology theories [9]. For the convenience of the reader we spend the rest of this section summarizing the relevant results from [9] in a convenient form. There are two models for rational $\mathbb{T}$-spectra, as derived categories of abelian categories.

Theorem 4.2. [9, 5.6.1, 6.5.1] *There are equivalences*

$$\mathbb{T}\text{-}\mathit{Spectra} \simeq D(\mathcal{A}_s) \simeq D(\mathcal{A}_t).$$

of triangulated categories.

The *standard* abelian category $\mathcal{A}_s$ has injective dimension 1, and the *torsion* abelian category $\mathcal{A}_t$ is of injective dimension 2. The derived category $D(\mathcal{A}_s)$ is formed by taking differential graded objects in $\mathcal{A}_s$ and inverting homology isomorphisms, and similarly for $D(\mathcal{A}_t)$. It is usually easiest to identify the model for a $\mathbb{T}$-spectrum in $D(\mathcal{A}_t)$, at least providing its model has homology of injective dimension 1. This is then transported to the standard category, where calculations are sometimes easier. We describe what we need about the categories in the following subsections.

4.A. **Rings of functions.** To describe the categories, we need some ingredients. The information is organized by isotropy group, and we let $\mathcal{F}$ denote the discrete set of finite subgroups of $\mathbb{T}$. On this we consider the constant sheaf $\mathcal{R}$ of rings with stalks $\mathbb{Q}[c]$ where c has degree -2. We need to consider the ring

$$R = \mathrm{map}(\mathcal{F}, \mathbb{Q}[c]) \cong \prod_{H \in \mathcal{F}} \mathbb{Q}[c]$$

of global sections, where maps and product are graded. For each subgroup H, we let $e_H \in R$ denote the idempotent with support H.

To avoid confusion about grading we introduce the requisite suspensions. In topology we may suspend by complex representations W; these enter the theory through the dimension function $w(H) := \dim_{\mathbb{C}}(W^H)$. Notice that w takes only finitely many values, and is equal to $w(\mathbb{T})$ for almost all finite subgroups H.

Definition 4.3. Suppose $w : \mathcal{F} \longrightarrow \mathbb{Z}$ is an almost constant function. We divide the set of finite subgroups into sets

$$\mathcal{F}_{w,i} = \{H \mid w(H) = i\}$$

on which w is constant; only finitely many of these are non-empty, and all but one are finite. We write $w(\mathbb{T})$ for the value of w on the infinite set.

Let $e_{w,i} \in R$ be the idempotent supported on $\mathcal{F}_{w,i}$, and introduce the suspension functor on R-modules by

$$\Sigma^w N = \bigoplus_i \Sigma^{2i} e_{w,i} N.$$

Now if $w : \mathcal{F} \longrightarrow \mathbb{Z}_{\geq 0}$ is zero almost everywhere, we write c^w for the *universal Thom class* of w, defined by $c^w(H) = c^{w(H)}$. Since it is not homogeneous, c^w is not an element of R, but nonetheless it is natural to consider the R-module

$$c^w R := \Sigma^{-w} R = \prod_H c^{w(H)} \mathbb{Q}[c],$$

viewed as an R-submodule of $\prod_H \mathbb{Q}[c, c^{-1}]$; since c^w is a generator in some sense, we call c^w a Thom class (further explanation is given at the end of the section). Classical Thom classes give rise to Euler classes by restriction to the coefficient ring. We now create a ring in which the Euler classes corresponding to the Thom classes c^w belong. First, let

$$\mathcal{E} = \{c^w \mid w : \mathcal{F} \longrightarrow \mathbb{Z}_{\geq 0} \text{ of finite support}\};$$

thinking of this as if it generates a multiplicatively closed subset, we make an adelic construction by forming the R-submodule

$$t_*^{\mathcal{F}} = \mathcal{E}^{-1} R := \lim_{\rightarrow\, w} \Sigma^w R = \bigcup_w c^{-w} R$$

of $\prod_H \mathbb{Q}[c, c^{-1}]$. Observe that $t_*^{\mathcal{F}}$ is a graded R-algebra. As a graded vector space $t_*^{\mathcal{F}}$ is $\bigoplus_H \mathbb{Q}$ in positive degrees and $\prod_H \mathbb{Q}$ in degrees zero and below.

Remark 4.4. (i) Note that if $w(\mathbb{T}) = 0$, there is a natural degree 0 *isomorphism*

$$c^w : t_*^{\mathcal{F}} \xrightarrow{\cong} \Sigma^w t_*^{\mathcal{F}},$$

which in the sth component is

$$c^{w(s)} : \mathbb{Q}[c, c^{-1}] \longrightarrow \Sigma^{2w(s)}\mathbb{Q}[c, c^{-1}].$$

It is natural to see this as multiplication by the Euler class.
(ii) Given a complex representation W of $\mathbb{T}$ with $W^{\mathbb{T}} = 0$ we may define an associated function $w : \mathcal{F} \longrightarrow \mathbb{Z}_{\geq 0}$ zero almost everywhere by $w(H) = \dim_{\mathbb{C}}(W^H)$. We sometimes write c^W for this element of R, and we note that $\mathcal{E}$ is generated as a multiplicative subset by elements of this form.
(iii) Viewing R as the ring of functions on the discrete space $\mathcal{F}$, the universal Euler classes can be used to define finite subsets. Indeed, we may view c^w as a non-homogeneous section of the structure sheaf, or as a homogeneous section of a line bundle $\mathcal{R}(-w)$ with global sections $\Sigma^{-w}R$. Now, for any finite subset $\mathcal{H} \subseteq \mathcal{F}$ we may consider its characteristic function $\chi(\mathcal{H})$. The associated universal Euler class $c^{\chi(\mathcal{H})}$ is the function vanishing to the first order on $\mathcal{H}$.

4.B. **Description of the abelian categories.** The objects of the standard model $\mathcal{A}_s$ are triples (N, β, V) where N is an R-module (called the *nub*), V is a graded rational vector space (called the *vertex*) and $\beta : N \longrightarrow t_*^{\mathcal{F}} \otimes V$ is a morphism of R-modules (called the *basing map*) which becomes an isomorphism when $\mathcal{E}$ is inverted. When no confusion is likely, we simply say that $N \longrightarrow t_*^{\mathcal{F}} \otimes V$ is an object of the standard abelian category. An object of $\mathcal{A}_s$ should be viewed as the module N with the additional structure of a trivialization of $\mathcal{E}^{-1}N$. A morphism $(N, \beta, V) \longrightarrow (N', \beta', V')$ of objects is given by an R-map $\theta : N \longrightarrow N'$ and a $\mathbb{Q}$-map $\phi : V \longrightarrow V'$ compatible under the basing maps.

Since the standard abelian category has injective dimension 1, homotopy types of objects of the derived category $D(\mathcal{A}_s)$ are classified by their homology in $\mathcal{A}_s$, so that homotopy types correspond to isomorphism classes of objects of the abelian category $\mathcal{A}_s$. In the sheaf theoretic approach [12], N is the space of global sections of a sheaf on the space of closed subgroups $\mathbb{T}$, the vertex V is the value of the sheaf at the subgroup $\mathbb{T}$ and the fact that the basing map $\beta : N \longrightarrow t_*^{\mathcal{F}} \otimes V$ is an isomorphism away from $\mathcal{E}$ is the manifestation of the patching condition for sheaves.

The objects of the torsion abelian category $\mathcal{A}_t$ are triples (V, q, T) where V is a graded rational vector space, T is an $\mathcal{E}$-torsion R-module and $q : t_*^{\mathcal{F}} \otimes V \longrightarrow T$ is a morphism of R-modules. The condition on T is equivalent to requiring (i) that T is the sum of its idempotent factors $T(H) = e_H T$ in the sense that $T = \bigoplus_H T(H)$ and (ii) that each $T(H)$ is a torsion $\mathbb{Q}[c]$-module. When no confusion is likely, we simply say that $t_*^{\mathcal{F}} \otimes V \longrightarrow T$ is an object of the torsion abelian category. In the sheaf theoretic approach, the module $T(H)$ is the cohomology of the structure sheaf with support at H. By contrast with the standard abelian category, the torsion abelian category has injective dimension 2. Thus not every object X of the derived category $D(\mathcal{A}_t)$ is determined up to equivalence by its homology $H_*(X)$ in the abelian category $\mathcal{A}_t$. We say that X is *formal* if it is equivalent to its homology (considered as a differential graded object with zero differential), and that it is *intrinsically formal* if it is equivalent to any object with the same homology. Evidently, an intrinsically formal object is formal. The Adams spectral sequence shows immediately that X is intrinsically formal if its homology has injective dimension 0 or 1 in $\mathcal{A}_t$. In general, if $H_*(X) = (t_*^{\mathcal{F}} \otimes V \longrightarrow T)$, the object X is equivalent to the fibre of a map $(t_*^{\mathcal{F}} \otimes V \longrightarrow 0) \longrightarrow (t_*^{\mathcal{F}} \otimes 0 \longrightarrow \Sigma T)$ (in the derived category) between objects in $\mathcal{A}_t$ of injective dimension 1. This map is classified by an element of $\mathrm{Ext}_R(t_*^{\mathcal{F}} \otimes V, \Sigma T)$, so that X is formal if the Ext group is zero in even degrees. Thus X is intrinsically formal if both V and T are in even degrees or ([9, 5.3.1]) if T is injective in the sense that each $T(H)$ is an injective $\mathbb{Q}[c]$-module.

Lemma 4.5. *An R-map $q : t_*^{\mathcal{F}} \otimes V \longrightarrow \bigoplus_s T_s$ is determined by its idempotent pieces $q_s : \mathbb{Q}[c, c^{-1}] \otimes V \longrightarrow T_s$.*

Conversely, any sequence of $\mathbb{Q}[c]$-maps q_s so that, for each $f \in V$, only finitely many of the values $q_s(c^0 \otimes f)$ are non-zero, determines an R-map q.

Proof: To see that the idempotent pieces determine q, note that if all idempotent pieces are zero we may argue that $q = 0$: if $q(1 \otimes v) \neq 0$ some idempotent piece would be non-zero, hence q vanishes on $R \otimes V$, and hence induces a map

$$\overline{q} : (t_*^{\mathcal{F}}/R) \otimes V = \bigoplus_s (\mathbb{Q}[c, c^{-1}]/\mathbb{Q}[c]) \otimes V \longrightarrow \bigoplus_s T_s,$$

which is the direct sum of its idempotent pieces.

The converse statement is easily checked. □

4.C. **Spheres, suspensions and Euler classes.** Spheres are important because they are invertible objects, and therefore play a role corresponding to that of line bundles in categories of sheaves. We introduce the appropriate apparatus to discuss them.

We described the suspension Σ^w on the category of R-modules in 4.3.

Definition 4.6. The suspension functor on objects of the standard abelian category $\mathcal{A}_s$ is defined by

$$\Sigma^w(N \longrightarrow t_*^{\mathcal{F}} \otimes V) = (\Sigma^w N \longrightarrow \Sigma^w t_*^{\mathcal{F}} \otimes V \xrightarrow{c^w} t_*^{\mathcal{F}} \otimes \Sigma^{2w(\mathbb{T})} V).$$

Thus, the basing map for the suspension is obtained by multiplying the original one by the appropriate Euler class, which is $c^{w(i)-w(\mathbb{T})}$ on $e_{w,i}N$.

Definition 4.7. [9, 5.8.2] The algebraic 0-sphere is the object

$$S^0 = (R \longrightarrow t_*^{\mathcal{F}} \otimes \mathbb{Q})$$

where R is the submodule of $t_*^{\mathcal{F}} \otimes \mathbb{Q}$ generated by $1 \otimes 1$.

Given an almost constant function $w : \mathcal{F} \longrightarrow \mathbb{Z}$ the algebraic w-sphere is the object of $\mathcal{A}_s$ defined by

$$S^w = \Sigma^w S^0 = (R(w) \longrightarrow t_*^{\mathcal{F}} \otimes \Sigma^{2w(\mathbb{T})}\mathbb{Q})$$

where

$$R(w) = \Sigma^w R = c^{-w} R \subseteq \Sigma^w t_*^{\mathcal{F}} \xrightarrow{\cong} \Sigma^{2w(\mathbb{T})} t_*^{\mathcal{F}}$$

as above. Note that different parts of this diagram have been shifted by different amounts, so that both the grading and the structure maps are different for different spheres.

If X is a $\mathbb{T}$-spectrum we write $M_s(X)$ and $M_t(X)$ for the models of X in $\mathcal{A}_s$ and $\mathcal{A}_t$ respectively. In fact, if $\Phi^{\mathbb{T}}X$ denotes the geometric fixed point spectrum of X, and $E\mathcal{F}$ denotes the universal almost free $\mathbb{T}$-space, we have

$$H_*(M_t(X)) = (t_*^{\mathcal{F}} \otimes V \longrightarrow T),$$

where

$$V = \pi_*(\Phi^{\mathbb{T}}X)),$$

and

$$T = \pi_*^{\mathbb{T}}(\Sigma E\mathcal{F}_+ \wedge X).$$

Since $\mathcal{A}_t$ is of injective dimension 2, this does not always determine $M_t(X)$. On the other hand, since $\mathcal{A}_s$ is of injective dimension 1, we may take $M_s(X)$ to be an object of the underlying abelian category $\mathcal{A}_s$ (i.e., to have zero differential). In fact,

$$M_s(X) \simeq H_*(M_s(X)) = (N \longrightarrow t_*^{\mathcal{F}} \otimes V)$$

where V is as above and N lives in a long exact sequence

$$\cdots \longrightarrow N \longrightarrow t_*^{\mathcal{F}} \otimes V \longrightarrow T \longrightarrow \cdots$$

This at least makes clear that V is to do with $\mathbb{T}$-fixed points of X, T is to do with the almost free part of X and N is an appropriate mixture. It also suggests the relationship between $\mathcal{A}_s$ and $\mathcal{A}_t$. This amount of detail is more than we need for the present paper. Finally, we need to record that spheres in the algebraic and topological contexts correspond.

Lemma 4.8. [9, 5.8.3] *Suppose W is a virtual complex representation, and let $w = \dim_{\mathbb{C}}(W)$.*
(i) The object modelling the sphere S^W in $\mathcal{A}_s$ is the algebraic sphere S^w:

$$M_s(S^W) = S^w = (R(w) \longrightarrow t_*^{\mathcal{F}}).$$

(ii) Algebraic and topological suspensions coincide in the sense that

$$M_s(\Sigma^W X) = \Sigma^w M_s(X).$$

Proof: Part (ii) follows from Part (i) since the algebraic suspension is tensor product with S^w and S^w is flat. □

Warning 4.9. We are modelling *complex* representations W. Thus if ϵ is the trivial representation of $\mathbb{T}$ on $\mathbb{C}$, we have $S^\epsilon = S^2$. We thus need to be careful when discussing a single suspension (smash product with the circle). We use the same method to resolve this conflict in algebra as in topology: an integer has its usual meaning, whereas the function $\mathcal{F} \longrightarrow \mathbb{Z}$ with constant value 1 will be denoted ϵ.

We are now in a position to justify calling the function c^w a universal Euler class when $w(\mathbb{T}) = 0$. In the topological context, the Euler class of a complex representation W with $W^{\mathbb{T}} = 0$ in a complex oriented cohomology theory is defined by pulling back a Thom class along $S^0 \longrightarrow S^W$; equivalently, in the associated homology theory we take the image of the Thom class under the map $S^{-W} \longrightarrow S^0$. In the algebraic context we do precisely the same. The Thom class of S^{-W} is the 'generator' of $R(-w)$, namely the 'element' c^w, which is the image of $1 \in t_*^{\mathcal{F}}$ under the isomophism $c^w : t_*^{\mathcal{F}} \longrightarrow t_*^{\mathcal{F}}$. The two obstructions to a universal Thom isomorphism are the two linked facts that c^w is not homogeneous and that the putative isomorphism is not compatible with basing maps.

Consider the subgroup $\mathbb{T}[n]$ of order n, and the representation z^n. If we take the K-theory Euler class we have

$$e(z^n) = 1 - z^n = \prod_{s|n} \phi_s,$$

where ϕ_s is the sth cyclotomic function, independent of n. Similarly, the dimension function corresponding to z^n, is the characteristic function $\mathrm{sub}(n)$ for the subgroups of $\mathbb{T}[n]$. Hence the universal Euler class defining the subgroups of $\mathbb{T}[n]$ is

$$c^{z^n} = c^{\mathrm{sub}(n)} = \prod_{s|n} c_s,$$

where c_s is the universal Euler class for the characteristic function of the singleton $\{\mathbb{T}[s]\}$. It is therefore natural to view c_s as a universal cyclotomic function.

5. Cohomology of spheres.

The main point of contact between topology and geometry is through the cohomology of spheres and line bundles. We therefore describe how this works in the standard model for $\mathbb{T}$-spectra. We shall only need to discuss $\mathbb{T}$-spectra with particularly nice algebraic models, so we begin by describing them.

5.A. **Rigidity.** Given a $\mathbb{T}$-spectrum E with torsion model $M_t(E)$ with homology $H_*(M_t(E)) = (t_*^{\mathcal{F}} \otimes V \longrightarrow T)$ in the abelian category $\mathcal{A}_t$, it is not hard to calculate $V = E_*^{\mathbb{T}}(\tilde{E}\mathcal{F})$ or $T = E_T^*(\Sigma^{-1} E\mathcal{F}_+)$. However, if this is to determine E we must show in addition that $M_t(E)$ is formal.

Definition 5.1. We say that a $\mathbb{T}$-equivariant cohomology theory E is *rigid* if the following two equivalent conditions hold

(1) $H_*M_t(E) = (t_*^{\mathcal{F}} \otimes V \xrightarrow{q} T)$ has surjective structure map q.

(2) $H_*M_s(E) = (N \xrightarrow{\beta} t_*^{\mathcal{F}} \otimes V)$ has injective structure map β.

We say that a rigid spectrum E is *even* if V, T and N are concentrated in even degrees.

Lemma 5.2. *If E is rigid then $M_t(E)$ is intrinsically formal, and if $H_*M_t(E) = (t_*^{\mathcal{F}} \otimes V \xrightarrow{q} T)$ then*

$$M_t(E) \simeq (t_*^{\mathcal{F}} \otimes V \xrightarrow{q} T)$$

and

$$M_s(E) \simeq (N \xrightarrow{\beta} t_*^{\mathcal{F}} \otimes V)$$

where

$$N = \ker(t_*^{\mathcal{F}} \otimes V \longrightarrow T),$$

and the basing map β is the inclusion. Furthermore we have the explicit injective resolution

$$0 \longrightarrow M_s(E) \simeq \begin{pmatrix} N \\ \downarrow \\ t_*^{\mathcal{F}} \otimes V \end{pmatrix} \longrightarrow \begin{pmatrix} t_*^{\mathcal{F}} \otimes V \\ \downarrow \\ t_*^{\mathcal{F}} \otimes V \end{pmatrix} \longrightarrow \begin{pmatrix} T \\ \downarrow \\ 0 \end{pmatrix} \longrightarrow 0$$

in $\mathcal{A}_s$.

Proof: To see that $M_t(E)$ is formal, it is only necessary to remark that T is the quotient of an $\mathcal{E}$-divisible group and therefore injective [9, 5.3.1]. $\square$

Lemma 5.3. *If E is rigid, the corresponding object $M_s(E) = (N \longrightarrow t_*^{\mathcal{F}} \otimes V)$ in $\mathcal{A}_s$ is flat.*

Proof: Tensor product on $\mathcal{A}_s$ is defined termwise. First, note that $t_*^{\mathcal{F}} \otimes V$ is exact for tensor product with objects P with $\mathcal{E}^{-1}P \cong t_*^{\mathcal{F}} \otimes W$ for some W, so the tensor product is exact on the vertex part.

For the nub, we use the fact that the category $\mathcal{A}_s$ is of flat dimension 1 by [9, 23.3.5], together with the fact that N is a submodule of $t_*^{\mathcal{F}} \otimes V$. $\square$

5.B. **Homomorphisms out of S^0.** For an object X of $\mathcal{A}_s$ there is an exact sequence

$$0 \longrightarrow \mathrm{Ext}_{\mathcal{A}_s}(S^{1+w}, M) \longrightarrow [S^w, M] \longrightarrow \mathrm{Hom}_{\mathcal{A}_s}(S^w, M) \longrightarrow 0,$$

so we shall need to calculate these Hom and Ext groups. For the present we restrict ourselves to the Hom groups. We avoid confusion about grading by restricting to the case $w = 0$ using $[S^w, M] = [S^0, \Sigma^{-w} M]$.

Lemma 5.4. *For an object $M = (N \xrightarrow{\beta} t_*^{\mathcal{F}} \otimes V)$ of the abelian category $\mathcal{A}_s$*

$$\mathrm{Hom}_{\mathcal{A}_s}(S^0, (N \longrightarrow t_*^{\mathcal{F}} \otimes V)) = N(c^0) := \{n \in N \mid \beta(n) \in c^0 \otimes V\}.$$

Proof: A homomorphism $f : S^0 \longrightarrow M$ of degree 0 is given by a square

$$\begin{array}{ccc} R & \xrightarrow{\theta} & N \\ \downarrow & & \downarrow \\ t_*^{\mathcal{F}} \otimes \mathbb{Q} & \xrightarrow{1 \otimes \phi} & t_*^{\mathcal{F}} \otimes V. \end{array}$$

Thus f is determined by the R-map θ, and $\mathrm{Hom}_R(R, N) = N$. On the other hand, the image of $1 \in R$ under the basing map is $1 \otimes 1$, which imposes the stated condition, since $\phi(1) \in V_0$. $\square$

5.C. **Cohomology of spheres.** The aim of the present section is to make explicit the calculation of $E_*^{\mathbb{T}}(S^W)$ in terms of $H_*(M_t(E)) = (q : t_*^{\mathcal{F}} \otimes V \longrightarrow T)$ assuming that E is rigid and even.

Lemma 5.5. *Suppose $w : \mathcal{F} \longrightarrow \mathbb{Z}$ is zero almost everywhere. If E is rigid and even then the wth suspension $\Sigma^w E$ is rigid and even.*

If $M_t(E) = (t_*^{\mathcal{F}} \otimes V \xrightarrow{q} T)$ then

$$M_t(\Sigma^w E) \simeq (t_*^{\mathcal{F}} \otimes V \xrightarrow{q^w} \Sigma^w T),$$

where the structure map is given by

$$q^w(c_s^{i(s)} \otimes \alpha) = q(c_s^{i(s)+w(s)} \otimes \alpha) \in e_s(\Sigma^w T)_{2n-2i(s)}$$

for $\alpha \in VA_{2n}$. Thus

$$M_s(\Sigma^w E) = (\Sigma^w N \longrightarrow t_*^{\mathcal{F}} \otimes V)$$

where

$$\Sigma^w N = \ker(t_*^{\mathcal{F}} \otimes V \xrightarrow{q^w} \Sigma^w T). \quad \square$$

Remark 5.6. A natural mnemonic is to write

$$q(xc^w \otimes \alpha) = q^w(x \otimes \alpha),$$

despite the fact that xc^w is not an element of $t^{\mathcal{F}}_*$.

We may now assemble the information to calculate the homology of spheres.

Corollary 5.7. *Suppose that E is rigid and even, so that $H_*(M_t(E)) = (q : t^{\mathcal{F}}_* \otimes V \longrightarrow T)$ is surjective and V and T are in even degrees. For any function $w : \mathcal{F} \longrightarrow \mathbb{Z}$ zero almost everywhere*

$$\widetilde{E}^{\mathbb{T}}_0(S^w) = \ker(q : c^w \otimes V_0 \longrightarrow (\Sigma^w T)_0)$$

and

$$\widetilde{E}^{\mathbb{T}}_{-1}(S^w) = \mathrm{cok}(q : c^w \otimes V_0 \longrightarrow (\Sigma^w T)_0)$$

Proof: To calculate the homology we use the short exact sequence

$$0 \longrightarrow \mathrm{Ext}_{\mathcal{A}_s}(S^1, M_s(\Sigma^w E)) \longrightarrow E^{\mathbb{T}}_0(S^w) \longrightarrow \mathrm{Hom}_{\mathcal{A}_s}(S^0, M_s(\Sigma^w E)) \longrightarrow 0.$$

We may calculate the Hom and Ext groups by applying $\mathrm{Hom}_{\mathcal{A}_s}(S^0, \cdot)$ to the injective resolution of $\Sigma^w E$ given in 5.2. □

Part 3. Background on elliptic curves.

In Part 3 we summarize relevant facts about elliptic curves, and make some easy deductions that we will need for the construction of rational $\mathbb{T}$-equivariant elliptic cohomology.

6. Elliptic curves.

In this section we record the well known facts about elliptic curves that will play a part in our construction. We use [24] as a basic reference for facts about elliptic curves, and [17] as background from algebraic geometry.

Let A be an elliptic curve (i.e., a smooth projective curve of genus 1 with a specified point e) over a field k of characteristic 0 and let $\mathcal{O} = \mathcal{O}_A$ be its sheaf of regular functions. Note that $\Gamma\mathcal{O} = k$, so the sheaf contains a great deal more information than its ring of global sections. A divisor on A is a finite $\mathbb{Z}$-linear combination of points defined over the algebraic closure $\overline{k}$ of k, and associated to any rational function f on A we have the divisor $\mathrm{div}(f) = \Sigma_P \mathrm{ord}_P(f)(P)$, where $\mathrm{ord}_P(f) \in \mathbb{Z}$ is the order of vanishing of f at P. If a divisor is fixed by $\mathrm{Gal}(\overline{k}/k)$ it is said to be defined over k, and all the divisors we consider will be of this sort. In the usual way, if D is a divisor on A, we write $\mathcal{O}(D)$ for the associated invertible sheaf. Its global sections are given by

$$\Gamma\mathcal{O}(D) = \{f \mid \mathrm{div}(f) \geq -D\} \cup \{0\},$$

so that for a point P, the global sections of $\mathcal{O}(-P)$ are the functions vanishing at P.

We also have $\mathcal{O}(D_1) \otimes_{\mathcal{O}} \mathcal{O}(D_2) = \mathcal{O}(D_1 + D_2)$.

Since the global sections functor is not right exact, we are led to consider cohomology, but since A is one-dimensional this only involves $H^0(A; \cdot) = \Gamma(\cdot)$ and $H^1(A; \cdot)$, which are related by Serre duality. This takes a particularly simple form since the canonical divisor is zero on an elliptic curve:

$$H^0(A; \mathcal{O}(D)) = H^1(A; \mathcal{O}(-D))^\vee,$$

where $(\cdot)^\vee = \mathrm{Hom}_k(\cdot, k)$ denotes vector space duality.

From the Riemann-Roch theorem we deduce that the canonical divisor is 0 and the cohomology of each line bundle:

$$\dim(H^0(A; \mathcal{O}(D)) = \begin{cases} \deg D & \text{if } \deg(D) \geq 1 \\ 0 & \text{if } \deg(D) \leq -1 \end{cases}$$

and

$$\dim(H^1(A; \mathcal{O}(D)) = \begin{cases} |\deg D| & \text{if } \deg(D) \leq -1 \\ 0 & \text{if } \deg(D) \geq 1. \end{cases}$$

For the trivial divisor one has

$$\dim(H^0(A; \mathcal{O})) = \dim(H^1(A; \mathcal{O})) = 1.$$

Now if $D = \Sigma_P n_P(P)$ is a divisor of degree 0, we may form the sum $S(D) = \Sigma_P n_P P$ in A, and D is linearly equivalent to $(S(D)) - (e)$. If $S(D) = e$ then the sheaf $\mathcal{O}(D)$ has the same cohomology as $\mathcal{O}$. Otherwise, since no function vanishes to order exactly 1 at P, we find

$$H^0(A; \mathcal{O}(D)) = H^1(A; \mathcal{O}(D)) = 0.$$

We may recover A from the graded ring $\Gamma(\mathcal{O}(*e)) = \{\Gamma\mathcal{O}(ne)\}_{n\geq 0}$. Indeed, this is the basis of the proof in [24, III.3.1] that any elliptic curve is a subvariety of $\mathbb{P}^2$ defined by a Weierstrass equation. We choose a basis $\{1, x\}$ of $\Gamma\mathcal{O}(2e)$ and a extend it to a basis $\{1, x, y\}$ of $\Gamma\mathcal{O}(3e)$. Now observe that since $\Gamma\mathcal{O}(6e)$ is 6-dimensional, there is a relation between the seven elements $1, x, x^2, x^3, y, xy$ and y^2: this is the Weierstrass equation, and it may be verified that A is the closure in $\mathbb{P}^2$ of the plane curve it defines. The graded ring $\Gamma(\mathcal{O}(*e))$ has generator Z of degree 1 corresponding to the constant function 1 in $\Gamma\mathcal{O}(e)$, X of degree 2 corresponding to x, and Y of degree 3 corresponding to y. These three variables satisfy the homogeneous form of the Weierstrass equation. The statement that A is the projective closure of the plane curve defined by the Weierstrass equation may be restated in terms of Proj:

$$A = \mathrm{Proj}(\Gamma(\mathcal{O}(*e))).$$

7. Torsion points and topology.

On the one hand, equivariant topology only gives counterparts to torsion points, but on the other it gives them greater importance. This gives two significant variations of the standard theory: we need to use a different topology and we need to invert different sets of morphisms in forming the derived category.

7.A. **The torsion point topology.** Because the topological model only gives counterparts of torsion points, we restrict sheaves to open sets which are complements of sets of points of finite order. This means that for us meromorphic functions are only allowed poles at points of finite order, and this entails a number of other small effects that need attention.

The divisor $A\langle n\rangle$ of points of exact order n will play a central role. Note that

$$A[n] = \sum_{s|n} A\langle s\rangle.$$

Definition 7.1. (i) Any divisor of the form $\sum_s a_s A\langle s\rangle$ (with $a_s \in \mathbb{Z}$) is called a *torsion point divisor.*
(ii) The *torsion point topology* on A is the topology whose proper closed sets are specified by a finite set F of positive integers

$$V_F = \bigcup_{s\in F} A\langle s\rangle.$$

The non-empty open sets are thus $U_F := A \setminus V_F$.

Since the sets V_F are closed in the Zariski topology, we have a change of topology map $i : A_{\mathrm{Zar}} \longrightarrow A_{\mathrm{tp}}$, and the usual adjoint pair of functors

$$i^{-1} : \mathrm{Shv}_A^{\mathrm{tp}} \rightleftarrows \mathrm{Shv}_A^{\mathrm{Zar}} : i_*$$

between categories of sheaves. The restriction of topology functor i_* is defined on Zariski presheaves $\mathcal{F}$ by $i_*(\mathcal{F})(V) := \mathcal{F}(V)$, which evidently takes sheaves to sheaves and is exact. The extension of topology functor i^{-1} is defined on torsion point presheaves $\mathcal{G}$ by $(i^{-1}\mathcal{G})(U) = \mathcal{G}(\hat{U})$, where $\hat{U} = U_F$ where $F := \{n \mid A\langle n\rangle \cap U = \emptyset\}$; this functor does not preserve sheaves, so to obtain the sheaf level functor we pass to associated sheaves.

Lemma 7.2. *The unit of the adjunction gives an isomorphism $i_*i^{-1}\mathcal{G} \cong \mathcal{G}$.* $\square$

To describe stalks it is convenient to use the notation

$$\mathcal{F}_\infty := \lim_{\to\, n} \mathcal{F}(A \setminus \bigcup_{p\le n} A\langle p\rangle),$$

for sections with poles at any points of finite order, and

$$\mathcal{F}_s := \lim_{\to\, n} \mathcal{F}(A \setminus \bigcup_{p\le n,\, p\neq s} A\langle p\rangle)$$

for sections regular on points of exact order s but with poles at any points of any other finite order. Note that these are not Zariski stalks, but if we use the corresponding notation for a torsion point sheaf $\mathcal{G}$ we find $\mathcal{G}_P = \mathcal{G}_s$, where s is the order of P. A short calculation then gives

$$(i_*\mathcal{F})_P = \begin{cases} \mathcal{F}_\infty & \text{if } P \text{ is of infinite order} \\ \mathcal{F}_s & \text{if } P \text{ is of order } s \end{cases}$$

and

$$(i^{-1}\mathcal{G})_P = \begin{cases} \mathcal{G}_\infty & \text{if } P \text{ is of infinite order} \\ \mathcal{G}_s & \text{if } P \text{ is of order } s, \end{cases}$$

so that i^{-1} preserves stalks.

Note that this means Zariski sheaves of the form $i^{-1}\mathcal{G}$ are very rare, since the stalks at points of the same order are identical. In particular, all stalks at points of infinite order are the same, suggesting there are no continuous families of sheaves of this sort.

Example 7.3. We may restrict the Zariski structure sheaf $\mathcal{O}_A^{\mathrm{Zar}}$ to the torsion point topology, and we take $\mathcal{O}_A^{\mathrm{tp}} := i_*\mathcal{O}_A^{\mathrm{Zar}}$.

Similarly, our ring of meromorphic functions is

$$\mathcal{K} = \{f \mid f \text{ has poles only at points of finite order }\},$$

with associated constant sheaf $\mathcal{O}(\infty tors)$. Note that functions vanishing at points of infinite order are not invertible in $\mathcal{K}$.

The local rings of the structure sheaf are thus

$$(\mathcal{O}_A^{\mathrm{tp}})_P = \begin{cases} \mathcal{K} & \text{if } P \text{ is of infinite order} \\ \{f \in \mathcal{K} \mid f \text{ is regular at points of exact order } s\ \} & \text{if } P \text{ is of finite order } s \end{cases}$$

Lemma 7.4. *The functors i_* and i^{-1} are both exact.*

Proof: The exactness of i^{-1} follows since it preserves stalks. For i_*, note that taking sections over $A \setminus F$ is exact for any non-empty set F of torsion points since it is affine; the stalks of i_* are calculated as direct limits of such functors. $\square$

Corollary 7.5. *For Zariski sheaves $\mathcal{F}$, the cohomology in the Zariski and torsion point topologies agree:*

$$H^*_{\mathrm{Zar}}(A; \mathcal{F}) = H^*_{\mathrm{tp}}(A; i_*\mathcal{F}).$$

Proof: Since $\mathcal{F}$ and $i_*\mathcal{F}$ have the same global sections, and i_* is exact, it suffices to note that if $\mathcal{I}$ is flabby then $i_*\mathcal{I}$ is a fortiori flabby too. $\square$

In future we will simply write $H^*(A; \mathcal{F})$ for the common value of cohomology. Note that this applies to the sheaves $\mathcal{O}_A^{\mathrm{Zar}}(D(V))$ of most concern to us, and we will usually omit notation for the topology, writing simply $\mathcal{O}(D(V))$.

7.B. **Torsion point equivalences.** The previous subsection dealt with the change of topology, but there is the second issue of what set of morphisms are inverted to form the derived category. In equivariant topology one does not usually invert all equivariant maps which are non-equivariant weak equivalences (since this gives only the homotopy theory of free actions). Instead, we invert only those equivariant maps which are equivalences in all fixed points.

We may transpose these considerations to sheaves of modules. More precisely, $\mathcal{O}_A^{\mathrm{Zar}}$ is a sheaf of rings in the Zariski topology and $\mathcal{O}_A^{\mathrm{tp}}$ is a sheaf of rings in the torsion point topology, and we may consider their respective categories of modules, $\mathcal{O}_A^{\mathrm{Zar}}$-mod and $\mathcal{O}_A^{\mathrm{tp}}$-mod. These are both abelian categories, and related by the adjoint pair

$$i^* : \mathcal{O}_A^{\mathrm{tp}}\text{-mod} \rightleftarrows \mathcal{O}_A^{\mathrm{Zar}}\text{-mod} : i_*\ ,$$

where

$$i^*N := i^{-1}N \otimes_{i^{-1}(\mathcal{O}_A^{\mathrm{tp}})} \mathcal{O}_A^{\mathrm{Zar}}.$$

Lemma 7.6. *The unit of the adjunction gives an isomorphism $i_*i^*N \cong N$, so $\mathcal{O}_A^{\mathrm{tp}}$-mod may be viewed as a subcategory of the category $\mathcal{O}_A^{\mathrm{Zar}}$-mod.* $\square$

Lemma 7.7. *The functor i^* is exact.*

Proof: It suffices to prove that $\mathcal{O}_A^{\mathrm{Zar}}$ is flat over $i^{-1}\mathcal{O}_A^{\mathrm{tp}}$, which we may verify at the level of stalks. This is straightforward since $\mathcal{O}_A^{\mathrm{Zar}}(U)$ is flat over $i^{-1}\mathcal{O}_A^{\mathrm{tp}}(U) = \mathcal{O}_A^{\mathrm{Zar}}(\hat{U})$ for any open set U. $\square$

Derived categories are formed from abelian categories by taking a category of differential graded objects and inverting a suitable collection of morphisms. If all homology isomorphisms are inverted we obtain $D(\mathcal{O}_A^{\mathrm{Zar}}\text{-mod})$ and $D(\mathcal{O}_A^{\mathrm{tp}}\text{-mod})$, but we wish to invert fewer morphisms. The torsion point homology isomorphisms are those which induce isomorphisms of $H^*(A; \cdot \otimes \mathcal{O}(D))$ for all torsion point divisors D, and we denote the derived categories obtained by inverting these $D_{tp}(\mathcal{O}_A^{\mathrm{Zar}}\text{-mod})$ and $D_{tp}(\mathcal{O}_A^{\mathrm{tp}}\text{-mod})$.

To actually construct the derived categories we use cellular approximation. This is determined by specifying a set of spheres $(\sigma_\alpha)_{\alpha \in A}$ which must be small objects. An object is *cellular* if it is built from the spheres σ_α using arbitrary coproducts and triangles. A map $X \longrightarrow Y$ is a *weak equivalence* if it induces an isomorphism of $[\sigma_\alpha, \cdot]_*$ for all α. A *cellular approximation* of an object X is then a weak equivalence $\Gamma X \longrightarrow X$ where ΓX is cellular. We then work with the actual homotopy category of cellular objects. For us the underlying category is the category of differential graded sheaves of $\mathcal{O}$-modules in the appropriate topology and the cells are the sheaves $\mathcal{O}(D)$ where D runs through torsion point divisors.

For clarity we display the relationship with the conventional derived category of sheaves on A.

Proposition 7.8. *The derived categories are related by functors in the commutative diagram*

$$\begin{array}{ccc} D_{tp}(\mathcal{O}_A^{\mathrm{tp}}\text{-}mod) & \longrightarrow & D(\mathcal{O}_A^{\mathrm{tp}}\text{-}mod) \\ i_* \uparrow\downarrow i^* & & i_* \uparrow\downarrow i^* \\ D_{tp}(\mathcal{O}_A^{\mathrm{Zar}}\text{-}mod) & \longrightarrow & D(\mathcal{O}_A^{\mathrm{Zar}}\text{-}mod), \end{array}$$

where the verticals are adjoint pairs with counits giving equivalences $i_* i^* N \simeq N$.

Proof: The horizontals are elementary, since any torsion point homology isomorphism is a homology isomorphism.

Since i_* and i^* are exact, they preserves homology isomorphisms, and therefore induces maps of derived categories. For torsion point homology isomorphisms we make additional arguments. Indeed, $i_*\mathcal{HOM}(M, N) = \mathcal{HOM}(i_*M, i_*N)$ so that, taking $M = \mathcal{O}(-D)$ we see that $i_*(N(D)) = (i_*N)(D)$ and so i_* preserves torsion point homology isomorphisms. Finally, $i^*(i_*M \otimes_{\mathcal{O}_A^{\mathrm{tp}}} N) \cong M \otimes_{\mathcal{O}_A^{\mathrm{Zar}}} i^*N$, so taking $M = \mathcal{O}(D)$ we see that i^* preserves torsion point homology isomorphisms as required. $\square$

As remarked before, there is a far greater change in character in the vertical maps changing the topology than in the horizontal maps changing the inverted morphisms. Even in $D(\mathcal{O}_A^{\mathrm{Zar}}\text{-mod})$ there are continuous families $\mathcal{O}(P)$ of distinct objects.

8. Coordinate data

Our main theorem constructs a cohomology theory of type A for an elliptic curve A. The construction depends on a choice of function vanishing at the identity, and the purpose of this section is to make clear the exact extent of this dependence.

8.A. **The coordinate.** Because the local ring $\mathcal{O}_e$ in the torsion point topology is not quite the usual Zariski local ring, we make explicit the the properties we need.

Lemma 8.1. *The ideal*

$$I_e = \{f \in \mathcal{O}_e \mid f(e) = 0\}$$

of functions vanishing at the identity in $\mathcal{O}_e$ is principal. The generators of I_e are exactly the functions t_e vanishing to first order at e whose zeroes and poles are all at points of finite order.

If t_e is a generator of I_e then for any non-zero $f \in \mathcal{K}$ there is an integer n such that $ft_e^n \in \mathcal{O}_e$ and ft_e^n does not vanish at e.

Proof: Suppose that t_e is a function whose zeroes and poles are at points of finite order with $t_e(e) = 0$. Certainly $t_e \in I_e$; on the other hand, if $f \in I_e$, then $f(e) = 0$ so that f/t_e is still regular at e, and only has poles at points of finite order. Hence $f = t_e \cdot f/t_e \in (t_e)$ and $I_e = (t_e)$ as required. To see that this exhausts the set of generators, we note that a function $s \in I_e$ with a zero at a point P of infinite order is not a generator. Indeed, I_e contains functions f which do not vanish at P, and whenever $f = sg$, the function g has a pole at P.

The final statement is clear since t_e is a uniformizing element in the Zariski local ring. $\square$

Definition 8.2. (i) A *coordinate* on A (at the identity) is a generator t_e of the ideal I_e in $\mathcal{O}_e$ of functions vanishing at e.

(ii) A *coordinate divisor* is a divisor Z_e of the form $\mathrm{div}(t_e)$ for some coordinate t_e. By Abel's theorem, a torsion point divisor $Z_e = \Sigma_P n_P(P)$ with $n_e = 1$ is a coordinate divisor if and only if $\Sigma_P n_P = 0$ and $\Sigma_P n_P P = 0$.

Remark 8.3. The ring $\mathcal{O}_e$ is not a local ring in the sense of commutative algebra: although I_e is maximal, not all functions outside I_e are invertible. However, the following lemma will provide the good behaviour we need.

Lemma 8.4. *For any $s \geq 0$ the quotient I_e^s/I_e^{s+1} is one dimensional over k, generated by the image of t_e^s. Hence $\mathcal{O}_e/I_e^s$ is s-dimensional, generated by the images of $1, t_e, \ldots, t_e^{s-1}$.* $\square$

We briefly discuss a special way of choosing coordinates.

Definition 8.5. A *Weierstrass parametrization* of an elliptic curve is a choice of two functions x_e with a pole of order 2 at the identity and nowhere else, and y_e with a pole of order 3 at the identity and nowhere else. Because we work with the torsion point topology, we also require that x_e and y_e only vanish at torsion points. This Weierstrass parametrization determines a coordinate $t_e = x_e/y_e$ of $\mathcal{O}_e$.

Remark 8.6. (i) The function x_e is specified up to scalar multiplication by a pair of non-identity points A, B of finite order with $A+B = e$ by the condition $\mathrm{div}(x_e) = -2(e)+(A)+(B)$. The function y_e is specified up to scalar multiplication by three non-identity points C, D, E of finite order with $C + D + E = e$ by $\mathrm{div}(y_e) = -3(e) + (C) + (D) + (E)$. This gives the coordinate divisor

$$\mathrm{div}(t_e) = (e) + (A) + (B) - (C) - (D) - (E).$$

(iii) One popular choice of Weierstrass parametrization involves choosing a point P of order 2. This determines a choice of x_e and y_e up to a constant multiple by the conditions

$$\mathrm{div}(x_e) = -2(e) + 2(P) \text{ and } \mathrm{div}(y_e) = -3(e) + (P) + (P') + (P'')$$

where $A[2]=\{e,P,P',P''\}$. Thus we obtain the coordinate divisor

$$\mathrm{div}(t_e)=(e)+(P)-(P')-(P''). \quad \square$$

8.B. **The cyclotomic functions.** Once we have chosen a coordinate, this determines the choice of a function defining the points of exact order s.

Lemma 8.7. *Given a choice t_e of coordinate on the elliptic curve A, for each $s\geq 2$, there is a unique function t_s with the properties*

(1) *t_s vanishes exactly to the first order on $A\langle s\rangle$,*
(2) *t_s is regular except at the identity $e\in A$ where it has a pole of order $|A\langle s\rangle|$,*
(3) *$t_e^{|A\langle s\rangle|}t_s$ takes the value 1 at e*

Furthermore, the function t_s only depends on the image of t_e in $\omega := I_e/I_e^2$, and multiplying t_e by a scalar λ multiplies t_s by $\lambda^{|A\langle s\rangle|}$.

Proof: Consider the divisor $A\langle s\rangle - |A\langle s\rangle|(e)$. Note that the sum of the points of $A\langle s\rangle$ in A is the identity: if $s\neq 2$ this is because points occur in inverse pairs, and if $s=2$ it is because the $A[2]$ is isomorphic to $C_2\times C_2$. It thus follows from the Riemann-Roch theorem that there is a function f with $A\langle s\rangle - |A\langle s\rangle|(e)$ as its divisor. This function (which satisfies the first two properties in the statement) is unique up to multiplication by a non-zero scalar. The third condition fixes the scalar, and replacing the coordinate t_e by $t_e+ft_e^2$ has no effect since $t_st_e^{2|A\langle s\rangle|}$ vanishes at e. $\square$

Remark 8.8. If we choose any finite collection $\pi=\{s_1,\ldots,s_s\}$ of orders ≥ 2, there is again a unique function t_π with analogous properties. Indeed, the good multiplicative property of the normalization means we may take

$$t_\pi=\prod_i t_{s_i}.$$

This applies in particular to the set $A[n]\setminus\{e\}$. $\square$

For some purposes, it is convenient to have a basis for functions with specified poles. We already have the basis $1,x,y,x^2,xy,\ldots$ if all the poles are at the identity. Multiplication by a function f induces an isomorphism

$$f\cdot : \Gamma\mathcal{O}(D)\xrightarrow{\cong}\Gamma\mathcal{O}(D-(f))$$

so we can translate the basis we have.

Lemma 8.9. *For the divisor $D=\Sigma_{s\geq 1}n(s)A\langle s\rangle$ let $t^*(D):=\prod_{b\geq 2}t_b^{n(b)}$. Multiplication by $t^*(D)$ gives an isomorphism*

$$t^*(D)\cdot : H^0(A;\mathcal{O}(D))\xrightarrow{\cong}H^0(A;\mathcal{O}(\deg(D)\cdot(e))).$$

A basis of $H^0(A;\mathcal{O}(D))$ is given by $1/t^*(D)$ if $\deg(D)=0$, and by the first $\deg(D)$ terms in the sequence

$$1/t^*(D), x/t^*(D), y/t^*(D), x^2/t^*(D), xy/t^*(D),\ldots$$

otherwise. $\square$

8.C. **Differentials.** On any elliptic curve we may choose an invariant differential, also characterized by the fact that it has no poles or zeroes. This is well defined up to scalar multiplication, and we would like to make a canonical choice. Since t_e vanishes to the first order at e, its differential is regular and non-vanishing at e, so we may take Dt to be the invariant differential agreeing with dt_e at e.

We shall be considering the space $\Omega \otimes_{\mathcal{O}} \mathcal{K}$ of meromorphic differentials: those which can be written in the form fDt for a meromorphic function f.

Warning 8.10. The differentials dt_s are not generally meromorphic. To give an explicit example, suppose A is defined by $y^2 = x^3 + ax + b$. In this case, the invariant differential is a scalar multiple of dx/y, and we may take t_2 to be a scalar multiple of y, so that the zeroes of dt_2 are those of $dy = (3x^2 + a)dx/y$. The four points at which $3x^2 + a$ vanishes will not generally be torsion points.

It would be nice to make a construction which depends only on the coordinate divisor and not the coordinate itself, but we only know how to do this for a generic curve. We shall see that for such a construction, it suffices to construct for each s a meromorphic differential with poles to the first order on each point of order s which does not change if t_e is multiplied by a scalar.

For $s = 1$ the expression Dt/t_e gives a suitable meromorphic differential. For $s \geq 2$, the situation is less straightforward. To start with, by the last clause of 8.7, the differential Dt/t_s does change if t_e is multiplied by a scalar. Our next attempt is to note that the differential dt_s is again regular and non-vanishing at each point P of exact order s, and its value at P is thus a nonzero multiple λ_P of that of Dt, but in general λ_P does depend on P. The differential $\lambda_P Dt/t_s$ is suitable, but it involves making a choice of a particular point P of order s. The alternative is to consider the average value

$$\lambda_s = \frac{1}{|A\langle s\rangle|} \sum_{P \in A\langle s\rangle} \lambda_P$$

of the scalars and use the differential $\lambda_s Dt/t_s$. Provided λ_s is non-zero, this gives a suitable differential depending only on the coordinate divisor. However, for each s there is a finite number of curves with $\lambda_s = 0$, so it is only for a generic curve that this is legitimate. To avoid this restriction we prefer to make a choice of coordinate rather than coordinate divisor.

9. Principal parts of functions on elliptic curves.

The point of this section is to analyze the sheaf $\mathcal{O}(\infty tors)/\mathcal{O}$ of principal parts of functions with poles at torsion points. We repeat that we are working with sheaves in the torsion point topology, so that $\mathcal{O}(\infty tors)$ is the constant sheaf corresponding to the ring $\mathcal{K}$ of functions with arbitrary poles at points of finite order.

For any effective torsion point divisor D we may use the short exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(aD) \longrightarrow Q(aD) \longrightarrow 0$$

of sheaves to define the quotient sheaf $Q(aD)$ for $0 \leq a \leq \infty$. The cohomology of $Q(\infty D)$ is the cohomology of A with support defined by D.

In fact we may reduce constructions to the case when the divisor $D = A\langle s\rangle$ for some s. Evidently, $Q(\infty A\langle s\rangle)$ is a skyscraper sheaf concentrated on $A\langle s\rangle$, so we may localize at $A\langle s\rangle$ to obtain

$$0 \longrightarrow \mathcal{O}_s \longrightarrow \mathcal{O}(\infty A\langle s\rangle)_s \longrightarrow Q(\infty A\langle s\rangle) \longrightarrow 0.$$

Because we use the torsion point topology,

$$\mathcal{O}(\infty A\langle s\rangle)_s = \mathcal{O}(\infty A\langle s\rangle)_{A\langle s\rangle} = \mathcal{O}(\infty tors) = \mathcal{K}.$$

Since t_s is an invertible meromorphic function vanishing to the first order on $A\langle s\rangle$, the sequence may be written

$$0 \longrightarrow \mathcal{O}_s \longrightarrow \mathcal{O}_s[1/t_s] \longrightarrow \mathcal{O}_s/t_s^{\infty} \longrightarrow 0.$$

This gives the basis of a Thom isomorphism for the homology of almost free spectra.

Lemma 9.1. *A choice of coordinate gives isomorphisms*

$$\mathcal{O}((a+r)A\langle s\rangle)/\mathcal{O}(rA\langle s\rangle) = Q((a+r)A\langle s\rangle)/Q(rA\langle s\rangle) \cong Q(aA\langle s\rangle),$$

induced by multiplication by t_s^r *and hence*

$$Q(\infty A\langle s\rangle) \otimes \mathcal{O}(rA\langle s\rangle) \cong Q(\infty A\langle s\rangle).$$

If $s \geq 2$ *the dependence is only through the image of* t_e *in* $\omega = I_e/I_e^2$.

Proof: Since the sheaves are all skyscraper sheaves over $A\langle s\rangle$, it suffices to observe that for any a, multiplication by t_s induces an isomorphism

$$t_s : \mathcal{O}((a+1)A\langle s\rangle)_s \xrightarrow{\cong} \mathcal{O}(aA\langle s\rangle)_s.$$

To see this, view the rings as subrings of the ring $\mathcal{K}$ of meromorphic functions. Since t_s vanishes on $A\langle s\rangle$ and its poles are at points of finite order other than s, the image lies in the stated subring. Multiplication by any non-zero function is injective, and to see the map is surjective, we observe that if $f \in \mathcal{K}$ has no pole of order more than a on $A\langle s\rangle$ then f/t_s is a meromorphic function no pole of order more than $a+1$ on $A\langle s\rangle$. □

Note that it is immediate from the Riemann-Roch formula that for $0 \leq a \leq \infty$ the cohomology group $H^0(A; Q(aA\langle s\rangle))$ is $a|A\langle s\rangle|$ dimensional, and $H^1(A; Q(aP)) = 0$.

Now we may assemble these sheaves. Indeed, we have a diagram

$$\begin{array}{ccccc} \mathcal{O} & \longrightarrow & \mathcal{O}(\infty D) & \longrightarrow & Q(\infty D) \\ \downarrow & & \downarrow & & \\ \mathcal{O} & \longrightarrow & \mathcal{O}(\infty(D+D')) & \longrightarrow & Q(\infty(D+D')) \end{array}$$

of sheaves, and hence a map $Q(\infty D) \longrightarrow Q(\infty(D+D'))$.

Proposition 9.2. *If* $s, s' \geq 1$ *are distinct, then the natural map*

$$Q(\infty A\langle s\rangle) \oplus Q(\infty A\langle s'\rangle) \xrightarrow{\cong} Q(\infty(A\langle s\rangle + A\langle s'\rangle))$$

is an isomorphism.

Proof: We apply the Snake Lemma to the diagram

$$\begin{array}{ccccc} \mathcal{O}\oplus\mathcal{O} & \longrightarrow & \mathcal{O}(\infty A\langle s\rangle)\oplus\mathcal{O}(\infty A\langle s'\rangle) & \longrightarrow & Q(\infty A\langle s\rangle)\oplus Q(\infty A\langle s'\rangle) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{O} & \longrightarrow & \mathcal{O}(\infty(A\langle s\rangle + A\langle s'\rangle)) & \longrightarrow & Q(\infty(A\langle s\rangle + A\langle s'\rangle)) \end{array}$$

in the abelian category of sheaves on A. The first vertical is obviously surjective with kernel $\mathcal{O}$. The kernel of the second vertical is also $\mathcal{O}$, since if f and f' are local sections of $\mathcal{O}(\infty A\langle s\rangle)$ and $\mathcal{O}(\infty A\langle s'\rangle)$ (i.e., f only has poles on $A\langle s\rangle$ and f' only on $A\langle s'\rangle$) then $f+f'=0$ implies that f and f' are regular. Finally we must show that $\mathcal{O}(\infty(A\langle s\rangle + A\langle s'\rangle))$ is the sheaf quotient of $\mathcal{O}\longrightarrow\mathcal{O}(\infty A\langle s\rangle)\oplus\mathcal{O}(\infty A\langle s'\rangle)$. However, this may be verified stalkwise, where it is clear. $\square$

Corollary 9.3. *(i) The natural map gives an isomorphism*

$$\bigoplus_s Q(\infty A\langle s\rangle) \xrightarrow{\cong} Q(\infty tors).$$

(ii) A choice of coordinate t_e gives an isomorphism

$$T_s : Q(\infty A\langle s\rangle)\otimes\mathcal{O}(A\langle s\rangle) \xrightarrow{\cong} Q(\infty A\langle s\rangle).$$

(iii) The sheaf $Q(\infty A\langle s\rangle)$ has no higher cohomology and its global sections are

$$\Gamma Q(\infty A\langle s\rangle) = \mathcal{K}/\{f \mid f \text{ is regular on } A\langle s\rangle\}. \quad \square$$

Part 4. **The construction.**

In Part 4 we show that the structure of the algebraic model for rational $\mathbb{T}$-equivariant cohomology theories matches the structure of sheaves of functions on an elliptic curve so neatly that the construction of a cohomology theory is effortless. Short as it is, this is the core of the paper.

10. A cohomology theory associated to an elliptic curve.

We are now ready to state and prove the main theorem.

Theorem 10.1. *Given an elliptic curve A over a field k of characteristic 0, and a coordinate t_e, there is an associated 2-periodic rational $\mathbb{T}$-equivariant cohomology theory $EA^*_{\mathbb{T}}(\cdot) = E(A,t_e)^*_{\mathbb{T}}(\cdot)$ of type A, so that for any representation W with $W^{\mathbb{T}}=0$ we have*

$$\widetilde{EA}^i_{\mathbb{T}}(S^W) = H^i(A;\mathcal{O}(-D(W)))$$

and

$$\widetilde{EA}^{\mathbb{T}}_{-i}(S^W) = H^i(A;\mathcal{O}(D(W)))$$

for $i=0,1$, where the divisor $D(W)$ is defined by taking

$$D(W) = \sum_n a_n A[n] \text{ when } W = \sum_n a_n z^n \text{ with } a_0 = 0.$$

This association is invariant under base extension and functorial for isomorphisms of the pair (A,t_e).

The construction is also natural for quotient maps $p : A \longrightarrow A/A[n]$ in the sense that if the multiplicity of $p(t_e)$ at e is 1 (for example if $\mathrm{div}(t_e)$ contains no points of order dividing n), there is a map $p^* : \inf^{\mathbb{T}}_{\mathbb{T}/\mathbb{T}[n]} E(A/A[n], p(t_e)) \longrightarrow E(A, t_e)$ of $\mathbb{T}$-spectra, where $E(A/A[n])$ is viewed as a $\mathbb{T}/\mathbb{T}[n]$-spectrum and inflated to a $\mathbb{T}$-spectrum.

Remark 10.2. (i) The elliptic curve can be recovered from the cohomology theory. Indeed, we may form the graded ring

$$\widetilde{EA}^{\mathbb{T}}_0(S^{*z}) := \{\widetilde{EA}^{\mathbb{T}}_0(S^{az})\}_{a\geq 0}$$

from the products $S^{az} \wedge S^{bz} \longrightarrow S^{(a+b)z}$, and the elliptic curve can be recovered from the cohomology theory via

$$A = \mathrm{Proj}(\widetilde{EA}^{\mathbb{T}}_0(S^{*z})),$$

as commented in Section 6. Furthermore, this reconstruction is functorial in that any multiplicative natural transformation of cohomology theories will induce a map of elliptic curves.
(ii) In fact the coordinate can also be recovered from the cohomology theory, by evaluating the theory on suitable spaces (see Proposition 16.1 below).
(iii) A Weierstrass parametrization of A can be specified by elements of homology:

$$x_e \in \widetilde{EA}^{\mathbb{T}}_0(S^{2z}) \text{ and } y_e \in \widetilde{EA}^{\mathbb{T}}_0(S^{3z}).$$

Remark 10.3. (i) We have not required that k is an algebraically closed field. To see the advantage of this, note that even for the multiplicative group, the individual points of order n are only defined over k if k contains appropriate roots of unity. However $\mathbb{G}_m[n]$ (defined by $1 - z^n$) and hence also $\mathbb{G}_m\langle n\rangle$ (defined by the cyclotomic polynomial $\phi_n(z)$) are defined over $\mathbb{Q}$. Hence equivariant K-theory itself is defined over $\mathbb{Q}$.
(ii) It is useful to generalize the construction to allow k to be an arbitrary $\mathbb{Q}$-algebra so as to include various universal cases. There is no obstacle to making the construction in this generality, provided functions t_e and t_s can be specified, but the analysis of the resulting cohomology theory is more problematic. Since the entire construction is invariant under base change (provided we use corresponding coordinate functions), the case of a field already gives significant information. The present methods are intrinsically restricted to $\mathbb{Q}$-algebras.

Remark 10.4. One use for the naturality is that any automorphism of the elliptic curve preserving the coordinate t_e induces an automorphism of the cohomology theory. For example if t_e is defined using a point P of order 2 as in Example 8.6 (iii), any rigid Galois automorphism fixing P gives an automorphism of the theory.

Proof: The basic ingredients of the torsion model of a the cohomology theory associated to an elliptic curve A are analogous to the affine case described in Appendix A. We will write down a rigid, even object

$$M_t(EA) = (t^{\mathcal{F}}_* \otimes VA \xrightarrow{q} TA)$$

of the torsion category $\mathcal{A}_t$ (i.e., the structure map q is surjective and VA and TA are in even degrees). By 5.2 this is intrinsically formal and therefore determines

$$M_s(EA) = (NA \longrightarrow t^{\mathcal{F}}_* \otimes VA)$$

with $NA = \ker(q)$, and the representing spectrum EA.

We divide the proof into three parts: (1) construction of VA and TA, (2) construction of the map q and (3) verification that the cohomology of spheres is correct.

(1) The vertex and nub: Exactly as in the affine case, the degree 0 part of the vertex

$$VA_0 = \Gamma\mathcal{O}(\infty tors) = \mathcal{K}$$

consists of rational functions whose poles are all at torsion points, however the torsion module is not simply the quotient of this by regular functions, but rather

$$TA_0 = \Gamma(\mathcal{O}(\infty tors)/\mathcal{O}) = \Gamma(Q(\infty tors)).$$

Now we use the splitting

$$Q(\infty tors) \cong \bigoplus_s Q(\infty A\langle s\rangle)$$

of 9.3 to separate points of different orders. This gives

$$TA_0 = \Gamma Q(\infty tors) \cong \bigoplus_s \Gamma Q(\infty A\langle s\rangle)$$

where

$$\Gamma Q(\infty A\langle s\rangle) = \mathcal{K}/\{f \mid f \text{ is regular on } A\langle s\rangle\} = \mathcal{K}/\mathcal{O}_s.$$

Both VA and TA are zero in odd degrees, and in other even degrees we take

$$VA_{2n} = \mathcal{K} \otimes_{\mathcal{O}} \Omega^n \cong VA_0 \otimes \omega^n \text{ and } TA_{2n} = \Gamma(\mathcal{K}/\mathcal{O} \otimes_{\mathcal{O}} \Omega^n) \cong TA_0 \otimes \omega^n,$$

where Ω is the sheaf of Kähler differentials and ω is the cotangent space at the identity, and where exponents refer to tensor powers (rather than exterior powers). We may now describe the R-module structure on TA. The direct sum splitting

$$TA = \bigoplus_s \Gamma Q(\infty A\langle s\rangle)$$

corresponds to the splitting

$$R \cong \prod_s \mathbb{Q}[c],$$

and

$$e_s TA = \Gamma Q(\infty A\langle s\rangle)$$

is a $\mathbb{Q}[c]$-module where c acts as multiplication by t_s/Dt, where t_s defines $A\langle s\rangle$ as described in 8.7. For $s = 1$ this structure does not change if t_e is multiplied by a non-zero scalar, so depends only on the coordinate divisor Z_e; for $s \geq 2$ this depends only on the image of t_2 in $\omega = I_e/I_e^2$. Since the order of any pole is finite, $e_s TA$ is a torsion $\mathbb{Q}[c]$-module.

Remark 10.5. The torsion module TA may be described without using the coordinate data. Indeed, we may define TA' by giving its idempotent pieces

$$e_s(TA')_{2n} = \mathcal{K}/\{f \in \mathcal{K} \mid \mathrm{ord}_s(f) \geq n\},$$

and define the $\mathbb{Q}[c]$-action to be projection. A $\mathbb{Q}[c]$-isomorphism $TA' \cong TA$ is given by the coordinate:

$$(\frac{Dt}{t_s})^n : e_s(TA')_{2n} \longrightarrow e_s(TA)_{2n}.$$

We have used TA rather than TA' because the coordinate data does need to be used somewhere, whilst differentials are used in a more uniform way in TA.

(2) The structure map q: By 4.5 a map q is determined by its idempotent summands, which can be easily written down.

Definition 10.6. We define
$$q : t_*^{\mathcal{F}} \otimes VA \longrightarrow TA = \bigoplus_s e_s TA$$
by specifying its sth component
$$q(c_s^{w(s)} \otimes \alpha) = \overline{(\frac{t_s}{Dt})^{w(s)}\alpha};$$
up to normalization, this picks out the part of α with poles of order $> w(s)$ on points of order s.

Remark 10.7. Any $\alpha \in V_{2n}$ may be written
$$\alpha = f \cdot (Dt)^{\otimes n}$$
for some meromorphic function $f \in \mathcal{K}$. The formula then becomes
$$q(c_s^{w(s)} \otimes f \cdot (Dt)^{\otimes n})_s = \overline{t_s{}^{w(s)} f} \cdot (Dt)^{\otimes(n-w(s))}.$$

Lemma 10.8. *The definition does determine an R-map $q : t_*^{\mathcal{F}} \otimes VA \longrightarrow TA$.*

Proof: Since any function is regular at all but finitely many points, the map q maps into the sum. Thus $q(c^u \otimes \alpha)$ is well defined, and we need to check that taken together they specify an R-map. For this, we apply 4.5. Taking $V = VA$ and $T = TA$ we note that 10.6 does determine maps q_s, and that they satisfy the condition. It follows that there is an R-map q with these idempotent pieces. □

(3) Cohomology: Now we can check that the resulting homology and cohomology of spheres agrees with the cohomology of the corresponding divisors on the elliptic curve. Because the use of differentials is uniform, it is enough to prove the result for representations W with $W^{\mathbb{T}} = 0$.

Since we have decided to use the isomorphism $[S^{-w}, M] = [S^0, \Sigma^w M]$, we need to identify the suspension of the representing object EA. Applying 5.5 in this case we obtain the following.

Lemma 10.9. *Suppose $w : \mathcal{F} \longrightarrow \mathbb{Z}$ is zero almost everywhere. The wth suspension of EA is given by*
$$\Sigma^w EA = (\Sigma^w NA \longrightarrow t_*^{\mathcal{F}} \otimes VA)$$
where
$$\Sigma^w NA = \ker(t_*^{\mathcal{F}} \otimes VA \xrightarrow{q^w} \Sigma^w TA)$$
and for $\alpha \in VA_{2n} = \mathcal{K} \otimes \omega^n$
$$q^w(c_s^{i(s)} \otimes \alpha) = \overline{\alpha(\frac{t_s}{Dt})^{w(s)+i(s)}} \in (\mathcal{K}/\mathcal{O}_s) \otimes \omega^{n-w(s)-i(s)} = e_s(\Sigma^w TA)_{2n-2i(s)}.$$

We also use the mnemonic

$$q(xc^w \otimes \alpha) = q^w(x \otimes \alpha),$$

despite the fact that xc^w is not an element of $t_*^{\mathcal{F}}$. $\square$

Consider the complex representation W with $W^{\mathbb{T}} = 0$ and the corresponding function $w(H) = \dim_{\mathbb{C}}(W^H)$. By 5.7 the homology is given by

$$\widetilde{EA}_0^{\mathbb{T}}(S^W) = \ker(q : c^w \otimes VA_0 \longrightarrow (\Sigma^w TA)_0)$$

and

$$\widetilde{EA}_{-1}^{\mathbb{T}}(S^W) = \mathrm{cok}(q : c^w \otimes VA_0 \longrightarrow (\Sigma^w TA)_0)$$

and similarly with W replaced by $-W$. Since the kernel and cokernel are vector spaces over k, it is no loss of generality to extend scalars to assume it is algebraically closed. This is convenient because it is simpler to treat separate points of order n one at a time.

The following two lemmas complete the proof. $\square$

Lemma 10.10. *If W is a representation with $W^{\mathbb{T}} = 0$ then*

$$\widetilde{EA}_0^{\mathbb{T}}(S^W) = H^0(A; \mathcal{O}(D(W))),$$

and if $W \neq 0$,

$$\widetilde{EA}_0^{\mathbb{T}}(S^{-W}) = 0.$$

Proof: By definition

$$q(c^w \otimes f)_s = \overline{(\frac{t_s}{Dt})^{w(s)} f}.$$

First note that Dt is regular and non-vanishing on $A\langle s\rangle$, so the differential can be ignored for the purpose of calculating the kernel. Since the function t_s vanishes to exactly the first order on $A\langle s\rangle$, the condition that f lies in the kernel is that $\mathrm{ord}_P(f) \geq -w(s)$ for each point P of exact order s. Since $D(W) = \Sigma_P w(s_P)(P)$ we have

$$\ker(q : c^w \otimes VA_0 \longrightarrow (\Sigma^w TA)_0) = \{f \in VA \mid \mathrm{div}(f) + D(W) \geq 0\}$$

as required.

Replacing W by $-W$, the second statement is immediate. $\square$

The calculation of the odd cohomology is less elementary.

Proposition 10.11. *If W is a representation with $W^{\mathbb{T}} = 0$ then*

$$\widetilde{EA}_{-1}^{\mathbb{T}}(S^{-W}) = H^1(A; \mathcal{O}(-D(W))),$$

and if $W \neq 0$,

$$\widetilde{EA}_0^{\mathbb{T}}(S^W) = 0.$$

Proof: We have to calculate $\mathrm{cok}(q : c^{-w} \otimes VA_0 \longrightarrow (\Sigma^w TA)_0)$. First we give the concrete description of $H^1(A; \mathcal{O}(-D(W)))$ using adèles from [22, Proposition II.3].

The exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}(-D(W)) \longrightarrow \mathcal{K} \longrightarrow Q(-D(W)) \longrightarrow 0.$$

induces a cohomology exact sequence ending

$$\mathcal{K} \xrightarrow{\phi} H^0(A; Q(-D(W))) \longrightarrow H^1(A; \mathcal{O}(-D(W))) \longrightarrow 0.$$

However the definition of $Q(-D(W))$ shows that it is a skyscraper sheaf concentrated on the support of $D(W)$. Its space of sections is $\mathbb{A}/\mathbb{A}(-D(W))$, where

$$\mathbb{A} = \{(x_s)_s \mid x_s \in \mathcal{K}, \text{ and almost all } x_s \in k\}$$

is the space of adèles and

$$\mathbb{A}(-D(W)) = \{(x_s) \in W \mid \mathrm{ord}_P(x_s) + \mathrm{ord}_s(-D(W)) \geq 0\}.$$

Thus $\mathrm{cok}(\phi) = \mathbb{A}/(\mathbb{A}(-D(W)) + \mathcal{K})$.

To complete the proof we construct an isomorphism m so that the left hand square in the diagram

$$\begin{array}{ccccccc}
\mathcal{K} & \xrightarrow{\phi} & H^0(A; Q(-D(W))) & \longrightarrow & H^1(A; \mathcal{O}(-D(W))) & \longrightarrow & 0 \\
=\downarrow & & m \downarrow\cong & & \downarrow & & \\
c^{-w} \otimes (VA)_0 & \xrightarrow{q} & (\Sigma^{-w}TA)_0 & \longrightarrow & \widetilde{EA}^{\mathbb{T}}_{-1}(S^{-W}) & \longrightarrow & 0
\end{array}$$

commutes; the result follows from the 5-lemma. Both the domain and codomain of m split into pieces corresponding to the divisors $A\langle s\rangle$. If $a_s = \dim_{\mathbb{C}}(W^{\mathbb{T}[s]})$, and we define m by taking the sth term

$$m_s : \mathbb{A}/\mathbb{A}(-a_s A\langle s\rangle) = \mathcal{K}/\mathcal{O}(-a_s A\langle s\rangle) \longrightarrow \mathcal{K}/\mathcal{O}_s \otimes \omega^{a_s} = e_s(\Sigma^{-w}TA)_0$$

to be

$$m_s(\overline{f}) = f \cdot (\frac{Dt}{t_s})^{a_s}.$$

Indeed, the definition is forced by the requirement that the square commute, but since the vanishing of t_s defines $A\langle s\rangle$, m_s is an isomorphism. □

Remark 10.12. It is possible to give a more explicit proof of 10.11 as follows. First, one checks any element $(g_1, g_2, \ldots) \in \bigoplus_s e_s TA$ is congruent (modulo the image of q^w) to one with $g_2 = g_3 = \cdots = 0$. Now, using 8.9, identify a subspace of the correct codimension in the image. Using divisors one sees the cokernel must be at least this big. Finally, the cokernel is naturally dual to $H^0(A; \mathcal{O}(D(W))$, and hence naturally isomorphic to $H^1(A; \mathcal{O}(-D(W)))$ by Serre duality.

Part 5. Properties of $\mathbb{T}$-equivariant elliptic cohomology.

Now that we have defined the cohomology theory $EA^*_{\mathbb{T}}(\cdot)$ associated to an elliptic curve A, we discuss some of its properties, including multiplicativity and a structure reflecting the addition on A.

11. Homotopical multiplicative properties.

For the rest of this section we identify EA with the corresponding object in $\mathcal{A}_s$, so that $EA = (NA \longrightarrow t^{\mathcal{F}}_* \otimes VA)$, and there is a short exact sequence

$$0 \longrightarrow NA \xrightarrow{\beta} t^{\mathcal{F}}_* \otimes VA \xrightarrow{q} TA \longrightarrow 0.$$

11.A. **The ring structure on EA.** Note that $VA = \bigoplus_n \mathcal{K} \otimes_{\mathcal{O}} \Omega^n$ has a commutative and associative product, which therefore induces such a product on $t^{\mathcal{F}}_* \otimes VA$.

Theorem 11.1. *The product of functions and differential forms induces a commutative and associative product on on the algebraic model for EA, so EA is a commutative ring spectrum up to homotopy. Using results of* [14] *we may choose EA to be a strictly commutative ring $\mathbb{T}$-spectrum.*

Proof: First, note that by 5.3 EA is flat, so that tensor product with EA models the smash product. It therefore suffices to show that the product on $t^{\mathcal{F}}_* \otimes VA$ restricts to a product on NA.

Suppose $a, b \in t^{\mathcal{F}}_* \otimes VA$; we must show that if $q(a) = 0$ and $q(b) = 0$ then $q(ab) = 0$. It suffices to concentrate on the component mapping into e_sTA for each s. The key to this is that for fixed s we may give VA the structure of a $\mathbb{Q}[c]$-module by letting c act as t_s/Dt. With this definition, c acts invertibly, so that we have a ring homomorphism

$$i_s : \mathbb{Q}[c, c^{-1}] \longrightarrow VA.$$

Now q_s factors as

$$\mathbb{Q}[c, c^{-1}] \otimes VA \xrightarrow{i_s \otimes 1} VA \otimes VA \longrightarrow VA \longrightarrow e_sTA.$$

The fact that $q_s(ab) = 0$ if $q_s(a) = 0$ and $q_s(b) = 0$ now follows since the product of two functions regular at a point is also regular there. $\square$

11.B. **Duality.** Now that we have a product structure we can tie up topological and geometric duality in a satisfactory way.

Lemma 11.2. *Spanier-Whitehead duality for spheres corresponds to Serre duality in the sense that the Serre duality pairing*

$$\begin{array}{ccc} H^1(A; \mathcal{O}(-D(W))) \otimes H^0(A; \mathcal{O}(D(W))) & \longrightarrow & H^1(A; \mathcal{O}) \\ \| & & \| \\ [S^0, S^{-W} \wedge \Sigma EA]^{\mathbb{T}} \otimes [S^0, S^W \wedge EA]^{\mathbb{T}} & & [S^0, \Sigma EA]^{\mathbb{T}} \end{array}$$

is induced by the algebraically obvious Spanier-Whitehead pairing

$$S^{-W} \wedge EA \wedge S^W \wedge EA \simeq S^{-W} \wedge S^W \wedge EA \wedge EA \longrightarrow S^0 \wedge EA \wedge EA \longrightarrow EA.$$

Proof: Both maps can be taken to be induced by multiplication of functions and a residue map (see [22, Chapter II]). $\square$

12. Reflecting the group structure of the elliptic curve.

The group multiplication on an affine algebraic group $\mathbb{G}$ gives its ring of functions $\mathcal{O}$ a diagonal, and thus $\mathcal{O}$ becomes a Hopf algebra. When we say that K-theory corresponds to the multiplicative group $\mathbb{G}_m$ we mean that not only is $K^0_{\mathbb{T}} = \mathbb{Z}[z, z^{-1}]$ the representing ring for $\mathbb{G}_m$ but also that the diagonal also has a topological source. Indeed, the multiplication map $\mu : \mathbb{T} \times \mathbb{T} \longrightarrow \mathbb{T}$ induces a map

$$K^0_{\mathbb{T}} \xrightarrow{\mu^*} K^0_{\mathbb{T}\times\mathbb{T}} = K^0_{\mathbb{T}} \otimes K^0_{\mathbb{T}},$$

which turns out to be the coproduct on the ring of functions on $\mathbb{G}_m$. The corresponding situation for formal groups and complex oriented theories is even more familiar.

When we work with an elliptic curve, we again expect the group structure on A to give additional structure on spaces of functions. However the structure is not just a coproduct, and we extract the relevant information from Mumford's work [20]. Indeed, choosing a line bundle L to control the behaviour of functions, the multiplication $\mu : A \times A \longrightarrow A$ would give a map $\mu^* : H^0(A; L) \longrightarrow H^0(A \times A; \mu^*(L))$, but since $\mu^*(L)$ does not decompose as a tensor product, this is not very helpful. Instead Mumford considers the map

$$\xi : A \times A \longrightarrow A \times A$$

given by $\xi(x, y) = (x + y, x - y)$. It then turns out that if we let $M = p_1^*L \otimes p_2^*L$, by the see-saw principle and the theorem of the square that $\xi^*M \cong M^2$ (see [20, p. 320]). Using the Künneth isomorphism, we obtain a map

$$\phi_L : H^0(A; L) \otimes H^0(A; L) = H^0(A \times A; M) \xrightarrow{\xi^*} H^0(A \times A; M^2) = H^0(A; L^2) \otimes H^0(A; L^2).$$

Applying this when $L = \mathcal{O}(D(W))$ for a representation W with $W^{\mathbb{T}} = 0$ we see that this is a map

$$\phi_W : \widetilde{EA}^{\mathbb{T}}_0(S^W) \otimes \widetilde{EA}^{\mathbb{T}}_0(S^W) \longrightarrow \widetilde{EA}^{\mathbb{T}}_0(S^{2W}) \otimes \widetilde{EA}^{\mathbb{T}}_0(S^{2W}).$$

By choosing W sufficiently large we can evidently find $\xi^*(f_1, f_2)$ for an arbitrary meromorphic functions f_1, f_2, and since $\xi^*(f_1, f_2)(x, y) = (f_1(x + y), f_2(x - y))$, we recover $f_1(x + y)$ by suitable restriction.

We now describe how ϕ_W should be realised at the level of spectra. The realization involves using $\mathbb{T}\times\mathbb{T}$-equivariant spectra, so proofs lie outside the scope of the present paper. However the picture is sufficiently compelling to merit a brief account.

Suppose there exists a $\mathbb{T} \times \mathbb{T}$-equivariant cohomology theory of type $A \times A$. Constructing such a theory is significantly easier than constructing a $\mathbb{T} \times \mathbb{T}$-equivariant theory for an arbitrary abelian surface. To the representation $w^i \otimes z^j$ of $\mathbb{T} \times \mathbb{T}$ we associate the divisor

$$D(w^i \otimes z^j) = \ker(A \times A \xrightarrow{(i,j)} A \times A),$$

and extend this to arbitrary representations so that

$$D(V \oplus W) = D(V) + D(W).$$

The 2-periodic theory $E(A \times A)^{\mathbb{T}\times\mathbb{T}}_*(\cdot)$ should then come with a spectral sequence

$$H^*(A \times A; \mathcal{O}_{A\times A}(D(W)) \Rightarrow \widetilde{E(A \times A)}^{\mathbb{T}\times\mathbb{T}}_*(S^W).$$

Since some line bundles have cohomology in degree 2, this does not determine $\widetilde{E(A \times A)}_*^{\mathbb{T}\times\mathbb{T}}(S^W)$ in general. However when $\mathcal{O}_{A\times A}(D(W))$ has no cohomology in dimension 2 we find

$$\widetilde{E(A \times A)}_0^{\mathbb{T}\times\mathbb{T}}(S^W) = H^0(A \times A; \mathcal{O}_{A\times A}(D(W))).$$

Next, the map $\xi : A \times A \longrightarrow A \times A$ is an isogeny with kernel

$$\Delta A[2] = \{(a, a) \mid a + a = e\}.$$

We also consider the corresponding group homomorphism

$$\hat{\xi} : \mathbb{T} \times \mathbb{T} \longrightarrow \mathbb{T} \times \mathbb{T},$$

defined by $\hat{\xi}(w, z) = (wz, w/z)$, which is surjective with kernel

$$\Delta\mathbb{T}[2] = \{(z, z) \mid z^2 = 1\}.$$

To minimize confusion, we identify the second $\mathbb{T}\times\mathbb{T}$ with $\overline{\mathbb{T} \times \mathbb{T}} = (\mathbb{T}\times\mathbb{T})/\Delta\mathbb{T}[2]$. The map ξ should correspond to a map

$$\xi_i^* : \inf_{\overline{\mathbb{T}\times\mathbb{T}}}^{\mathbb{T}\times\mathbb{T}} E(A \times A) \longrightarrow E(A \times A)$$

(i for inflation) of $\mathbb{T} \times \mathbb{T}$-spectra or, adjointly, to a map

$$\xi_f^* : E(A \times A) \longrightarrow E(A \times A)^{\Delta\mathbb{T}[2]}$$

(f for fixed point) of $\overline{\mathbb{T} \times \mathbb{T}}$-spectra.

Lemma 12.1. *For any representation $\overline{W}$ of $\overline{\mathbb{T} \times \mathbb{T}}$, the map ξ_f induces*

$$\xi_f^* : E(A \times A)_0^{\overline{\mathbb{T}\times\mathbb{T}}}(S^{\overline{W}}) \longrightarrow E(A \times A)_0^{\mathbb{T}\times\mathbb{T}}(S^{\overline{W}}).$$

Proof: The map ξ_f^* induces

$$\xi_f^* : [S^0, S^{\overline{W}} \wedge E(A \times A)]_0^{\overline{\mathbb{T}\times\mathbb{T}}} \longrightarrow [S^0, S^{\overline{W}} \wedge E(A \times A)^{\Delta\mathbb{T}[2]}]_0^{\overline{\mathbb{T}\times\mathbb{T}}},$$

so it suffices to identify the domain and codomain. By definition $\widetilde{E(A \times A)}_0^{\overline{\mathbb{T}\times\mathbb{T}}}(S^{\overline{W}}) = [S^0, S^{\overline{W}} \wedge E(A \times A)]_0^{\overline{\mathbb{T}\times\mathbb{T}}}$ so we turn to the codomain and calculate

$$\begin{aligned}
[S^0, S^{\overline{W}} \wedge E(A \times A)^{\Delta\mathbb{T}[2]}]_0^{\overline{\mathbb{T}\times\mathbb{T}}} &= [S^{-\overline{W}}, E(A \times A)^{\Delta\mathbb{T}[2]}]_0^{\overline{\mathbb{T}\times\mathbb{T}}} \\
&= [S^{-\overline{W}}, E(A \times A)]_0^{\mathbb{T}\times\mathbb{T}} \\
&= [S^0, S^{\overline{W}} \wedge E(A \times A)]_0^{\mathbb{T}\times\mathbb{T}} \\
&= \widetilde{E(A \times A)}_0^{\mathbb{T}\times\mathbb{T}}(S^{\overline{W}}).
\end{aligned}$$

□

To model $M = p_1^*L \otimes p_2^*L$ with $L = \mathcal{O}(D(W))$ for a representation W of $\mathbb{T}$ we take $\overline{W} = (W \otimes 1) \oplus (1 \otimes W)$. Direct sum of representations corresponds to tensor product of line bundles and to sums of divisors, so if

W corresponds to the line bundle L and the divisor $D(W)$,

then

$\overline{W}$ corresponds to the line bundle $p_1^*L \otimes p_2^*L$ and the divisor $[D(W) \times A] + [A \times D(W)]$.

Viewed as a representation of $\mathbb{T}\times\mathbb{T}$ by pullback along $\hat{\xi}$ we find
$$\hat{\xi}^*(\overline{W}) = \hat{\xi}_1^* W \oplus \hat{\xi}_2^* W.$$
In particular if $W = z^n$ we find
$$\hat{\xi}^*(\overline{W}) = (w^n \otimes z^n) \oplus (w^n \otimes z^{-n}).$$
Finally, we need to observe that for any n, the bundles associated to
$$(w^n \otimes z^n) \oplus (w^n \otimes z^{-n}) \text{ and } (w^{2n} \otimes 1) \oplus (1 \otimes z^{2n})$$
are isomorphic: this is precisely the same argument as showed $\xi^* M \cong M^2$ above. With $L = \mathcal{O}(D(W))$, we thus expect a commutative diagram
$$\begin{tikzcd}
H^0(A;L)^{\otimes 2} \arrow[r, phantom, "="] & H^0(A\times A; p_1^*L \otimes p_2^*L) \arrow[r, "\xi^*"] \arrow[d] & H^0(A\times A; p_1^*L^2 \otimes p_2^*L^2) \arrow[r, phantom, "="] \arrow[d] & H^0(A;L^2)^{\otimes 2} \\
& E\widetilde{(A\times A)}_0^{\overline{\mathbb{T}\times\mathbb{T}}}(S^{\overline{W}}) \arrow[r, "\xi_f^*"] & E\widetilde{(A\times A)}_0^{\mathbb{T}\times\mathbb{T}}(S^{\overline{W}}). &
\end{tikzcd}$$

13. The completion theorem.

By formal completion around the identity, we may associate a formal group $\hat{A}$ to an elliptic curve A. In favourable circumstances there is a (non-equivariant) 2-periodic complex oriented cohomology theory $E\hat{A}^*(\cdot)$ associated to $\hat{A}$, and a Borel theory
$$E\hat{A}^*_{\mathbb{T}}(X) := E\hat{A}^*(E\mathbb{T} \times_{\mathbb{T}} X).$$

The purpose of this section is to make explicit the relationship between the equivariant theory $EA^*_{\mathbb{T}}(X)$ associated to the elliptic curve A and the Borel theory associated to the formal group $\hat{A}$.

Proposition 13.1. *The cohomology of $E\mathbb{T}$ is concentrated in even degrees, and in degree 0 it is the completion of $\mathcal{O}_e$ at the ideal I_e of functions vanishing at e:*
$$EA^0_{\mathbb{T}}(E\mathbb{T}) = \lim_{\leftarrow k} \mathcal{O}_e / I_e^k.$$

Proof: Indeed, we may make the calculation
$$\widetilde{EA}^*_{\mathbb{T}}(E\mathbb{T}_+) = [E\mathbb{T}_+, EA]^*_{\mathbb{T}} = [E\mathbb{T}_+, EA \wedge E\mathbb{T}_+]^*_{\mathbb{T}} = \mathrm{Hom}^*_{\mathbb{Q}[c]}(\mathbb{Q}[c]^\vee, e_1 TA).$$
Now, shifting into degree 0 we replace the action by c with the action by t_e and find this is
$$\mathrm{Hom}^*_{\mathbb{Q}[t_e]}(\mathbb{Q}[t_e]^\vee, \mathcal{K}/\mathcal{O}_e) = \lim_{\leftarrow k}(\mathrm{ann}(\mathcal{K}/\mathcal{O}_e, t_e^k), t_e) = \lim_{\leftarrow k}(\mathcal{O}_e(k(e))/\mathcal{O}_e, t_e).$$
Now multiplication by powers of t_e gives an isomormphism between the inverse system $(\mathcal{O}_e(k(e))/\mathcal{O}_e, t_e)$ and the inverse system $(\mathcal{O}_e/I_e^k, \mathit{projection})$. $\square$

Since the formal group law on $\hat{A}$ comes from $f(a+b) = F(f(a), f(b))$ when f is a coordinate function, the formal group law for $E\hat{A}$ can be inferred from the map ξ^* for EA described in Section 12.

There is another less natural approach involving comparison with the Borel theory of the periodic theory represented by
$$HP = \bigvee_{n\in\mathbb{Z}} \Sigma^{2n} H.$$

This has coefficients

$$HP^0_{\mathbb{T}} = \mathbb{Q}[[y]].$$

Lemma 13.2. *There is an equivalence $EA \wedge E\mathbb{T}_+ \simeq HP \wedge E\mathbb{T}_+$, and therefore*

$$EA^*_{\mathbb{T}}(X \times E\mathbb{T}) \cong HP^*(E\mathbb{T} \times_{\mathbb{T}} X),$$

so that in the notation above, $E\hat{A} \simeq HP$.

Remark 13.3. The additional information in $E\hat{A}$ is in the comparison with EA, and hence in the relationship between the formal group law and the addition on A.

Proof: First, to see the equivalence we need only show the two theories give homology of $E\mathbb{T}$ isomorphic as $\mathbb{Q}[c]$-modules [9, 4.4.1]. Since, both theories are 2-periodic and $EA^{\mathbb{T}}_*(E\mathbb{T})$ and $HP_*(B\mathbb{T})$ are divisible, it suffices to observe that the two theories have isomorphic non-equivariant coefficients.

Now for a based space Y,

$$F(E\mathbb{T}_+ \wedge Y, EA) \simeq F(E\mathbb{T}_+ \wedge Y, E\mathbb{T}_+ \wedge EA) \simeq F(E\mathbb{T}_+ \wedge Y, E\mathbb{T}_+ \wedge HP) \simeq F(E\mathbb{T}_+ \wedge Y, HP).$$

□

14. The homology and cohomology of universal spaces.

From the point of view of equivariant topology, the completion theorem of the previous section is just one example of a family of calculations. For other universal spaces we obtain analogous results by the same proof. For simplicity we restrict the statement to the value on a point.

Suppose then that π is a finite set of positive integers and let $\mathcal{F}(\pi)$ denote the family of subgroups with orders dividing elements of π and $A[\pi]$ denote the set of points with orders dividing elements of π.

Theorem 14.1. *(i)* (Completion theorem.) *The cohomology of $E\mathcal{F}(\pi)$ is in even degrees and*

$$EA^0_{\mathbb{T}}(E\mathcal{F}(\pi)) = H^0(A; \mathcal{O}^{\wedge}_{A[\pi]})$$

where $A[\pi]$ is the set of points with orders dividing elements of π. Since $\mathcal{O}^{\wedge}_{A[\pi]}$ is a skyscraper sheaf, this is just the sum of the completed local rings at the points of $A[\pi]$.

(ii) (Local cohomology theorem.) *The homology of $E\mathcal{F}(\pi)$ is in odd degrees and*

$$EA^{\mathbb{T}}_1(E\mathcal{F}(\pi)) = H^1_{A[\pi]}(\mathcal{O}),$$

where the cohomology on the right is $A[\pi]$-local cohomology.

Proof: The proof of Part (i) follows that of 13.1.

For Part (ii) we may use the model

$$S(\infty V(\pi)) = E\mathcal{F}(\pi) \text{ where } V(\pi) = \bigoplus_{n|\pi} z^n.$$

The cofibre sequence

$$S(\infty V(\pi))_+ \longrightarrow S^0 \longrightarrow S^{\infty V(\pi)}$$

and the fact that the Euler class of z^n defines $A[n]$ give the result. □

The calculation of the cohomology of $E\mathcal{F}(n) = E(\mathbb{T}/\mathbb{T}[n])$ corresponds to the fact that one may obtain a $\mathbb{T}[n]$-equivariant formal group law in the sense of [5] by formal completion of the curve A along $A[n]$, as described in [10].

15. The Hasse square.

We want to combine the localization and completion theorems to give a method of calculation of elliptic cohomology in terms of Borel theories combined using the geometry of the curve.

The localization theorem is elementary.

Lemma 15.1. (Localization theorem) *For any $\mathbb{T}$-space X we have*

$$EA^{\mathbb{T}}_*(X \wedge \tilde{E}\mathcal{F}) = H_*(X^{\mathbb{T}}; \Omega^*_A \otimes_{\mathcal{O}} \mathcal{K}),$$

where the grading on the right is that for homology with graded coefficients (i.e., total degree). A similar result holds in cohomology for finite complexes X.

Proof: Since $\lim_{\to V} \mathcal{O}(D(V)) = \mathcal{K}$, and $\tilde{E}\mathcal{F} = \lim_{\to V^{\mathbb{T}}=0} S^V$ we have

$$EA^{\mathbb{T}}_{2d}(\tilde{E}\mathcal{F}) = (\Omega^1_A)^{\otimes d} \otimes_{\mathcal{O}} \mathcal{K}.$$

□

We want to apply the completion theorem for the family of all finite subgroups. To do this for arbitrary complexes it is convenient to introduce the notation

$$H^*_{\mathbb{T}}(X^C; I) := \mathrm{Hom}_{H^*(B\mathbb{T}_+)}(H^{\mathbb{T}}_*(X^C); I)$$

for any $H^*(B\mathbb{T}_+)$-module I, where the grading is that of homomorphisms of $H^*(B\mathbb{T}_+)$-modules. If I is injective, this is a cohomology theory in X, and if $H^{\mathbb{T}}_*(X^C) = H_*(X^C) \otimes H_*(B\mathbb{T}_+)$ then $H^*_{\mathbb{T}}(X^C; I) = H^*(X^C; \mathrm{Hom}_{H^*(B\mathbb{T}_+)}(H_*(B\mathbb{T}_+), I))$.

Lemma 15.2. *For any $\mathbb{T}$-space X*

$$EA^*_{\mathbb{T}}(X \wedge E\mathcal{F}_+) = \prod_C H^*_{\mathbb{T}}(X^C; T_C A \otimes \omega^*_A).$$

If $H^{\mathbb{T}}_*(X^C) = H_*(X^C) \otimes H_*(B\mathbb{T}_+)$ for all C then

$$EA^*_{\mathbb{T}}(X \wedge E\mathcal{F}_+) = \prod_C H^*(X^C; \mathcal{O}^{\wedge}_C \otimes \omega^*_A),$$

where $\mathcal{O}^{\wedge}_C$ is the ring obtained as the formal completion of $\mathcal{O}$ at $A\langle s\rangle$ if C is of order s.

Proof: The first statement amounts to the fact that $EA \wedge \Sigma E\mathcal{F}_+$ is injective, with coefficients $TA \otimes \omega^*_A$. Now we use the fact that there is a rational splitting $E\mathcal{F}_+ \simeq \bigvee_C E\langle C\rangle$ corresponding to $TA \simeq \bigoplus_C T_C A$, and that $[X, E\langle C\rangle \wedge Y]^{\mathbb{T}} = [X^C, E\langle C\rangle \wedge Y]^{\mathbb{T}}$. Passing to the summand corresponding to C, the $H^*(B\mathbb{T}_+)$-module structure on rings of functions is through t_s/Dt. The second statement follows since the short exact sequence

$$0 \longrightarrow \mathcal{K}_C \longrightarrow \mathcal{K} \longrightarrow T_C A \longrightarrow 0$$

gives an isomorphism

$$\mathrm{Hom}_{H^*(B\mathbb{T}_+)}(H_*(B\mathbb{T}_+), T_C A \otimes \omega_A^*) = \mathrm{Ext}_{H^*(B\mathbb{T}_+)}(H_*(B\mathbb{T}_+), \mathcal{K}_C \otimes \omega_A^*) = \mathcal{O}_C^\wedge \otimes \omega_A^*.$$

$\square$

We express the homotopy level Hasse square via the associated Mayer-Vietoris long exact sequence.

Proposition 15.3. *(Hasse square) For any $\mathbb{T}$-space X there is a long exact sequence*

$$\cdots \longrightarrow EA^n_{\mathbb{T}}(X) \longrightarrow H^n(X^{\mathbb{T}}; \mathcal{K} \otimes_{\mathcal{O}} \Omega_A^*) \times \prod_C H^n_{\mathbb{T}}(X^C; T_C A \otimes \omega_A^*) \longrightarrow H^n(X^{\mathbb{T}}; \mathcal{K}_{\mathcal{F}} \otimes \omega_A^*)$$
$$\longrightarrow EA^{n+1}_{\mathbb{T}}(X) \longrightarrow \cdots,$$

natural in X, where $\mathcal{K}_{\mathcal{F}} = \prod_C \mathcal{O}_C^\wedge \otimes \mathcal{K}$. If $H_*^{\mathbb{T}}(X^C) = H_*(X^C) \otimes H_*(B\mathbb{T}_+)$ then

$$H^n_{\mathbb{T}}(X^C; T_C A \otimes \omega_A^*) \cong H^n(X^C; \mathcal{O}_C^\wedge \otimes \omega_A^*).$$

Remark 15.4. Since X is a space, two of the maps in the above long exact sequence give a diagram of rings

$$\begin{array}{ccc} EA^*_{\mathbb{T}}(X) & \longrightarrow & H^*(X^{\mathbb{T}}; \mathcal{K} \otimes_{\mathcal{O}} \Omega_A^*) \\ \downarrow & & \downarrow \\ \prod_C H^*_{\mathbb{T}}(X^C; T_C A \otimes \omega_A^*) & \longrightarrow & H^*(X^{\mathbb{T}}; \mathcal{K}_{\mathcal{F}} \otimes \omega_A^*). \end{array}$$

When the connecting homomorphism in the long exact sequence is zero, this is a pullback diagram of rings. For example, this applies if both $H^*(X^{\mathbb{T}})$ and $H^*_{\mathbb{T}}(X^C)$ are in even degrees for all C.

Proof: Any $\mathbb{T}$-spectrum E occurs in the Tate homotopy pullback square

$$\begin{array}{ccc} E & \longrightarrow & E \wedge \tilde{E}\mathcal{F} \\ \downarrow & & \downarrow \\ F(E\mathcal{F}_+, E) & \longrightarrow & F(E\mathcal{F}_+, E) \wedge \tilde{E}\mathcal{F} \end{array}$$

where $\mathcal{F}$ is the family of proper subgroups, and applying $F(X, \cdot)$ we obtain the homotopy pullback square

$$\begin{array}{ccc} F(X, E) & \longrightarrow & F(X, E \wedge \tilde{E}\mathcal{F}) \\ \downarrow & & \downarrow \\ F(X \wedge E\mathcal{F}_+, E) & \longrightarrow & F(X, F(E\mathcal{F}_+, E) \wedge \tilde{E}\mathcal{F}). \end{array}$$

Note that $[X, Y \wedge \tilde{E}\mathcal{F}]^{\mathbb{T}}_* = [X^{\mathbb{T}}, \Phi^{\mathbb{T}} Y]_*$, so that both the right hand terms can be expressed in terms of the geometric fixed points of X. Now take $E = EA$ and apply the localization theorem 15.1 to see that $\pi^{\mathbb{T}}_*(F(X, EA \wedge \tilde{E}\mathcal{F})) = H^*(X^{\mathbb{T}}; \mathcal{K} \otimes \omega_A^*)$ and the completion theorem 14.1(i) to see that

$$\pi^{\mathbb{T}}_*(F(X \wedge E\mathcal{F}_+, EA)) = EA^*(X \wedge E\mathcal{F}_+) = \prod_C H^*_{\mathbb{T}}(X^C; \mathcal{O}_C^\wedge).$$

$\square$

16. Recovering the coordinate.

By showing that the coordinate used in Section 10 can be recovered from the cohomology theory we show that it is necessary to make such a choice.

To give a full algebraic model of Type A theories in the sense of 3.1 we would need to show that if $E^*_{\mathbb{T}}(\cdot)$ is a cohomology theory of Type A then there is a unique coordinate so that $E^*_{\mathbb{T}}(\cdot) = E(A, t_e)^*_{\mathbb{T}}(\cdot)$. However it certainly requires certain additional structure on the cohomology theory to do this. First, we need to asssume that the theory is multiplicative (this will mean it is specified by a collection of differentials ω_s vanishing to first order at points of exact order s). However to relate the points of different orders we need to take into account the group structure on A and its reflection in cohomology. We restrict ourselves to showing the required uniqueness for theories constructed by the procedure of Section 10.

Proposition 16.1. *If EA is constructed as in Section 10, the coordinate t_e may be recovered from the cohomology theory.*

Proof: First we will recover the coordinate *divisor*, by concentrating on point with trivial isotropy, and then return to find a suitable coordinate with this divisor by considering isotropy of order 2 and 3.

We evaluate the cohomology on suitable objects $B = (M \longrightarrow t^{\mathcal{F}}_* \otimes U)$ of $\mathcal{A}_s$ (depending on a number n and a representation W). These are certain wide spheres in the sense of [9, 23.3], but we give a self-contained description here.

Since our concern is mainly with what happens at the identity, we separate the behaviours at and away from e using idempotents. Indeed, we adopt the convention that $M' = e_1 M$, $M'' = (1 - e_1)M$ and so forth. Away from the identity we take B to be an ordinary wedge of spheres

$$B'' = (S^0 \vee \Sigma^2 S^{-W})''$$

with $W^{\mathbb{T}} = 0$. By choosing suitable representations W this allows us to permit poles away from the identity, for which we write

$$\Omega(W)'' := \{\alpha \in \Omega \otimes_{\mathcal{O}} \mathcal{K} \mid \mathrm{ord}_s(\alpha) \geq -\dim_{\mathbb{C}}(W^{C_s}) \text{ for } s \geq 2\}.$$

The interesting part of B is what happens at the identity

$$B' = (M' \longrightarrow \mathbb{Q}[c, c^{-1}] \otimes U).$$

First we take $U = \mathbb{Q} \oplus \Sigma^2\mathbb{Q}$ with basis b_0, b_2 in degrees 0 and 2 (as forced by B''). Now take M' to be the $\mathbb{Q}[c]$-submodule of $\mathbb{Q}[c, c^{-1}] \otimes U$ generated by $a_0 = 1 \otimes b_0$ and $a_{2n+2} = c^{-(n+1)} \otimes b_0 + c^{-n} \otimes b_2$.

Lemma 16.2. *The cohomology of the object B defined above (depending on W and n) is given by*

$$\widetilde{EA}^0_{\mathbb{T}}(B) = \{(\lambda, \alpha) \in k \times \Omega(W)'' \mid \mathrm{ord}_e(\lambda \frac{Dt}{t_e} + \alpha) \geq n\}.$$

Remark 16.3. Since B has geometric $\mathbb{T}$-fixed points $S^0 \vee S^2$, the identification with a subset of $\mathcal{K} \times (\Omega \otimes_{\mathcal{O}} \mathcal{K})$ is intrinsic.

Proof: Consider a map $B \longrightarrow EA$ given by the diagram

$$\begin{array}{ccc} M & \stackrel{\theta}{\longrightarrow} & NA \\ \downarrow & & \downarrow \\ t_*^{\mathcal{F}} \otimes U & \stackrel{1\otimes\phi}{\longrightarrow} & t_*^{\mathcal{F}} \otimes VA \end{array}$$

Since $NA \subseteq t_*^{\mathcal{F}} \otimes VA$, the map is determined by the R-map $\theta : M \longrightarrow NA$. Since $M \subseteq t_*^{\mathcal{F}} \otimes (\mathbb{Q} \oplus \Sigma^2\mathbb{Q})$ the map θ is determined by $f = \phi(b_0) \in VA_0 = \mathcal{K}$ and $\alpha = \phi(b_2) \in VA_2 = \Omega \otimes_{\mathcal{O}} \mathcal{K}$. However in order for (f, α) to determine such a map we need to know the generators of M map into $NA = \ker(q)$.

Exactly as in 10.10, the condition away from e is that f is regular away from e and $\alpha \in \Omega(W)''$. The condition at e imposes the two conditions that $\theta(c^0 \otimes f) \in NA'_0$ and that $\theta(c^{-(n+1)} \otimes f + c^{-n} \otimes \alpha) \in NA'_{2n+2}$. The first of these shows $f = \lambda$ is constant, and the second gives the stated condition on α. $\square$

Now fix $\lambda = 1$ (say), and consider the set

$$\Lambda_{n,W}(t_e) = \{\alpha \in \Omega(W)'' \mid \mathrm{ord}_e(\frac{Dt}{t_e} + \alpha) \geq n\}.$$

Finally, suppose t and $\bar{t}$ are two choices of coordinate with $\Lambda_{n,W}(t) = \Lambda_{n,W}(\bar{t})$, then provided the two sets are non-empty (as we may assume by choice of W), we deduce

$$\mathrm{ord}_e(\frac{Dt}{t} - \frac{D\bar{t}}{\bar{t}}) \geq n.$$

Expressing t and $\bar{t}$ in terms of a fixed coordinate t_0 we have $t = ut_0$ and $\bar{t} = \bar{u}t_0$ the condition is equivalent to requiring that $\frac{u(e)}{u} - \frac{\bar{u}(e)}{\bar{u}}$ vanishes to order n. Now, since this is true for all n and u and $\bar{u}$ are both non-zero at e, it follows that $u/\bar{u}$ is the scalar $u(e)/\bar{u}(e)$. This shows that EA determines the coordinate divisor.

Now choose a coordinate t_0 with the appropriate divisor, and consider which multiple $t = \mu t_0$ gives the correct cohomology theory. For this we use a similar argument to the above with $n = 0$, once with the idempotent e_1 replaced by e_2, and once with e_1 replaced with e_3. Using e_2, we may pick out α satisfying the condition

$$\mathrm{ord}_2(\frac{Dt}{t_2} + \alpha) \geq 0,$$

(this determines μ^3). Using e_3, we may pick out α satisfying the condition

$$\mathrm{ord}_3(\frac{Dt}{t_3} + \alpha) \geq 0$$

(this determines μ^8). These two together give μ as required. $\square$

Part 6. **Categories of modules.**

We are working towards a comparison between the derived category $D_{tp}(\mathcal{O}_A^{\mathrm{tp}}\text{-mod})$ of tp-sheaves of $\mathcal{O}_A$-modules and a category of EA-modules. Before this can be useful, we need to describe methods of calculation, and settle a number of technical difficulties.

17. Algebraic categories of modules.

To start with, we work entirely in the algebraic category $\mathcal{A}_s$ with the strictly commutative ring EA in $\mathcal{A}_s$.

17.A. **Modules over EA.** We may consider the category EA-mod of left modules over over the algebraic model of EA. In fact a left EA-module $M = (P \longrightarrow t_*^{\mathcal{F}} \otimes W)$ is given by a map $EA \otimes M \longrightarrow M$, or more explicitly, a diagram

$$\begin{tikzcd}
NA \otimes_R P \arrow[r] \arrow[d] & P \arrow[dd] \\
(t_*^{\mathcal{F}} \otimes VA) \otimes_R (t_*^{\mathcal{F}} \otimes W) \arrow[d, "="] & \\
t_*^{\mathcal{F}} \otimes (VA \otimes W) \arrow[r] & t_*^{\mathcal{F}} \otimes W
\end{tikzcd}$$

From examples we see that we do not wish to require the structure map $P \longrightarrow t_*^{\mathcal{F}} \otimes W$ to be monomorphic, so we view the NA-module P as the basic object. Compatibility with the VA-module structure on W imposes a further condition.

Thus an EA-module is given by a suitably restricted NA-module P. We view P as a module of sections over the algebra NA of regular sections.

It is worth making this more explicit for special types of object. If $M = e(W)$, then the module structure is simply the structure of a $\mathcal{K}$-module on W.

If M is torsion so that $M = f(T)$ then $P = \bigoplus_s T_s$ where each module $T_s P$ is a module over NA_s, which is spanned by elements $c_s^i \otimes f$ with $t_s^i f$ regular on $A\langle s \rangle$. Furthermore, the action of NA_s factors through $\mathcal{O}_s = \{f \mid f \text{ is regular on } A\langle s \rangle\}$.

17.B. **Homological algebra of the category of modules.** The purpose of this section is to describe the derived category $D_{\mathbb{T}}(EA\text{-mod})$ of the algebraic category of modules, where the subscript $\mathbb{T}$ refers to the fact that only the counterparts of equivariant equivalences are inverted. We classify its objects up to isomorphism and give a means of calculating maps. Since the tp-derived category is formed by inverting maps which are homology isomorphisms for all twists, the maps are calculated in terms of the corresponding relative Ext groups, which we now describe.

With sheaves it is convenient to work with flabby objects rather than injective objects because we invert cohomology isomorphisms (i.e., isomorphisms of the derived functors of global sections, or $\mathrm{Ext}^*_{\mathcal{O}}(\mathcal{O}, \cdot)$). There are enough flabby objects for homological dimension to be visible at the level of abelian categories. We will work with a corresponding class of EA-modules.

First we introduce the relevant test objects, namely the spheres and torsion modules

$$\mathcal{T} := \{EA \wedge S^V \mid V \text{ a complex representation}\} \cup \{M \mid \Phi^{\mathbb{T}} M = 0\}.$$

The tp-*flabby* objects are then given by

$$\mathcal{I}_F := \{I \mid \mathrm{Ext}^s_{EA}(T, I) = 0 \text{ for all } T \in \mathcal{T}, s \geq 1\}.$$

We next form an injective class by a process of saturation; the tp-*monomorphisms* are

$$\mathcal{M} := M(\mathcal{I}_F) := \{f : X \longrightarrow Y \mid f^* : \mathrm{Hom}_{EA}(Y, I) \longrightarrow \mathrm{Hom}_{EA}(X, I) \text{ is epi for all } I \in \mathcal{I}_F\},$$

and the tp-*injectives* by

$$\mathcal{I} := I(\mathcal{M}) := \{I \mid f^* : \mathrm{Hom}_{EA}(Y, I) \longrightarrow \mathrm{Hom}_{EA}(X, I) \text{ is epi for all } f \in \mathcal{M}\}.$$

First we need some examples of tp-flabby objects.

Lemma 17.1. *If W is any $\mathcal{K}$-module, then $e(W)$ is a* tp-*flabby EA-module.*

If $I = \bigoplus_s I_s$ with I_s a divisible $\mathcal{O}_s$-module, then $f(I)$ is a tp-*flabby EA-module.*

Proof: First note that modules of the form $e(W)$ admit injective resolutions of the same form, and similarly for those of form $f(I)$. This means we can settle the question by considering just Hom.

Next, we note that the case $U = 0$ of the condition holds (i.e., $\mathrm{Ext}^s_{EA}(EA, N) = 0$ for $s > 0$ for N of the specified forms). Indeed,

$$\mathrm{Hom}_{EA}(EA, N) = \mathrm{Hom}_{\mathcal{A}_s}(S^0, N),$$

so it suffices for N to be injective in $\mathcal{A}_s$, which is certainly the case for both $N = e(W)$ and $N = f(I)$ with I_s being c-divisible.

For the modules $e(W)$ we use the adjunction

$$\mathrm{Hom}_{EA}(M, e(W)) = \mathrm{Hom}_{\mathcal{K}}(V, W)$$

where V is the vertex of M. The result when $\Phi^{\mathbb{T}} M \simeq 0$ is clear since it has zero vertex. The vertex of $S^U \wedge EA$ is independent of U, the result follows from the case $U = 0$.

For the modules $f(I)$, we use the adjunction

$$\mathrm{Hom}_{EA}(M, f(I)) = \prod_s \mathrm{Hom}_{\mathcal{O}_s}(M_s, I_s)$$

where M_s is the sth idempotent summand of the nub of M. The result is clear since I_s is injective by hypothesis. □

Lemma 17.2. *The objects $\mathcal{I}$ and the morphisms $\mathcal{M}$ form an injective class and a monomophic class.*

Proof: By definition $\mathcal{I} = I(\mathcal{M})$, and by saturation $\mathcal{M} = M(\mathcal{I})$. It remains to show that for any EA-module N there is a map $f : N \longrightarrow F$ in $\mathcal{M}$ with $F \in \mathcal{I}$.

For an arbitrary EA-module $N = (L \longrightarrow t^{\mathcal{F}}_* \otimes V)$ we have a map $N \longrightarrow \mathcal{E}^{-1}N = e(V)$. The kernel K is of the form $f(T)$ for a torsion module N, and we may embed this in a divisible module I, giving a short exact sequence

$$0 \longrightarrow N \xrightarrow{i} e(V) \oplus f(I) \longrightarrow f(J) \longrightarrow 0,$$

where $f(J)$ is divisible and hence also tp-flabby.

The fact that the map i is tp-monomorphic follows since $f(J)$ is a test object. □

This means that we can do relative homological algebra, and form $\mathrm{Ext}^*_{EA,\mathrm{tp}}(M, N)$. Better still, the proof supplied tp-injective resolutions of length 1.

Corollary 17.3. *The* tp-*injective dimension of any EA-module is ≤ 1, so that $\mathrm{Ext}^s_{EA,\mathrm{tp}}(M, N) = 0$ for $s \geq 2$. Furthermore $\mathrm{Hom}_{EA,\mathrm{tp}}(M, N) = \mathrm{Hom}_{EA}(M, N)$.* □

This makes the category very accessible to calculation.

Theorem 17.4. *(i) All objects of $D_{\mathbb{T}}(EA\text{-mod})$ are formal, in that $M \simeq H_*(M)$. Thus homotopy types in $D_{\mathbb{T}}(EA\text{-mod})$ correspond to isomorphism classes of EA-modules.*

(ii) For EA-modules M and N there is a short exact sequence

$$0 \longrightarrow \mathrm{Ext}^1_{EA,\mathrm{tp}}(\Sigma H_*(M), H_*(N)) \longrightarrow [M,N]_{EA} \longrightarrow \mathrm{Hom}_{EA}(H_*(M), H_*(N)) \longrightarrow 0.$$

The method of proof is standard, and slightly simplified by the fact that any EA-module can be considered as an object of $D_{\mathbb{T}}(EA\text{-mod})$ by using the zero differential.

We consider the map

$$\nu : [M,N] \longrightarrow \mathrm{Hom}_{EA}(H_*(M), H_*(N))$$

given by taking homology. We will show that it is an isomorphism for good tp-flabby modules N. We have seen that any EA-module may be embedded in a good tp-flabby module with tp-flabby quotient. Now, for an arbitrary differential graded EA-module N we choose a tp-resolution

$$0 \longrightarrow H_*(N) \longrightarrow I_0 \longrightarrow I_1 \longrightarrow 0$$

of its homology, where I_0 and I_1 are both good tp-flabby modules. Now let $N \longrightarrow I_0$ be the map corresponding to the first map in the resolution and note that the mapping cone has homology I_1. Up to isomorphism we therefore have a cofibre sequence

$$N \longrightarrow I_0 \longrightarrow I_1,$$

and applying $[M, \cdot]_{EA}$ we obtain Part (ii) of the theorem. Part (i) now follows, since if $H_*(M) \cong H_*(M')$ we may lift this isomorphism to a map $M \longrightarrow M'$, which, being a homology isomorphism, is an equivalence. In particular tp-flabby objects are classified by their homology, so it was reasonable to call the cofibre I_1. It remains to prove that our good tp-flabby modules have the right properties.

Lemma 17.5. *If N is one of the modules $e(W)$ and $f(T)$ in 17.1, the map ν is an isomorphism.*

Proof: By definition the functor $\mathrm{Hom}_{EA}(\cdot, H_*(N))$ is exact when N is tp-flabby, so we have a natural transformation of cohomology theories and it suffices to check it is an isomorphism on a collection of EA-modules which generate all modules using direct sums and cofibre sequences. By Adams's projective resolution argument, it suffices to use the objects $EA \wedge S^V$, since they are small and detect weak equivalences. The objects $EA \wedge S^V$ are extended by construction, and we have

$$\pi^{\mathcal{A}}_*(EA \wedge S^V) \cong \pi^{\mathcal{A}}_*(EA) \otimes \pi^{\mathcal{A}}_*(S^V),$$

and hence a commutative diagram

$$\begin{tikzcd}
{[EA \wedge S^V, N]_{EA}} \arrow[r] \arrow[d, "\cong"'] & \mathrm{Hom}_{EA}(\pi^{\mathcal{A}}_*(EA \wedge S^V), \pi^{\mathcal{A}}_*(N)) \arrow[d, "\cong"] \\
{[S^V, N]} \arrow[r] & \mathrm{Hom}_{\mathcal{A}_s}(\pi^{\mathcal{A}}_*(S^V), \pi^{\mathcal{A}}_*(N)).
\end{tikzcd}$$

The result follows from the fact that the objects are injective in $\mathcal{A}_s$ together with the corresponding statements there [9, 5.6.7,5.6.8]. $\square$

18. Homotopy modules.

The equivalence of [9] is only defined at the homotopy level and the equivalence of [23] is not known to be monoidal at the model category level. The results of [14] do show that we may choose EA to be a strictly commutative ring spectrum, and hence there is a model category of EA-module $\mathbb{T}$-spectra, but since this is not yet published, it seems worth including a brief account of what can be said about modules up to homotopy: this section will discuss how good a model of $D_{\mathbb{T}}(EA\text{-mod})$ can be obtained by working with rings and modules up to homotopy.

Modules up to homotopy have notoriously bad formal behaviour, but the low homological dimension of the algebraic categories means we can nonetheless obtain some useful information. The idea is to use the category of homotopy modules and homotopy module maps as a model for the homotopy category of modules. To see the effectiveness of this, we continue to work in the algebraic category.

At the level of objects, the model is good.

Lemma 18.1. *Every homotopy EA-module is represented by a strict EA-module. Two homotopy EA-modules are equivalent if and only if their strict representives are equivalent.*

Proof: Any EA-module M is obviously a homotopy module. Since the orginal module may be recovered via the action of $EA = \pi^{\mathcal{A}}_*(EA)$ on $\pi^{\mathcal{A}}_*(M)$, the forgetful map is injective on objects. Furthermore, every object of $\mathcal{A}_s$ is formal, so there is an equivalence $M \simeq \pi^{\mathcal{A}}_* M$, and the action passes to $\pi^{\mathcal{A}}_*(M)$. Thus any homotopy module M is equivalent to the strict module $\pi^{\mathcal{A}}_*(M)$, and the forgetful map is surjective. $\square$

Given two homotopy modules M, N, we define the group of homotopy module maps by

$$[M,N]_{Ho(EA)} := \{ f \in [M,N] \mid f \text{ is a module map up to homotopy } \}.$$

The main point to make is that this is a subset of the maps ignoring EA-module structure, so that it is unlikely to model phenomena of positive filtration. As usual the cofibre of a homotopy module map has no canonical structure as a homotopy module. Taking homotopy module maps need not be exact, even if applied to a cofibre sequence of strict modules.

The best we can do is to attempt to detect homotopy module maps. Given homotopy modules, we choose strict modules M, N representing them, and to simplify the notation, we assume they have zero differential. We then have a diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}^1_{EA,\mathrm{tp}}(\Sigma M, N) \arrow[r] \arrow[dd] & {[M,N]_{EA}} \arrow[r] \arrow[d] & \mathrm{Hom}_{EA}(M,N) \arrow[r] \arrow[dd] & 0 \\
 & & {[M,N]_{Ho(EA)}} \arrow[d] & & \\
0 \arrow[r] & \mathrm{Ext}^1_{\mathcal{A}_s}(\Sigma M, N) \arrow[r] & {[M,N]} \arrow[r] & \mathrm{Hom}_{\mathcal{A}_s}(M,N) \arrow[r] & 0.
\end{tikzcd}$$

Since every module map in homology is represented by a strict module map $M \longrightarrow N$, it is represented by a homotopy module map. Subtracting this, the remaining issue is how to decide when a map inducing zero in homology is a homotopy module map. Certainly it suffices for it to be in the image of $\mathrm{Ext}^1_{EA,\mathrm{tp}}(\Sigma M, N) \longrightarrow \mathrm{Ext}^1_{\mathcal{A}_s}(\Sigma M, N)$. When no other elements of $\mathrm{Ext}^1_{\mathcal{A}_s}(\Sigma M, N)$ represent homotopy module maps, the forgetful map

$$[M, N]_{EA} \longrightarrow [M, N]_{Ho(EA)}$$

is surjective, but even then its kernel is

$$\ker \left[\ \mathrm{Ext}^1_{EA,\mathrm{tp}}(\Sigma M, N) \longrightarrow \mathrm{Ext}^1_{\mathcal{A}_s}(\Sigma M, N) \ \right],$$

which may be non-trivial.

Part 7. An equivalence between derived categories of sheaves and spectra.

Having shown that the structure sheaf $\mathcal{O}_A$ of the elliptic curve A gives rise to a commutative ring EA in $\mathcal{A}_s$, we show in this part that this extends to an equivalence between their derived categories of modules.

The discussion of modules up to homotopy in Section 18 shows how much of the resulting information can be transported to the category of spectra without using further technology. However, the results of [14] show that the strictly commutative ring in $\mathcal{A}_s$ gives a *strictly* commutative ring $\mathbb{T}$-spectrum, and using this additional technology, the present account applies without change to categories of equivariant EA-module *spectra.*

19. Sheaves from spectra.

We describe a natural construction of a sheaf over A from a $\mathbb{T}$-spectrum. In Section 22 we show how it is related to Grojnowski's construction [15].

19.A. Sheaves associated to R-modules.

An object of $\mathcal{A}_s$ is a *based* R-module in a suitable sense, but it will clarify the later construction to begin with a construction on arbitrary R-modules N.

Note first that we have defined suspension functors $\Sigma^w N$ for any almost constant function $w : \mathcal{F} \longrightarrow \mathbb{Z}$, and if $w(s) \leq w'(s)$ for all s there is a map $\Sigma^w N \longrightarrow \Sigma^{w'} N$ which is multiplication by $c^{w'(s)-w(s)}$ on the sth idempotent summand.

Recall that, for any finite set π of positive integers, V_π is the set of points of A whose orders are in π, and $U_\pi = A \setminus V_\pi$.

Definition 19.1. Suppose N is an R-module and let

$$\mathcal{E}_\pi^{-1} N := \lim_{\rightarrow \, w(\pi)=0} \Sigma^w N.$$

Now define a presheaf $\tilde{N}$ of R-modules on A by taking

$$\tilde{N}(U_\pi) := \mathcal{E}_\pi^{-1} N$$

Lemma 19.2. *The presheaf $\tilde{N}$ is a sheaf.*

Proof: First note that, since $\mathcal{E}_\pi^{-1} N = \mathcal{E}_\pi^{-1} R \otimes_R N$, we have $\tilde{N} = \tilde{R} \otimes_R N$.

Now since any cover has a finite subcover, it suffices to check the sheaf condition on $U_{\pi\cap\pi'} = U_\pi \cup U_{\pi'}$. Since $\mathcal{E}_\pi^{-1} R$ is flat for any π, it suffices to deal with the special case $N = R$, where we have an exact sequence

$$0 \longrightarrow \mathcal{E}_{\pi\cap\pi'}^{-1} R \longrightarrow \mathcal{E}_\pi^{-1} R \oplus \mathcal{E}_{\pi'}^{-1} R \longrightarrow \mathcal{E}_{\pi\cup\pi'}^{-1} R.$$

□

19.B. **Construction of the sheaf.** We begin in earnest by defining a functor

$$\mathcal{M}_A : \mathcal{A}_s \longrightarrow \text{sheaves}/A$$

at the level of abelian categories. We will show that it restricts to a functor

$$\mathcal{M}_A : EA\text{-mod} \longrightarrow \mathcal{O}_A\text{-mod}.$$

When π is the set of divisors of n we think of V_π as defined by the Euler class of z^n. This motivates some corresponding definitions in equivariant topology. For each subgroup H we need the space $E\langle H\rangle = \text{cofibre}(E[\subset H]_+ \longrightarrow E[\subseteq H]_+)$, whose distinguishing feature is that its K-fixed points are contractible unless $K = H$, and S^0 if $K = H$. We consider the set $\mathcal{F}\langle\pi\rangle$ of subgroups of $\mathbb{T}$ with order in π and then form the space

$$E\langle\pi\rangle := e_{\mathcal{F}\langle\pi\rangle} E\mathcal{F}_+ = \bigvee_{H \in \mathcal{F}(\pi)} E\langle H\rangle,$$

where $e_{\mathcal{F}\langle\pi\rangle} \in \text{map}(\mathcal{F}, \mathbb{Q})$ is the idempotent with support $\mathcal{F}\langle\pi\rangle$. We may then form the space $\tilde{E}\langle\pi\rangle$ using the cofibre sequence

$$E\langle\pi\rangle \longrightarrow S^0 \longrightarrow \tilde{E}\langle\pi\rangle.$$

The space $E\langle\pi\rangle$ is modelled in $\mathcal{A}_s$ by

$$T(\pi) = (\bigoplus_{H \in \mathcal{F}\langle\pi\rangle} \mathbb{I}(H) \longrightarrow 0)$$

and the space $\tilde{E}\langle\pi\rangle$ by

$$L(\pi) = (R(\infty\pi) \longrightarrow t_*^{\mathcal{F}})$$

where $R(\infty\pi) \subseteq t_*^{\mathcal{F}}$ consists of elements with poles only on $\mathcal{F}\langle\pi\rangle$.

Next, we associate a sheaf $\mathcal{M}_A(X)$ in the torsion-point topology with an object $X = (P \longrightarrow t_*^{\mathcal{F}} \otimes W)$ of $\mathcal{A}_s$. First recall the notation

$$P(c^0) = \{p \in P \mid \beta(p) \in c^0 \otimes W\} = \text{Hom}_{\mathcal{A}_s}(S^0, X).$$

Continuing the analogy with sections, we write

$$P(\infty\pi) = P \otimes_R R(\infty\pi)$$

so that

$$X \otimes L(\pi) = (P(\infty\pi) \longrightarrow t_*^{\mathcal{F}} \otimes W).$$

Definition 19.3. For any object $X = (P \longrightarrow t_*^{\mathcal{F}} \otimes W)$ of $\mathcal{A}_s$ the presheaf $\mathcal{M}_A(X)$ is defined by

$$\mathcal{M}_A(X)(U_\pi) = \text{Hom}_{\mathcal{A}_s}(S^0, X \otimes L(\pi)) = P(\infty\pi)(c^0).$$

The restriction associated to $U_{\pi'} \subseteq U_\pi$ is induced by the map $L(\pi) \longrightarrow L(\pi')$ which is the identity on the vertex.

Lemma 19.4. *The presheaf $\mathcal{M}_A(X)$ is in fact a sheaf.*

Proof: It suffices to consider the cover of $U_{\pi\cap\pi'}$ by U_π and $U_{\pi'}$, and we need to show there is an exact sequence

$$0 \longrightarrow \mathcal{M}_A(X)(U_{\pi\cap\pi'}) \longrightarrow \mathcal{M}_A(X)(U_\pi) \oplus \mathcal{M}_A(X)(U_{\pi'}) \longrightarrow \mathcal{M}_A(X)(U_{\pi\cup\pi'}).$$

This is obtained from the short exact sequence

$$0 \longrightarrow L(\pi\cap\pi') \longrightarrow L(\pi) \oplus L(\pi') \longrightarrow L(\pi\cup\pi') \longrightarrow 0.$$

Indeed, since $L(\pi\cup\pi')$ is flat, we obtain the desired exact sequence by applying the functor $\mathrm{Hom}_{\mathcal{A}_s}(S^0, X\otimes\cdot)$. □

Lemma 19.5. *If X has vertex V then*

$$\mathcal{M}_A(X)(\infty tors) = V.$$

Proof: Since $\mathcal{M}_A(X)(U_\pi) = \mathrm{Hom}_{\mathcal{A}_s}(S^0, X\otimes L(\pi))$ and S^0 is small, we find

$$\mathcal{M}_A(X)(\infty tors) = \lim_{\substack{\to \\ U_\pi}} \mathrm{Hom}_{\mathcal{A}_s}(S^0, X\otimes L(\pi)) = \mathrm{Hom}_{\mathcal{A}_s}(S^0, \lim_{\substack{\to \\ U_\pi}} X\otimes L(\pi)) = V.$$

□

For torsion free spectra we can also identify stalks.

Lemma 19.6. *If X is torsion free then the stalk of $\mathcal{M}_A(X)$ at a point of order s is given by*

$$\mathcal{M}_A(X)_s = \ker(c^0\otimes V \longrightarrow T \longrightarrow e_sT).$$

Remark 19.7. It is natural to refer to $\mathcal{M}_A(X)_s$ as the space of X-meromorphic functions regular at s.

Proof: To calculate the stalk we take a direct limit over U_π containing points of order s, which are U_π with $s\notin\pi$. For a torsion free X

$$\mathcal{M}_A(X)(U_\pi) = \ker(c^0\otimes V \longrightarrow T \longrightarrow \bigoplus_{r\notin\pi} e_rT).$$

□

Since direct sums commute with tensor products and S^0 is small, we deduce a useful formal property.

Lemma 19.8. *The functor $\mathcal{M}_A$ preserves arbitrary direct sums.* □

19.C. **The sheaf associated to an EA-module.** We show that applying the functor to an EA-module gives a sheaf of $\mathcal{O}_A$-modules.

Lemma 19.9. *(i) The functor $\mathcal{M}_A$ takes EA to the structure sheaf:*

$$\mathcal{M}_A(EA) = \mathcal{O}_A.$$

(ii) The functor $\mathcal{M}_A$ takes EA-modules to $\mathcal{O}_A$-modules, and therefore induces a functor

$$\mathcal{M}_A : EA\text{-}mod \longrightarrow \mathcal{O}_A\text{-}mod.$$

Proof: Part (i) is clear from our construction of elliptic cohomology.

For Part (ii), we need to show that there are structure maps $\mathcal{O}(U_\pi) \otimes \mathcal{M}_A(X)(U_\pi) \longrightarrow \mathcal{M}_A(X)(U_\pi)$, or in other words,

$$NA(\infty\pi)(c^0) \otimes P(\infty\pi)(c^0) \longrightarrow P(\infty\pi)(c^0).$$

However we need only note that, $L(\pi)$ (like S^0) is idempotent in the sense that $L(\pi) \otimes L(\pi) = L(\pi)$ so that the required map is the composite

$$\begin{aligned}\mathrm{Hom}_{\mathcal{A}_s}(S^0, EA \otimes L(\pi)) \otimes \mathrm{Hom}_{\mathcal{A}_s}(S^0, X \otimes L(\pi)) &\xrightarrow{\otimes} \mathrm{Hom}_{\mathcal{A}_s}(S^0 \otimes S^0, EA \otimes L(\pi) \otimes X \otimes L(\pi)) \\ = \mathrm{Hom}_{\mathcal{A}_s}(S^0, EA \otimes X \otimes L(\pi)) &\longrightarrow \mathrm{Hom}_{\mathcal{A}_s}(S^0, X \otimes L(\pi)).\end{aligned}$$

Compatibility with restriction is clear since the restriction associated to $U_{\pi'} \subseteq U_\pi$ is induced by a map $L(\pi) \longrightarrow L(\pi')$. □

One more special value plays an important role.

Lemma 19.10. *The EA-module $S^W \wedge EA$ is taken to the corresponding line bundle*

$$\mathcal{M}_A(S^W \wedge EA) = \mathcal{O}_A(D(W)).$$

Proof: This follows directly from the parallel between topological suspension 4.6 and algebraic twisting by line bundles. □

20. $\mathbb{T}$-spectra from $\mathcal{O}_A$-modules.

In this section we adapt the construction of EA given in Section 10 to associate an object of $\mathcal{A}_s$ to an $\mathcal{O}_A$-module, and hence provide a functor

$$\mathcal{S}_A : \mathcal{O}_A\text{-mod} \longrightarrow EA\text{-mod}.$$

In the construction of EA we made fundamental use of the fact that the sheaf $\mathcal{O}_A$ is torsion free in the sense that $\mathcal{O}(D)$ is a submodule of $\mathcal{O}(\infty tors) = \mathcal{K}$ for all torsion point divisors D. As a result, the nub is a submodule of $t_*^{\mathcal{F}} \otimes VA$, where VA_0 consists of the space $\mathcal{K}$ of meromorphic functions. For an $\mathcal{O}$-module $\mathcal{Y}$, it often happens for a non-zero sheaf $\mathcal{Y}$, that the sheaf $\mathcal{Y}(\infty tors)$ of meromorphic sections is zero, so that the earlier construction would give zero. The construction we give here does specialize to construct EA, but also deals with torsion sheaves.

20.A. **The construction.** In topology, the object of $\mathcal{A}_s$ associated to a $\mathbb{T}$-spectrum X is obtained from the map

$$X \wedge DE\mathcal{F}_+ \longrightarrow X \wedge DE\mathcal{F}_+ \wedge \tilde{E}\mathcal{F}$$

by taking equivariant homotopy groups. The key facts are

- $X \wedge DE\mathcal{F}_+ \wedge \tilde{E}\mathcal{F} \simeq \Phi^{\mathbb{T}} X \wedge DE\mathcal{F}_+ \wedge \tilde{E}\mathcal{F}$ and
- the cofibre of the map is the $\mathbb{T}$-free object $X \wedge \Sigma E\mathcal{F}_+$
- there is a cofibre sequence

$$DE\mathcal{F}_+ \longrightarrow DE\mathcal{F}_+ \wedge \tilde{E}\mathcal{F} \longrightarrow \Sigma E\mathcal{F}_+.$$

We make the analogous construction on sheaves, by starting with an analogue of the above cofibre sequence. Indeed, we consider $t_*^{\mathcal{F}} \otimes VA$ as the constant sheaf of R-modules, and $Q(\infty tors)$ as the sum of skyscraper sheaves for the modules e_sTA. We then define a sheaf $\mathcal{D}$ by the short exact sequence of sheaves

$$\mathcal{D} \longrightarrow t_*^{\mathcal{F}} \otimes VA \xrightarrow{q} Q(\infty tors) \otimes_{\mathcal{O}} \Omega^*$$

of R-modules. Surjectivity of q follows from the corresponding fact for R-modules. We also note that neither $t_*^{\mathcal{F}} \otimes VA$ nor $Q(\infty tors)$ have higher sheaf cohomology. Thus $\mathcal{D}$ encapsulates all the cohomology of spheres.

Remark 20.1. Unlike the topological case, it appears that $\mathcal{D}$ is not the dual of anything. In particular

$$\Sigma \mathrm{Hom}_{\mathcal{O}}(Q(\infty tors), \mathcal{O}) \simeq \mathrm{Hom}_{\mathcal{O}}(Q(\infty tors), Q(\infty tors))$$

is a proper completion of $\mathcal{D}$. The 0th idempotent piece of its space of sections is of uncountable dimension, so it is different from $\mathcal{D}$.

The next step in the construction is to tensor the basic short exact sequence with the $\mathcal{O}$-module $\mathcal{Y}$ to form

$$\mathcal{D} \otimes_{\mathcal{O}} \mathcal{Y} \longrightarrow t_*^{\mathcal{F}} \otimes VA \otimes_{\mathcal{O}} \mathcal{Y} \xrightarrow{q} Q(\infty tors) \otimes_{\mathcal{O}} \mathcal{Y}.$$

To understand the central term we note that $VA_0 = \mathcal{K} = \mathcal{O}(\infty tors)$.

Lemma 20.2. *For any $\mathcal{O}$-module $\mathcal{Y}$ the sheaf $\mathcal{Y}(\infty tors) = \mathcal{Y} \otimes_{\mathcal{O}} \mathcal{O}(\infty tors)$ is constant, and its cohomology is entirely in degree zero.* □

Similarly, the essential thing about the last term is that its cohomology is $\mathcal{E}$-torsion.

Lemma 20.3. *The R-module*

$$H^i(\mathcal{Y} \otimes_{\mathcal{O}} Q(\infty tors) \otimes_{\mathcal{O}} \Omega^*)$$

is $\mathcal{E}$-torsion for $i = 0$ or 1.

Proof: Consider the decomposition $Q(\infty tors) = \bigoplus_s Q(\infty A\langle s \rangle)$: the sth term is a direct limit of terms $Q(kA\langle s \rangle)$ whose cohomology is annihilated by inverting t_s. □

Corollary 20.4. *The map $\mathcal{D} \longrightarrow t_*^{\mathcal{F}} \otimes VA$ induces an isomorphism*

$$\mathcal{E}^{-1}H^i(\mathcal{D} \otimes_{\mathcal{O}} \mathcal{Y}) \cong H^i(\mathcal{D} \otimes_{\mathcal{O}} \mathcal{O}(\infty tors) \otimes_{\mathcal{O}} \mathcal{Y}) = \begin{cases} t_*^{\mathcal{F}} \otimes \mathcal{Y}(\infty tors) \otimes_{\mathcal{O}} \Omega^* & \text{for } i = 0 \\ 0 & \text{for } i = 1. \end{cases}$$

□

Definition 20.5. We now define the functor

$$\mathcal{S}_A : \mathcal{O}_A\text{-mod} \longrightarrow \mathcal{A}_s$$

at the level of abelian categories. The object

$$\mathcal{S}_A(\mathcal{Y}) = (N\mathcal{Y} \longrightarrow t_*^{\mathcal{F}} \otimes V\mathcal{Y})$$

of $\mathcal{A}_s$ associated to a sheaf $\mathcal{Y}$ in degree 0 is

$$H^*(\mathcal{Y} \otimes_{\mathcal{O}} \mathcal{D}) \longrightarrow H^*(\mathcal{Y} \otimes_{\mathcal{O}} \mathcal{D} \otimes_{\mathcal{O}} \mathcal{O}(\infty tors)).$$

To be explicit the nub is

$$N\mathcal{Y}_{ev} = H^0(\mathcal{Y} \otimes_{\mathcal{O}} \mathcal{D})$$

in even degrees, and

$$N\mathcal{Y}_{od} = \Sigma^{-1} H^1(\mathcal{Y} \otimes_{\mathcal{O}} \mathcal{D})$$

in odd degrees. The vertex is entirely in even degrees and

$$V\mathcal{Y}_0 = \mathcal{Y}(\infty tors).$$

Remark 20.6. The fact that this is indeed an object of $\mathcal{A}_s$ follows from 20.4. Furthermore, for any $\mathcal{O}$-module $\mathcal{Y}$, the vertex $V\mathcal{Y}$ is entirely in even degrees. The odd degree part of the nub is entirely $\mathcal{E}$-torsion

Since tensor product is compatible with passage to stalks, we may describe the divisible torsion part.

Corollary 20.7. *The sheaf $\mathcal{Y} \otimes_{\mathcal{O}} Q(\infty tors)$ is a sum of skyscraper sheaves. Indeed, the stalk at a point of order s is*

$$\mathcal{Y} \otimes_{\mathcal{O}} Q(\infty tors)_s = \mathcal{Y}_s \otimes_{\mathcal{O}_s} e_s TA_0. \quad \square$$

Since direct sums commute with tensor products and $\mathcal{O}$ is small, we deduce a useful formal property.

Lemma 20.8. *The functor $\mathcal{S}_A$ preserves arbitrary direct sums.* $\square$

20.B. **Module structure.** The formal nature of the construction gives a module structure rather simply.

Lemma 20.9. *(i) The functor $\mathcal{S}_A$ takes $\mathcal{O}_A$ to the structure ring spectrum*

$$\mathcal{S}_A(\mathcal{O}_A) = EA.$$

(ii) The functor $\mathcal{S}_A$ takes $\mathcal{O}_A$-modules to EA-modules, and therefore induces a functor

$$\mathcal{S}_A : \mathcal{O}_A\text{-}mod \longrightarrow EA\text{-}mod.$$

Proof: (i) It is built into the definition that, $\mathcal{S}_A(\mathcal{O}_A) = EA$, and we proved in 11.1 that EA is a ring.

(ii) The sheaf level construction preserves tensor products, and there is a map

$$H^i(\mathcal{Y}) \otimes H^i(\mathcal{Z}) \longrightarrow H^i(\mathcal{Y} \otimes_{\mathcal{O}} \mathcal{Z}).$$

$\square$

One more special value will be important.

Lemma 20.10. *The functor $\mathcal{S}_A$ takes the basic line bundles to spheres:*

$$\mathcal{S}_A(\mathcal{O}_A(D(V))) = S^V \wedge EA.$$

Proof: The correspondence between line bundles and suspensions has been built into the framework 5.5. Thus, if we take $\mathcal{Y} = \mathcal{O}_A(D(V))$ we note first that $\mathcal{Y}(\infty tors) = \mathcal{K}$ and $\mathcal{Y} \otimes_{\mathcal{O}} Q(\infty tors) = \mathcal{K}/\mathcal{O}(D)$. By construction $\mathcal{M}_A(\mathcal{S}_A^{ev}\mathcal{Y})(U_\pi)$ is the space $\mathcal{O}(D)(U_\pi)$ of functions regular away from π. $\square$

20.C. **The functor $\mathcal{S}_A$ on torsion free sheaves.** Whenever $\mathcal{Y}$ is torsion free in the sense that it is a subsheaf of the constant sheaf $\mathcal{Y}(\infty tors)$ then the spectrum $\mathcal{S}_A(\mathcal{Y})$ can be constructed exactly as we originally constructed EA.

Definition 20.11. If $\mathcal{Y}$ is an $\mathcal{O}$-module we define an object

$$\mathcal{S}_A^t(\mathcal{Y}) = (t_*^{\mathcal{F}} \otimes V\mathcal{Y} \longrightarrow T\mathcal{Y})$$

of $\mathcal{A}_t$. Here

$$V\mathcal{Y}_0 = \mathcal{Y}(\infty tors) = \lim_{\rightarrow\ \pi} \mathcal{Y}(U_\pi),$$

and

$$T\mathcal{Y}_0 = H^0(A; Q\mathcal{Y}),$$

where $Q\mathcal{Y}$ is defined by the exact sequence

$$0 \longrightarrow \mathcal{Y} \longrightarrow \mathcal{Y}(\infty tors) \longrightarrow Q\mathcal{Y} \longrightarrow 0.$$

These are made periodic with differentials as usual:

$$V\mathcal{Y} = V\mathcal{Y}_0 \otimes \omega^* \text{ and } T\mathcal{Y} = T\mathcal{Y}_0 \otimes \omega^*.$$

Now the structure map is defined exactly as before, using the differentials Dt/t_s.

Lemma 20.12. *If $\mathcal{Y}$ is torsion free then*

$$\mathcal{S}_A(\mathcal{Y}) \cong (N\mathcal{Y} \longrightarrow t_*^{\mathcal{F}} \otimes V\mathcal{Y})$$

where $N\mathcal{Y} = \ker(q : t_*^{\mathcal{F}} \otimes V\mathcal{Y} \longrightarrow T\mathcal{Y})$.

Proof: This is immediate from the defining triangle

$$\mathcal{Y} \otimes_{\mathcal{O}} \mathcal{D} \longrightarrow \mathcal{Y} \otimes_{\mathcal{O}} \mathcal{D} \otimes_{\mathcal{O}} \mathcal{O}(\infty tors) \xrightarrow{q} \mathcal{Y} \otimes_{\mathcal{O}} Q(\infty tors) \otimes_{\mathcal{O}} \Omega^*.$$

Note first that $\mathcal{Y}$ is flat, being a submodule of the flat module $\mathcal{Y}(\infty tors)$, so that this is a short exact sequence of sheaves. It therefore induces a long exact sequence in cohomology. Since the cohomology of $\mathcal{Y}(\infty tors)$ is in even degrees, it therefore suffices to show that the map q induces a surjective map in cohomology. $\square$

21. Equivalence of EA-modules and $\mathcal{O}$-modules.

We now have functors relating the algebraic model of spectra and sheaves over an elliptic curve. In this section we show that these can be combined to give an equivalence between suitable derived categories.

21.A. **The derived categories.** We recall the constructions in parallel. In both cases we form the derived categories by a process of cellular approximation as in Subsection 7.B.

In the topological case, the category $D_{\mathbb{T}}(EA\text{-mod})$ from Subsection 17.B is formed from the category of differential graded EA-modules. It is natural to use the cells $EA \wedge \mathbb{T}/H_+$ where H runs through the set of closed subgroups of $\mathbb{T}$. However the cofibre sequence

$$\mathbb{T}/\mathbb{T}[n]_+ \longrightarrow S^0 \longrightarrow S^{z^n}$$

shows that it is equivalent to use the cells $EA \wedge S^V$ as V runs through complex representations. With either of these collections of cells, a map $X \longrightarrow Y$ of EA-modules is a weak equivalence if and only if it induces an isomorphism $\pi^H_*(\cdot)$ for all closed subgroups H, which is the usual notion of an equivariant weak equivalence of $\mathbb{T}$-spectra (and equivalent to being a homology isomorphism in $\mathcal{A}_s$).

In the algebraic case, the category $D_{tp}(\mathcal{O}_A^{\mathrm{tp}}\text{-mod})$ from Subsection 7.B is formed from the category of differential graded sheaves of $\mathcal{O}_A^{\mathrm{tp}}$-modules. Motivated by the topological case, we use the cells $\mathcal{O}(D(V))$ for representations V. It is equivalent to use the line bundles $\mathcal{O}(D)$ where D runs through torsion point divisors as was done previously. A map $X \longrightarrow Y$ is then a weak equivalence if it induces an isomorphism of $H^*(A; \mathcal{O}(-D) \otimes_{\mathcal{O}} (\cdot))$ for all torsion point divisors D.

21.B. **The equivalence.** We are now equipped to state our second main theorem.

Theorem 21.1. *The functors $\mathcal{M}_A : EA\text{-}mod \longrightarrow \mathcal{O}_A\text{-}mod$ and $\mathcal{S}_A : \mathcal{O}_A\text{-}mod \longrightarrow EA\text{-}mod$ relating the categories of algebraic EA-module $\mathbb{T}$-spectra and sheaves of $\mathcal{O}$-modules defined in 19.3 and 20.5 induce an equivalence*

$$D_{\mathbb{T}}(EA\text{-}mod) \simeq D_{tp}(\mathcal{O}_A^{\mathrm{tp}}\text{-}mod)$$

of associated derived categories.

Remark 21.2. Neither functor preserves infinite products, so this not an adjoint pair or a Quillen equivalence.

We begin at the level of abelian categories.

Lemma 21.3. *There is a natural transformation of functors $\mathcal{M}_A\mathcal{S}_A \longrightarrow 1$ which is a natural isomorphism on the line bundles $\mathcal{O}(D(V))$ for any complex representation V with $V^{\mathbb{T}} = 0$.*

Proof: Suppose then that $\mathcal{Y}$ is a module, and let

$$\mathcal{S}_A^{ev}\mathcal{Y} = \left[H^0(\mathcal{Y} \otimes_{\mathcal{O}} \mathcal{D}) \longrightarrow H^0(\mathcal{Y} \otimes_{\mathcal{O}} \mathcal{D}(\infty tors))\right]$$

be the even summand of $\mathcal{S}_A\mathcal{Y}$. Now

$$\mathcal{M}_A(\mathcal{S}_A^{ev}\mathcal{Y})(U_\pi) = \mathrm{Hom}_{\mathcal{A}_s}(S^0, \mathcal{S}_A^{ev}\mathcal{Y} \otimes L(\pi));$$

this contains the torsion part of $N^{ev}\mathcal{Y}$ and maps to

$$\ker(c^0 \otimes \mathcal{Y}(\infty tors) \longrightarrow H^0(\mathcal{Y} \otimes_{\mathcal{O}} Q(\infty tors)) \otimes R(\pi)),$$

which is $\mathcal{Y}(U_\pi)$. This defines the map $\mathcal{M}_A\mathcal{S}_A\mathcal{Y} \longrightarrow \mathcal{Y}$.

Now consider the sheaf $\mathcal{Y} = \mathcal{O}(D)$. Combining 19.10 and 20.10, we see that $\mathcal{M}_A\mathcal{S}_A\mathcal{Y} \cong \mathcal{Y}$ in this case, and since $\mathcal{O}(D)$ is torsion free, the natural transformation is the identity. $\square$

Lemma 21.4. *There is a natural transformation of functors $\mathcal{S}_A\mathcal{M}_A \longrightarrow 1$ which is a natural isomorphism on the spheres $EA \wedge S^W$ for any complex representation W.*

Proof: We suppose $X = (N \stackrel{\beta}{\longrightarrow} t_*^{\mathcal{F}} \otimes V)$ is an EA-module concentrated in even degrees and construct a diagram

$$\begin{array}{ccc} H^0(\mathcal{D} \otimes_{\mathcal{O}} \mathcal{M}_A(X)) & \stackrel{\eta_n}{\longrightarrow} & N \\ \downarrow & & \downarrow \\ H^0((t_*^{\mathcal{F}} \otimes VA) \otimes_{\mathcal{O}} \mathcal{M}_A(X)) & \stackrel{\eta_v}{\longrightarrow} & t_*^{\mathcal{F}} \otimes V. \end{array}$$

Since $(t_*^{\mathcal{F}} \otimes VA) \otimes_{\mathcal{O}} \mathcal{M}_A(X)$ is the constant sheaf at $t_*^{\mathcal{F}} \otimes V$, we take η_v to be the identity, and it remains to give a compatible definition for η_n. For this we use the structure map $EA \wedge X \longrightarrow X$ of the EA-module X.

The sheaf $\mathcal{D} \otimes_{\mathcal{O}} \mathcal{M}_A(X)$ is associated to the presheaf given by a tensor product of modules over each open set. By 19.2, the presheaf $\tilde{N}$ is a sheaf with global sections N, so it suffices to construct a map at the presheaf level. More concretely, we need maps

$$\mathcal{D}(U_\pi) \otimes_{\mathcal{O}(U_\pi)} \mathcal{M}_A(X)(U_\pi) \longrightarrow \tilde{N}(U_\pi) = \mathcal{E}_\pi^{-1} N$$

compatible under restriction. Now the domain is the tensor product of $\mathcal{D}(U_\pi)$ and $\mathcal{M}_A(X)(U_\pi) = \mathrm{Hom}_{\mathcal{A}_s}(S^0, X \otimes L(\pi))$. The former can be identified with functions f in NA regular away from π and the latter with elements $x \in \mathcal{E}_\pi^{-1} N$ with $\beta(x) \in c^0 \otimes V$. We map this to $f \cdot x$ in N, and notice that this association is $\mathcal{O}(U_\pi)$ bilinear.

Now if we take $X = EA \wedge S^W$ we find $\mathcal{M}_A(X) = \mathcal{O}(D(W))$ by 19.10 and $\mathcal{S}_A\mathcal{M}_A(X) = X$ by 20.10. We may check the natural transformation is an isomorphism stalkwise. This is obvious for $W = 0$, and for any other value, both $\mathcal{M}_A(X)_s$ and N_s are free on the single element $t_s^{-w(s)}$. $\square$

We may now complete the proof of 21.1.

Proof: We have defined the pair of functors $\mathcal{M}_A$ and $\mathcal{S}_A$ at the level of abelian categories, and hence they preserve actual homotopies at the level of differential graded categories. Accordingly they induce functors at the level of derived categories by replacing objects with approximations using spheres or torsion point line bundles. Since both functors take sphere objects to cellular objects the derived functor construction preserves composites. Hence the functors $\mathcal{M}_A$ and $\mathcal{S}_A$ on derived categories again provide an equivalence. $\square$

Corollary 21.5. *For an EA-module X, there is a short exact sequence*

$$0 \longrightarrow \Sigma H^1(A; \mathcal{M}_A(X)) \longrightarrow \pi_*^{\mathbb{T}}(X) \longrightarrow H^0(A; \mathcal{M}_A(X)) \longrightarrow 0.$$

Proof: Indeed, from the equivalence of categories the cohomology of $\mathcal{M}_A X$ is equal to the homotopy of X. The exact sequence for the cohomology of an object $\mathcal{Y}$ in the derived category of sheaves is obtained from the Adams resolution $\mathcal{Y} \longrightarrow \mathcal{I}_0 \longrightarrow \mathcal{I}_1$, with $\mathcal{I}_j$ flabby. $\square$

22. Relation to Grojnowski's construction.

The first construction of a $\mathbb{T}$-equivariant elliptic cohomology was given by Grojnowski [15]. It is defined for analytic elliptic curves A, and takes values in $\mathbb{Z}/2$-graded sheaves over

A. We first describe Grojnowski's construction and then show that it is related to the sheaf $\mathcal{M}_A F(X, EA)$ in the torsion point topology in the simplest possible way.

22.A. **Grojnowski's construction.** The construction works with an *analytic* elliptic curve A over $\mathbb{C}$, presented as

$$p : \mathbb{C} \longrightarrow \mathbb{C}/\Lambda = A$$

for a lattice $\Lambda \subseteq \mathbb{C}$. To each *finite* $\mathbb{T}$-space X it associates a sheaf $\mathrm{Groj}(X)$ over A in the analytic topology.

An open set U of A is *small* if $p^{-1}U$ is the disjoint union of connected components V such that $p|_V : V \xrightarrow{\cong} U$ is an isomorphism. The construction works with the analytic topology, because the description needs to deal with small open sets. Accordingly we let $\mathcal{O}^{an}$ denote the sheaf of analytic functions on A with the analytic topology.

Next, for a point $a \in A$ we write

$$X^a = \begin{cases} X^{\mathbb{T}[s]} & \text{if } a \text{ is of exact order } s \\ X^{\mathbb{T}} & \text{if } a \text{ is of infinite order,} \end{cases}$$

and we say that a is *generic* if $X^a = X^{\mathbb{T}}$ and a is *special* otherwise.

Finally, we say that an open cover $\{U_a\}_{a \in A}$ of A is *adapted to* X if the following five conditions are satisfied

- $a \in U_a$
- each U_a is small
- if a is special and $a \neq b$ then $a \notin U_a \cap U_b$
- if a and b are both special and $a \neq b$ then $U_a \cap U_b = \emptyset$
- if b is generic, then $U_a \cap U_b$ is non-empty for at most one special a

For any finite $\mathbb{T}$-complex X, there is a cover adapted to X, and any two admit a common refinement. We say that the cover is N-discrete if there is at most one point of order dividing N in any U_a. For any finite $\mathbb{T}$-complex X and any N, there is an N-discrete cover adapted to X, and any two admit a common refinement.

Definition 22.1. (Grojnowski) Given an open cover $\{U_a\}_a$ adapted to X we define $\mathbb{Z}/2$-graded sheaves $\mathrm{Groj}(X)_a$ over U_a by

$$\mathrm{Groj}(X)_a(U) = H^*(E\mathbb{T} \times_{\mathbb{T}} X^a) \otimes_{\mathbb{C}[z]} \mathcal{O}^{an}_A(U - a),$$

where $U - a$ is obtained by translating U by $-a$, and where $\mathcal{O}^{an}(U - a)$ is a $\mathbb{C}[z]$-module since z can be viewed as an analytic function on $U - a$ using p to identify it with a neighbourhood of $0 \in \mathbb{C}$.

These sheaves are compatible on intersections. Indeed, since the cover is adapted to X, we need only observe that the localization theorem gives an isomorphism

$$\mathrm{Groj}(X)_a|_U \cong H^*(E\mathbb{T} \times_{\mathbb{T}} X^{\mathbb{T}}) \otimes_{\mathbb{C}[z]} \mathcal{O}^{an}_A(U - a)$$

when $a \notin U$. The cocycle condition is easily checked, so the sheaves patch to give a sheaf $\mathrm{Groj}(X)$ of $\mathcal{O}^{an}$-algebras. This is independent of the adapted cover, since a refinement induces an isomorphism.

If X has a $\mathbb{T}$-fixed basepoint x_0, the inclusion and projection induce a decomposition

$$\mathrm{Groj}(X) = \widetilde{\mathrm{Groj}}(X) \oplus \mathrm{Groj}(x_0),$$

defining the reduced theory.

Remark 22.2. (i) It is easy to adapt this to give a 2-periodic sheaf valued theory. Indeed, we need only replace $\mathcal{O}^{an}$ by $\Omega^*_{an} = \bigoplus_n \Omega^n_{an}$, and declare that $c \in H^2(B\mathbb{T})$ acts as z/dz. We will do this without change of notation, to allow comparison with our 2-periodic constructions.

(ii) The functor $\mathrm{Groj}(X)$ is exact. Indeed a cofibre sequence $X' \longrightarrow X \longrightarrow X''$ induces a long exact sequence in Borel cohomology of a-fixed points, for each a. Since z is not a zero-divisor as an analytic function, $\otimes_{\mathbb{C}[z]}\mathcal{O}^{an}(U-a)$ preserves exactness. Finally, exactness of sequences of sheaves is detected stalkwise.

22.B. **The derived $\mathcal{M}_A$ functor.** Grojnowski's functor preserves weak equivalences, so we need to apply a homotopy invariant version of the functor $\mathcal{M}_A$. We therefore take $\mathcal{M}_A F(X, EA)$, applying the function spectrum functor rather than the Hom functor. The context makes clear that $\mathcal{M}_A$ is to be interpreted as the total derived functor of the abelian category level functor.

We remark that this gives an exact functor. First note that a cofibre sequence $X' \longrightarrow X \longrightarrow X''$ of based $\mathbb{T}$-spaces induces a fibre sequence $F(X', EA) \longleftarrow F(X, EA) \longleftarrow F(X'', EA)$ in the homotopy category of EA-modules. Applying the total derived functor $\mathcal{M}_A$ we get a triangle in the derived category of $\mathbb{Z}/2$-graded sheaves.

22.C. **Comparison.** In order to make the comparison we need to use the map

$$j : \mathrm{tp} \longrightarrow \mathrm{an}$$

including the sets open in the torsion point topology amongst all open sets. Any sheaf in the analytic topology is a sheaf in the torsion point topology by restriction and a sheaf $\mathcal{Y}$ in the torsion point topology gives a sheaf $j_*\mathcal{Y}$ in the analytic topology via

$$(j_*\mathcal{Y})(U) = \lim_{\rightarrow\, U_\pi \supseteq U} \mathcal{Y}(U_\pi).$$

We also use the map $i : j_*\mathcal{O} \longrightarrow \mathcal{O}^{an}$ of sheaves of rings, giving a map i_* converting $j_*\mathcal{O}$-modules into $\mathcal{O}^{an}$-modules by taking tensor products.

Theorem 22.3. *The 2-periodic version of Grojnowksi's sheaf associated to a finite based $\mathbb{T}$-space X is equivalent to the sheaf arising from the function spectrum $F(X, EA)$:*

$$\widetilde{\mathrm{Groj}}(X) \simeq i_* j_* \mathcal{M}_A(F(X, EA)).$$

Proof: First we construct a natural map

$$\nu_X : i_* j_* \mathcal{M}_A(F(X, EA)) \longrightarrow \widetilde{\mathrm{Groj}}(X)$$

of $\mathcal{O}^{an}$-algebras.

This corresponds to a map

$$\nu'_X : j_* \mathcal{M}_A F(X, EA) \longrightarrow i^* \widetilde{\mathrm{Groj}}(X)$$

of $j_*\mathcal{O}$-algebras. For this we choose a cover $\{U_a\}_{a \in A}$ adapted to X and construct a system of maps

$$\nu'_{X,a} : (j_* \mathcal{M}_A F(X, EA))|_{U_a} \longrightarrow \widetilde{\mathrm{Groj}}(X)_a = H^*(E\mathbb{T}_+ \wedge_{\mathbb{T}} X^a) \otimes_{\mathbb{C}[z]} \mathcal{O}^{an}(U_a - a)$$

compatible as a varies.

Choose g so that all points of order $\geq g$ are generic, and let $N = g!$. Now choose an N-discrete cover $\{U_a\}_{a \in A}$ adapted to X.

Lemma 22.4. *The map $X^a \longrightarrow X$ induces an isomorphism*

$$j_*\mathcal{M}_A F(X, EA)(U_a) \cong j_*\mathcal{M}_A F(X^a, EA)(U_a).$$

Proof: Write $\pi \cap U = \emptyset$ if U contains no points with order in π, so that $\pi \cap U = \emptyset$ if and only if $U \supseteq U_\pi$. We have

$$j_*\mathcal{M}_A F(X, EA)(U_a) = \lim_{\substack{\rightarrow \\ \pi \cap U_a = \emptyset}} \mathcal{M}_A F(X, EA)(U_\pi) = [X, \tilde{E}(H \mid |H| \cap U_a = \emptyset) \wedge EA]^0_{\mathbb{T}}$$

If a is of order s, the quotient X/X^a is built from cells $\mathbb{T}/\mathbb{T}[n]$ with n special and $n \neq s$. Hence $n \cap U_a = \emptyset$, and so the cell makes no contribution to the cohomology. $\square$

Now we may define the natural transformation as a composite

$$\begin{aligned} j_*\mathcal{M}_A F(X, EA)(U_a) = j_*\mathcal{M}_A F(X^a, EA)(U_a) &\longrightarrow j_*\mathcal{M}_A F(E\mathbb{T}_+ \wedge X^a, EA)(U_a) \\ &\xrightarrow{\alpha} H^*(E\mathbb{T}_+ \wedge_{\mathbb{T}} X^a) \otimes_{\mathbb{C}[z]} \mathcal{O}^{an}(U_a - a). \end{aligned}$$

To define α, we use the fact that EA is almost ordinary (in the sense of 13.2), so that

$$F(E\mathbb{T}_+ \wedge X^a, EA) \simeq F(E\mathbb{T}_+ \wedge X^a, E\mathbb{T}_+ \wedge EA) \simeq F(E\mathbb{T}_+ \wedge X^a, E\mathbb{T}_+ \wedge HP).$$

Composing with projection $HP \longrightarrow H$, we may now complete the definition, since there are maps

$$\mathcal{M}_A F(E\mathbb{T}_+ \wedge X^a, H)(U_\pi) \xrightarrow{\alpha} H^*(E\mathbb{T}_+ \wedge_{\mathbb{T}} X^a) \otimes_{\mathbb{C}[z]} \mathcal{O}^{an}(U_a - a)$$

for each π so that $U_\pi \supseteq U$.

There are at least two ways to see that ν_X is an isomorphism for all X. Most directly, we can show that ν_X is an isomorphism on stalks. Passing to limits, over neighbourhoods U_a of a, we find

$$\begin{aligned} j_*\mathcal{M}_A F(X, EA)_a &= \lim_{\substack{\rightarrow \\ U_a}} [X, \tilde{E}(H \mid |H| \cap U_\pi = \emptyset) \wedge EA]^0_{\mathbb{T}} \\ &= [X, \tilde{E}(H \mid |H| \neq s) \wedge EA]^*_{\mathbb{T}} \\ &= EA^*_{\mathbb{T}}(X^a) \otimes \mathcal{O}_s \end{aligned}$$

The completion theorem 13.1 shows what happens when we pass to $E\mathbb{T}_+ \wedge X^a$ and then we extend to analytic germs.

The alternative is to use the fact that both sides are sheaf valued cohomology theories in X. It suffices to check that the natural map is an isomorphism for a class of X sufficient to generate a thick category containing the suspension spectra of all finite complexes.

By definition it suffices to deal with the homogeneous spaces $\mathbb{T}/\mathbb{T}[k]_+$, and the cofibre sequences

$$\mathbb{T}/\mathbb{T}[k]_+ \longrightarrow S^0 \longrightarrow S^{z^k}$$

show it suffices to check that ν is an isomorphism for the spheres S^V. In this case all is well since since $\widetilde{\mathrm{Groj}}(S^V) = \mathcal{O}^{an}(-D(V))$, and $\mathcal{M}_A F(S^V, EA) = \mathcal{O}(-D(V))$. $\square$

In practical terms this gives a means for calculating the cohomology of X using a spectral sequence from the sheaf cohomology of the Grojnowksi sheaf.

Corollary 22.5. *There is a short exact sequence*

$$0 \longrightarrow \Sigma H^1(A; \widetilde{\mathrm{Groj}}X) \longrightarrow EA^*_{\mathbb{T}}(X) \longrightarrow H^0(A; \widetilde{\mathrm{Groj}}X) \longrightarrow 0.$$

Proof: This follows from 21.5 and the fact that the cohomology is unchanged by i_*j_*. $\square$

Appendix A. The affine case: $\mathbb{T}$-equivariant cohomology theories from additive and multiplicative groups.

The algebraic models of equivariant K-theory and Borel cohomology are easily described [9, 13.1, 13.4]. In this section we express the models as special cases of the general functorial construction of a cohomology theory $E\mathbb{G}^*_{\mathbb{T}}(\cdot)$ associated to a one dimensional affine group scheme $\mathbb{G}$ equipped with a coordinate.

The additive group scheme $\mathbb{G}_a$ and the multiplicative group scheme $\mathbb{G}_m$ are affine, and therefore the construction of associated cohomology theories is considerably simpler than that for elliptic curves. It turns out that the associated 2-periodic $\mathbb{T}$-equivariant theories are concentrated in even degrees and

$$(E\mathbb{G}_a)^0_{\mathbb{T}}(X) = H^{ev}(E\mathbb{T} \times_{\mathbb{T}} X)$$

and

$$(E\mathbb{G}_m)^0_{\mathbb{T}}(X) = K^0_{\mathbb{T}}(X),$$

and models for these theories were given in [9]. We will repeat the answer here in our present language.

There are some features that differ from the elliptic case. Once again, we must specify a coordinate y on $\mathbb{G}$, which is a function whose vanishing defines e, or equivalently, a generator of the augmentation ideal $(y) = \ker(\mathcal{O} \longrightarrow k)$. However here we may use the differential dy to generate meromorphic differentials. Next, we must choose functions defining the points of order s for each s. By definition $\mathbb{G}[n]$ is given by the vanishing of $[n](y)$. The cyclotomic functions ϕ_s are defined recursively by $[n](y) = \prod_{s|n} \phi_s$. Once again, $d\phi_s$ need not generate the Kähler differentials. For example if $s = 3$ and we consider the multiplicative group with $y = 1 - z$, then $\phi_3 = 1 + z + z^2$, and $d\phi_3 = (1 + 2z)dz$. Since the zero of $1 + 2z$ is not a point of finite order, the function $1 + 2z$ is not invertible.

Theorem A.1. *Given a commutative 1-dimensional affine group scheme $\mathbb{G}$ over a ring containing $\mathbb{Q}$, and a coordinate y on $\mathbb{G}$ there is a 2-periodic cohomology theory $E\mathbb{G}^*_{\mathbb{T}}(\cdot)$ of type $\mathbb{G}$. Since $\mathbb{G}$ is affine, the cohomology theory is complex oriented, $E\mathbb{G}^*_{\mathbb{T}}$ is in even degrees and $\mathbb{G} = \mathrm{spec}(E\mathbb{G}^0_{\mathbb{T}})$. The construction is natural for isomorphisms of $(\mathbb{G}, y)$.*

The construction is also natural for quotient maps $p : \mathbb{G} \longrightarrow \mathbb{G}/\mathbb{G}[n]$ in the sense that there is a map $p^* : \mathrm{infl}^{\mathbb{T}}_{\mathbb{T}/\mathbb{T}[n]} E(\mathbb{G}/\mathbb{G}[n]) \longrightarrow E\mathbb{G}$ of $\mathbb{T}$-spectra, where $E\mathbb{G}$ is viewed as a $\mathbb{T}/\mathbb{T}[n]$-spectrum and inflated to a $\mathbb{T}$-spectrum, and the coordinate on $\mathbb{G}/\mathbb{G}[n]$ is $\Pi_{a \in \mathbb{G}[n]} T_a y$, where y is the coordinated on $\mathbb{G}$ and T_a denotes translation by a.

Proof: The construction was motivated in Section 2. The idea is that all the ingredients described in Section 4 are implicit in the definition of the type (3.1).

We will write down a rigid even object

$$M_t(E\mathbb{G}) = (t^{\mathcal{F}}_* \otimes V\mathbb{G} \xrightarrow{q} T\mathbb{G})$$

of the torsion category $\mathcal{A}_t$ (i.e., the structure map q will be surjective and $V\mathbb{G}$ and $T\mathbb{G}$ will be in even degrees). By 5.2 this is intrinsically formal and therefore determines

$$M_s(E\mathbb{G}) = (N\mathbb{G} \longrightarrow t^{\mathcal{F}}_* \otimes V\mathbb{G})$$

with $N\mathbb{G} = \ker(q)$, and the representing spectrum $E\mathbb{G}$.

Writing $\mathcal{O} = \mathcal{O}_{\mathbb{G}}$ for the ring of functions on $\mathbb{G}$, in degree 0 we take

$$V\mathbb{G}_0 = \mathcal{O}(\infty tors)$$

and

$$T\mathbb{G}_0 = \mathcal{O}(\infty tors)/\mathcal{O}.$$

For other degrees we twist by ω, taking

$$V_{2n} = V_0 \otimes \omega^n \text{ and } T_{2n} = T_0 \otimes \omega^n.$$

According to 4.5, the map $q : t_*^{\mathcal{F}} \otimes V\mathbb{G} \longrightarrow T\mathbb{G}$ may be described compactly by giving its idempotent summands. We take

$$q(c^{w(s)} \otimes \alpha)_s := \overline{(\frac{\phi_s(y)}{dy})^{w(s)}\alpha}$$

A choice of coordinate y gives a generator $dy^{\otimes n}$ of ω^n, and multiplication by dy gives an isomorphism $\omega^n \longrightarrow \omega^{(n+1)}$. Now $\alpha \in V_{2n}$ can be written

$$\alpha = f \cdot (dy)^{\otimes n}$$

for some function $f \in \mathcal{O}(\infty tors)$ and

$$q(c^{w(s)} \otimes f \cdot (dy)^{\otimes n})_s := \overline{\phi_s(y)^{w(s)} f} \cdot [(dy)^{\otimes(n-w(s))}].$$

Since any function f only has finitely many poles, we see that this does map into the direct sum $T\mathbb{G} = \bigoplus_s e_s T\mathbb{G}$.

We must explain how $T\mathbb{G}$ is a module over R, and why q is a map of R-modules. We make $T\mathbb{G}$ into a module over R by letting c_s act as $\phi_s(y)/dy$ on $e_s T\mathbb{G}$. Since poles are of finite order, $T\mathbb{G}$ is a $\mathcal{E}$-torsion module. The definition of the map q shows it is an R-map.

Finally, we must show that the homotopy groups of the resulting object are as required in 3.1. By 5.2 we have $M_s(E\mathbb{G}) = (\beta : N\mathbb{G} \longrightarrow t_*^{\mathcal{F}} \otimes V\mathbb{G})$, where $N\mathbb{G} = \ker(t_*^{\mathcal{F}} \otimes V\mathbb{G} \longrightarrow T\mathbb{G})$, and we need to calculate

$$[S^W, E\mathbb{G}]_*^{\mathbb{T}} = [S^w, M_s(E\mathbb{G})]_*.$$

Since q is epimorphic, β is monomorphic, and $T\mathbb{G}$ is injective. Thus by 5.2 we have the explicit injective resolution

$$0 \longrightarrow M_s(E\mathbb{G}) = \begin{pmatrix} N\mathbb{G} \\ \downarrow \\ t_*^{\mathcal{F}} \otimes V\mathbb{G} \end{pmatrix} \longrightarrow \begin{pmatrix} t_*^{\mathcal{F}} \otimes V\mathbb{G} \\ \downarrow \\ t_*^{\mathcal{F}} \otimes V\mathbb{G} \end{pmatrix} \longrightarrow \begin{pmatrix} T\mathbb{G} \\ \downarrow \\ 0 \end{pmatrix} \longrightarrow 0.$$

Now, applying 5.4 with $w(\mathbb{T}) = 0$ we obtain the exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{A}_s}(S^w, M_s(E\mathbb{G})) \longrightarrow c^{-w} \otimes V\mathbb{G}_0 \xrightarrow{q^{-w}} (\Sigma^{-w} T\mathbb{G})_0 \longrightarrow \mathrm{Ext}_{\mathcal{A}_s}(S^w, M_s(E\mathbb{G})) \longrightarrow 0.$$

Hence $\mathrm{Ext}(S^w, M_s(E\mathbb{G})) = 0$ since $q : c^{-w} \otimes V\mathbb{G} \longrightarrow T\mathbb{G}$ is surjective. Indeed, any torsion element $t \in (\Sigma^w T\mathbb{G})_0 \cong \mathcal{O}(\infty tors)/\mathcal{O}$ lifts to $f \in \mathcal{O}(\infty tors)$ and hence to $1/c^W \otimes \chi(W) f$. It is immediate from the definition that if $w(\mathbb{T}) = 0$,

$$\mathrm{Hom}(S^w, M_s(E\mathbb{G})) = \{c^{-w} \otimes f \mid f/\chi(W) \text{ regular }\}.$$

By construction the divisor associated to the function $\chi(V)$ is $D(V)$, so $f/\chi(V)$ is regular if and only if $f \in \mathcal{O}(-D(V))$ as required.

For the statement about isogenies, note that if y is a coordinate on $\mathbb{G}$ then its norm $\Pi_{a\in\mathbb{G}[n]}T_a y$ is a coordinate on $\mathbb{G}/\mathbb{G}[n]$ (where T_a denotes translation by a). Using these coordinates, we obtain equivariant spectra $E\mathbb{G}/\mathbb{G}[n]$ and $E\mathbb{G}$. As a first step to maps between them, note that we have maps $p_V^*: V(\mathbb{G}/\mathbb{G}[n]) \longrightarrow V\mathbb{G}$ and $p_T^*: T(\mathbb{G}/\mathbb{G}[n]) \longrightarrow T\mathbb{G}$ corresponding to pullback of functions. However p_V^* and p_T^* do not give a map of $\mathbb{T}$-spectra $E(\mathbb{G}/\mathbb{G}[n]) \longrightarrow E\mathbb{G}$; for example the non-equivariant part of $E(\mathbb{G}/\mathbb{G}[n])$ corresponds to functions on $\mathbb{G}/\mathbb{G}[n]$ with support at the identity, and these pull back to functions on $\mathbb{G}$ supported on $\mathbb{G}[n]$, which correspond to the part of $E\mathbb{G}$ with isotropy contained in $\mathbb{T}[n]$. The answer is to view the circle of equivariance of $E\mathbb{G}$ as $\mathbb{T}/\mathbb{T}[n]$, and then to use the inflation functor studied in Chapters 10 and 24 of [9] to obtain a $\mathbb{T}$-spectrum.

□

Remark A.2. In the above proof we made use of the fact that the Euler class $\chi(W)$ exists as a function in $\mathcal{K}$. This should be contrasted with the elliptic case, where the Euler class is given by different functions at different points. This corresponds to the fact that elliptic cohomology is not complex orientable, so that the bundle specified by W is not trivializable.

We make the construction explicit in four cases. Because the differentials occur in the same way for all s, this has been omitted in the examples, and the map q translated to degree 0.

Example A.3. *(The additive group.)* The ring of functions on $\mathbb{G}_a$ is $\mathbb{Q}[x]$, and the group structure is defined by the coproduct $x \longmapsto 1\otimes x + x\otimes 1$. We choose x as a coordinate about the identity, zero. The group $\mathbb{G}_a[n]$ of points of order dividing n is defined by the vanishing of $\chi(z^n) = nx$, so the identity is the only element of finite order over $\mathbb{Q}$-algebras. This case becomes rather degenerate in that it only detects isotropy 1 and $\mathbb{T}$.

The cohomology theory associated to $\mathbb{G}_a$ is 2-periodic Borel cohomology. This is complex orientable, concentrated in even degrees and in each even degree is the map

$$t_*^{\mathcal{F}} \otimes \mathcal{O}(\infty tors) = t_*^{\mathcal{F}} \otimes \mathbb{Q}[x, x^{-1}] \longrightarrow \mathbb{Q}[x, x^{-1}]/\mathbb{Q}[x] = \mathcal{O}(\infty tors)/\mathcal{O}$$
$$s/e(V) \otimes f \longmapsto s \cdot \overline{f/\chi(V)}.$$

Here $\mathcal{O} = \mathbb{Q}[x]$ and $\chi(z^n) = nx$. The ring $\mathcal{O}(\infty tors) = \mathbb{Q}[x, x^{-1}]$ of functions with poles only at points of finite order is obtained by inverting the Euler class of z. □

Example A.4. *(The multiplicative group.)*[9, 13.4.4] The multiplicative group is defined by $\mathbb{G}_m(k) = \text{Units}(k)$ with group structure given by the product. Accordingly, the ring of functions on $\mathbb{G}_m$ is $\mathcal{O} = R(\mathbb{T}) = \mathbb{Q}[z, z^{-1}]$, and the group structure is defined by the coproduct $z \longmapsto z \otimes z$. We choose $y = 1 - z$ as a coordinate about the identity element, 1. The coproduct then takes the more familiar form $y \longmapsto 1 \otimes y + y \otimes 1 - y \otimes y$. The group $\mathbb{G}_m[n]$ of points of order dividing n is defined by the vanishing of $\chi(z^n) = 1 - z^n$.

The cohomology theory associated to $\mathbb{G}_m$ is equivariant K-theory. This is complex oriented, concentrated in even degrees and in each even degree is the map

$$t_*^{\mathcal{F}} \otimes \mathcal{O}(\infty tors) \longrightarrow \mathcal{O}(\infty tors)/\mathcal{O}$$
$$s/e(V) \otimes f \longmapsto s \cdot \overline{f/\chi(V)}.$$

Here $\mathcal{O} = \mathbb{Q}[z, z^{-1}]$ and $\chi(z^n) = 1 - z^n$. The ring $\mathcal{O}(\infty tors)$ of functions with poles only at points of finite order is obtained by inverting all Euler classes. $\square$

Example A.5. *(The non-split one dimensional torus.)* The ring of functions on the non-split (non-deployé) torus $\mathbb{G}_{nd}$ is $\mathcal{O} = \mathbb{Q}[a, b]/(a^2+b^2 = 1)$. Once one adjoins an element i with $i^2 = -1$, this becomes equivalent to the multiplicative group (also known as the standard torus). Indeed, we may take $z = a + ib$ to see the equivalence. From the usual multiplication rule for complex numbers we see that the coproduct is given by $a \longmapsto a \otimes a - b \otimes b$ and $b \longmapsto a \otimes b + b \otimes a$. The ideal $(1 - a, b)$ of functions vanishing at 0 is not principal, so there is no coordinate in the previous sense.

Since there is no coordinate, a cohomology theory of type $\mathbb{G}_{nd}$ cannot be complex orientable. For example the map $S^0 \longrightarrow S^z$ induces the inclusion $\mathcal{O} \longleftarrow \mathcal{O}(-(e))$ of functions vanishing at the identity. Hence the cohomology of S^z would not be a free module of rank 1.

It is standard that $\mathbb{G}_{nd}$ can be revovered from $\mathbb{G}_m$ over $\mathbb{Q}(i)$ using an action of C_2. Indeed, C_2 acts on $\mathcal{O} = \mathbb{Q}(i)[z, z^{-1}]$ by the Galois action on $\mathbb{Q}(i)$ and by exchanging z with z^{-1}. Thus $a = (z + z^{-1})/2$ and $b = i(z - z^{-1})/2$ are fixed. The coordinate $y = 1 - z$ is not fixed, although $1 - a = z^{-1}(1 - z)^2$ is fixed. Because the coordinate y is not fixed, the action of C_2 on $\mathcal{O}$ does not extend to an action on $K\mathbb{Q}(i)$.

We may construct a theory of type $\mathbb{G}_{nd}$ in the usual way. We let S denote the multiplicative set of functions f with zeroes only at points of finite order, and take $\mathcal{K} = S^{-1}\mathcal{O}$. Now take $V_0 = \mathcal{K}$ and $T_0 = \bigoplus_s H^1_{\mathbb{G}_{nd}\langle s\rangle}(\mathcal{O})$. Here $H^1_{\mathbb{G}_{nd}\langle s\rangle}(\mathcal{O})$ is the local cohomology for the ideal of functions vanishing at points of order exactly s. Now as before we define

$$q : t_*^{\mathcal{F}} \otimes V_0 \longrightarrow T_0.$$

For this we need to know that $\mathbb{G}_{nd}\langle s\rangle$ is *essentially* defined by a principal ideal (generated by ϕ_s say), so that we may define

$$q(c_s^{w(s)} \otimes f)_s = \overline{(\phi_s)^{w(s)} f}.$$

The point is that even though $\mathbb{G}_{nd}\langle s\rangle$ is not itself defined by a principal ideal, it is in the appropriate local ring. For instance the ideal of functions vanishing at the identity is $(1-a, b)$. This is not a principal ideal, but at the level of local cohomology we have

$$H^1_{(1-a,b)}(\mathcal{O}) = H^1_{(1-a,b)}(\mathcal{O}_{(1-a,b)}) = H^1_{(1-a)}(\mathcal{O}_{(1-a,b)}),$$

where the second equality follows since

$$(1 - a, b) = \sqrt{(1 - a)} \text{ in } \mathcal{O}_{(a-1,b)},$$

as one sees explicitly from the equation $(1 - a)(1 + a) = b^2$. We therefore take $\phi_1 = 1 - a$ and define ϕ_s recursively by the equation

$$n^*\phi_1 = \prod_{s|n} \phi_s.$$

Example A.6. *(Formal groups.)* By way of completeness we also record the analogue for formal groups. This completes the circle by establishing the universality of the motivation described in Section 2. However, since we must work over $\mathbb{Q}$, there is little difference from the additive group above. Suppose given a commutative one dimensional formal group $\widehat{\mathbb{G}}$ over a ring k containing $\mathbb{Q}$, with a coordinate y. We may identify the ring of functions on $\widehat{\mathbb{G}}$ with $k[[x]]$, and the group structure is the coproduct $x \longmapsto F(x\otimes 1, 1\otimes x)$. The group $\widehat{\mathbb{G}}[n]$ of points of order dividing n is defined by the vanishing of $\chi(z^n) = [n](x)$ so the identity is the only element of finite order over $\mathbb{Q}$-algebras. We may now make the direct analogue of the construction in A.1. This case becomes rather degenerate in that it only detects isotropy 1 and $\mathbb{T}$.

The cohomology theory associated to the formal group of a complex oriented 2-periodic cohomology theory E is the 2-periodic Borel cohomology of E. This is concentrated in even degrees and in each even degree is the map

$$t^{\mathcal{F}}_* \otimes \mathcal{O}(\infty tors) = t^{\mathcal{F}}_* \otimes E^0((x)) \longrightarrow E^0((x))/E^0[[x]] = \mathcal{O}(\infty tors)/\mathcal{O}$$

$$s/e(V) \otimes f \longmapsto s \cdot \overline{f/\chi(V)}.$$

Here $\mathcal{O} = E^0[[x]]$ and $\chi(z^n) = [n](x)$. The ring $\mathcal{O}(\infty tors) = E^0[[x]][1/x] = E^0((x))$ of functions with poles only at points of finite order is obtained by inverting the Euler class of z. $\square$

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