Norm estimates for weighted composition operators on spaces of holomorphic functions

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Abstract

This paper shows that the boundedness of a weighted composition operator on the Hardy–Hilbert space on the disc or half-plane implies its boundedness on a class of related spaces, including weighted Bergman spaces. The methods used involve the study of lower-triangular and causal operators.

Keywords: weighted composition operator, Hardy space, weighted Bergman space, Zen space, lower-triangular operator, causal operator.
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1 Introduction

The main subject of this paper is the estimation of norms of weighted composition operators acting on spaces related to the classical Hardy and Bergman spaces of the unit disc $\mathbb{D}$ and right half-plane $\mathbb{C}_+$ (and in particular, showing that under certain circumstances the operators are bounded). Recall that if $\varphi : \mathbb{D} \to \mathbb{D}$ is holomorphic, then the *composition operator* C_φ on a space

$\mathcal{X}$ of functions on $\mathbb{D}$ is defined by $C_\varphi f(z) = f(\varphi(z))$ for $f \in \mathcal{X}$ and $z \in \mathbb{D}$. Furthermore, if h is a holomorphic function defined on $\mathbb{D}$, then the *weighted composition operator* $T_h C_\varphi$ is defined by $T_h C_\varphi f(z) = h(z) f(\varphi(z))$. Similar definitions apply for spaces of holomorphic functions on $\mathbb{C}_+$.

Recall that $H^2 = H^2(\mathbb{D})$ is the space of all holomorphic functions $f(z) = \sum_{n=0}^\infty c_n z^n$ in $\mathbb{D}$ such that

$$\|f\| := \left(\sum_{n=0}^\infty |c_n|^2 \right)^{1/2}$$

is finite, and that $H^\infty(\mathbb{D})$ is the space of bounded analytic functions on $\mathbb{D}$ with the supremum norm.

It is a classical theorem of Littlewood (see, for example, [5]) that composition operators are always bounded on the Hardy space H^2, and hence that $T_h C_\varphi$ is bounded if in addition $h \in H^\infty(\mathbb{D})$. However, this is not a necessary and sufficient condition for boundedness, although clearly we require $h \in H^2(\mathbb{D})$ (see [9] for more details).

Weighted composition operators on Hardy and Bergman spaces have been much studied in recent years (see, for example, the articles [1, 2, 3, 4, 6, 9, 11] and the survey [16]). In particular, their boundedness on $H^2(\mathbb{D})$ is characterised in [3, 6, 11], as is explained further in Section 2. However, less is known about their behaviour on more general spaces of functions.

For $\beta = (\beta_n)_{n \geq 0}$ a real positive decreasing sequence, we define the space $H^2(\beta)$ as the space of analytic functions $f(z) = \sum_{n=0}^\infty c_n z^n$ in $\mathbb{D}$ such that the norm

$$\|f\|_\beta := \left(\sum_{n=0}^\infty |c_n|^2 \beta_n^2 \right)^{1/2}$$

is finite. Note that $H^2 \subseteq H^2(\beta)$. An important example here is the Bergman space, where we take $\beta_n = 1/\sqrt{n+1}$ for each n. Such spaces are discussed in [5].

In Section 2 we prove a result which gives as a corollary the boundedness of a weighted composition operator on the spaces $H^2(\beta)$, assuming only that it is bounded on H^2. Then in Section 3 we consider the continuous case, where the natural analogue of $H^2(\beta)$ is a class of spaces of functions defined on $\mathbb{C}_+$, known as Zen spaces. Once again, we give criteria for boundedness of weighted composition operators on these spaces.

Here are some of the tools we shall require in the sequel. We recall that a complex Banach space $\mathcal{X}$ has a 1-unconditional basis $(x_n)_{n\geq 0}$ if every vector $x \in \mathcal{X}$ can be written as a unique convergent sum $x = \sum_{n\geq 0} a_n x_n$ with $(a_n)_{n\geq 0}$ a complex sequence, and if in addition we always have

$$\left\| \sum_{n\geq 0} a_n x_n \right\| = \left\| \sum_{n\geq 0} |a_n| x_n \right\|.$$

Examples are the canonical basis for ℓ^p, where $1 \leq p < \infty$.

If $\mathcal{X}$ has a basis (x_n), then an operator $T : \mathcal{X} \to \mathcal{X}$ is said to be *lower-triangular* if Tx_n lies in the closed linear span of $\{x_k : k \geq n\}$ for each n, and it is *diagonal* if there exist scalars $(d_n)_{n\geq 0}$ such that $Tx_n = d_n x_n$ for each n.

2 The discrete case

A somewhat less general form of the following result appears in [14], where it is applied to anti-analytic Toeplitz operators on Dirichlet-type spaces. We include the proof for completeness.

Theorem 2.1. *Let $\mathcal{X}$ be a complex Banach space with 1-unconditional basis $(x_j)_{j\geq 0}$. Let $A : \mathcal{X} \to \mathcal{X}$ be a lower-triangular and bounded operator. Let D be a (possibly unbounded) diagonal operator on $\mathcal{X}$, i.e., $Dx_j = d_j x_j$, where $(d_j)_{j\geq 0}$ is an increasing sequence of positive reals. Then $\|D^{-1}AD\| \leq \|A\|$.*

Proof. Let $\Omega(z) := D^{-z}AD^z$ where D^z is the diagonal operator associated with $(d_j^z)_{j\geq 0}$. It follows that $\Omega(z) = (w_{ij}(z))_{i,j\geq 0}$ with $w_{ij}(z) = d_i^{-z} a_{ij} d_j^z$, so that $w_{ij}(z) = 0$ if $i < j$ and $w_{ij}(z) = a_{ij}\left(\frac{d_j}{d_i}\right)^z$ otherwise. Since the sequence $(d_j)_{j\geq 0}$ is increasing, we get

$$\sup_{\Re(z)>0} \|\Omega_N(z)\| < \infty,$$

where $\Omega_N(z) = (w_{ij}(z))_{0\leq i,j\leq N}$. We recall the maximum principle, in the form that for a complex function g that is bounded and holomorphic in the open right half-plane $\mathbb{C}_+$ and continuous on the closed half-plane, we have that $|g(\zeta)| \leq \sup_{z\in i\mathbb{R}} |g(z)|$ for all $\zeta \in \mathbb{C}_+$. (This can be proved either by a Phragmén–Lindelöf argument, or using the theory of the Hardy space H^∞.)

We apply this principle to the vector-valued function $\Omega_N(z)$, holomorphic and bounded in the right-half plane (the vector-valued case follows easily from the scalar case using the Hahn–Banach theorem). It implies that, for all z with a nonnegative real part,

$$\|\Omega_N(z)\| \leq \sup_{\Re(z)=0} \|\Omega_N(z)\| = \|\Omega_N(0)\| \leq \|A\|,$$

since $\|D^z\| = 1$ when the real part of z is 0. Taking $z = 1$ and letting N tend to ∞, we get the desired result. $\square$

Before stating and proving a corollary for Hardy space operators, we give some more notation.

Let $d = (d_n)_{n\geq 0}$ be a sequence of increasing and positive reals, and let $D : H^2 \to H^2(1/d)$ be the isometry defined by

$$D(\sum_{n\geq 0} a_n z^n) = \sum_{n\geq 0} a_n d_n z^n, \tag{1}$$

where, obviously, $1/d = (1/d_n)_{n\geq 0}$.

Corollary 2.2. *Let T be a bounded operator on H^2 given by a lower-triangular matrix with respect to the basis $(z^n)_{n\geq 0}$. Let $\beta = (\beta_n)_{n\geq 0}$ be a positive decreasing sequence. Then T acts as a bounded operator on $H^2(\beta)$ and satisfies*

$$\|T\|_{H^2(\beta)} \leq \|T\|.$$

Proof. Let (d_n) be the increasing sequence given by $d_n = 1/\beta_n$ for all integers $n \geq 0$, and let D be the unitary diagonal operator defined in (1). Then $A := DTD^{-1}$ is a lower-triangular bounded operator on $H^2(\beta)$ whose norm is $\|T\|_{H^2}$, since D is unitary. Applying Theorem 2.1, we have the result, since

$$\|T\|_{H^2(\beta)} = \|D^{-1}AD\|_{H^2(\beta)} \leq \|A\|_{H^2(\beta)} = \|T\|_{H^2}.$$

$\square$

Examples of bounded linear operators T on H^2 which are lower-triangular are weighted composition operators of the form $T = T_h C_\varphi$ where $C_\varphi(f) = f \circ \varphi$, $\varphi : \mathbb{D} \to \mathbb{D}$ is holomorphic with $\varphi(0) = 0$, and where $T_h f = hf$ with $h \in H^2$ such that T is bounded.

Indeed, Corollary 2.2 is known to hold for the unweighted case $h = 1$ (see [5, Cor. 3.3], where the proof relies on the theory of Hadamard–Schur products).

Note also that the inequality $\|T\|_{H^2(\beta)} \leq \|T\|$ can be strict, for example, in the case of certain Toeplitz operators on Bergman spaces, or, more simply, certain rank-one operators of the form $f \mapsto \langle f, 1\rangle z$.

In the general case, necessary and sufficient conditions for boundedness of $T_h C_\varphi$ on H^2 have been given as follows:

(i) $\mu_{h,\varphi}$ is a Carleson measure on $\overline{\mathbb{D}}$, where

$$\mu_{h,\varphi}(E) = \int_{\varphi^{-1}E\cap\mathbb{T}} |h|^2 \, dm,$$

for measurable subsets $E \subseteq \mathbb{T}$ (see [3]);

(ii) one has

$$\sup_{|w|<1} \left\| \frac{(1-|w|^2)^{1/2} h}{1-\overline{w}\varphi} \right\|_{H^2} < \infty,$$

(see [11, 6]).

In fact, there is a slightly more general class of operators A for which we get the boundedness on $H^2(1/d)$, where $d = (d_n)_{n\geq 0}$ is an increasing sequence of positive reals, provided that A is bounded on H^2.

For $a \in \mathbb{D}$ we write ψ_a for the involutive automorphism $\psi_a(z) = \frac{a-z}{1-\bar{a}z}$.

Theorem 2.3. *Let φ be a holomorphic self-mapping of the open unit disc $\mathbb{D}$ and let $h \in H^2$ such that $A := T_h C_\varphi$ is a bounded operator on H^2. Let $(d_n)_{n\geq 0}$ be an increasing sequence of positive reals such that $H^2(1/d)$ is automorphism-invariant. Then A is also a bounded operator on $H^2(1/d)$ satisfying*

$$\|A\|_{H^2(1/d)} \leq \|C_{\psi_{\varphi(0)}}\|_{H^2(1/d)} \|C_{\psi_{\varphi(0)}}\|_{H^2} \|A\|_{H^2}.$$

Proof. Note that the case $\varphi(0) = 0$ is covered by Corollary 2.2.

With $a := \varphi(0)$ we have $\psi_a \circ \varphi(0) = 0$. Note that $C_\varphi C_{\psi_a} = C_{\psi_a \circ \varphi}$. Therefore we have

$$\|T_h C_\varphi C_{\psi_a}\|_{H^2(1/d)} \leq \|T_h C_\varphi C_{\psi_a}\|_{H^2}$$

by Corollary 2.2. Thus, since C_{ψ_a} is self-inverse, we have

$$\begin{aligned} \|T_h C_\varphi\|_{H^2(1/d)} &\leq \|C_{\psi_a}\|_{H^2(1/d)} \|T_h C_\varphi C_{\psi_a}\|_{H^2} \\ &\leq \|C_{\psi_a}\|_{H^2(1/d)} \|T_h C_\varphi\|_{H^2} \|C_{\psi_a}\|_{H^2}, \end{aligned}$$

as required.

□

Lower and upper estimates of the norm of C_{ψ_a} on $H^2(1/d)$ may be found in [10].

3 The continuous case

3.1 Causal operators

We write down a continuous analogue of the discrete (matrix) theorem on weighted ℓ^2 spaces.

Definition 3.1. *We say that $A : L^2(0,\infty) \to L^2(0,\infty)$ is a* causal operator *(or* lower-triangular operator*), if for each $T > 0$ the closed subspace $L^2(T,\infty)$ is invariant for A. Also A is* strictly causal*, if for some $\alpha > 0$ we have $AL^2(T,\infty) \subseteq L^2(T+\alpha,\infty)$ for all $T > 0$.*

Theorem 3.2. *Suppose that $A : L^2(0,\infty) \to L^2(0,\infty)$ is a causal operator, and D the operator of multiplication by a strictly positive monotonically increasing function d. Then $\|D^{-1}AD\| \le \|A\|$.*

Proof. The restriction of $D^{-1}AD$ to a dense subspace, which is denoted $\bigcup_{N=1}^{\infty} X_N$, will be shown to be bounded; then the operator can then be extended continuously to $L^2(0,\infty)$.

Suppose first that A is strictly causal. For $\operatorname{Re} z \ge 0$ we define $\Omega(z) = D^{-z}AD^z$, where D^z is multiplication by the complex function d^z.

For each integer $N \ge 1$ we consider the subspace X_N spanned by the characteristic functions of the intervals $(k/2^N, (k+1)/2^N)$ for $0 \le k < 2^{2N}$. These subspaces increase with N and have dense union. Let $P_N : L^2(0,\infty) \to X_N$ denote the orthogonal projection.

Now for each N consider $\Omega_N(z) = \Omega(z)P_N$ and note that

$$\sup_{\operatorname{Re} z \ge 0} \|\Omega_N(z)\| < \infty.$$

To see this, let e_k denote the characteristic function of the interval

$$(k/2^N, (k+1)/2^N)$$

for $0 \le k < 2^{2N}$. This is mapped to $d^{-z}Ad^z e_k$. Now

$$\|Ad^z e_k\| \le \|A\| d((k+1)/2^N)^{(\operatorname{Re} z)} \|e_k\|$$

since d is increasing, and $Ad^z e_k$ is supported on $[k/2^N + \alpha, \infty)$ so that

$$\|d^{-z} A d^z e_k\| \leq d(k/2^N + \alpha)^{-(\mathrm{Re}\, z)} \|A\| d((k+1)/2^N)^{(\mathrm{Re}\, z)},$$

which is bounded independently of z as soon as $2^{-N} \leq \alpha$. (This is where we use the strict causality.)

Since X_N is finite-dimensional, we see that $\Omega_N(z)$ is bounded independently of z, because $\|\Omega_N(z)\| \leq \|\Omega(z)_{|X_N}\|$.

Next, by the maximum principle we have that

$$\|\Omega_N(1)\| \leq \sup_{\mathrm{Re}\, z \geq 0} \|\Omega_N(z)\| \leq \sup_{\mathrm{Re}\, z = 0} \|\Omega(z)\| = \|A\|.$$

Now, since the union of the X_N is dense, we have the result.

Now, to deduce the result for arbitrary causal operators A, let S_α denote the right shift by α. Since $S_\alpha A$ is strictly causal for each $\alpha > 0$ we have

$$\|D^{-1} S_\alpha A D\| \leq \|S_\alpha A\| = \|A\|.$$

Now $\|D^{-1} S_\alpha f\| = \|d_\alpha^{-1} f\|$ where $d_\alpha(t) = d(t + \alpha)$, and $|d_\alpha^{-1} f|$ increases to $|d^{-1} f|$ almost everywhere as $\alpha \to 0$ because the monotonically decreasing function d^{-1} is continuous almost everywhere; hence the result follows by the monotone convergence theorem. □

3.2 Zen spaces

We are going to prove norm estimates for operators on $L^2(w(t)\, dt)$, where $w(t)$ is decreasing (it will correspond to $1/d(t)^2$) and associated spaces of analytic functions obtained using the Laplace transform.

We refer now to [13]. Take ν a positive Borel measure on $[0, \infty)$ and for $t \geq 0$ define

$$w(t) = 2\pi \int_0^\infty e^{-2xt}\, d\nu(x).$$

(Note that $w(t)$ decreases with t and is non-zero.) The Laplace transform then defines an isometric map from $L^2(w(t)\, dt)$ into a generalized Hardy–Bergman space

$$A^2_\nu = \{f \in \mathrm{Hol}(\mathbb{C}_+) : \sup_{\varepsilon > 0} \int_0^\infty \int_{-\infty}^\infty |f(x + iy + \varepsilon)|^2\, d\nu(x)\, dy < \infty\}.$$

One such example is the Hardy space $H^2(\mathbb{C}_+)$, where $\nu = \delta_0$, a Dirac mass at 0, and $w(t) = 2\pi$ for all t; thus the norm on $H^2(\mathbb{C}_+)$ is given by

$$\|f\| = \left(\sup_{\varepsilon>0} \int_{-\infty}^{\infty} |f(\varepsilon + iy)|^2 \, dy \right)^{1/2}.$$

Similarly, the Bergman space corresponds to the case $d\nu = dx$ (Lebesgue measure), and $w(t) = \pi/t$. In the cases above, and in many others, the Laplace transform is surjective (see [12]) and as in [13] we shall use the terminology *Zen spaces* for such spaces of holomorphic functions.

3.3 Weighted composition operators

We have the Nevanlinna representation of a holomorphic function $\varphi : \mathbb{C}_+ \to \mathbb{C}_+$, namely

$$\varphi(s) = as + ib + \int_{-\infty}^{\infty} \frac{1 - its}{s - it} \, d\mu(t), \tag{2}$$

where $a \geq 0$, $b \in \mathbb{R}$, and μ is a measure on $\mathbb{R}$; see for example [15]. Note that a composition operator C_ϕ is causal as soon as $a \geq 1$, since if $g(s) = e^{-sT} h(s)$ then e^{-saT} divides $g \circ \varphi$.

Applying Theorem 3.2 to Zen spaces we have the following result.

Theorem 3.3. *Let h be a holomorphic function on $\mathbb{C}_+$ and $\varphi : \mathbb{C}_+ \to \mathbb{C}_+$ a causal holomorphic function such that the weighted composition operator $T_h C_\varphi$ is bounded on $H^2(\mathbb{C}_+)$. Then $T_h C_\phi$ is bounded on each Zen space A^2_ν, with $\|T_h C_\phi\|_{A^2_\nu} \leq \|T_h C_\varphi\|_{H^2(\mathbb{C}_+)}$.*

Proof. The proof is similar to the proof of Corollary 2.2; namely, by considering the following commutative diagram:

$$\begin{array}{ccc} L^2(0,\infty) & \xrightarrow{D} & L^2(0,\infty; w(t)\,dt) \\ \big\downarrow {\scriptstyle D^{-1}AD} & & \big\downarrow {\scriptstyle A} \\ L^2(0,\infty) & \xrightarrow{D} & L^2(0,\infty; w(t)\,dt) \end{array}$$

where D is the isometric operator of multiplication by $d(t) = 1/\sqrt{w(t)}$, and A is the operator induced on $L^2(0,\infty; w(t)\,dt)$ by $T_h C_\varphi$ acting on A^2_ν. $\square$

As in the discrete case we can remove the causality assumption, as follows. For $a > 0$ we define $\psi_a : \mathbb{C}_+ \to \mathbb{C}_+$ by $\psi_a(s) = as$ for $s \in \mathbb{C}_+$. Then $C_{\psi_{1/a}} T_h C_\varphi = T_{h\circ\psi_{1/a}} C_{\psi_{1/a}} C_\varphi$ is causal and so we obtain the following result (the case $a = 0$ is excluded, and we explain why in Remark 3.6).

Corollary 3.4. *Let φ be a holomorphic self-mapping of $\mathbb{C}_+$ given by (2) with $a > 0$, and h a holomorphic function on $\mathbb{C}_+$ such that $A := T_h C_\varphi$ is a bounded operator on $H^2(\mathbb{C}_+)$. Let A^2_ν be a Zen space. Then A is also a bounded operator on A^2_ν satisfying*

$$\|A\|_{A^2_\nu} \le \|C_{\psi_a}\|_{A^2_\nu} \|C_{\psi_{1/a}}\|_{H^2(\mathbb{C}_+)} \|A\|_{H^2(\mathbb{C}_+)}.$$

It is possible to give an exact expression of the norm of C_{ψ_a} in all cases, as follows:

Proposition 3.5. *Let $a > 0$ and let C_{ψ_a} denote the composition operator on A^2_ν induced by $\psi_a(s) = as$. Then*

$$\|C_{\psi_a}\|_{A^2_\nu} = \frac{1}{\sqrt{a}} \left(\sup_{x>0} \frac{w(ax)}{w(x)} \right)^{1/2}.$$

Proof. Note that if $g(t) = f(t/a)/a$, then

$$\int_0^\infty g(t) e^{-st}\, dt = \frac{1}{a} \int_0^\infty f(t/a) e^{-st}\, dt = \int_0^\infty f(x) e^{-asx}\, dx,$$

and so the norm of C_ψ is the same as the norm of the composition operator $f \mapsto g$. The result follows since

$$\|g\|^2 = \int_0^\infty a^{-2} |f(t/a)|^2 w(t)\, dt = \int_0^\infty a^{-1} |f(x)|^2 w(ax)\, dx.$$

$\square$

Remark 3.6. In [7] it is shown that a composition operator C_ϕ on $H^2(\mathbb{C}_+)$ is bounded if and only if $\operatorname{n.t.lim}_{z\to\infty} \phi(z) = \infty$ and $\lambda := \operatorname{n.t.lim}_{z\to\infty} z/\phi(z)$ exists (the angular derivative) with $0 < \lambda < \infty$. Here n.t.lim denotes the nontangential limit, taken with z restricted to the sector

$$C_K := \{z = x + iy : x > 0, |y|/x \le K\}$$

for some $K > 0$. Under these conditions, $\|C_\phi\| = \sqrt{\lambda}$. Then in [8] composition operators on weighted Bergman spaces, the Zen spaces A^2_ν with $d\nu(x) = x^\alpha \, dx$ for $\alpha > -1$, are considered, and the corresponding result is proved with the norm now being $\lambda^{(2+\alpha)/2}$. For causality we have $\lambda \leq 1$, and then

$$\|C_\phi\|_{A^2_\nu} \leq \|C_\phi\|_{H^2},$$

as claimed in Theorem 3.3.

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