This is the author’s post-print version of an article published in the IMA Journal of Numerical Analysis, 33 (2)

White Rose Research Online URL for this paper:
http://eprints.whiterose.ac.uk/id/eprint/77779

Published article:


http://dx.doi.org/10.1093/imanum/drs022
Finite element analysis for a coupled bulk-surface partial differential equation

CHARLES M. ELLIOTT AND TOM RANNER†
Mathematics Institute, University of Warwick, Coventry CV4 7AL, UK

[Received on 17 January 2012; revised on 28 June 2012]

In this paper, we define a new finite-element method for numerically approximating the solution of a partial differential equation in a bulk region coupled to a surface partial differential equation posed on the boundary of the bulk domain. The key idea is to take a polyhedral approximation of the bulk region consisting of a union of simplices, and use piecewise polynomial boundary faces as an approximation of the surface. Two finite element spaces are defined, one in the bulk region and one on the surface, by taking the set of all continuous functions which are also piecewise polynomial on each bulk simplex or boundary face. We study this method in the context of a model elliptic problem; in particular, we look at well-posedness of the system using a variational formulation, derive perturbation estimates arising from domain approximation, and apply these to find optimal order error estimates. A numerical experiment is described which demonstrates the order of convergence.

Keywords: Surface finite elements; error analysis; bulk-surface elliptic equations

1. Introduction

Coupled bulk-surface partial differential equations arise in many applications; for example, they arise naturally in fluid dynamics and biological applications. This paper studies mathematically a finite element approach to a sample elliptic problem. The method is based on taking a polyhedral approximation of the domain. Given a sufficiently smooth boundary we go on to show error bounds of order $h^k$ in the $H^1$ norm and order $h^{k+1}$ in the $L^2$ norm, where $k$ is the polynomial degree in the underlying finite element space.

1.1 The coupled system

For a bounded domain $\Omega \subset \mathbb{R}^N \ (N = 2, 3)$ with boundary $\Gamma$, we seek solutions $u: \Omega \rightarrow \mathbb{R}$ and $v: \Gamma \rightarrow \mathbb{R}$ of the system

\[
\begin{align*}
-\Delta u + u &= f \text{ in } \Omega, \\
(\alpha u - \beta v) + \frac{\partial u}{\partial n} &= 0 \text{ on } \Gamma, \\
-\Delta_{\Gamma} v + v + \frac{\partial u}{\partial n} &= g \text{ on } \Gamma.
\end{align*}
\]

Here we assume that $\alpha$ and $\beta$ are given positive constants and $f$ and $g$ are known functions on $\Omega$ and $\Gamma$ respectively. We denote by $\Delta_{\Gamma}$ the Laplace–Beltrami operator on $\Gamma$ and by $n$ the outward pointing normal to $\Gamma$.

†Email: T.Ranner@warwick.ac.uk
1.2 Applications

In recent times there has been a great deal of attention paid to problems involving diffusion on a surface, for example Dziuk & Elliott (2007b) and references therein. Of particular interest is cell biology, see, for example, Sbalzarini et al. (2006) and Schwartz et al. (2005). Indeed, cellular metabolism and signalling are mediated in part by trans-membrane receptors that can diffuse in the cell membrane; see Alberta et al. (2002). There are also examples where this surface diffusion is coupled to diffusion in the bulk. For example, fluorescence loss in photobleaching (FLIP) where surface diffusion of a signalling molecule, G-protein RAC, cycles between the cytoplasm (bulk) and cell membrane (surface); see Novak et al. (2007).

The coupling on the surface (1.1b, 1.1c) has been used by Novak et al. (2007). It can be viewed as a linearisation of the more general equation

$$\frac{\partial u}{\partial n} + L(u, v) = 0$$

where $L_u > 0$ and $L_v < 0$ which has been used in Booty & Siegel (2010); Kwon & Derby (2001); Medvedev & Stuchebrukhov (2011); Rätz & Röger (2012) for example. We leave the numerical analysis of more general couplings, the parabolic case, and evolving domains to future work.

1.3 Outline of paper

The paper is laid out as follows. In the second section we will derive a variational form for the equations. The third section looks at existence, uniqueness and regularity of variational solutions. The fourth section focuses on the approximations we make to the geometry of the problem. In the fifth section we develop the finite element method and in the sixth section we will look for error bounds for this method. In the final section we will show some numerical results.

2. Derivation of variational form

2.1 Surface properties

Throughout we will use the notation from Deckelnick et al. (2005). We will assume that $\Gamma$ is a compact $(N - 1)$ dimensional hypersurface without boundary and that $\Gamma$ is $C^2$, so there exists a distance function $d: \mathbb{R}^N \rightarrow \mathbb{R}$ defined by

$$d(x) = \begin{cases} \inf\{|x-y| : y \in \Gamma\} & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \Gamma, \\ \inf\{|x-y| : y \in \Gamma\} & \text{if } x \notin \overline{\Omega}. \end{cases}$$

Since $|\nabla d(x)| \equiv 1$ in a neighbourhood about $\Gamma$, we can define the normal to $\Gamma$ for almost every $x \in \Gamma$ by

$$n(x) = \nabla d(x).$$

It follows that there exists a narrow band $U = \{x \in \mathbb{R}^N : |d(x)| < \delta_{\Gamma}\}$ about $\Gamma$, such that $d \in C^2(U)$, for which we can also define the normal projection $x \mapsto p(x)$ from $U$ onto $\Gamma$ given by the unique solution of

$$x = p(x) + d(x)n(p(x)). \quad (2.1)$$
This is possible by the assumptions above; see, for example, Hildebrant (1982). Note that \( p(x) \) is the closest point to \( x \) on \( \Gamma \), so \( p \) is also the closest point operator. Since this decomposition is unique, we can extend \( n \) to a vector field on all of \( U \) so that \( n(x) = n(p(x)) \).

For a function \( \xi : \Gamma \to \mathbb{R} \) we define its surface gradient to be

\[
\nabla_{\Gamma} \xi := \nabla \xi - (\nabla \xi \cdot n)n,
\]

where \( \nabla \xi \) denotes the gradient with respect to ambient coordinates of an arbitrary extension to \( U \) of \( \xi \). Alternatively, we can denote this relation as

\[
\nabla_{\Gamma} \xi = P \nabla \xi
\]

where \( P \) is symmetric. The Laplace–Beltrami operator is given by the surface divergence of the surface gradient, that is:

\[
\Delta_{\Gamma} \xi := \nabla_{\Gamma} \cdot \nabla_{\Gamma} \xi.
\]

We denote by \( \mathcal{H} = \nabla_{\Gamma} \cdot n \) the mean curvature of \( \Gamma \). For facts about tangential gradients see Gilbarg & Trudinger (1983), chapter 16.

We denote by \( d_o \) the \( (N-1) \) dimensional surface measure on \( \Gamma \). The formula for integration by parts on \( \Gamma \) is given by

\[
\int_{\Gamma} (\nabla_{\Gamma}) \xi \, d_o = - \int_{\Gamma} \xi \, \mathcal{H} n_i \, d_o.
\]

This gives us a surface Green’s formula; for a surface without boundary,

\[
\int_{\Gamma} (-\Delta_{\Gamma}) \xi \, d_o = \int_{\Gamma} \nabla_{\Gamma} \cdot \nabla_{\Gamma} \xi \, d_o. \tag{2.2}
\]

### 2.2 Variational form

We take functions \( \eta, \xi \) in a suitable space of test functions, multiply (1.1a) by \( \eta \) and (1.1c) by \( \xi \), and integrate by parts to get

\[
\int_{\Omega} \nabla u \cdot \nabla \eta + u \eta \, dx - \int_{\Gamma} \eta \, \frac{\partial u}{\partial n} \, d\sigma = \int_{\Omega} f \eta \, dx, \tag{2.3a}
\]

\[
\int_{\Gamma} \nabla_{\Gamma} v \cdot \nabla_{\Gamma} \xi + v \xi \, d\sigma + \int_{\Gamma} \frac{\partial u}{\partial n} \xi \, d\sigma = \int_{\Gamma} g \xi \, d\sigma. \tag{2.3b}
\]

The boundary condition (1.1b) gives us that

\[
- \int_{\Gamma} \frac{\partial u}{\partial n} \, d\sigma = \int_{\Gamma} (\alpha u - \beta v) \eta \, d\sigma \quad \text{and} \quad \int_{\Gamma} \frac{\partial u}{\partial n} \xi \, d\sigma = - \int_{\Gamma} (\alpha u - \beta v) \xi \, d\sigma. \tag{2.4}
\]

We substitute these into (2.3) to get

\[
\int_{\Omega} \nabla u \cdot \nabla \eta + u \eta \, dx + \int_{\Gamma} (\alpha u - \beta v) \eta \, d\sigma = \int_{\Omega} f \eta \, dx \tag{2.5a}
\]

\[
\int_{\Gamma} \nabla_{\Gamma} v \cdot \nabla_{\Gamma} \xi + v \xi \, d\sigma - \int_{\Gamma} (\alpha u - \beta v) \xi \, d\sigma = \int_{\Gamma} g \xi \, d\sigma. \tag{2.5b}
\]

We now take a weighted sum of (2.5a) and (2.5b) to obtain the variational form

\[
\alpha \int_{\Omega} (\nabla u \cdot \nabla \eta + u \eta) \, dx + \beta \int_{\Gamma} (\nabla_{\Gamma} v \cdot \nabla_{\Gamma} \xi + v \xi) \, d\sigma
\]

\[
+ \int_{\Gamma} (\alpha u - \beta v)(\alpha \eta - \beta \xi) \, d\sigma = \alpha \int_{\Omega} f \eta \, dx + \beta \int_{\Gamma} g \xi \, d\sigma. \tag{2.6}
\]
To help with notation later, we will write \( a((u, v), (\eta, \xi)) \) for the left hand side of this equation and \( l((\eta, \xi)) \) for the right hand side. Following this, we split the equations to develop the usual Lax–Milgram techniques (Evans, 1998) to the variational form described in section 2 in order to find a unique solution to (2.10).

We will test this variational form over the space \( H^1(\Omega) \times H^1(\Gamma) \) which we define to be

\[
H^1(\Omega) \times H^1(\Gamma) := \{ (\eta, \xi) \mid \eta \in H^1(\Omega), \xi \in H^1(\Gamma) \}.
\]

We equip this space with the inner product

\[
\langle (\eta, \xi), (\eta', \xi') \rangle_{H^1(\Omega) \times H^1(\Gamma)} = \langle \eta, \eta' \rangle_{H^1(\Omega)} + \langle \xi, \xi' \rangle_{H^1(\Gamma)},
\]

and induced norm given by

\[
\| (\eta, \xi) \|_{H^1(\Omega) \times H^1(\Gamma)} = \left( \| \eta \|_{H^1(\Omega)}^2 + \| \xi \|_{H^1(\Gamma)}^2 \right)^{\frac{1}{2}}.
\]

One may define higher order spaces if \( \Gamma \) is more regular: to define \( H^l(\Omega) \times H^l(\Gamma) \), we require \( \Gamma \) to be \( C^{l, \kappa} \) with \( l \leq j + \kappa \) and \( \kappa = 0, 1 \). See Wloka (1987) for details of how to define the surface Sobolev spaces.

Hence the variational formulation of the problem is to find \((u, v) \in H^1(\Omega) \times H^1(\Gamma)\) such that

\[
a((u, v), (\eta, \xi)) = l((\eta, \xi)) \quad \text{for all} \quad (\eta, \xi) \in H^1(\Omega) \times H^1(\Gamma). \tag{2.10}
\]

### 3. Existence, uniqueness and regularity

In this section, we apply the usual Lax–Milgram techniques (Evans, 1998) to the variational form developed in section 2 in order to find a unique solution to (2.10). Following this, we split the equations to show regularity with respect to the bulk and surface variables independently. To apply these techniques we must show that \( a \) is bounded and coercive and \( l \) is bounded over \( H^1(\Omega) \times H^1(\Gamma) \).

To see that \( a \) is bounded, notice that

\[
a((w, y), (\eta, \xi)) \leq \alpha \| w \|_{H^1(\Omega)} \| \eta \|_{H^1(\Omega)} + \beta \| y \|_{H^1(\Gamma)} \| \xi \|_{H^1(\Gamma)}
\]

\[
\quad + \int_{\Gamma} (\alpha w - \beta y)(\alpha \eta - \beta \xi) \, ds
\]

\[
\leq \sqrt{2} \max\{\alpha, \beta\} \| (w, y) \|_{H^1(\Omega) \times H^1(\Gamma)} \| (\eta, \xi) \|_{H^1(\Omega) \times H^1(\Gamma)}
\]

\[
\quad + 2c_f^2 \max\{\alpha, \beta\}^2 \| (w, y) \|_{H^1(\Omega) \times H^1(\Gamma)} \| (\eta, \xi) \|_{H^1(\Omega) \times H^1(\Gamma)}
\]

\[
\leq c \| (w, y) \|_{H^1(\Omega) \times H^1(\Gamma)} \| (\eta, \xi) \|_{H^1(\Omega) \times H^1(\Gamma)}.
\]

Here, \( c_f \) is the constant from the Trace Theorem, see Evans (1998). Coercivity of \( a \) is immediate since we have

\[
a((\eta, \xi), (\eta, \xi)) = \alpha \| \eta \|_{H^1(\Omega)}^2 + \beta \| \xi \|_{H^1(\Gamma)}^2 + \| \alpha \eta - \beta \xi \|_{L^2(\Gamma)}^2
\]

\[
\geq \sqrt{2} \min\{\alpha, \beta\} \| (\eta, \xi) \|_{H^1(\Omega) \times H^1(\Gamma)}^2.
\]

Hence \( a \) is coercive if \( \alpha, \beta > 0 \). By the Cauchy-Schwarz inequality \( l \) is clearly bounded.
THEOREM 3.1 (Existence and uniqueness) Given \( f \in H^{-1}(\Omega), g \in H^{-1}(\Gamma) \), and \( \alpha, \beta > 0 \), there exists a unique pair \( (u, v) \in H^1(\Omega) \times H^1(\Gamma) \) such that

\[
a((u, v), (\eta, \xi)) = l((\eta, \xi)) \quad \text{for all } (\eta, \xi) \in H^1(\Omega) \times H^1(\Gamma).
\] (3.3)

Furthermore, if \( \Gamma \) is \( C^2 \), we can achieve bounds in the \( H^2 \)-norms by setting restriction the bilinear forms by setting \( \eta = 0 \) and \( \xi = 0 \) in turn.

For \( \eta = 0 \), we get

\[
\beta \int_{\Gamma} \nabla \Gamma v \cdot \nabla \Gamma \xi + v\xi \, d\sigma + \int_{\Gamma} \beta^2 v\xi \, d\sigma = \beta \int_{\Gamma} g\xi \, d\sigma + \int_{\Gamma} \alpha \beta u\xi \, d\sigma.
\] (3.4)

This is exactly the variational form of the equation

\[
-\beta \Delta \Gamma v + (\beta + \beta^2) v = \beta g + \alpha \beta u \quad \text{on } \Gamma.
\] (3.5)

Hence by surface elliptic theory (Aubin, 1982), if \( \Gamma \) is \( C^3 \), we have that \( v \in H^2(\Gamma) \). Since, by the Trace Theorem, \( u \in L^2(\Gamma) \), we have the bound:

\[
\|v\|_{H^2(\Gamma)} \leq c \left( \|g\|_{L^2(\Gamma)} + \|v\|_{L^2(\Gamma)} + \|u\|_{H^1(\Omega)} \right).
\] (3.6)

For \( \xi = 0 \), we get

\[
\alpha \int_{\Omega} \nabla u \cdot \nabla \eta + u\eta \, dx + \int_{\Gamma} \alpha^2 u\eta \, d\sigma = \alpha \int_{\Omega} f\eta \, dx + \int_{\Gamma} \alpha \beta v\eta \, d\sigma.
\] (3.7)

This equation arises as the variational form of the equations

\[
-\alpha \Delta u + \alpha u = \alpha f \quad \text{in } \Omega
\]

\[
\frac{\partial u}{\partial n} + \alpha u = \beta v \quad \text{on } \Gamma
\] (3.8a, 3.8b)

By regularity theory of elliptic problems with Robin boundary data (see Gilbarg & Trudinger (1983); Ladyzhenskaia & Uraltseva (1968)), if \( \Gamma \) is \( C^3 \), we have the following result:

\[
\|u\|_{H^2(\Omega)} \leq c \left( \|f\|_{L^2(\Omega)} + \|v\|_{H^{1/2}(\Gamma)} \right).
\] (3.9)

THEOREM 3.2 (Regularity) If \( \Gamma \) is \( C^3 \), \( f \in L^2(\Omega), g \in L^2(\Gamma) \) and \( \alpha, \beta > 0 \) and \( (u, v) \) solve the variational problem (2.6), then

\[
u \in H^2(\Omega) \quad \text{and} \quad v \in H^2(\Gamma),
\]

and

\[
\|(u, v)\|_{H^2(\Omega) \times H^2(\Gamma)} \leq c \left( \|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma)} \right).
\] (3.10)

4. Domain perturbation and estimates

4.1 Domain approximation

The first step we take in discretising the system (1.1) is to take \( k \)th order approximations \( \Omega_h^{(k)} \) and \( \Gamma_h^{(k)} \) of \( \Omega \) and \( \Gamma \). We follow ideas taken from Lenoir (1986); Bernardi (1989) and Dubois (1990) in order
to define the triangulation of our bulk domain and use results of Dziuk (1988); Dziuk & Elliott (2007b) and Demlow (2009) to make estimates about the perturbation of the boundary of this domain. To prove the results in this section, we will assume $\Gamma$ is $C^{k+1}$. The higher order surface finite element spaces, used here, are described in Heine (2005).

Let $\hat{\Omega}_h$ be a polyhedral approximation of $\Omega$ and $\tilde{\Gamma}_h = \partial \hat{\Omega}_h$. We suppose that the faces of $\tilde{\Gamma}_h$ are $(N-1)$-simplices whose vertices lie on $\Gamma$ so that $\tilde{\Gamma}_h$ is an interpolant of $\Gamma$. We take a quasi-uniform triangulation $\mathcal{H}_h$ of $\hat{\Omega}_h$ (Brenner & Scott, 2002) consisting of closed simplices, either triangles in $\mathbb{R}^2$ or tetrahedra in $\mathbb{R}^3$.

We define $h = \max \{\text{diam}(T) : T \in \mathcal{H}_h\}$ and assume that $h$ is sufficiently small so that $\tilde{\Gamma}_h \subseteq U$ so that for all $x \in \tilde{\Gamma}_h$, there exists a unique point $p = p(x) \in \Gamma$ defined by (2.1). Finally, we assume that for each $T \in \mathcal{H}_h$, $T \cap \tilde{\Gamma}_h$ has at most one face of $\tilde{\Gamma}_h$.

4.1.1 Exact triangulation. In order to define our computational domains, we first define an exact triangulation of $\Omega$. For each simplex $T \in \mathcal{H}_h$, we define an affine function $F_T : \mathbb{R}^N \to \mathbb{R}^N$ which maps the unit $N$-simplex $\hat{T}$ onto $T$ (mapping the vertices of $\hat{T}$ onto the vertices of $T$) which we write as

$$F_T(\hat{x}) = A_T \hat{x} + b_T. \quad (4.1)$$

We say that a closed set $T^\varepsilon$ is a curved $N$-simplex if there exists a $C^1$-mapping $F_T^\varepsilon$ that maps $\hat{T}$ onto $T^\varepsilon$ that is of the form

$$F_T^\varepsilon = F_T + \Phi_T, \quad (4.2)$$

where $F_T$ is the affine map from (4.1) and $\Phi_T$ is a $C^1$-mapping from $\hat{T}$ to $\mathbb{R}^N$ satisfying

$$C_T := \sup_{\hat{x} \in \hat{T}} |D\Phi_T(\hat{x})A_T^{-1}| \leq C < 1. \quad (4.3)$$

From this definition we immediately have the following results:

**Proposition 4.1** If the map $F_T^\varepsilon$ exists, then it is a $C^1$-diffeomorphism from $\hat{T}$ onto $T^\varepsilon$ and satisfies

$$\sup_{\hat{x} \in \hat{T}} |DF_T^\varepsilon(\hat{x})| \leq (1 + C_T) |A_T|,$$

$$\sup_{x \in T^\varepsilon} |D(F_T^\varepsilon)^{-1}(x)| \leq (1 - C_T)^{-1} |A_T^{-1}|,$$

$$(1 - C_T)^N |\det A_T| \leq |\det DF_T^\varepsilon(\hat{x})| \leq (1 + C_T)^N |\det DF_T|$$

for all $\hat{x} \in \hat{T}$.

There are several ways of defining such a $\Phi_T$ given in the literature. Zlamal (1973, 1974) and Scott (1973) considered problems with finite element spaces defined over curved spaces. Scott gives an explicit construction of an exact triangulation in two dimensions which was generalised by Lenoir (1986). Here, in this paper, we use a construction based on Dubois (1990) which uses the normal projection (2.1). We will adopt the notation of Bänsch & Deckelnick (1999); Deckelnick et al. (2009).

Bearing in mind our assumptions on the triangulation, each $T \in \mathcal{H}_h$ is either an internal simplex, with at most one node on $\tilde{\Gamma}_h$, in which case we set $\Phi_T = 0$; or $T$ has more than one node on the boundary. We denote by $l$ the number of nodes of $T$ that lie in $\tilde{\Gamma}_h$ and denote by $\psi_1, \ldots, \psi_{l+1}$ the vertices of $T$, ordered so that $\psi_1, \ldots, \psi_l$ lie on $\tilde{\Gamma}_h$. For each point $x \in T$, we define barycentric coordinates by

$$x = \sum_{j=1}^{N+1} \lambda_j \psi_j$$
and write \( \hat{x} = (\lambda_1, \ldots, \lambda_N) \) for the coordinates in \( \hat{T} \). We next introduce

\[
\lambda^* = \lambda^*(\hat{x}) = \sum_{j=1}^{i} \lambda_j \quad \hat{\sigma} = \{ \hat{x} \in \hat{T} : \lambda^*(\hat{x}) = 0 \}.
\]

In three dimensions this falls into the following cases:

1. \( T \cap \hat{T}_h \) is an edge of a tetrahedron \( (l = 2) \), then \( \hat{\sigma} \) is the inverse image of the edge spanned by \( \psi_3, \psi_4 \) under \( F_T \).

2. \( T \cap \hat{T}_h \) is a face of a tetrahedron \( (l = 3) \), then \( \hat{\sigma} \) is the point \( F_T^{-1}(\psi_4) \).

For \( \hat{x} \not\in \hat{\sigma} \), we denote the projection of \( x \) onto \( \tau \) by \( y = y(\hat{x}) \in \tau \) by

\[
y = \sum_{j=1}^{l} \frac{\lambda_j}{\lambda^*} \psi_j \in \tau.
\]

Then using the normal projection \( p(y) \in \Gamma \) of \( y \) given by (2.1) and we define \( \Phi_T \) by

\[
\Phi_T(\hat{x}) = \begin{cases} 
(\lambda^*)^{l+2} (p(y) - y) & \text{if } \hat{x} \not\in \hat{\sigma} \\
0 & \text{if } \hat{x} \in \hat{\sigma}.
\end{cases}
\] (4.4)

We now follow a sequence of Lemmas from Bernardi (1989) to show that \( \Phi_T \) satisfies (4.3).

**Lemma 4.1** The mapping \( y \) is of class \( C^{k+1} \) on \( \hat{T} \setminus \hat{\sigma} \) and satisfies

\[
\|D^m_T y\|_{L^\infty(\hat{T} \setminus \hat{\sigma})} \leq \frac{c h}{(\lambda^*)^m}, \quad \text{for } 1 \leq m \leq k + 1. \] (4.5)

Lemma 4.2 The mapping $p(y)$ is of class $C^{k+1}$ on $\hat{T} \setminus \hat{\sigma}$ and we have the bound

$$||D^m_x (p(y) - y)||_{L^\infty(\hat{T} \setminus \hat{\sigma})} \leq \frac{c h^2}{(\lambda^*)^m}.$$  \hfill (4.6)

Proof. We remark, using Bernardi (1989) equation (2.9),

$$||D^m_x (p(y) - y)||_{L^\infty(\hat{T} \setminus \hat{\sigma})} \leq c \sum_{r=1}^m \left( ||D^r_y (p(y) - y)||_{L^\infty(\tau)} \prod_{q=1}^m ||D^q_x||_{L^\infty(\hat{T} \setminus \hat{\sigma})} \right)$$

where $\underline{i} = (i_1, \ldots, i_m) \in \mathbb{N}^m$ with

$$\sum_{q=1}^m i_q = r \quad \text{and} \quad \sum_{q=1}^m q_i = m.$$

We notice that $p(y) = y$ if $y = \psi_j$ for any $0 \leq j \leq l$, so $y|_\tau$ can be seen as a linear interpolant of $p(y)$ on $\tau$. Hence, from our geometric assumptions on $T$ (Dziuk, 1988), $||D^r_y (p(y) - y)||_{L^\infty(\tau)} \leq c h^{2-r}$ for $0 \leq r \leq 2$.

Using (4.5) we see, if $m \leq 2$

$$||D^m_x (p(y) - y)||_{L^\infty(\hat{T} \setminus \hat{\sigma})} \leq c \sum_{r=1}^m h^{2-r} h^{\sum_{q=1}^m i_q} (\lambda^*)^{-\sum_{q=1}^m q_i} \leq \frac{c h^2}{(\lambda^*)^m},$$

and if $m > 2$,

$$||D^m_x (p(y) - y)||_{L^\infty(\hat{T} \setminus \hat{\sigma})} \leq c \left( \sum_{r=1}^2 h^{2-r} h^{\sum_{q=1}^m i_q} (\lambda^*)^{-\sum_{q=1}^m q_i} + \sum_{r=3}^m h^{\sum_{q=1}^m q_i} (\lambda^*)^{-\sum_{q=1}^m q_i} \right)$$

$$\leq \frac{c h^2}{(\lambda^*)^m}.$$  \hfill \Box

Proposition 4.2 The mapping $\Phi_T (\hat{x}) = (\lambda^*)^{k+2} (p(y) - y)$ is of class $C^{k+1}$ on $\hat{T}$ and satisfies

$$||D^m \Phi_T||_{L^\infty(\hat{T})} \leq c h^2 \quad \text{for} \ 0 \leq m \leq k+1.$$  \hfill (4.7)

Furthermore, $\Phi_T$ satisfies (4.3).

Proof. Using the Leibniz formula, we have for any $\hat{x}$ in $\hat{T} \setminus \hat{\sigma}$,

$$D_m \Phi_T (\hat{x}) = D^m_x ((\lambda^*)^{k+2} (p(y) - y)) = \sum_{r=0}^m \binom{m}{r} (k+2) \ldots (k+3-r)(\lambda^*)^{k+2-r} (D^r_x\lambda^*)^{r} D_x^{m-r} (p(y) - y),$$

so that applying (4.6)

$$||D^m_x ((\lambda^*)^{k+2} (p(y) - y))||_{L^\infty(\hat{T} \setminus \hat{\sigma})} \leq c \sum_{r=0}^m (\lambda^*)^{k+2-r} \frac{ch^2}{(\lambda^*)^{m-r}} \leq c h^2 (\lambda^*)^{k+2-m}. $$
The mapping $\Phi_T$ is of class $C^{k+1}$ on $\hat{T} \setminus \hat{\sigma}$ with derivatives of order less than or equal to $k+1$ tending to zero when $\hat{x}$ tends to a point in $\hat{\sigma}$. Hence, it can be extended to a $C^{k+1}$-mapping on $\hat{T}$ (Gilbarg & Trudinger, 1983) which satisfies (4.7).

Since $\frac{|\partial \hat{x}_i|}{\partial x_j} \leq c/h$ (Ciarlet & Raviart, 1972a, page 239), we know that

$$|A_T^{-1}| = \frac{c}{h}.$$

This result together with (4.7) shows

$$C_T \leq \sup_{\hat{x} \in \hat{T}} |D\Phi_T(\hat{x})| |A_T^{-1}| \leq ch,$$

hence $\Phi_T$ satisfies (4.3) for $h$ small enough.

**Remark 4.1** Note that we could have chosen $\Phi_T(\hat{x}) = \lambda^* (p(y) - y)$. However this function is not $C^1(T)$, and the interpolation theory of Bernardi (1989) would be unavailable. Our construction is a combination of ideas from Lenoir (1986) and Dubois (1990).

We will call the exact triangulation, defined by $F_{Te}^T$ above, $T_{k}^e$. Notice that under this construction, simplices in $T_{k}^e$, which have more than one vertex on the boundary, can have more than one curved face. See Figure 2 for an example.

### 4.1.2 Computational domain.

We can now define our computational domains $\Omega_h^{(k)}$ and $\Gamma_h^{(k)}$. Let $T \in \mathcal{T}_h$ and $\phi_1^k, \ldots, \phi_n^k$ be a Lagrangian basis of degree $k$ on $\hat{T}$ corresponding to the nodal points $\hat{x}^1, \ldots, \hat{x}^n$. Then for $\hat{x} \in \hat{T}$, we can define a parametrisation of a polynomial simplex $T^{(k)}$ by

$$F_T^{(k)}(\hat{x}) = \sum_{j=1}^{n_k} F_{Te}^{j}(\hat{x}) \phi_j^k(\hat{x}).$$
Proof. Using (4.9), we can write
\[ T^{(k)} := \{ F_T^{(k)}(\hat{x}) : \hat{x} \in \hat{T} \} \quad \text{and} \quad \mathcal{T}_h^{(k)} := \{ T^{(k)} | T \in \hat{T}_h \}. \] (4.8)

Then \( T_h^{(k)} \) is the boundary of the domain \( \Omega_h^{(k)} \) with the triangulation \( \mathcal{T}_h^{(k)} \). This construction admits quasi-uniform triangulations \( \mathcal{T}_h^{(k)} \) and \( \mathcal{T}_h^{(k)} \) for \( \Omega_h^{(k)} \) and \( T_h^{(k)} \) respectively. Notice that, like the exact simplices in \( \mathcal{T}_h^e \), the simplices in \( \mathcal{T}_h^{(k)} \) can have curved (polynomial) faces.

4.2 Bulk estimates

We define a function \( G_h : \Omega_h^{(k)} \to \Omega \) locally by \( G_h|_{T^{(k)}} := F_T \circ (F_T^{(k)})^{-1} \) for each \( T^{(k)} \in \mathcal{T}_h^{(k)} \). This is a homeomorphism, which when restricted to interior simplices (those with at most one vertex on the boundary) is the identity.

We use the notation \( DG_h \) for the gradient of \( G_h \) where \( (DG_h)_{ij} = \frac{\partial}{\partial x_j} (G_h)_i \), and \( DG_h^i \) for its transpose. Also, we will write \( DG_h^{-1} \) for \( D(G_h^{-1}) = (DG_h)^{-1} \). We denote by \( J_h | T \) the absolute value of the determinant of \( DG_h | T \).

We denote by \( B_h \) the union of elements in \( \mathcal{T}_h^{(k)} \) which have more than one vertex on the boundary \( T_h^{(k)} \) and \( B_h^e \) the associated exact elements in \( \mathcal{T}_h^e \). Notice that \( B_h \) is the region where \( G_h \) is different from the identity.

Let us use the notation that for a fixed \( \hat{x} \in \hat{T} \), we denote \( F_T^{(k)}(\hat{x}) = x \), then one may write that
\[ G_h(x) = F_T^{-1} \circ (F_T^{(k)})^{-1}(x) = F_T^{-1}(\hat{x}) + (F_T^{-1}(\hat{x}) - F_T^{(k)}(\hat{x})). \] (4.9)

**Lemma 4.3** If \( \Gamma \) is \( C^{k+1} \) then \( G_h|_{T} \in C^{k+1}(T^{(k)}) \) for each \( T^{(k)} \in \mathcal{T}_h^{(k)} \) and we have that \( \| G_h \|_{W^{k+1,\infty}(T^{(k)})} \) is bounded independently of \( h \).

**Proof.** Using (4.9), we can write \( G_h \) as
\[ G_h(x) = F_T(\hat{x}) + \Phi_T(\hat{x}). \]
Since \( x \mapsto \hat{x} \) is smooth, \( G_h \) is the sum of an affine function and a \( C^{k+1} \) function, so \( G_h \) is of class \( C^{k+1} \) on \( T^{(k)} \). To achieve the bound independently of \( h \), we use (4.3). \( \square \)

**Proposition 4.3** (Geometric bulk estimates) Let \( T \in \mathcal{T}_h^{(k)} \) be a boundary simplex (one which has more than one vertex on the boundary \( \Gamma_h^{(k)} \)), and \( T^e \) the associated exact triangle in \( \mathcal{T}_h^e \). Under the assumption that \( \mathcal{T}_h \) is quasi-uniform, for sufficiently small \( h \), we have that
\[ \| DG_h^i | T - 1d |_{L^{\infty}(T)} \leq ch^k, \] (4.10a)
\[ \| J_h | T - 1 |_{L^{\infty}(T)} \leq ch^k. \] (4.10b)

**Proof.** We will bound
\[ \left| \frac{\partial}{\partial x_j} (G_h)_i - \delta_{ij} \right|, \]
which will show the estimates above. We start by taking the $x_j$ derivative of $G_h$ to get

$$
\frac{\partial}{\partial x_j}(G_h)_i = \sum_l \frac{\partial (F_T^{(k)})^{-1}(x)_j}{\partial x_j} \frac{\partial (F_T^{(k)}(\hat{\xi}))_l}{\partial \hat{\xi}_l},
$$

where we have used the substitution $F_T^{(k)}(\hat{\xi}) = x$. We note that this means

$$
\sum_l \frac{\partial (F_T^{(k)})^{-1}(x)_j}{\partial x_j} \frac{\partial (F_T^{(k)}(\hat{\xi}))_l}{\partial \hat{\xi}_l} = \frac{\partial (F_T^{(k)}(\hat{\xi}))_j}{\partial x_j} = \delta_{ij}
$$

Hence

$$
\frac{\partial}{\partial x_j}(G_h)_i - \delta_{ij} = \sum_l \frac{\partial (F_T^{(k)})^{-1}(x)_j}{\partial x_j} \frac{\partial (F_T^{(k)}(\hat{\xi}))_l}{\partial \hat{\xi}_l} (F_T^{(k)}(\hat{\xi}) - F_T^{(k)}(\hat{\xi}))_l;
$$

It is classical (Ciarlet & Raviart, 1972a, Lemma 7, page 238) that

$$
\left| \frac{\partial}{\partial x_j} \left( (F_T^{(k)})^{-1}(\hat{\xi})_j \right) \right| \leq \left| \frac{\partial \hat{\xi}_l}{\partial x_j} \right| \leq \frac{c}{h},
$$

and from standard interpolation theory, we see that

$$
\left| \frac{\partial}{\partial \hat{\xi}_l} \left( F_T^{(k)}(\hat{\xi}) - F_T^{(k)}(\hat{\xi})_l \right) \right| \leq c \left\| D^{k+1}_l (F_T^{(k)}) \right\|_{L^\infty(T_k)}.
$$

However we may use the fact that $|D^{m+1}_l x_j| \leq ch^m$ (Ciarlet & Raviart, 1972a, page 239) and change coordinates to see

$$
\left\| D^{k+1}_l (F_T^{(k)}) \right\|_{L^\infty(T_k)} \leq ch^{k+1} \left\| (F_T^{(k)} \circ (F_T^{(k)})^{-1}) \right\|_{W^{k+1, \infty}(T_k)} = ch^{k+1} \left\| G_h \right\|_{W^{k+1, \infty}(T_k)}.
$$

From Lemma 4.3, we know $\left\| G_h \right\|_{W^{k+1, \infty}(T_k)}$ is bounded independently of $h$, this shows that

$$
\left| \frac{\partial}{\partial x_j} (G_h)_i - \delta_{ij} \right| \leq ch^k.
$$

We can now lift a function defined on $\Omega_h^{(k)}$ onto a function defined on $\Omega$.

**Definition 4.4** For a function $\eta_h : \Omega_h^{(k)} \rightarrow \mathbb{R}$, we define its lift $\eta_h^\ell : \Omega \rightarrow \mathbb{R}$ by

$$
\eta_h^\ell := \eta_h \circ G_h^{-1}.
$$

For a function $\eta : \Omega \rightarrow \mathbb{R}$, we can also define an inverse lift $\eta^{-\ell} : \Omega_h^{(k)} \rightarrow \mathbb{R}$ by

$$
\eta^{-\ell} := \eta \circ G_h.
$$

In this case, it follows that $(\eta^{-\ell})^\ell = \eta$. 

\[\square\]
We also have equivalence of norms via this lifting process:

**Proposition 4.5** Let \( \eta_h : \Omega_h^{(k)} \to \mathbb{R} \) and let \( \eta_h^\delta : \Omega \to \mathbb{R} \) be its lift. Then there exists constants \( c_1, c_2 \) independent of \( h \) such that

\[
\begin{align*}
&c_1 \left\| \eta_h^\delta \right\|_{L^2(\Omega)} \leq c_2 \left\| \eta_h \right\|_{L^2(\Omega_h^{(k)})} \leq c_2 \left\| \eta_h^\delta \right\|_{L^2(\Omega)} \quad \text{(4.11a)} \quad \text{and}
&c_1 \left\| \nabla \eta_h^\delta \right\|_{L^2(\Omega)} \leq c_2 \left\| \nabla \eta_h \right\|_{L^2(\Omega_h^{(k)})} \leq c_2 \left\| \nabla \eta_h^\delta \right\|_{L^2(\Omega)} \quad \text{(4.11b)}.
\end{align*}
\]

**Proof.** We can write integrals over \( \Omega_h^{(k)} \) in the following way

\[
\int_{\Omega_h^{(k)}} \eta_h(x) \, dx = \int_{\Omega_h^\delta} \eta_h^\delta(y) \frac{1}{f_h(y)} \, dy,
\]

and the gradient on \( \Omega_h^{(k)} \) as

\[
\nabla \eta_h(x) = DG_h^\delta(y) \nabla \eta_h^\delta(y).
\]

The results follows simply from applying the previous proposition.

In the following error analysis, we will require the following narrow band trace inequality.

**Lemma 4.4** Let \( \mathcal{N}_\delta \subseteq U \) be the band of width \( \delta < \delta^\delta \) given by

\[
\mathcal{N}_\delta = \{ x \in \Omega : -\delta < d(x) < 0 \}.
\]

It holds that for \( \eta \in H^1(\Omega) \)

\[
\left\| \eta \right\|_{L^2(\mathcal{N}_\delta)} \leq c \delta^{\frac{1}{2}} \left\| \eta \right\|_{H^1(\Omega)}. \quad \text{(4.12)}
\]

**Proof.** First, we may assume that \( \eta \in C^1(\Omega) \), since the more general result will follow by a density argument. Note that \( d \in C^2(\mathcal{N}_\delta) \) and \( |\nabla d| \equiv 1 \) on \( \mathcal{N}_\delta \). We can apply the coarea formula to integrals over \( \mathcal{N}_\delta \) as follows:

\[
\int_{\mathcal{N}_\delta} \eta(y)^2 \, dy = \int_{\mathcal{N}_\delta} \eta(y)^2 |\nabla d(y)| \, dy = \int_{-\delta}^{0} \int_{\Gamma_s} \eta^2 |\nu| \, d\sigma \, dx.
\]

Here \( \Gamma_s \) denotes the \( C^2 \) hypersurface which is the inverse image of \( s \) under \( d \), namely \( \Gamma_s = \{ x \in \mathcal{N}_\delta : d(x) = s \} \). Next, we wish to apply a trace inequality type argument to bound the right hand side of this equation. We follow the proof of the trace inequality (Theorem 1.5.1.10) from Grisvard (2011). Let the vector field \( D : \Omega \to \mathbb{R}^N \) be an extension of \( \nabla d \) of class \( C^1 \) on \( \Omega \), equal to \( \nabla d \) on \( \mathcal{N}_\delta \), with the bound \( \| D \|_{C^1(\Omega)} \leq c \| d \|_{C^2(\mathcal{N}_\delta)} \). Setting \( \mathcal{O}_s = \{ x \in \Omega : d(x) < s \} \), see Figure 3, we have that

\[
\int_{\mathcal{O}_s} \nabla (\eta^2) \cdot D \, dx = 2 \int_{\mathcal{O}_s} \eta \nabla \eta \cdot D \, dx.
\]

On the other hand applying Green’s theorem, using the notation \( n_s \) for the normal to \( \Gamma_s \), we obtain

\[
\int_{\Omega_s} \nabla (\eta^2) \cdot D \, dx = \int_{\Gamma_s} \eta^2 D \cdot n_s \, d\sigma - \int_{\Omega_s} \eta^2 \nabla \cdot D \, dx.
\]
Since $D \cdot n_s = 1$ on $\Gamma$, combining these two equations we have that
\[
\int_{\Gamma_s} \eta^2 D \cdot n_s \, d\sigma = 2 \int_{\Omega_s} \eta \nabla \eta \cdot D \, dx + \int_{\Omega_s} \eta^2 \nabla \cdot D \, dx,
\]
which means that
\[
\int_{\Gamma_s} \eta^2 \, d\sigma \leq 2 \max_{\Omega_s} |D| \int_{\Omega_s} |\nabla \eta| \, dx + \max_{\Omega_s} |\nabla \cdot D| \int_{\Omega_s} \eta^2 \, dx.
\]
Since we have that $\Omega_s \subseteq \Omega$, applying a Young’s inequality gives
\[
\int_{\Gamma_s} \eta^2 \, d\sigma \leq c \|D\|_{C^1(\bar{\Omega})} \int_{\Omega} |\nabla \eta|^2 + \eta^2 \, dx.
\]
Hence we have that
\[
\int_{\gamma_s} \eta^2 \, dy \leq c \delta \|\eta\|^2_{H^1(\Omega)}.
\] (4.13)

**4.3 Surface estimates**

We have the following geometric estimates for the surface $\Gamma_h$. They follow since $\Gamma_h$ can be viewed as an interpolant of $\Gamma$. Details can be found in Dziuk (1988); Dziuk & Elliott (2007a,b); Demlow (2009).

**Proposition 4.6** (Geometric surface estimates) Under the above assumptions on $\Gamma$ and $\Gamma_h$ we have that
\[
\|d\|_{L^\infty(\Gamma_h)} \leq ch^{k+1}.
\]
Let $\mu_h$ be the quotient of the measures on the surface and the approximate surface, so that $d\sigma = \mu_h \, d\sigma_h$. Then we have the estimate
\[
\sup_{\Gamma_h} |1 - \mu_h| \leq c h^{k+1}.
\] (4.14)

Let $P$ and $P_h$ denote the projections onto the tangent spaces of $\Gamma$ and $\Gamma_h$ respectively. We introduce the notation
\[
\mathcal{Q}_h = \frac{1}{\mu_h} [(\text{Id} - d\mathcal{H})P_h P (\text{Id} - d\mathcal{H})]
\] (4.15)
then we have the estimate that
\[
|\text{Id} - \mu_h \mathcal{Q}_h| \leq c h^{k+1}.
\] (4.16)
A proof can be found in Dziuk (1988); Dziuk & Elliott (2007a) for the linear case and Demlow (2009) for higher orders.

We use the closest point operator (2.1) to define the lift and inverse lift of surface functions.

**Definition 4.7** Given \( \xi_h : \Gamma_h^{(k)} \to \mathbb{R} \), we define its lift, denoted by \( \xi^\ell_h : \Gamma \to \mathbb{R} \), by

\[
\xi^\ell_h(p(x)) := \xi_h(x).
\]

Similarly for a function \( \xi : \Gamma \to \mathbb{R} \), we define its inverse lift, written \( \xi^{-\ell} : \Gamma^{(k)}_h \to \mathbb{R} \), by

\[
\xi^{-\ell}(x) := \xi(p(x)).
\]

It can be shown that the following norms are equivalent via this lifting process.

**Proposition 4.8** Let \( \xi_h : \Gamma_h^{(k)} \to \mathbb{R} \) and let \( \xi^\ell_h : \Gamma \to \mathbb{R} \) be its lift. Then there exists constants \( c_1, c_2 \) independent of \( h \) such that

\[
c_1 \| \xi^\ell_h \|_{L^2(\Gamma)} \leq \| \xi_h \|_{L^2(\Gamma^{(k)}_h)} \leq c_2 \| \xi^\ell_h \|_{L^2(\Gamma)} \quad (4.17a)
\]

\[
c_1 \| \nabla \Gamma \xi^\ell_h \|_{L^2(\Gamma)} \leq \| \nabla \Gamma \xi_h \|_{L^2(\Gamma^{(k)}_h)} \leq c_2 \| \nabla \Gamma \xi^\ell_h \|_{L^2(\Gamma)} . \quad (4.17b)
\]

A proof is given in Dziuk (1988); Dziuk & Elliott (2007a) for \( k = 1 \) and Demlow (2009) for any \( k > 1 \).

5. Finite Element Method

In this work we will use piecewise polynomial finite element functions of the same degree as the approximation of the domain. This leads to so-called isoparametric elements which will give the optimal rate of convergence. One could also implement this method with different order finite element functions, but would lead to suboptimal convergence.
5.1 Isoparametric finite element spaces

We use this section to define the finite element spaces \( V_h \) and \( S_h \) that our finite element method will be based on. We recall that the computational domains \( \Omega_h \) and \( \Gamma_h \) are defined elementwise by a parametrisation \( F^{(k)}_T : \hat{T} \to T^{(k)} \subset \Omega_h^{(k)} \) as in (4.8). In both the bulk and surface cases, we define the finite element functions to be continuous functions which are piecewise polynomials of degree \( k \) with respect to the barycentric coordinates of the reference element in dimension \( N \) and \( N-1 \). An important part of the construction is that the trace of a function on \( \Gamma_h^{(k)} \) in \( V_h \) lies in \( S_h \).

More precisely for the bulk finite element functions:

\[
V_h = \{ \eta_h \in C(\Omega_h^{(k)}): \eta_h|_T = \hat{\eta}_h \circ (F^{(k)}_T)^{-1} \text{ with } \hat{\eta}_h \in P_k(\hat{T}) \text{ for all } T \in \mathcal{T}_h \}
\]

For the surface finite element functions, we introduce

\[
S_h = \{ \xi_h \in C(\Gamma_h^{(k)}): \xi_h|_\tau = \hat{\xi}_h \circ (F^{(k)}_T)^{-1} \text{ with } \hat{\xi}_h \in P_k(\hat{T}) \text{ for all } T \in \mathcal{T}_h \text{ with } \tau = T \cap \Gamma_h \neq \emptyset \}.
\]

We have used the notation \( \hat{T} = (F^{(k)}_T)^{-1}(\tau) \) for the face of the reference element \( \hat{T} \) corresponding to \( \tau \), and \( P_k(\omega) \) for the space of polynomials of degree \( k \) on \( \omega \).

From now on we will assume \( k \) is fixed and write \( \Omega_h, \Gamma_h, \mathcal{T}_h \) for \( \Omega_h^{(k)}, \Gamma_h^{(k)}, \mathcal{T}_h^{(k)} \) without ambiguity.

5.2 Description of the method

We define approximate data \( f_h, g_h \) using the appropriate inverse lifts. That is

\[
f_h = f^{-\ell} J_h \quad \quad g_h = g^{-\ell} \mu_h. \tag{5.1}
\]

The approximate problem is then to find \((u_h, v_h) \in V_h \times S_h \) such that

\[
\alpha \int_{\Omega_h} \nabla u_h \cdot \nabla \eta_h + u_h \eta_h \, dx + \beta \int_{\Gamma_h} \nabla u_h \cdot \nabla \xi_h + v_h \xi_h \, d\sigma_h \\
+ \int_{\Gamma_h} (\alpha u_h - \beta v_h)(\alpha \eta_h - \beta \xi_h) \, d\sigma_h = \alpha \int_{\Omega_h} f_h \eta_h \, dx + \beta \int_{\Gamma_h} g_h \xi_h \, d\sigma_h, \tag{5.2}
\]

for all \((\eta_h, \xi_h) \in V_h \times S_h \),

where \( \nabla u_h \) is the surface gradient on \( \Gamma_h \).

**Remark 5.1** This choice of \( f_h \) and \( g_h \) is not fully practical for arbitrary \((f, g) \in L^2(\Omega) \times L^2(\Gamma) \) as the right hand side integrals would need to be calculated via some numerical integration rule. We are not concerned in analysing such errors in this paper and will assume that it is possible to calculate these integrals exactly. For general results on numerical integration in the context of curved domains see Ciarlet & Raviart (1972b); Barrett & Elliott (1987).

**Remark 5.2** To implement the method we use exact quadrature rules to calculate mass and stiffness matrices on reference elements using the transformation (4.8).
We introduce the bilinear and linear forms on $V_h \times S_h$:

$$
a_h((w_h, y_h), (\eta_h, \xi_h)) = \alpha \int_{\Omega_h} \nabla w_h \cdot \nabla \eta_h + w_h \eta_h \, dx \\
+ \beta \int_{\Gamma_h} \nabla I_h y_h \cdot \nabla I_h \xi_h + y_h \xi_h \, d\sigma_h, \\
+ \int_{\Gamma_h} (\alpha w_h - \beta y_h)(\alpha \eta_h - \beta \xi_h) \, d\sigma_h
$$

$$
l_h((\eta_h, \xi_h)) = \alpha \int_{\Omega_h} f_h \eta_h \, dx + \beta \int_{\Gamma_h} g_h \xi_h \, d\sigma_h,
$$

so that we can write (5.2) as: find $(u_h, v_h) \in V_h \times S_h$ such that

$$
a_h((u_h, v_h), (\eta_h, \xi_h)) = l_h((\eta_h, \xi_h)) \quad \text{for all } (\eta_h, \xi_h) \in V_h \times S_h. \quad (5.3)
$$

**Theorem 5.1** The finite element method defined in (5.2) has a unique solution $(u_h, v_h) \in V_h \times S_h$ for all $h$ which satisfies the bound

$$
\|(u_h, v_h)\|_{H^1(\Omega_h) \times H^1(\Gamma_h)} \leq c \|(f, g)\|_{L^2(\Omega) \times L^2(\Gamma)}. \quad (5.4)
$$

**Proof.** It is clear that the equations have a unique solution since $a_h$ is also coercive – this follows from the same reasoning as (3.2). To show the bound we use the coercivity of $a_h$, the equivalence of norms shown in (4.17a, 4.11a) and (4.14, 4.10) to see for $h$ small enough

$$
\|(u_h, v_h)\|_{H^1(\Omega_h) \times H^1(\Gamma_h)} \leq c \|(f, g)\|_{L^2(\Omega) \times L^2(\Gamma)} \leq c \|(f, g)\|_{L^2(\Omega) \times L^2(\Gamma)},
$$

□

### 5.3 Lifted finite element spaces

In order to prove error bounds, we define the lifted finite element spaces that lift of finite element functions live in. In particular this allows us to define $(u_h^L, v_h^L)$, the lifts of the finite element solution, defined on the same domain as the solutions of the continuous problem. We define the lift of the finite element spaces as

$$
V_h^L = \{ \eta_h^L : \eta_h \in V_h \} \subseteq H^1(\Omega)
$$

$$
S_h^L = \{ \xi_h^L : \xi_h \in S_h \} \subseteq H^1(\Gamma). \quad (5.5)
$$

It is important to note that the trace on $\Gamma$ of functions in $V_h^L$ live in $S_h^L$.

**Proposition 5.2** (Approximation property) For the lifted finite element spaces $V_h^L, S_h^L$ defined above there exists an interpolation operator $I_h^L : H^{k+1}(\Omega) \times H^{k+1}(\Gamma) \rightarrow V_h^L \times S_h^L$ such that for $2 \leq m \leq k + 1$,

$$
\|(w, y) - I_h(w, y)\|_{L^2(\Omega) \times L^2(\Gamma)} + h \|(w, y) - I_h(w, y)\|_{H^1(\Omega) \times H^1(\Gamma)} \leq c h^m \|(w, y)\|_{H^m(\Omega) \times H^m(\Gamma)} \quad (5.6)
$$

for all $(w, y) \in H^2(\Omega) \times H^2(\Gamma)$.
Proof. We start by defining the interpolation operator \( \tilde{I}_h : H^2(\Omega) \times H^2(\Gamma) \to V_h \times S_h \) so that \((w,y)\) and \(\tilde{I}_h(w,y)\) agree at the nodes of \(\Omega_h\) and \(\Gamma_h\). We use both lifts to define \(I_h(w,y) = \left( \tilde{I}_h(w,y) \right)^\ell\). The error bounds follow from given interpolation theory; see Bernardi (1989), Corollary 4.1, for the bulk and Demlow (2009) for the surface.

Using the fact that
\[
\nabla (w_h^\ell) = \nabla (w_h \circ G_h^{-1}) = DG_h^{-1}(\nabla w_h)^\ell,
\]
writing \(DG_h^{-1}\) for \((DG_h^{-1})^\ell\) and from Dziuk (1988),
\[
(P_h(Id - d\mathcal{K}))\nabla \Gamma(y_h^\ell) = (\nabla \Gamma y_h)^\ell,
\]
we have that
\[
a_h^\ell((w_h,y_h),(\eta_h,\xi_h)) = \alpha \int_\Omega (DG_h^\ell \nabla w_h^\ell \cdot DG_h^\ell \nabla \eta_h + w_h^\ell \eta_h) \frac{1}{J_h} \, dx
\]
\[
+ \beta \int_\Gamma \partial_h^\ell \nabla \Gamma y_h^\ell \cdot \nabla \Gamma \xi_h + y_h^\ell \xi_h \frac{1}{\mu_h} \, do
\]
\[
+ \int_\Gamma (\alpha w_h^\ell - \beta y_h^\ell)^\ell \frac{1}{\mu_h} \, do = a_h^\ell((w_h^\ell,y_h^\ell),(\eta_h,\xi_h^\ell)).
\]
for all \((w_h,y_h),(\eta_h,\xi_h)\) \(\in V_h \times S_h\) with lifts \((w_h^\ell,y_h^\ell),(\eta_h,\xi_h^\ell)\) \(\in V_h^\ell \times S_h^\ell\).

Whereas for the right hand side, we immediately have that \(I_h((\eta_h,\xi_h)) = l((\eta_h^\ell,\xi_h^\ell))\) since
\[
\int_{\Omega_h} f_h \eta_h \, dx = \int_{\Omega_h} (f - f_h) \eta_h \, dx = \int_{\Omega} (f - f_h) \eta_h \frac{1}{J_h} \, dx = \int_{\Omega} f \eta_h \frac{1}{J_h} \, dx = \int_{\Omega} f \eta_h^\ell \, dx,
\]
and
\[
\int_{\Gamma_h} g_h \xi_h \, do_h = \int_{\Gamma_h} (g - g_h) \xi_h \, do_h = \int_{\Gamma} (g - g_h) \xi_h \frac{1}{\mu_h} \, do = \int_{\Gamma} g \mu_h \xi_h \frac{1}{\mu_h} \, do = \int_{\Gamma} g \xi_h^\ell \, do.
\]
Hence we may rewrite (5.3) as: Find \((u_h^\ell,v_h^\ell)\) \(\in V_h^\ell \times S_h^\ell\)
\[
a_h^\ell((u_h^\ell,v_h^\ell),(\eta_h,\xi_h)) = l((\eta_h^\ell,\xi_h^\ell)) \text{ for all } (\eta_h,\xi_h) \in V_h \times S_h. \tag{5.7}
\]

We will make use of the fact that \(a_h^\ell\) now make sense for all function pairs in \(H^1(\Omega) \times H^1(\Gamma)\) in the following.

6. Error analysis

In this section, we wish to compare the error of the solutions \((u,v)\) of the exact problem (1.1) to the solutions \((u_h,v_h)\) of the approximate problem (5.2) defined in section 5.

One of the problems we have to overcome is the fact that the two problems are posed over different domains. However the lift operators we have defined will help us.

In order to derive optimal order estimates for \(k > 1\), we must assume higher regularity of the smooth solution \((u,v)\) of (2.10) and the surface \(\Gamma\). We require \((u,v) \in H^{k+1}(\Omega) \times H^{k+1}(\Gamma)\) which requires \(\Gamma\) to be \(C^{k+2}\) (Wloka, 1987).
THEOREM 6.1 Let \((u, v) \in H^{k+1}(\Omega) \times H^{k+1}(\Gamma)\) be the solution of the variational problem (2.10) and let \((u_h, v_h) \in V_h \times S_h\) be the solution of the finite element scheme given by (5.2). Denote by \(u_h^e\) and \(v^e_h\) the lifts of \(u_h\) and \(v_h\), respectively. Then we have the following error bounds:

\[
\left\| (u - u_h^e, v - v^e_h) \right\|_{H^1(\Omega) \times H^1(\Gamma)} \leq C_1 h^k,
\]

(6.1)

where

\[
C_1 = c(\left\| (u, v) \right\|_{H^{k+1}(\Omega) \times H^{k+1}(\Gamma)} + \left\| (f, g) \right\|_{L^2(\Omega) \times L^2(\Gamma)}),
\]

and

\[
\left\| (u - u_h^e, v - v^e_h) \right\|_{L^2(\Omega) \times L^2(\Gamma)} \leq C_2 h^{k+1},
\]

(6.2)

where

\[
C_2 = c(\left\| (u, v) \right\|_{H^{k+1}(\Omega) \times H^{k+1}(\Gamma)} + \left\| (f, g) \right\|_{L^2(\Omega) \times L^2(\Gamma)}).
\]

6.1 Geometric errors

Part of the error of the finite element method comes from the fact that there is a so-called ‘variational crime’, that is we are using different bilinear forms in the exact and approximate formulations and \(V_h \not\subseteq H^1(\Omega)\) and \(S_h \not\subseteq H^1(\Gamma)\). These errors come from the change in geometry of the computational domain.

LEMMA 6.1 For \((w, y), (\eta, \xi) \in V_h \times S_h\) we have

\[
\left| a((w, y), (\eta, \xi)) - a_h^e((w, y), (\eta, \xi)) \right| \leq ch^k \left\| w \right\|_{H^1(\Omega)} \left\| \eta \right\|_{H^1(\Omega)} + ch^{k+1} \left\| (w, y) \right\|_{H^1(\Omega) \times H^1(\Gamma)} \left\| (\eta, \xi) \right\|_{H^1(\Omega) \times H^1(\Gamma)}.
\]

(6.3)

Proof. To prove this lemma we will split the forms \(a\) and \(a_h^e\) into bulk, surface and cross terms. That is

\[
a^{(\Omega)}(w, \eta) = \alpha \int_{\Omega} \nabla w \cdot \nabla \eta + w \eta \, dx
\]

\[
a^{(\Gamma)}(y, \xi) = \beta \int_{\Gamma} \nabla y \cdot \nabla \xi + y \xi \, dx
\]

\[
a^{(\times)}((w, y), (\eta, \xi)) = \int_{\Gamma} (\alpha w - \beta y)(\alpha \eta - \beta \xi) \, dx.
\]

We define \(a_h^{(\times)}\) similarly.

Given \(w, \eta \in V_h\), for the bulk term we see that

\[
\left| \int_{\Omega_h} \nabla w \cdot \nabla \eta \, dx - \int_{\Omega} \nabla w \cdot \nabla \eta \, dx \right| = \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3,
\]

where

\[
\mathcal{A}_1 = \int_{\Omega} (DG^1_h - \text{Id}) \nabla w \cdot DG^1_h \nabla \eta \frac{1}{J_h} \, dx,
\]

\[
\mathcal{A}_2 = \int_{\Omega} \nabla w \cdot (DG^1_h - \text{Id}) \nabla \eta \frac{1}{J_h} \, dx,
\]

\[
\mathcal{A}_3 = \int_{\Omega} \nabla w \cdot \nabla \eta \left( \frac{1}{J_h^2} - 1 \right) \, dx.
\]
Making use of the fact that
\[ \frac{1}{J_h} - 1 = 0 \quad \text{and} \quad DG_h^t - \text{Id} = 0, \quad \text{in} \ \Omega \setminus B_h^t, \]
we actually have
\[ \mathcal{A}_1 = \int_{B_h^t} (DG_h^t - \text{Id}) \nabla w \cdot \nabla \eta \frac{1}{J_h} \, dx, \]
\[ \mathcal{A}_2 = \int_{B_h^t} \nabla w \cdot (DG_h^t - \text{Id}) \nabla \eta \frac{1}{J_h} \, dx, \]
\[ \mathcal{A}_3 = \int_{B_h^t} \nabla w \cdot \nabla \eta \left( \frac{1}{J_h} - 1 \right) \, dx. \]

Using Proposition 4.3, we see the three terms \( \mathcal{A}_j \) are bounded by
\[ ch^k \| \nabla w \|_{L^2(B_h^t)} \| \nabla \eta \|_{L^2(\Omega)}. \]

Similarly,
\[ \left| \int_{\Gamma_h} w^{-t} \eta^{-t} \, ds - \int_{\Gamma} w \eta \, ds \right| = \left| \int_{\Omega} w \eta (\frac{1}{J_h} - 1) \, dx \right| \leq ch^k \| w \|_{L^2(\Omega)} \| \eta \|_{L^2(\Omega)}. \]

Given \( y, \xi \in S_h^t \), for the surface terms we see that, using Proposition 4.6,
\[ \left| \int_{\Gamma_h} \nabla \eta \cdot \nabla \xi - \int_{\Gamma} \nabla \eta \cdot \nabla \xi \, ds \right| = \left| \int_{\Gamma} (\mu_h^t \mathcal{O}_h^t) \nabla \eta \cdot \nabla \xi \, ds \right| \leq ch^{k+1} \| \nabla \eta \|_{L^2(\Gamma)} \| \nabla \xi \|_{L^2(\Gamma)}, \]
and
\[ \left| \int_{\Gamma_h} \frac{1}{\mu_h^t} y^{-t} \xi^{-t} \, ds - \int_{\Gamma} y^{-t} \xi^{-t} \, ds \right| = \left| \int_{\Gamma} y \xi (\frac{1}{\mu_h^t} - 1) \, ds \right| \leq ch^{k+1} \| y \|_{L^2(\Gamma)} \| \xi \|_{L^2(\Gamma)}. \]

Using the previous result we also have that
\[ \left| \int_{\Gamma_h} (\alpha w^{-t} - \beta y^{-t})(\alpha \eta^{-t} - \beta \xi^{-t}) \, ds - \int_{\Gamma} (\alpha w - \beta y)(\alpha \eta - \beta \xi) \, ds \right| \]
\[ = \left| \int_{\Gamma} (\alpha w - \beta y)(\alpha \eta - \beta \xi) \left( \frac{1}{\mu_h^t} - 1 \right) \, ds \right| \leq c h^{k+1} \| (w, y) \|_{L^2(\Gamma) \times L^2(\Gamma)} \| (\eta, \xi) \|_{L^2(\Gamma) \times L^2(\Gamma)} \]
\[ \leq c h^{k+1} \| (w, y) \|_{H^1(\Omega) \times H^1(\Gamma)} \| (\eta, \xi) \|_{H^1(\Omega) \times H^1(\Gamma)}. \]

This shows (6.3). \( \square \)
We remark briefly that since $B_h^p$ is contained in $\Omega$ we also have for functions $(\eta, \xi) \in H^1(\Omega) \times H^1(\Gamma)$
\[
|a((w, y), (\eta, \xi)) - a_h((w, y), (\eta, \xi))| \leq ch^k ||(w, y)||_{H^1(\Omega) \times H^1(\Gamma)} ||(\eta, \xi)||_{H^1(\Omega) \times H^1(\Gamma)}.
\] (6.4)

Finally, we remark we can use Lemma 4.4 for integrals over $B_h^p$.

**Lemma 6.2** For $\eta \in H^1(\Omega)$,
\[
||\eta||_{L^2(B_h^p)} \leq ch^{\frac{1}{2}} ||\eta||_{H^1(\Omega)}.
\] (6.5)

**Proof.** We may apply Lemma 4.4 to the domain $\mathcal{A}_h$. We can choose $\delta$ such that $\delta > ch > h > 0$, since the width of $B_h^p$ is just one element. Hence
\[
||\eta||_{L^2(B_h^p)} \leq ||\eta||_{L^2(\mathcal{A}_h)} \leq c\delta^{\frac{1}{2}} ||\eta||_{H^1(\Omega)} \leq ch^{\frac{1}{2}} ||\eta||_{H^1(\Omega)}.
\]

\[\square\]

### 6.2 Proof of error bounds

Let $(u, v) \in H^{k+1}(\Omega) \times H^{k+1}(\Gamma)$ be the solution of the variational problem (2.6) and let $(u_h, v_h) \in V_h \times S_h$ be the solution of the finite element scheme given by (5.2). Denote by $u_h^f$ and $v_h^f$ the lifts of $u_h$ and $v_h$ respectively. Define $F_h : H^1(\Omega) \times H^1(\Gamma) \to \mathbb{R}$ by
\[
F_h((\eta, \xi)) := a(u - u_h^f, v - v_h^f, (\eta, \xi)).
\] (6.6)

**Lemma 6.3** If $(\eta, \xi) = (\eta_h^f, \xi_h^f) \in V_h^f \times S_h^f$, then $F_h$ is bounded by
\[
|F_h((\eta_h^f, \xi_h^f))| \leq ch^k ||(u_h^f, v_h^f)||_{H^1(\Omega) \times H^1(\Gamma)} ||(\eta_h^f, \xi_h^f)||_{H^1(\Omega) \times H^1(\Gamma)}.
\] (6.7)

If $(\eta, \xi) \in H^2(\Omega) \times H^2(\Gamma)$ then, we can improve the bound on $F_h$ to
\[
|F_h(\eta, \xi)| \leq (eh^{k+1} ||(u_h^f, v_h^f)||_{H^1(\Omega) \times H^1(\Gamma)} + ch^k ||(u_h^f - u, v_h^f - v)||_{H^1(\Omega) \times H^1(\Gamma)}
+ ch^{k+1} ||(u, v)||_{H^2(\Omega) \times H^2(\Gamma)} ||(\eta, \xi)||_{H^2(\Omega) \times H^2(\Gamma)}.
\] (6.8)

**Proof.** First, we notice that if $(\eta, \xi) = (\eta_h^f, \xi_h^f) \in V_h^f \times S_h^f$, using the fact that $(u, v)$ satisfies (2.6) and $(u_h^f, v_h^f)$ satisfies (5.7), $F_h$ can be written as
\[
F_h((\eta_h^f, \xi_h^f)) = a((u - u_h^f, v - v_h^f), (\eta_h^f, \xi_h^f))
= l((\eta_h^f, \xi_h^f)) - a_h((u_h^f, v_h^f), (\eta_h^f, \xi_h^f))
= \left(l((\eta_h^f, \xi_h^f)) - a_h((u_h^f, v_h^f), (\eta_h^f, \xi_h^f))\right)
- \left(a((u_h^f, v_h^f), (\eta_h^f, \xi_h^f)) - a_h((u_h^f, v_h^f), (\eta_h^f, \xi_h^f))\right)
- \left(a((u_h^f, v_h^f), (\eta_h^f, \xi_h^f)) - a_h((u_h^f, v_h^f), (\eta_h^f, \xi_h^f))\right).
\]
Applying the result from (6.4) gives (6.7).

To show the second result, we assume \((\eta, \xi) \in H^2(\Omega) \times H^2(\Gamma)\) and introduce the interpolant \(I_h(\eta, \xi) \in V_h^\ell \times S_h^\ell\) of \((\eta, \xi)\), so that

\[
F_h((\eta, \xi)) = a((u - u_h^\ell, v - v_h^\ell), (\eta, \xi)) \\
= a((u - u_h^\ell, v - v_h^\ell), (\eta, \xi) - I_h(\eta, \xi)) + a((u - u_h^\ell, v - v_h^\ell), I_h(\eta, \xi)).
\]

Then, again we can use the fact that \((u, v)\) satisfies (2.6) and \((u_h^\ell, v_h^\ell)\) satisfies (5.7), so that

\[
F_h((\eta, \xi)) = a((u - u_h^\ell, v - v_h^\ell), (\eta, \xi) - I_h(\eta, \xi)) + \left(a_h^\ell((u_h^\ell, v_h^\ell), I_h(\eta, \xi)) - a((u_h^\ell, v_h^\ell), I_h(\eta, \xi))\right).
\]

Hence we have that

\[
F_h((\eta, \xi)) = a((u - u_h^\ell, v - v_h^\ell), (\eta, \xi) - I_h(\eta, \xi)) \\
+ \left(a_h^\ell((u_h^\ell, v_h^\ell), I_h(\eta, \xi) - (\eta, \xi)) - a((u_h^\ell, v_h^\ell), I_h(\eta, \xi) - (\eta, \xi))\right) \\
+ \left(a_h^\ell((u_h^\ell - u, v_h^\ell), (\eta, \xi) - a((u_h^\ell - u, v_h^\ell), (\eta, \xi))\right) \\
+ \left(a_h^\ell((u, v), (\eta, \xi)) - a((u, v), (\eta, \xi))\right).
\]

We bound each of the terms on the right hand side of (6.9) in turn. For the first term we apply (6.1) together with the Approximation Property (Proposition 5.2) to see

\[
\left|a((u - u_h^\ell, v - v_h^\ell), (\eta, \xi) - I_h(\eta, \xi))\right| \leq C_1 h^k \|\eta, \xi\|_{H^2(\Omega) \times H^2(\Gamma)}.
\]

For the second term, we use the geometric bound (6.4), again, with the Approximation Property (Proposition 5.2) to get

\[
\left|a_h^\ell((u_h^\ell, v_h^\ell), I_h(\eta, \xi) - (\eta, \xi)) - a((u_h^\ell, v_h^\ell), I_h(\eta, \xi) - (\eta, \xi))\right| \\
\leq ch^k \|(u_h^\ell, v_h^\ell)\|_{H^1(\Omega) \times H^1(\Gamma)} ch \|\eta, \xi\|_{H^2(\Omega) \times H^2(\Gamma)}.
\]

A bound for the third term follows by applying the geometric bound (6.4)

\[
\left|a_h^\ell((u_h^\ell - u, v_h^\ell - v), (\eta, \xi) - a((u_h^\ell - u, v_h^\ell - v), (\eta, \xi))\right| \\
\leq ch^k \|(u_h^\ell - u, v_h^\ell - v)\|_{H^1(\Omega) \times H^1(\Gamma)} \|(\eta, \xi)\|_{H^2(\Omega) \times H^2(\Gamma)}
\]

Finally, for the fourth term, we simply apply (6.3) followed by the result from Lemma 6.2 to see

\[
\left|a_h^\ell((u, v), (\eta, \xi))\right| - a((u, v), (\eta, \xi))\right| \\
\leq ch^k \|u\|_{H^2(\Omega)} \|\eta\|_{H^2(\Gamma)} + ch^k+1 \|(u, v)\|_{H^1(\Omega) \times H^1(\Gamma)} \|(\eta, \xi)\|_{H^2(\Omega) \times H^2(\Gamma)} \\
\leq ch^k \|(u, v)\|_{H^2(\Omega) \times H^2(\Gamma)} \|(\eta, \xi)\|_{H^2(\Omega) \times H^2(\Gamma)}.
\]

Adding together the previous four result into (6.9) gives (6.8).
Remark 6.1 Notice that for \((\eta, \xi) = (\eta_h, \xi_h) \in V_h \times S_h\) in the absence of domain perturbation then
\[ F_h((\eta_h, \xi_h)) = 0, \]
where this is simply Galerkin orthogonality, whereas, in the absence of the bulk equations then the bound would be order \(h^{k+1}\), Demlow (2009).

Proof of Theorem 6.1. The error estimate (6.1) follows simply by combining the approximation property (Proposition 5.2) with the bound on \(F\) from above (6.7). We rewrite the error as
\[
a((u - u_h^e, v - v_h^e), (u - u_h^e, v - v_h^e))
\[
= a((u - u_h^e, v - v_h^e), (u, v) - I_h(u, v))
\[
+ a((u - u_h^e, v - v_h^e), I_h(u, v) - (u_h^e, v_h^e))
\[
= a((u - u_h^e, v - v_h^e), (u, v) - I_h(u, v)) + F_h(I_h(u, v) - (u_h^e, v_h^e)).
\]
The result follows from application of a Cauchy inequality and the coercivity of the bilinear form \(a(\cdot, \cdot)\). To show the given value of \(C_1\) we use (5.4) from Theorem 5.1 and (4.17, 4.11) to bound \(\|u_h^e, v_h^e\|_{H^1(\Omega) \times H^1(\Gamma)}\).

We will use an Aubin-Nitsche duality argument to show the \(L^2\) bound. For \(\zeta = (\xi_1, \xi_2) \in L^2(\Omega) \times L^2(\Gamma)\), we define the dual problem: Find \(\zeta \in H^1(\Omega) \times H^1(\Gamma)\) such that
\[
a((\eta, \xi), \zeta) = \langle \xi, (\eta, \xi) \rangle_{L^2(\Omega) \times L^2(\Gamma)}\]
for all \((\eta, \xi) \in H^1(\Omega) \times H^1(\Gamma)\). (6.10)
Here, \(\langle \cdot, \cdot \rangle_{L^2(\Omega) \times L^2(\Gamma)}\) denotes the sum of the \(L^2\) inner products between \(w\) and \(\eta\) on \(\Omega\) and \(y\) and \(\xi\) on \(\Gamma\). Similar to Theorem 3.2, one can show the following regularity result for the dual problem:
\[
\|\zeta\|_{H^1(\Omega) \times L^2(\Gamma)} \leq C \|\xi\|_{L^2(\Omega) \times L^2(\Gamma)}.\]
(6.11)
We write the error,
\[
e = (u - u_h^e, v - v_h^e) \in L^2(\Omega) \times L^2(\Gamma),
\]
as the data for the dual problem and test with \((\eta, \xi) = e\) so that
\[
\|e\|_{L^2(\Omega) \times L^2(\Gamma)}^2 = a(e, \zeta) = F_h(\zeta).
\]
Hence, using (6.8) combined with the \(H^1\) error bound (6.1) the dual regularity result (6.11), we have
\[
\|e\|_{L^2(\Omega) \times L^2(\Gamma)}^2 = F_h(\zeta) \leq C_2 h^{k+1} \|e\|_{L^2(\Omega) \times L^2(\Gamma)}^2,
\]
with \(C_2\) as in the statement of the theorem.

7. Numerical Results
We have implemented the above finite element method using the ALBERTA finite element toolbox (Schmidt et al., 2005).

The data was chosen, with \(\alpha = \beta = 1\), so that the exact solution was,
\[
u(x_1, x_2, x_3) = (\alpha + x_1(1 - 2x_1) + x_2(1 - 2x_2)) \exp(-x_1(x_1 - 1)x_2(x_2 - 1)).\]
We calculate the right hand side by setting $(f_h, g_h) = \tilde{I}_h(f, g)$. We ran two simulations: one with $k = 1$, one with $k = 2$. We present the error calculated after solving the matrix system at each mesh size in Tables 1–4. A plot of the solution is provided in Figure 5.

**Acknowledgements**

The work of C. M. Elliott was supported by the UK Engineering and Physical Sciences Research Council EPSRC Grant EP/G010404 and the work of T. Ranner was supported by an EPSRC PhD studentship. The authors also thank the Centre for Scientific Computing at the University of Warwick for supplying valuable computation time.

**References**


Table 1. Error table for the case $k = 1$ - bulk errors, $\|u - u_h\|$.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$L^2$ error</th>
<th>eoc</th>
<th>$H^1$ error</th>
<th>eoc</th>
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<td>-</td>
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Table 2. Error table for the case $k = 1$ - surface errors, $\|v - v_h\|$.

<table>
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<th>$H^1$ error</th>
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Table 3. Error table for the case $k = 2$ - bulk errors, $\|u - u_h\|$.

<table>
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<th>$L^2$ error</th>
<th>eoc</th>
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Table 4. Error table for the case $k = 2$ - surface errors, $\|v - v_h\|$.

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