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Contemporary Mathematics

Spectra for commutative algebraists.

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Abstract. The article is designed to explain to commutative algebraists what spectra are, why they were originally defined, and how they can be useful for commutative algebra.

Contents

0. Introduction.

This article grew out of a short series of talks given as part of the MSRI emphasis year on commutative algebra. The purpose is to explain to commutative algebraists what spectra (in the sense of homotopy theory) are, why they were originally defined, and how they can be useful for commutative algebra. An account focusing on applications in commutative algebra rather than foundations can be found in [**13**], and an introduction to the methods of proof can be found in another article in the present volume [**14**].

Historically, it was only after several refinements that spectra sufficiently rigid for the algebraic applications were defined. We will follow a similar path, so it

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will take some time before algebraic examples can be explained. Accordingly, we begin with an overview to explain where we are going. We only intend to give an outline and overview, not a course in homotopy theory: detail will be at a minimum, but we give references at the appropriate points for those who wish to pursue the subject further. As general background references we suggest [**23**] for general homotopy theory leading towards spectra, [**12**] for simplicial homotopy theory and [**19**] for Quillen model categories. A very approachable introduction to spectra is given in [**1**], but most of the applications to commutative algebra have only become possible because of developments since it was written. The main foundational sources for spectra are collected at the start of the bibliography, with letters rather than numbers for their citations.

1. Motivation via the derived category.

Traditional commutative algebra considers commutative rings R and modules over them, but some constructions make it natural to extend further to considering chain complexes of R-modules; the need to consider robust, homotopy invariant properties leads to the derived category $D(R)$. Once we admit chain complexes, it is natural to consider the corresponding multiplicative objects, differential graded algebras. Although it may appear inevitable, the real justification for this process of generalization is the array of naturally occurring examples.

The use of spectra is a logical extension of this process: they allow us to define flexible generalizations of the derived category. Ring spectra extend the notion of rings, module spectra extend the notion of chain complexes, and the homotopy category of module spectra extends the derived category. Many ring theoretic constructions extend to ring spectra, and thus extend the power of commutative algebra to a vast new supply of naturally occurring examples. Even for traditional rings, the new perspective is often enlightening, and thinking in terms of spectra makes a number of new tools available. Once again the only compelling justification for this inexorable process of generalization is the array of naturally occurring examples, some of which we will be described later in this article.

We now rehearse some of the familiar arguments for the derived category of a ring in more detail, so that it can serve as a model for the case of ring spectra.

1.A. Why consider the derived category? The category of R-modules has a lot of structure, but it is rather rigid, and not well designed for dealing with homological invariants and derived functors. The derived category $D(R)$ is designed for working with homological invariants and other properties which are homotopy invariant: it inherits structure from the module category, but in an adapted form.

- **Modules:** Conventional R-modules give objects of the derived category. It therefore contains many familiar objects. On the other hand, it contains many other objects (chain complexes), but all objects of the derived category are constructed from modules.
- **Homological invariants:** Tor, Ext, local cohomology and other homological invariants are represented in $D(R)$ and the derived category $D(R)$ provides a flexible environment for manipulating them. Indeed, one may view the derived category as the universal domain for homological invariants. After the construction of the derived category, homological invariants reappear as pale shadows of the objects which represent them.

This is one reason for including so many new objects in the derived category. Because the homological invariants are now embodied, they may be very conveniently compared and manipulated.

The derived category inherits a lot of useful structure from the category of modules.

Triangulation: In the abelian category of R-modules, kernels, cokernels and exact sequences allow one to measure how close a map is to an isomorphism. Passing to the derived category, short exact sequences give triangles, and the use of triangles gives a way to internalize the deviation from isomorphism.

Sums, products: We work with the unbounded derived category and therefore have all sums and products.

Homotopy direct and inverse limits: In the module category it is useful to be able to construct direct and inverse limits of diagrams of modules. However these are not homotopy invariant constructions: if one varies the diagram by a homotopy, the resulting limit need not be homotopy equivalent to the original one.

The counterparts in the derived category are homotopy direct and inverse limits. Perhaps the most familiar case is that of a sequence

$$\cdots \longrightarrow X_{n-1} \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X_{n+1} \longrightarrow \cdots .$$

One may construct the direct limit as the cokernel of the map

$$1 - f : \bigoplus_n X_n \longrightarrow \bigoplus_n X_n.$$

The *homotopy* direct limit is the next term in the triangle (the mapping cone of $1-f$). Because the direct limit over a sequence is exact, the construction is homotopy invariant, and the direct limit itself provides a model for the homotopy direct limit. Similarly one may construct the inverse limit as the kernel of the map

$$1 - f : \prod_n X_n \longrightarrow \prod_n X_n,$$

and in fact the cokernel is the first right derived functor of the inverse limit. The *homotopy* inverse limit is the previous term in the triangle (the mapping fibre of $1-f$). Because the inverse limit functor is not usually exact, one obtains a short exact sequence

$$0 \longrightarrow \lim_{\leftarrow n}{}^1 H_{i+1}(X_n) \longrightarrow H_i(\operatorname{holim}_{\leftarrow n} X_n) \longrightarrow \lim_{\leftarrow n} H_i(X_n) \longrightarrow 0.$$

One useful example is that this allows one to split all idempotents. Thus if e is an idempotent self-map of X, the corresponding summand is both the homotopy direct limit and the homotopy inverse limit of the sequence $(\cdots \longrightarrow X \xrightarrow{e} X \xrightarrow{e} X \longrightarrow \cdots)$.

1.B. How to construct the derived category. The steps in the construction of the derived category $D(R)$ of a ring or differential graded (DG) ring R may be described as follows. We adopt a somewhat elaborate approach so that it provides a template for the corresponding process for spectra.

Step 0: *Start with graded sets with cartesian product.* This provides the basic environment within which the rest of the construction takes place. However, we need to move to an additive category.

Step 1: *Form the category of graded abelian groups.* This provides a more convenient and algebraic environment. Next we need additional multiplicative structure.

Step 2: *Construct and exploit the tensor product.*

- **Step 2a:** Construct the tensor product $\otimes_{\mathbb{Z}}$.
- **Step 2b:** Define differential graded (DG) abelian groups.
- **Step 2c:** Find the DG-abelian group $\mathbb{Z}$ and recognize DG-abelian groups as DG-$\mathbb{Z}$-modules.

Step 3: *Form the categories of differential graded rings and modules.* First we take a DG-$\mathbb{Z}$-module R with the structure of a ring in the category of DG-$\mathbb{Z}$-modules, and then define modules over it. This constructs the algebraic objects behind the derived category. Finally, we pass to homotopy invariant structures.

Step 4: *Invert homology isomorphisms.* In one sense this is a purely categorical process, but to avoid set theoretic difficulties and to make it accessible to computation, we need to *construct* the category with homology isomorphisms inverted. One way to do this is to restrict to complexes of R-modules which are projective in a suitable sense, and then pass to homotopy; a flexible language for expressing this is that of model categories.

We may summarize this process in the picture

$$\begin{array}{llcl}
(0) & \text{Graded sets} & & \\
 & \downarrow & & \\
(1) & \mathbb{Z}\text{-modules} & & \\
 & \downarrow & & \\
(2) & DG\text{-}\mathbb{Z}\text{-modules} & \longrightarrow & (4)\ \mathrm{Ho}(\mathbb{Z}\text{-mod}) = D(\mathbb{Z}) \\
 & \downarrow & & \\
(3) & R\text{-modules} & \longrightarrow & (4)\ \mathrm{Ho}(R\text{-mod}) = D(R)
\end{array}$$

One of the things to note about this algebraic situation is that there is no direct route from the derived category $D(\mathbb{Z})$ of $\mathbb{Z}$-modules to the derived category $D(R)$ of R-modules. We need R to be an actual DG ring (rather than a ring object in $D(\mathbb{Z})$), and to consider actual R-modules (rather than module objects in $D(\mathbb{Z})$). The technical difficulties of this step for spectra took several decades to overcome.

The rest of the article will sketch how to parallel this development for spectra, with the ring R replaced by a ring *spectrum* and modules over R replaced by module *spectra* over the ring spectrum. First we give a very brief motivation for considering spectra in the first place, and it will not be until Section 4 that it becomes possible to explain what we mean by ring spectra. For the present we speak very informally, not starting to give definitions until Section 3.

2. Why consider spectra?

We will answer the question from the point of view of an algebraic topologist.

To avoid changing later, all our spaces will come equipped with specified basepoints. We write $[X, Y]_{\text{unst}}$ for the set of based homotopy classes of based maps from X to Y and we write $H^*(X)$ for the reduced cohomology of X with integer coefficients. The subscript unst is short for 'unstable'; this is to contrast with 'stable' maps of spectra, described below.

2.A. First Answer. Spectra describe a relatively well behaved part of homotopy theory [**30**]. We will see later that spaces give rise to spectra and, for highly connected spaces, homotopy classes of maps of spaces and of the corresponding spectra coincide.

To be more precise, we need the suspension functor $\Sigma Y := Y \wedge S^1$ where the smash product of based spaces is $X \wedge Y := X \times Y/(X \times \{y_0\} \cup \{x_0\} \times Y)$. If X is a CW-complex, the suspension ΣX is a CW-complex with cells corresponding to those of X, but one dimension higher. Now we define the morphisms in the *Spanier-Whitehead category* by

$$[X, Y] := \lim_{\substack{\rightarrow \\ k}} [\Sigma^k X, \Sigma^k Y]_{\mathrm{unst}},$$

where the limit is over the suspension maps

$$\Sigma : [\Sigma^k X, \Sigma^k Y]_{\mathrm{unst}} \longrightarrow [\Sigma^{k+1} X, \Sigma^{k+1} Y]_{\mathrm{unst}}.$$

An element of this direct limit is called a 'stable' map. In fact $[X, Y]$ is an abelian group, because the first suspension coordinate allows addition by concatenation, and the second suspension coordinate gives room to move the terms added past each other. It will transpire that when X is finite dimensional the group $[X, Y]$ gives the maps from the spectrum associated to X to the spectrum associated to Y. Furthermore, it turns out that the above limit is achieved, and hence the maps of spectra give a very well behaved piece of homotopy theory. To explain this, write *bottom*(Y) for the lowest dimension of a cell in Y and *dim*(Y) for the highest. The Freudenthal suspension theorem states that suspension gives an isomorphism

$$\Sigma : [X, Y]_{\mathrm{unst}} \xrightarrow{\cong} [\Sigma X, \Sigma Y]_{\mathrm{unst}} \text{ if } \dim X \leq 2 \cdot \text{ bottom}(Y) - 2.$$

Thus if X is finite dimensional all the maps in the direct limit system are eventually isomorphic.

One reason for considering stable maps is that the suspension isomorphism

$$H^n(X) \cong H^{n+1}(\Sigma X) \cong H^{n+2}(\Sigma^2 X) \cong \ldots$$

for reduced cohomology shows that it is stable maps that are relevant to cohomology. More precisely, if a stable map $f : X \longrightarrow Y$ is represented by a continuous function $g : \Sigma^k X \longrightarrow \Sigma^k Y$, then f induces a map f^* in cohomology so that the diagram

$$\begin{array}{ccc} H^n(Y) & \xrightarrow{f^*} & H^n(X) \\ \cong\downarrow & & \downarrow\cong \\ H^{n+k}(\Sigma^k Y) & \xrightarrow{g^*} & H^{n+k}(\Sigma^k X) \end{array}$$

commutes.

2.B. Second answer. Cohomological invariants are represented. Indeed (Brown representability [**8**]) any contravariant homotopy functor $E^*(\cdot)$ on spaces which satisfies the Eilenberg-Steenrod axioms (Homotopy, Excision/Suspension, Exactness) and the wedge axiom, is represented by a spectrum E in the sense that

$$E^*(X) = [X, E]^*.$$

This equation introduces Adams's convenient abbreviation whereby the name of the functor $E^*(\cdot)$ on the left has been used to provide the name for the representing spectrum E on the right. The convention is also used in the reverse direction to name the cohomology theory represented by a spectrum which already has a name.

In effect, this gives a way of *constructing* spectra, and hence a way of arguing geometrically with cohomology theories. For example the Yoneda lemma shows that natural transformations of cohomology theories which commute with suspension (*stable cohomology operations*) are represented:

$$\text{Stable cohomology operations}(E^*(\cdot), F^*(\cdot)) = [E, F]^*.$$

In particular the stable operations between $E^*(\cdot)$ and itself form the ring $E^*E = [E, E]^*$.

2.C. Third answer. Naturally occurring invariants occur as homotopy groups of spectra. For example various sorts of bordism, and algebraic K-theory. Similarly, many invariants in geometric topology are defined as homotopy groups of classifying spaces, and very often these spaces are the infinite loop spaces associated to spectra. This applies to Quillen's algebraic K-groups, originally defined as the homotopy groups of the space $BGL(R)^+$: there is a spectrum $K(R)$ with $K_*(R) = \pi_*(K(R))$. Examples from geometric topology include the Whitehead space $Wh(X)$, Waldhausen's K-theory of spaces $A(X)$ [**33**] and the classifying space of the stable mapping class group $B\Gamma_\infty^+$ [**32**]. We will give further details of some of these constructions later.

2.D. Fourth answer. This, finally, is relevant to commutative algebraists. Many of the invariants described above are not just groups, but also rings. In many cases this additional structure is reflected geometrically in the sense that the representing spectra have a product making them into rings (or even commutative rings) in a suitable category of spectra. These spectra *with an appropriate tensor product* provide a context like the derived category.

Several familiar algebraic constructions on rings can then also be applied to ring spectra to give new spectra. For example Hochschild homology and cohomology extends to topological Hochschild homology and cohomology, André-Quillen cohomology of commutative rings extends to topological André-Quillen cohomology of commutative ring spectra, and algebraic K-theory of rings extends to K-theory of ring spectra. We will give further details of some of these constructions later.

3. How to construct spectra (Step 1).

The counterpart to the use of graded sets in Step 0 of algebra is the use of based spaces. This section deals with the Step 1 transition to an additive category (corresponding to the formation of abelian groups in the algebraic case). Based on the discussion of the Freudenthal suspension theorem, the definition of a spectrum is fairly natural. For the present, we take a fairly naive approach, perhaps best reflected in [**Adams**], although the approach in the first few sections of [**LMS(M)**] is more appropriate for later developments.

We begin with the first approximation to a spectrum.

DEFINITION 3.1. A *spectrum* E is a sequence of based spaces E_k for $k \geq 0$ together with structure maps

$$\sigma \colon \Sigma E_k \to E_{k+1}.$$

A map of spectra $f\colon E \to F$ is a sequence of maps so that the squares

$$\begin{array}{ccc} \Sigma E_k & \xrightarrow{\Sigma f_k} & \Sigma F_k \\ \downarrow & & \downarrow \\ E_{k+1} & \xrightarrow{f_{k+1}} & F_{k+1} \end{array}$$

commute for all k.

Remark 3.2. May and others would call this a 'prespectrum', reserving 'spectrum' for the best sort of prespectrum. To avoid conflicts, we will instead add adjectives to restrict the type of spectrum.

Example 3.3. If X is a based space we may define the suspension spectrum $\Sigma^\infty X$ to have kth term $\Sigma^k X$ with the structure maps being the identity.

Remark: It is possible to make a definition of homotopy immediately, but this does not work very well for arbitrary spectra. Nonetheless it will turn out that for finite CW-complexes K, maps out of a suspension spectrum are given by

$$[\Sigma^\infty K, E] = \varinjlim_k [\Sigma^k K, E_k]_{\mathrm{unst}}.$$

In particular

$$\pi_n(E) := [\Sigma^\infty S^n, E] = \varinjlim_k [S^{n+k}, E_k]_{\mathrm{unst}}.$$

For example if $E = \Sigma^\infty L$ for a based space L, we obtain the *stable homotopy groups*

$$\pi_n(\Sigma^\infty L) = \varinjlim_k [\Sigma^k S^n, \Sigma^k L]_{\mathrm{unst}},$$

which coincides with the group of maps $[S^n, L]$ in the Spanier-Whitehead category. By the Freudenthal suspension theorem, this is the common stable value of the groups $[\Sigma^k S^n, \Sigma^k L]_{\mathrm{unst}}$ for large k. Thus spectra have captured stable homotopy groups.

Construction 3.4. We can suspend spectra by any integer r, defining $\Sigma^r E$ by

$$(\Sigma^r E)_k = \begin{cases} E_{k-r} & k-r \geq 0 \\ pt & k-r < 0. \end{cases}$$

Notice that if we ignore the first few terms, Σ^r is inverse to Σ^{-r}. Homotopy groups involve a direct limit and therefore do not see these first few terms. Accordingly, once we invert homotopy isomorphisms, the suspension functor becomes an equivalence of categories. Because suspension is an equivalence, we say that we have a *stable* category.

Example 3.5. In particular we have sphere spectra. We write $\mathbb{S} = \Sigma^\infty S^0$ for the 0-sphere because of its special role, and then define

$$S^r = \Sigma^r \mathbb{S} \quad \text{for all integers } r.$$

Note that S^r now has meaning for a space and a spectrum for $r \geq 0$, but since we have an isomorphism $S^r \cong \Sigma^\infty S^r$ of spectra for $r \geq 0$ the ambiguity is not important. We extend this ambiguity, by often suppressing Σ^∞.

EXAMPLE 3.6. *Eilenberg-MacLane spectra.* An Eilenberg-MacLane space of type (R,k) for a group R and $k \geq 0$ is a CW-complex $K(R,k)$ with $\pi_k(K(R,k)) = R$ and $\pi_n(K(R,k)) = 0$ for $n \neq k$; any two such spaces are homotopy equivalent. It is well known that each cohomology group is represented by an Eilenberg-MacLane space. Indeed, for any CW-complex X, we have $H^k(X;R) = [X, K(R,k)]_{\mathrm{unst}}$. In fact, this sequence of spaces, as k varies, assembles to make a spectrum.

To describe this, first note that the suspension functor Σ is defined by smashing with the circle S^1, so it is left adjoint to the loop functor Ω defined by $\Omega X := \mathrm{map}(S^1, X)$ (based loops, with a suitable topology). In fact there is a homeomorphism

$$\mathrm{map}(\Sigma W, X) = \mathrm{map}(W \wedge S^1, X) \cong \mathrm{map}(W, \mathrm{map}(S^1, X)) = \mathrm{map}(W, \Omega X)$$

This passes to homotopy, so looping shifts homotopy in the sense that $\pi_n(\Omega X) = \pi_{n+1}(X)$. We conclude that there is a homotopy equivalence

$$\tilde{\sigma} : K(R,k) \xrightarrow{\simeq} \Omega K(R,k+1),$$

and hence we may obtain a spectrum

$$HR = \{K(R,k)\}_{k \geq 0}$$

where the bonding map

$$\sigma \colon \Sigma K(R,k) \to K(R,k+1)$$

is adjoint to $\tilde{\sigma}$. Thus we find

$$[\Sigma^r \Sigma^\infty X, HR] = \lim_{\to\, k} [\Sigma^r \Sigma^k X, K(R,k)]_{\mathrm{unst}} = \lim_{\to\, k} H^k(\Sigma^r \Sigma^k X; R) = H^{-r}(X;R).$$

In particular the Eilenberg-MacLane spectrum has homotopy in a single degree like the spaces from which it was built:

$$\pi_k(HR) = \begin{cases} R & k = 0 \\ 0 & k \neq 0. \end{cases}$$

EXAMPLE 3.7. The classification of smooth compact manifolds provided an important motivation for the construction of spectra. Although this may seem too geometric for applications to commutative algebra, rather mysteriously the spectra that arise this way are amongst those with the most algebraic behaviour.

If we consider two n-manifolds to be equivalent if they together form the boundary of an $(n+1)$-manifold (they are 'cobordant') we obtain the set Ω_n^O of cobordism classes of n-manifolds. The superscript O stands for 'orthogonal', and refers to the fact that a bundle over a manifold admits a Riemannian metric and hence the normal bundle of an n-manifold embedded in Euclidean space has a reduction to the orthogonal group. The set Ω_n^O is a group under disjoint union, and taking all n together we obtain a graded commutative ring with product induced by cartesian product of manifolds. The group Ω_n^O may be calculated as the nth homotopy group of a spectrum MO. The idea is that a manifold M^n is determined up to cobordism by specifying an embedding in $\mathbb{R}^{N+n}$ and considering its normal bundle ν. Collapsing the complement of the normal bundle defines the so-called *Thom space* M^ν of ν and the *Pontrjagin-Thom collapse map* $S^{N+n} \longrightarrow M^\nu$. On the other hand, the normal bundle is N-dimensional and thus classified by a map $\xi_\nu : M \longrightarrow BO(N)$, where $BO(N)$ is the classifying space for $O(N)$-bundles with universal bundle γ_N over it. Taking the Thom spaces and composing with the collapse map, we have

$$S^{N+n} \longrightarrow M^{\nu} \longrightarrow BO(N)^{\gamma_N}.$$

By embedding $\mathbb{R}^{N+n}$ in $\mathbb{R}^{N+N'+n}$ these maps for different N may be compared, and as N gets large, the resulting class in

$$\lim_{\to N}[S^{N+n}, BO(N)^{\gamma_N}]_{\mathrm{unst}}$$

is independent of the embedding, and only depends on the cobordism class of M. Furthermore, the manifold M can be recovered up to cobordism by taking the transverse inverse image of the zero section. This motivates the definition of the cobordism spectrum MO.

We take $MO(n) := BO(n)^{\gamma_n}$ and the bonding map is

$$\Sigma MO(n) = BO(n)^{\gamma_n \oplus 1} = BO(n)^{i^*\gamma_{n+1}} \to BO(n+1)^{\gamma_{n+1}} = MO(n+1).$$

The motivating discussion of the Pontrjagin-Thom construction thus proves

$$\pi_n MO = \lim_{\to N}[S^{n+N}, MO(N)]_{\mathrm{unst}} \cong \Omega_n^O.$$

It is by this means that Thom calculated the group Ω_n^O of cobordism classes of n-manifolds [**31**].

There are many variants of this depending on the additional structure on the manifold. Of particular importance are manifolds with a complex structure on their stable normal bundle. The group of bordism classes of these is Ω_*^U (the superscript now refers to the fact that the stable normal bundle has a reduction to a unitary group), and again this is given by the homotopy groups of the Thom spectrum MU, and this allowed Milnor to calculate the complex cobordism ring

$$\Omega_*^U = \pi_* MU = \mathbb{Z}[x_1, x_2, \ldots]$$

where x_i has degree $2i$ [**24**]. The spectrum MU plays a central role in stable homotopy, both conceptually and computationally. It provides a close link with various bits of algebra, and in particular with commutative algebra. The root of this connection is Quillen's theorem [**25**] that the polynomial ring is isomorphic to Lazard's universal ring for one dimensional commutative formal group laws for geometric reasons.

Example 3.8. The theory of vector bundles gives rise to topological K-theory. Indeed, the unreduced complex K-theory of an unbased compact space X is given by

$$K(X) = Gr(\mathbb{C}\text{-bundles over } X),$$

where Gr is the Grothendieck group completion. The reduced theory is defined by $K^0(X) = \ker(K(X) \longrightarrow K(pt))$, and represented by the space $BU \times \mathbb{Z}$ in the sense that

$$K^0(X) = [X, BU \times \mathbb{Z}]_{\mathrm{unst}}.$$

The suspension isomorphism allows one to define $K^{-n}(X)$ for $n \geq 0$, but to give $K^n(X)$ we need Bott periodicity [**7, 2**]. In terms of the cohomology theory, Bott periodicity states $K^{i+2}(X) \cong K^i(X)$, and in terms of representing spaces it states

$$\Omega^2(BU \times \mathbb{Z}) \simeq BU \times \mathbb{Z}.$$

Hence we may define the representing spectrum K by giving it $2n$th term $BU \times \mathbb{Z}$ and 2-fold bonding maps adjoint to the Bott periodicity equivalence $BU \times \mathbb{Z} \xrightarrow{\simeq} \Omega^2(BU \times \mathbb{Z})$. We then find

$$[\Sigma^\infty X, K] = \lim_{\rightarrow k}[\Sigma^{2k} X, BU \times \mathbb{Z}]_{\mathrm{unst}} = [X, BU \times \mathbb{Z}]_{\mathrm{unst}} = K^0(X)$$

Remark 3.9. (a) Spectra with the property $\Omega E_{k+1} \simeq E_k$ for all k are called Ω-*spectra* (sometimes pronounced 'loop spectra'). As we saw for K-theory, it is then especially easy to calculate $[\Sigma^\infty X, E]$ since

$$[\Sigma^{k+1} X, E_{k+1}]_{\mathrm{unst}} \cong [\Sigma^k X, \Omega E_{k+1}]_{\mathrm{unst}} \cong [\Sigma^k X, E_k]_{\mathrm{unst}}$$

and all maps in the limit system are isomorphisms.

In particular

$$\pi_n(E) = \pi_n(E_0) \quad \text{for } n \geq 0$$

and in fact more generally

$$\pi_n(E) = \pi_{n+k}(E_k) \quad \text{for } n + k \geq 0.$$

(b) If X is a Ω-spectrum, the 0th term X_0 has the remarkable property that it is equivalent to a k-fold loop space for each k (indeed, $X_0 \simeq \Omega^k X_k$). Spaces with this property are called Ω^∞*-spaces* (sometimes pronounced 'infinite loop spaces'). The space X_0 does not retain information about negative homotopy groups of X, but if $\pi_i(X) = 0$ for $i < 0$ (we say X is *connective*), and we retain information about *how* it is a k-fold loop space for each k we have essentially recovered the spectrum X. The study of Ω^∞-spaces is equivalent to the category of connective spectra in a certain precise sense.

To get the best formal behaviour, we impose an even stronger condition than being a Ω-spectrum.

Definition 3.10. A *May spectrum* is a spectrum so that the adjoint bonding maps

$$\tilde{\sigma} \colon X_k \xrightarrow{\cong} \Omega X_{k+1}$$

are all homeomorphisms.

Remark 3.11. Spectra in this strong sense are rather rare in nature, but there is a left adjoint

$$L \colon \text{Spectra} \to \text{May spectra}$$

to the inclusion of May spectra in spectra. On reasonable spectra (including those for which the bonding maps are cofibrations) it is given by

$$(LE)_k = \lim_{\rightarrow s} \Omega^s E_{k+s}.$$

For instance

$$(L\Sigma^\infty X)_k = \lim_{\rightarrow s} \Omega^s \Sigma^{k+s} X$$

and we have a version of the Q-construction $(L\Sigma^\infty X)_0 = QX$.

In general we will omit mention of the functor L, for example writing $\Sigma^\infty X$ for the spectrum associated to the suspension spectrum and $\mathbb{S}$ for the 0-sphere May spectrum.

One can then proceed with homotopy theory of May spectra very much as with spaces or forming the derived category. One wants to invert π_*-isomorphisms and work with

$$\text{Spectra}[(\pi_*\text{-isos})^{-1}].$$

To avoid set-theoretic difficulties with categories of fractions, we construct this homotopy category directly. First we define cells and spheres using shifted suspension spectra and then CW-spectra. Since cells are compact in a suitable sense, it is elementary to form CW-approximations. For any spectrum E we may construct a spectrum ΓE from cells, together with a map

$$\Gamma E \to E$$

which is a weak equivalence. It is then a formality that Γ provides a functor in the homotopy category, and it is called the *CW-approximation functor*. Using this construction, we find

$$\text{Spectra}[(\pi_*\text{-isos})^{-1}] \simeq \text{Ho(CW-Spectra)}$$

and this is usually just called the *homotopy category of spectra*.

This has good formal properties like the derived category. It is triangulated, has products, sums and internal homs (function spectra).

4. The smash product (Step 2).

We have now completed Step 1 by constructing a suitable additive category, and we now proceed to Step 2 and endow the category of spectra with additional structure, especially that of an associative and commutative smash product. This is made a little harder because it is necessary to restrict or otherwise adapt the category of spectra that we have found so far.

We would like to form a smash product $E \wedge F$ of spectra E and F from the terms $E_k \wedge F_l$ in some way. In the first instance, we have a doubly indexed collection of spaces, and to make a spectrum out of it we would need to somehow combine all possibilities or select from them. If done too naively, we lose all hope of associativity of the result. There are several approaches to avoiding this problem. We describe three: the EKMM approach, the approach via symmetric spectra, and that via orthogonal spectra. We emphasize that these all give derived categories which are equivalent in a very strong sense [**MMSS**], but as usual each has its own advantages and disadvantages. In each case there is a sphere spectrum $\mathbb{S}$ which is a ring (using the smash product) and the spectra are modules over $\mathbb{S}$.

We begin with the EKMM approach for the same reason one starts homotopy theory with spaces rather than simplicial sets, but (partly because of what is omitted in this account) I suspect that commutative algebraists will prefer the symmetric spectra described in Subsection 4.B below.

4.A. Method 1: EKMM spectra. The acronym refers to Elmendorf, Kriz, Mandell and May [**EKMM**]. They call their category of spectra $\mathbb{S}$-modules, where $\mathbb{S}$ is the sphere spectrum, but this name also describes other categories, so we refer to 'EKMM spectra'.

First, there is a partial solution based on not making choices, sometimes called *coordinate free spectra*. We extend the notation for spheres and suspensions to permit arbitrary real vector spaces, so that S^V denotes the one-point compactification of V and $\Sigma^V X := X \wedge S^V$.

Definition 4.1. (i) A *universe* is a countable dimensional real inner product space. An *indexing space* in a universe $\mathcal{U}$ is a finite dimensional sub inner product space $V \subseteq \mathcal{U}$.

(ii) A *spectrum* E *indexed on* $\mathcal{U}$ is a collection of spaces E_V where V runs through indexing spaces V in $\mathcal{U}$ together with a transitive system of bonding maps

$$\sigma_{V,W} : \Sigma^{W-V} E_V \to E_W$$

whenever $V \subseteq W$, where $W - V$ denotes the orthogonal complement of V in W.

(iii) Such a spectrum is a *May spectrum* if all adjoint bonding maps $\tilde{\sigma} \colon E_V \xrightarrow{\cong} \Omega^{W-V} E_W$ are homeomorphisms.

Remark 4.2. (a) From any cofinal sequence of indexing spaces one may fill in gaps by using suspensions. Hence we consider a spectrum to be specified by such a cofinal sequence.

For example, if we choose a cofinal sequence $\mathbb{R} \subseteq \mathbb{R}^2 \subseteq \cdots \subseteq \mathcal{U}$ with n corresponding to $\mathbb{R}^n$ we can convert a spectrum as in 3.1 into a spectrum indexed on $\mathcal{U}$.

(b) We may also change universes. If $f : \mathcal{U} \longrightarrow \mathcal{V}$ is an isometry, we may use f to convert a spectrum E indexed on $\mathcal{U}$ to a spectrum $f_* E$ indexed on $\mathcal{V}$, by taking $(f_* E)(V) := E(f^{-1} V)$. $\square$

Definition 4.3. Given a spectrum E indexed on $\mathcal{U}$ and a spectrum F indexed on $\mathcal{V}$, one may define the external smash product $E \overline{\wedge} F$ indexed on $\mathcal{U} \oplus \mathcal{V}$ by taking

$$(E \overline{\wedge} F)(U \oplus V) := E(U) \wedge F(V)$$

on the cofinal sequence of indexing spaces of the form $U \oplus V$.

The merit of the definition is that no choices are involved. Thus if G is a spectrum indexed on $\mathcal{W}$, there is a coherent natural associativity isomorphism

$$(E \overline{\wedge} F) \overline{\wedge} G \cong E \overline{\wedge} (F \overline{\wedge} G)$$

of spectra indexed on $\mathcal{U} \oplus \mathcal{V} \oplus \mathcal{W}$.

The problem is that if E and E' are both indexed on $\mathcal{U}$ then $E \overline{\wedge} E'$ is indexed on $\mathcal{U} \oplus \mathcal{U}$ rather than on $\mathcal{U}$ itself. The old fashioned solution is to *choose* an isometric isomorphism

$$i \colon \mathcal{U} \oplus \mathcal{U} \xrightarrow{\cong} \mathcal{U},$$

and use it to index $E \overline{\wedge} E'$ on $\mathcal{U}$: we define

$$E \wedge_i E' := i_*(E \overline{\wedge} E').$$

This depends on i, but because the space $\mathcal{L}(2) := \mathcal{L}(\mathcal{U} \oplus \mathcal{U}, \mathcal{U})$ of linear isometries is contractible, the choice of i is relatively unimportant, and because the spaces $\mathcal{L}(n) := \mathcal{L}(\mathcal{U}^{\oplus n}, \mathcal{U})$ are contractible for $n \geq 1$, this gives a coherently commutative and associative operation *in the homotopy category*. This method of internalizing the smash product is quite useful, but to obtain the good properties before passing to homotopy we must work a little harder.

The EKMM solution is to use *all* choices. The key to this is the twisted half-smash product construction, which we only describe in general terms.

Construction 4.4. Given
(i) a space A,
(ii) a map $\alpha : A \to \mathcal{L}(\mathcal{U}, \mathcal{V})$, and
(iii) a spectrum E indexed on $\mathcal{U}$,
we may form the *twisted half-smash product* $A \ltimes E$. This is a spectrum indexed on $\mathcal{V}$ formed by assembling the spectra $\alpha(a)_* E$ for all $a \in A$.

The twisted half-smash product is natural for maps of A and E. It is also homotopy invariant in the strong sense that the homotopies need not be compatible with the structure maps α.

Example 4.5. (a) If we choose the one point space, we recover the earlier change of universe construction. If $A = \{i\} \subseteq \mathcal{L}(\mathcal{U}, \mathcal{V})$ then $\{i\} \ltimes E = i_* E$

(b) If we take $A = \mathcal{L}(\mathcal{U}^{\oplus 2}, \mathcal{U})$ and let α be the identity we obtain a canonical way to internalize a smash product. We may take

$$E \wedge' E' := \mathcal{L}(\mathcal{U}^{\oplus 2}, \mathcal{U}) \ltimes (E \overline{\wedge} E').$$

By the naturality, all choices of $\wedge_i$ are contained in this, but it is still a bit too big to be associative. □

Restricting attention to spectra with a little extra structure, one may remove some flab from this smash product and make an associative one.

Definition 4.6. An $\mathbb{L}$-spectrum is a May spectrum E with an action $\mathbb{L}E \longrightarrow E$, where $\mathbb{L}$ is the functor defined by $\mathbb{L}E := \mathcal{L}(1) \ltimes E$. We may view this as a continuous family of maps $f_* E \to E$ where $f \in \mathcal{L}(\mathcal{U}, \mathcal{U})$, compatible with composition.

There are plenty of examples of $\mathbb{L}$-spectra. For example the sphere spectrum $\mathbb{S}$ is an $\mathbb{L}$-spectrum, as is any suspension spectrum. In general, any spectrum E, is homotopy equivalent to the $\mathbb{L}$-spectrum $\mathbb{L}E$ (since $\mathcal{L}(1)$ is contractible).

Definition 4.7. The smash product of $\mathbb{L}$-spectra M, N is then defined by

$$M \wedge_{\mathcal{L}} N := \mathcal{L}(2) \ltimes_{\mathcal{L}(1) \times \mathcal{L}(1)} (M \overline{\wedge} N).$$

More precisely, it is the coequalizer

$$(\mathcal{L}(2) \times \mathcal{L}(1) \times \mathcal{L}(1)) \ltimes (M \overline{\wedge} N) \rightrightarrows \mathcal{L}(2) \ltimes (M \overline{\wedge} N) \longrightarrow M \wedge_{\mathcal{L}} N$$

using the maps

$$(\theta, \varphi, \psi) \longmapsto (\theta \circ (\varphi \oplus \psi), M, N)$$

and

$$(\theta, \varphi, \psi) \longmapsto (\theta, \varphi_* M, \psi_* N).$$

This finally gives a good smash product.

Proposition 4.8. *(Hopkins) The smash product $\wedge_{\mathcal{L}}$ is commutative and associative.*

Remark 4.9. This proposition is a formal consequence of two key features of $\mathcal{L}$:

(1) $\mathcal{L}(i+j) \cong \mathcal{L}(2) \times_{\mathcal{L}(1)\times\mathcal{L}(1)} \mathcal{L}(i) \times \mathcal{L}(j)$
(2) $\mathcal{L}(2)/\mathcal{L}(1) \times \mathcal{L}(1) = *$.

Building on these, we may also rearrange the iterated product

$$M_1 \wedge_{\mathcal{L}} \cdots \wedge_{\mathcal{L}} M_n \cong \mathcal{L}(j) \ltimes_{\mathcal{L}(1)^j} (M_1 \overline{\wedge} \cdots \overline{\wedge} M_n).$$

This is useful in recognizing monoids and commutative monoids.

It is convenient to ensure that $\mathbb{S}$ is itself the unit for the smash product, so we restrict attention to the category of $\mathbb{S}$-modules (i.e., $\mathbb{L}$-spectra for which the natural weak equivalence $\mathbb{S} \wedge_{\mathcal{L}} M \xrightarrow{\simeq} M$ is actually an isomorphism). Since every $\mathbb{L}$-spectrum E is weakly equivalent to the $\mathbb{S}$-module $\mathbb{S} \wedge_{\mathcal{L}} E$, and since the smash product preserves $\mathbb{S}$-modules, this is no real restriction.

4.B. Method 2: symmetric spectra. This method is due to Jeff Smith, with full homotopical details published in [**HSS**]. It gives a more elementary and combinatorial construction of a symmetric monoidal category of spectra, but the construction of the *homotopy* category is much more indirect and requires fluency with Quillen model categories. This is directly analogous to the situation for spaces. Most people find it more intuitive to work with actual topological spaces with homotopies being continuous one-parameter families of maps, and to restrict to CW-complexes to obtain a well-behaved homotopy category. However one may construct the homotopy category using simplicial sets instead. This gives a purely combinatorial model with some superior formal properties, but the construction of the homotopy category requires considerable work. Because of these superior properties, it is usual to base symmetric spectra on simplicial sets (i.e., in Step 0) rather than on topological spaces.

Definition 4.10. (a) A *symmetric sequence* is a sequence

$$E_0, E_1, E_2, \ldots,$$

of pointed simplicial sets with basepoint preserving action of the symmetric group Σ_n on E_n.

(b) We may define a tensor product $E \otimes F$ of symmetric sequences E and F by

$$(E \otimes F)_n := \bigvee_{p+q=n} (\Sigma_n)_+ \wedge_{\Sigma_p \times \Sigma_q} (X_p \wedge Y_q),$$

where the subscript $+$ indicates the addition of a disjoint basepoint.

It is elementary to check that this has the required formal behaviour.

Lemma 4.11. *The product $\otimes$ is symmetric monoidal with unit*

$$(S^0, *, *, *, \ldots).$$

Example 4.12. The sphere is the symmetric sequence $\mathbb{S} := (S^0, S^1, S^2, \ldots)$. Here $S^1 = \Delta^1/\partial\Delta^1$ is the simplicial circle and the higher simplicial spheres are defined by taking smash powers, so that $S^n = (S^1)^{\wedge n}$; this also explains the actions of the symmetric groups.

It is easy to check that *the sphere is a commutative monoid* in the category of symmetric sequences.

Definition 4.13. A *symmetric spectrum* E is a left $\mathbb{S}$-module in symmetric sequences.

Unwrapping the definition, we see that this means E is given by

(1) a sequence $E_0, E_1, E_2, \ldots$ of simplicial sets,

(2) maps $\sigma\colon S^1 \wedge X_n \to X_{n+1}$, and

(3) basepoint preserving left actions of Σ_n on X_n which are compatible with the actions in the sense that the composite maps $S^p \wedge X_n \to X_{n+p}$ are $\Sigma_p \times \Sigma_n$ equivariant.

Definition 4.14. The smash product of symmetric spectra is

$$E \wedge_{\mathbb{S}} F := \mathrm{coeq}(E \otimes \mathbb{S} \otimes F \rightrightarrows E \otimes F).$$

Proposition 4.15. *The tensor product over* $\mathbb{S}$ *is a symmetric monoidal product on the category of symmetric spectra.*

It is now easy to give the one example most important to commutative algebraists.

Example 4.16. For any abelian group M, we define the Eilenberg-MacLane symmetric spectra. For a set T we write $M \otimes T$ for the T-indexed sum of copies of M; this is natural for maps of sets and therefore extends to an operation on simplicial sets. We may then define the Eilenberg-MacLane symmetric spectrum $HM := (M \otimes S^0, M \otimes S^1, M \otimes S^2, \ldots)$. It is elementary to check that if R is a commutative ring, then HR is a monoid in the category of $\mathbb{S}$-modules, and if M is an R-module, HM is a module over HR.

We will not spoil the impression of immediate accessibility of symmetric spectra by explaining how to form the associated homotopy category: one needs to restrict to a good class of symmetric spectra and then invert a certain collection of weak equivalences. The weak equivalences are not just homotopy isomorphisms, so this involves some work in the framework of model categories.

4.C. Method 3: orthogonal spectra. Combining the merits of EKMM spectra and symmetric spectra there is a third option [**MMSS, MM**].

For this we let $\mathcal{I}$ denote the category of finite dimensional real inner product spaces; the set of morphisms between a pair of objects forms a topological space, and the composition maps are continuous. For example $\mathcal{I}(U,U)$ is the orthogonal group $O(U)$.

Definition 4.17. An $\mathcal{I}$*-space* is a continuous functor $X : \mathcal{I} \longrightarrow \mathbf{Spaces}_*$ to the category of based spaces.

Notice the large amount of naturality we require: for example $O(U)$ acts on $X(U)$, and an isometry $U \longrightarrow V$ gives a splitting $V = U \oplus V'$ so that $X(U) \longrightarrow X(V) = X(U \oplus V')$ is also $O(U)$-equivariant.

A very important example is the functor $\mathbb{S}$ which takes an inner product space V to its one point compactification S^V.

There is a natural external smash product of $\mathcal{I}$-spaces, so that if X and Y are $\mathcal{I}$-spaces we may form

$$X \overline{\wedge} Y : \mathcal{I} \times \mathcal{I} \longrightarrow \mathbf{Spaces}_*$$

by taking $(X \overline{\wedge} Y)(U,V) := X(U) \wedge Y(V)$.

Definition 4.18. An *orthogonal spectrum* is an $\mathcal{I}$-space X together with a natural map

$$\sigma : X\overline{\wedge}\mathbb{S} \longrightarrow X \circ \oplus$$

so that the evident unit and associativity diagrams commute. Decoding this, we see that the basic structure consists of maps

$$\sigma_{U,V} : X(U) \wedge S^V \longrightarrow X(U \oplus V),$$

and this commutes with the action of $O(U) \times O(V)$.

One may define the objects which play the role of rings without defining the smash product.

Definition 4.19. An *$\mathcal{I}$-functor with smash product* (or $\mathcal{I}$-FSP) is an $\mathcal{I}$-space X with a unit $\eta : \mathbb{S} \longrightarrow X$ and a natural map $\mu : X\overline{\wedge}X \longrightarrow X \circ \oplus$. We require that μ is associative, that η is a unit (and central) in the evident sense. For a commutative $\mathcal{I}$-FSP we impose a commutativity condition on μ.

Note that the unit is given by maps

$$\eta_V : S^V \longrightarrow X(V)$$

and the product μ is given by maps

$$\mu_{U,V} : X(U) \wedge X(V) \longrightarrow X(U \oplus V).$$

Thus, by composition we obtain maps

$$X(U) \wedge S^V \longrightarrow X(U) \wedge X(V) \longrightarrow X(U \oplus V),$$

and one may check that these give an $\mathcal{I}$-FSP the structure of an orthogonal spectrum.

Remark 4.20. The notion of $\mathcal{I}$-FSP is closely related to the FSPs introduced by Bökstedt in algebraic K-theory before a symmetric monoidal smash product was available. An FSP is a functor from simplicial sets to simplicial sets with unit and product. The restriction of an FSP to (simplicial) spheres is analogous to a $\mathcal{I}$-FSP and gives rise to a ring in symmetric spectra. □

To define a smash product one first defines the smash product of $\mathcal{I}$-spaces by using a Kan extension to internalize the product $\overline{\wedge}$ described above. Now observe that $\mathbb{S}$ is a monoid for this product and define the smash product of orthogonal spectra to be the coequalizer

$$X \wedge_{\mathbb{S}} Y := \mathrm{coeq}(X \wedge \mathbb{S} \wedge Y \rightrightarrows X \wedge Y).$$

The monoids for this product are essentially the same as $\mathcal{I}$-FSPs.

As for symmetric spectra, a fair amount of model categorical work is necessary to construct the associated homotopy category, but orthogonal spectra have the advantage that the weak equivalences are the homotopy isomorphisms.

5. Brave new rings.

Once we have a symmetric monoidal product on our chosen category of spectra we can implement the dream of the introduction: choose a ring spectrum $\mathbf{R}$ (i.e., a monoid in the category of spectra), form the category of $\mathbf{R}$-modules or $\mathbf{R}$-algebras and then pass to homotopy. We may then attempt to use algebraic methods and intuitions to study $\mathbf{R}$ and its modules. We use bold face for ring spectra to remind the reader that although the methods are familiar, we are not working in a conventional algebraic context. The 'brave new ring' terminology is due to Waldhausen, and nicely captures both the wonderful possibilities and the denaturing effect of inappropriate generality. Some in the new wave prefer the term 'spectral ring'.

In turning to examples, we remind the reader that the equivalence results of [**MMSS**] mean that we are free to choose the category most convenient for each particular application.

EXAMPLE 5.1. If we are prepared to use symmetric spectra, we already have the example of the Eilenberg-MacLane spectrum $\mathbf{R} = HR$ for a classical commutative ring R.

The construction of the Eilenberg-MacLane symmetric spectra gives a functor R-modules $\longrightarrow HR$-modules and passage to homotopy groups gives a functor $\mathrm{Ho}(HR\text{-mod}) \to R$-modules. It is much less clear that there are similar comparisons of derived categories but in fact the derived categories are equivalent.

THEOREM 5.2. *(Shipley [**29**]) There is a Quillen equivalence between the category of R-modules and the category of HR-modules, and hence in particular a triangulated equivalence*

$$D(R) = Ho(R\text{-}modules) \simeq Ho(HR\text{-}modules) = D(HR)$$

of derived categories. More generally, one may associate a ring spectrum HR to any DG ring R, so that $H_*(R) = \pi_*(HR)$, and the same result holds. □

Thus working with spectra does recover the classical algebraic derived category. However there are plenty more examples.

EXAMPLE 5.3. For any space X and a commutative ring k we may form the function spectrum $\mathbf{R} = \mathrm{map}(\Sigma^\infty X, Hk)$. It is obviously an Hk-module, but using the diagonal on X it is also a commutative Hk-algebra. Certainly

$$\pi_*(\mathrm{map}(\Sigma^\infty X, Hk)) = H^*(X; k),$$

and $\mathbf{R}$ should be viewed as a commutative substitute for the DG algebra of cochains $C^*(X; k)$. Similarly, a map $Y \longrightarrow X$ makes the substitute for $C^*(Y; k)$ into an $\mathbf{R}$-module. The commutative algebra of this ring spectrum $\mathbf{R}$ is extremely interesting ([**9**]) and discussed briefly in Section 7.

EXAMPLE 5.4. If G is a group or a monoid. Then

$$\mathbf{R} = \Sigma^\infty G_+$$

is a monoid, commutative if G is abelian. The case $G = \Omega X$ for a space X is important in geometric topology (here one should use Moore loops to ensure that G is strictly associative).

Example 5.5. We may apply the algebraic K-theory functor to any ring spectrum $\mathbf{R}$ to form a spectrum $K(\mathbf{R})$. If $\mathbf{R}$ is a commutative ring spectrum so is $K(\mathbf{R})$.

This generalizes the classical case in the sense that $K(HR) = K(R)$ (where the right hand side is the version of algebraic K-theory based on finitely generated free modules). Another important example comes from geometric topology: $K(\Sigma^\infty \Omega X_+)$ is Waldhausen's $A(X)$ [**EKMM**, VI.8.2]. The spectrum $A(X)$ embodies a fundamental step in the classification of manifolds [**33**]. The calculation of its homotopy groups can often be approached using the methods described for algebraic K-theory in Subsection 6.B.

To import many of the classical examples we need to decode what is needed to make a commutative $\mathbb{S}$-algebra in the EKMM sense, using Remark 4.9.

Lemma 5.6. [**EKMM**, II.3.6] *A commutative $\mathbb{S}$-algebra is essentially the same as an E_∞-ring spectrum i.e., a spectrum X with maps*

$$\mathcal{L}(\mathcal{U}^k, \mathcal{U}) \wedge X^{\wedge k} \to X$$

with suitable compatibility properties. More precisely, if X is an E_∞-ring spectrum, the weakly equivalent EKMM-spectrum $\mathbb{S} \wedge_{\mathcal{L}} X$ is a commutative $\mathbb{S}$-algebra.

Remark 5.7. (i) The space $\mathcal{L}(\mathcal{U}^k, \mathcal{U})$ is Σ_k-free and contractible, and taken together these spaces form the *linear isometries operad*. Any other sequence

$$\mathcal{O}(0), \mathcal{O}(1), \mathcal{O}(2), \ldots$$

of contractible spaces with free actions of symmetric groups and similar compositions is called an E_∞-operad [**22**]. Up to suitable equivalence, it does not depend which E_∞-operad is used, so that although the linear isometries operad is rather special because of 4.9, using it results in no real loss of generality.

(ii) This method allows an obstruction theoretic approach to constructing $\mathbb{S}$-algebra structures, where the obstruction groups are based on a topological version of Hochschild cohomology (or a topological version of André-Quillen cohomology in the commutative case).

Corollary 5.8. *The following spectra are commutative $\mathbb{S}$-algebras: the bordism spectra MO and MU, the K-theory spectrum K and its connective cover ku.*

Proof for MO**:** We may use the Grassmann model for the classifying space $BO(N)$. In fact for a universe $\mathcal{U}$ we may take $BO(N) = Gr_N(\mathcal{U})$, the space of N-dimensional subspaces of $\mathcal{U}$. Noting that $\mathcal{U} \cong U \oplus \mathcal{U}$ for any indexing subspace U, we have natural maps

$$\begin{array}{ccc}
MO(U)_{\mathcal{U}} \wedge MO(V)_{\mathcal{U}} & & MO(U \oplus V)_{\mathcal{U} \oplus \mathcal{U}} \\
\| & & \| \\
Gr_{|U|}(U \oplus \mathcal{U})^{\gamma_{|U|}} \wedge Gr_{|V|}(V \oplus \mathcal{U})^{\gamma_{|V|}} & \longrightarrow & Gr_{|U \oplus V|}(U \oplus V \oplus \mathcal{U} \oplus \mathcal{U})^{\gamma_{|U \oplus V|}}
\end{array}$$

A choice of isometry $\mathcal{U}^{\oplus 2} \to \mathcal{U}$ gives a map $MO(U \oplus V)_{\mathcal{U} \oplus \mathcal{U}} \longrightarrow MO(U \oplus V)_{\mathcal{U}}$, and assembling these we obtain a map

$$\mathcal{L}(\mathcal{U}^{\oplus 2}, \mathcal{U}) \wedge MO_{\mathcal{U}} \wedge MO_{\mathcal{U}} \to MO_{\mathcal{U}}$$

and similarly for other numbers of factors.

Another way to construct MO as a commuative $\mathbb{S}$-algebra is as an $\mathcal{I}$-FSP. Indeed we may take $MO'(V)$ to be the Thom space of the tautological bundle over $Gr_{|V|}(V \oplus V)$, and then the structure maps are constructed just as above. The inclusions

$$Gr_{|V|}(V \oplus V)^{\gamma_V} \longrightarrow Gr_{|V|}(V \oplus \mathcal{U})^{\gamma_V}$$

give rise to a map $MO' \longrightarrow MO$ of the associated spectra. Since the maps of spaces become more and more highly connected as the dimension of V increases, this shows that $MO' \simeq MO$.

Conclusion: There are many examples of commutative $\mathbb{S}$-algebras.

6. Some algebraic uses of ring spectra.

The main purpose of this article is to introduce spectra, but we want to end by showing they are useful in algebra. Our principal example of commutative algebra is in the next section, but we mention a number of other applications briefly here.

6.A. Topological Hochschild homology and cohomology. Given a k-algebra R with R flat over k, we may define the Hochshild homology and cohomology using homological algebra over $R^e := R \otimes_k R^{op}$, by taking

$$HH_*(R|k) := \mathrm{Tor}_*^{R^e}(R, R)$$

and

$$HH^*(R|k) := \mathrm{Ext}_{R^e}^*(R, R);$$

we have included k in the notation for emphasis, but it is often omitted. We may make precisely parallel definitions for ring spectra. In doing so, we emphasize that all Homs of ring spectra in this article are derived Homs (sometimes written $\mathbf{R}$Hom) and all tensors of ring spectra are derived (sometimes written $\otimes^{\mathbf{L}}$). Because of this, it is no longer necessary to make a flatness hypothesis. If $\mathbf{R}$ is a $\mathbf{k}$-algebra spectrum we may define the topological versions using homological algebra over the ring spectrum $\mathbf{R}^e := \mathbf{R} \wedge_{\mathbf{k}} \mathbf{R}^{op}$, defining the Hochschild homology spectrum by

$$THH_\bullet(\mathbf{R}|\mathbf{k}) := \mathbf{R} \wedge_{\mathbf{R}^e} \mathbf{R}$$

and the topological Hochschild cohomology spectrum by

$$THH^\bullet(\mathbf{R}|\mathbf{k}) := \mathrm{Hom}_{\mathbf{R}^e}(\mathbf{R}, \mathbf{R}).$$

The $\bullet$ subscript and superscript indicates whether homology or cohomology is intended. When $\mathbf{k}$ is omitted in the notation for THH, it is assumed to be the sphere spectrum $\mathbf{k} = \mathbb{S}$; in this case THH was first defined by Bökstedt by other means before good categories of spectra were available. We may obtain purely algebraic topological Hochschild homology and cohomology groups by taking homotopy, so that $THH_*(\mathbf{R}|\mathbf{k}) = \pi_*(THH_\bullet(\mathbf{R}|\mathbf{k}))$ and $THH^*(\mathbf{R}|\mathbf{k}) = \pi_*(THH^\bullet(\mathbf{R}|\mathbf{k}))$. Alternative notations such as $THH(\mathbf{R}|\mathbf{k}) = THH_\bullet(\mathbf{R}|\mathbf{k}) = THH^{\mathbf{k}}(\mathbf{R})$ and $THC(\mathbf{R}|\mathbf{k}) = THH^\bullet(\mathbf{R}|\mathbf{k}) = THH_{\mathbf{k}}(\mathbf{R})$ also occur in the literature, but unfortunately THC may be confused with cyclic homology.

Under flatness hypotheses to ensure $\pi_*(\mathbf{R}^e) = (\pi_*(\mathbf{R}))^e$, there are spectral sequences

$$HH_*(\pi_*(\mathbf{R})|\pi_*(\mathbf{k})) \Rightarrow \pi_*(THH_\bullet(\mathbf{R}|\mathbf{k}))$$

and

$$HH^*(\pi_*(\mathbf{R})|\pi_*(\mathbf{k})) \Rightarrow \pi_*(THH^\bullet(\mathbf{R}|\mathbf{k})).$$

In particular if $\mathbf{R} = HR$ and $\mathbf{k} = Hk$ for a conventional rings R and k with R flat over k, the spectral sequences collapse for dimensional reasons to show that the Hochschild homology and cohomology of R is equal to the topological Hochschild homology and cohomology of HR.

Two uses of the Hochschild groups are to provide invariants for algebraic K-theory and to provide an obstruction theory for extensions of rings; both of these applications have parallel versions in the topological theory. We briefly describe some applications below. There is also a topological version of André-Quillen cohomology [**27, 3, 4**] which can be used to give an obstruction theory for extensions of commutative ring spectra.

6.B. Algebraic K-theory and traces. The algebraic K-theory $K_*(R)$ of a ring R is notoriously hard to calculate, and one method is to use trace maps to attempt to detect K-theory. Bökstedt, Hesselholt, Madsen and others have calculated the p-complete algebraic K-theory of suitable p-adic rings [**6, 17, 21**] using spectral refinements of classical traces. The relevant constructions were first made using Bökstedt's FSPs.

The classical Dennis trace map $K_*(R) \longrightarrow HH_*(R)$ lands in the Hochschild homology of R, and Bökstedt has given a topological version, which is a map $K(R) \longrightarrow THH_\bullet(HR|\mathbb{S})$ of spectra. Taking homotopy of Bökstedt's map gives a refinement of the Dennis trace. However there is more structure to exploit: the cyclic structure of the Hochschild complex gives a circle action on $THH_\bullet(HR|\mathbb{S})$ and the geometry of Bökstedt's map shows it has equivariance properties. The fixed point spectra $THH_\bullet(HR|\mathbb{S})^C$ for finite cyclic groups C are related in the usual way, but also by maps arising from the special 'cyclotomic' nature of the Hochschild complex; taking both structures into account, one may construct a *topological cyclic homology spectrum* $TC(R)$ from these fixed point spectra. The construction of $TC(R)$ from $THH_\bullet(HR|\mathbb{S})$ can be modelled algebraically, and this makes the homotopy groups $TC_*(R)$ relatively accessible to calculation. Because the relationships between fixed point sets correspond to structures in algebraic K-theory, Bökstedt, Hsiang and Madsen [**5**] are able to construct a map

$$trc : K(R) \longrightarrow TC(R)$$

of spectra. Again we may take homotopy to give the *cyclotomic trace* $K_*(R) \longrightarrow TC_*(R)$. This is a very strong invariant, and for certain classes of rings R it is actually a p-adic isomorphism. Indeed, McCarthy [**20**] has shown that the cyclotomic trace always induces a profinite isomorphism of *relative* K-theory. From the known p-adic algebraic K-theory of perfect fields k of characteristic $p > 0$, Madsen and Hesselholt [**17**] deduce that the cyclotomic trace is a p-adic isomorphism in degrees ≥ 0 whenever R is an algebra over the Witt vectors $W(k)$ which is finite as a module. This, combined with calculations of $TC_*(R)$ has been used to calculate $K_*(R)^\wedge_p$ for many complete local rings R, including $R = \mathbb{Z}^\wedge_p$ and truncated polynomial rings $k[x]/(x^n)$, and to prove the Beilinson-Lichtenbaum conjectures on the K-theory of Henselian discrete valuation fields of mixed characteristic [**16**].

6.C. Topological equivalence. Two rings are said to be *derived equivalent* if their derived categories are equivalent as triangulated categories. The best known example is that of Morita equivalence, showing that a ring is derived equivalent to the ring of $n \times n$ matrices over it. Since useful invariants can be constructed from the derived category, the freedom to replace a ring by a derived equivalent ring can be very useful.

For ring spectra, it is natural to consider also the stronger condition that the module categories are Quillen equivalent (this implies that their derived categories are triangulated equivalent, but it is usually a stronger condition). We then say that the ring spectra are *Quillen equivalent*.

Just as any derived equivalence of rings is given by tensoring with a complex of bimodules, any Quillen equivalence between ring spectra is given by smashing with a bimodule spectrum [**28**]. In particular, any Quillen equivalence between DG algebras is given by smashing with a bimodule spectrum, but Dugger and Shipley [**10**] have given an example to show that it need not be given by tensoring with a complex of bimodules. Based on work of Schlichting, they have also given an example to show that derived equivalent ring spectra need not be Quillen equivalent (although derived equivalence and Quillen equivalence agree for ungraded rings).

Two DG algebras are *quasi-isomorphic* if they are related by a chain of homology isomorphisms. Similarly, two ring spectra are *topologically equivalent* if they are related by a chain of homotopy isomorphisms. If the DG algebras are quasi-isomorphic, the associated ring spectra are topologically isomorphic, but Dugger and Shipley have given an example to show that topological equivalence does not imply quasi-isomorphism. Perhaps the best way to think about this is that there is a ring map $\mathbb{S} \longrightarrow H\mathbb{Z}$; viewing a DG $\mathbb{Z}$-algebra as an $H\mathbb{Z}$-algebra, we may view it as a $\mathbb{S}$-algebra by restriction. It is then not surprising that an equivalence of $\mathbb{S}$-algebras need not be an equivalence of $H\mathbb{Z}$-algebras. Since topological equivalence implies Quillen equivalence, this shows that viewing DG algebras as ring spectra can have useful consequences.

There is an obstruction theory for extensions of rings based on Hochschild cohomology, and a parallel theory for extensions of ring spectra based on topological Hochschild cohomology. The Dugger-Shipley example is based on the comparison between algebraic and topological Hochschild cohomology.

7. Local ring spectra.

Finally we turn to the spectral analogue of a commutative Noetherian local ring R with residue field k. In effect we are extending the idea of trying to do commutative algebra entirely in the derived category. When notions can be reformulated in these terms, we gain considerable flexibility.

We consider a map $\mathbf{R} \longrightarrow \mathbf{k}$ of commutative ring spectra, viewed as an analogue of the map from a commutative local ring R to its residue field k. One example is to take $\mathbf{R} = HR \longrightarrow Hk = \mathbf{k}$, and we refer to this as the local algebra example. A second example is to take $\mathbf{R} = C^*(X; k)$ (in the sense of Example 5.3) for a space X and $\mathbf{k} = Hk$, and we refer to this as the example of cochains on a space.

We study the map $\mathbf{R} \longrightarrow \mathbf{k}$ with the eyes of commutative algebra, and illustrate the possibilities with results from [**9**]. The projects of Waldhausen [**34**] and Rognes [**26**] to give an analysis of chromatic stable homotopy theory by using commutative algebra and Galois theory are beyond the scope of these notes.

7.A. Regularity. Serre's characterization of regularity states that if R is a commutative Noetherian local ring with residue field k, then R is regular if and only if k has a finite free resolution by R-modules. In the derived category we can construct the resolution as a complex of R-modules in finitely many steps from R

by completing triangles and passing to direct summands (we say "k is finitely built from R"). This leads to our definition.

DEFINITION 7.1. We say that $\mathbf{R} \longrightarrow \mathbf{k}$ is *regular* if $\mathbf{k}$ can be finitely built from $\mathbf{R}$.

Serre's characterization shows that R is regular if and only if $\mathbf{R} = HR \longrightarrow Hk = \mathbf{k}$ is regular as a ring spectrum, so in the commutative algebra example the new notion agrees with the classical one.

Regularity is an interesting condition for many other examples. In the case of cochains on a space, we consider the special case with $k = \mathbb{F}_p$ for some prime p, and X p-complete. Thus $\mathbf{R} = C^*(X; \mathbb{F}_p)$ and $\mathbf{k} = H\mathbb{F}_p$. It is not hard to see that $\mathbf{R}$ is regular if and only if $H_*(\Omega X; \mathbb{F}_p)$ is finite dimensional.

If G is a finite p-group, $X = BG$ is p-complete and $\Omega BG \simeq G$, so that $\mathbf{R}$ is regular in this case. More generally, for p-complete spaces X, regularity of $\mathbf{R}$ is equivalent to X being the classifying space of a p-compact group in the sense [**11**] of Dwyer-Wilkerson.

7.B. The Gorenstein condition. A commutative Noetherian local ring R is Gorenstein if and only if $\mathrm{Ext}^*_R(k, R)$ is one dimensional as a k-vector space. In the derived category we can restate this as saying that the homology of the (right derived) Hom complex $\mathrm{Hom}_R(k, R)$ is equivalent to a suspension of k. This suggests the definition for ring spectra.

DEFINITION 7.2. We say that $\mathbf{R} \longrightarrow \mathbf{k}$ is *Gorenstein* if there is an equivalence of $\mathbf{R}$-modules $\mathrm{Hom}_{\mathbf{R}}(\mathbf{k}, \mathbf{R}) \simeq \Sigma^a \mathbf{k}$ for some integer a.

Evidently, R is Gorenstein if and only if $\mathbf{R} = HR \longrightarrow Hk = \mathbf{k}$ is Gorenstein in the new sense. However there is an interesting new phenomenon for spectra. We note that $\mathrm{Hom}_{\mathbf{R}}(\mathbf{k}, \mathbf{R})$ admits a *right* action by the (derived) endomorphism ring $\mathcal{E} = \mathrm{Hom}_{\mathbf{R}}(\mathbf{k}, \mathbf{k})$, whereas $\mathbf{k}$ is naturally a *left* $\mathcal{E}$-module. Thus if $\mathbf{R}$ is Gorenstein, $\mathbf{k}$ acquires new structure: that of a right $\mathcal{E}$-module. We want to say that $\mathbf{R}$ is *orientable* if this right action is trivial, but we must pause to give meaning to the notion of triviality.

7.C. Orientability. We say that an $\mathbf{R}$-module I is a *Matlis lift* of $\mathbf{k}$ if it is built from $\mathbf{k}$ using triangles and (arbitrary) coproducts and in addition $\mathrm{Hom}_{\mathbf{R}}(\mathbf{k}, I) \simeq \mathbf{k}$ as $\mathbf{R}$-modules. For example, if R is a local ring, the injective hull $I(k)$ of k is a Matlis lift of k. If $\mathbf{R}$ is a $\mathbf{k}$-algebra then $I = \mathrm{Hom}_{\mathbf{k}}(\mathbf{R}, \mathbf{k})$ is a Matlis lift of $\mathbf{k}$ if it is built from $\mathbf{k}$.

In general there may be several Matlis lifts, or no Matlis lifts at all, but in many circumstances there is a preferred one. The above examples will cover the cases we consider, and we assume that a Matlis lift has been chosen. We use this to define what we mean by the trivial action of $\mathcal{E}$ on $\mathbf{k}$ (i.e., namely the right action of $\mathcal{E}$ on $\mathrm{Hom}_{\mathbf{R}}(\mathbf{k}, I)$).

DEFINITION 7.3. A Gorenstein ring spectrum $\mathbf{R}$ is *orientable* if $\mathrm{Hom}_{\mathbf{R}}(\mathbf{k}, \mathbf{R}) \simeq \Sigma^a \mathrm{Hom}_{\mathbf{R}}(\mathbf{k}, I)$ as right $\mathcal{E}$-modules.

It turns out that for local rings R, there is a unique right $\mathcal{E}$-module structure on $\mathbf{k}$, and hence every Gorenstein commutative ring is orientable as a ring spectrum, and the notion of orientability is irrelevant to classical commutative algebra.

However things are more interesting for the cochain algebra $\mathbf{R} = C^*(X;k)$ on a space X. By Poincaré duality, such a ring spectrum $\mathbf{R}$ is orientably Gorenstein if X is a compact connected manifold orientable over k. More generally, if $k = \mathbb{Z}/2^n$ the ring spectrum $\mathbf{R}$ is Gorenstein if X is a compact connected manifold, and $\mathbf{R}$ is orientable if and only if the manifold X is orientable over k. Indeed, right actions of $\mathcal{E}$ on $\mathbf{k}$ correspond to group homomorphisms $\pi_1(X) \longrightarrow k^\times$, and the Gorenstein action of $\mathcal{E}$ corresponds to the orientation homomorphism for the tangent bundle.

Similarly, the ring spectrum is orientably Gorenstein if $X = BG$ is the classifying space of a finite group, and more generally if G is a compact Lie group with the property that the (adjoint) action of component group on the Lie algebra T_eG is trivial over k. More generally, if $k = \mathbb{Z}/2^n$, the ring spectrum is Gorenstein, and it is orientable if and only if the adjoint action is trivial over k.

To exploit the Gorenstein condition to give structural information, we need to discuss Morita equivalences.

7.D. Morita equivalences. Continuing to let $\mathcal{E} = \mathrm{Hom}_{\mathbf{R}}(\mathbf{k},\mathbf{k})$ denote the (derived) endomorphism ring, we consider the relationship between the derived categories of left $\mathbf{R}$-modules and of right $\mathcal{E}$-modules. We have the adjoint pair

$$T : D(\text{mod-}\mathcal{E}) \rightleftarrows D(\mathbf{R}\text{-mod}) : E$$

defined by

$$T(X) := X \otimes_{\mathcal{E}} \mathbf{k} \text{ and } E(M) := \mathrm{Hom}_{\mathbf{R}}(\mathbf{k}, M).$$

It is easy to see that if $\mathbf{k}$ is finitely built from $\mathbf{R}$, this gives an equivalence between the derived category of $\mathbf{R}$-modules built from $\mathbf{k}$ and the derived category of $\mathcal{E}$-modules. However a variant will be more useful to us. For this we say that $\mathbf{k}$ is *proxy-small* if there is a finite $\mathbf{R}$-module K so that $\mathbf{k}$ is built from K and K is finitely built from $\mathbf{k}$. In the local algebra example with $\mathbf{k} = Hk$ (and k a field), we may take K to correspond to the Koszul complex and see that $\mathbf{k}$ is always proxy-small. The variant then reads as follows.

LEMMA 7.4. *Provided* $\mathbf{k}$ *is proxy-small, the counit* $TEM \longrightarrow M$ *is an equivalence if* M *is built from* $\mathbf{k}$. □

7.E. The local cohomology theorem. Now suppose that $\mathbf{R} \longrightarrow \mathbf{k}$ is proxy-small, and note that it is a formality that we can construct a $\mathbf{k}$-cellular approximation $\Gamma\mathbf{R}$ to $\mathbf{R}$. By definition, the $\mathbf{k}$-cellular approximation is an $\mathbf{R}$-module $\Gamma\mathbf{R}$ built from $\mathbf{k}$ with a map $\Gamma\mathbf{R} \longrightarrow \mathbf{R}$ inducing an equivalence $\mathrm{Hom}_{\mathbf{R}}(\mathbf{k},\mathbf{R}) \simeq \mathrm{Hom}_{\mathbf{R}}(\mathbf{k},\Gamma\mathbf{R})$. In the local algebra example, we can take $\Gamma\mathbf{R}$ to be given by the stable Koszul complex. We may use a stable Koszul complex as a model for the $\mathbf{k}$-cellular approximation more generally, for example if $\pi_*(\mathbf{R})$ is a Noetherian local ring with residue field $\pi_*(\mathbf{k})$, and this shows its homotopy is calculated using local cohomology in that there is a spectral sequence

$$H^*_{\mathfrak{m}}(\pi_*(\mathbf{R})) \Rightarrow \pi_*(\Gamma\mathbf{R}).$$

We can then deduce a valuable duality property from the Gorenstein condition. Indeed, if $\mathbf{R} \longrightarrow \mathbf{k}$ is orientably Gorenstein, we have the equivalences

$$E\Gamma\mathbf{R} = \mathrm{Hom}_{\mathbf{R}}(\mathbf{k},\Gamma\mathbf{R}) \simeq \mathrm{Hom}_{\mathbf{R}}(\mathbf{k},\mathbf{R}) \simeq \Sigma^a \mathrm{Hom}_{\mathbf{R}}(\mathbf{k}, I) = E\Sigma^a I$$

of right $\mathcal{E}$-modules. Now applying Morita theory, we conclude

$$\Gamma\mathbf{R} \simeq TE\Gamma\mathbf{R} \simeq TE\Sigma^a I \simeq \Sigma^a I.$$

For example if $\mathbf{R}$ is a $\mathbf{k}$-algebra with $\mathbf{k} = Hk$ for a field k, we can take $I = \mathrm{Hom}_{\mathbf{k}}(\mathbf{R}, \mathbf{k})$ and conclude that there is a spectral sequence

$$H^*_{\mathfrak{m}}(\pi_*(\mathbf{R})) \Rightarrow H^*(\mathrm{Hom}_{\mathbf{k}}(\mathbf{R}, \mathbf{k})) = \mathrm{Hom}_k(\pi_*\mathbf{R}, k).$$

In particular *if* $\pi_*(\mathbf{R})$ *is Cohen-Macaulay*, this spectral sequence collapses to show *it is also Gorenstein.* In fact one may apply Grothendieck's dual localization process to this spectral sequence and hence conclude that *in any case* $\pi_*(\mathbf{R})$ *is generically Gorenstein* [**15**].

For example we have seen that $C^*(BG)$ is regular if G is a finite p-group; it follows that it is Gorenstein. Since $\pi_*(C^*(BG)) = H^*(BG)$, there is a spectral sequence

$$H^*_{\mathfrak{m}}(H^*(BG)) \Rightarrow H_*(BG),$$

showing that the group cohomology ring $H^*(BG)$ has very special properties, such as being generically Gorenstein.

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