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BIBO stability of some classes of delay systems and fractional systems

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Abstract

A new test is presented for the BIBO stability of delay systems of neutral type with a single delay, specified in terms of their transfer functions, enabling us to decide on some cases that were previously open. Next, a class of fractional systems is considered, and a method is given for determining the stability intervals for such systems.

Keywords: Delay system, fractional system, BIBO stability, H^∞ stability, asymptotic stability.

1 Introduction

This paper deals with various stability notions of linear time-invariant systems, specified in the frequency domain by their transfer functions. The class of systems that we shall consider contains delay systems of neutral type, as well as fractional delay systems of neutral and retarded type: these notions will be defined below. The three versions of stability that we shall consider (in decreasing strength) are BIBO (i.e., bounded-input bounded-output) stability, H^∞ stability (i.e., finite L^2–L^2 gain), and asymptotic stability (no poles in the closed right-hand half-plane $\overline{\mathbb{C}_+}$). Here $\mathbb{C}_+$ denotes the open right-hand half-plane $\mathbb{C}_+ = \{s \in \mathbb{C} : \operatorname{Re} s > 0\}$.

Our principal reference for infinite-dimensional systems is the book by Curtain and Zwart [6], and for delay systems [1]. For fractional systems we

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mention also the work of Matignon [10] and the book [12]. More on transfer-function and operatorial approaches to systems theory in general can be found in [5] and [15].

Spectral-based techniques for obtaining stability results for delay systems by making the link with the spectrum of the associated difference equation may be found in [11], although they do not provide answers to the questions we are addressing in this paper. See also [18].

Note that BIBO stability is equivalent to the condition that the impulse response function h satisfies $h \in L^1(0,\infty)$; that is,

$$\|h\|_1 := \int_0^\infty |h(t)|\, dt < \infty,$$

or, more generally, if the impulse response is a measure μ, that

$$\int_0^\infty d|\mu|(t) < \infty.$$

Moreover, H^∞ stability is equivalent to the condition that the transfer function G, the Laplace transform of the impulse response, is a bounded analytic function in $\mathbb{C}_+$. As usual, we write

$$\|G\|_\infty := \sup\{|G(s)| : s \in \mathbb{C}_+\}.$$

In Section 2 we shall be analyse linear systems with transfer functions of the form

$$G(s) = \frac{r(s)}{p(s) + q(s)e^{-hs}}, \qquad s \in \mathbb{C}_+,$$

where $h > 0$ and p, q and r are polynomials. (In fact we need consider just the case $h = 1$, since the general case reduces to this by a trivial change of variable.)
More generally, they may be *quasi-polynomials*, that is, of the form $a_0 s^{\alpha_0} + \cdots + a_n s^{\alpha_n}$, where $0 \le \alpha_0 < \ldots < \alpha_n$. Throughout this note, we regard s^α as being a single-valued holomorphic function defined on the cut plane $\{s = re^{i\theta} : r > 0 : -\pi < \theta < \pi\}$ as $s^\alpha = r^\alpha e^{i\alpha\theta}$, with the obvious convention that $0^\alpha = 0$.

If $\deg p > \deg q$, the system is said to be of *retarded type*; if $\deg p = \deg q$, it is of *neutral type*, and if $\deg p < \deg q$, it is of *advanced type*. (See, for example [1, 15].)

Thus in Section 2, we give a new test for BIBO stability of delay systems of neutral type, and use it to give answers to some delicate questions raised in [4, 14].

In Section 3 we shall consider fractional systems, those in which the exponential e^{-sh} is replaced by a term of the form $\exp(-hs^\alpha)$ with $0 < \alpha < 1$. These occur, for example, in the heat equation, where transfer functions such as $(\sinh x_0\sqrt{s})/(\sinh\sqrt{s})$, with $0 < x_0 < 1$, are encountered: see, for example, [6]. Here the main issue turns out to be the location of the 'small' poles (the large ones are asymptotically determinable, and lie in the left-hand half-plane). Fractional systems also arise in the theory of transmission lines: see, for example, [17].

Thus we develop a generalization of the Walton–Marshall test [16], which finds stability intervals for delay systems with variable delay. The theory is motivated by an example before being stated in detail.

2 Delay systems

Stability questions are well understood for delay systems of retarded and advanced type: in this section we shall concentrate on systems of neutral type, which are far more difficult to analyse. We shall necessarily assume that the system is *proper*, i.e., $\deg r \leq \deg p$.

As a motivating example which has been considered in several other papers, we consider

$$G_k(s) = \frac{1}{(s+1)^k(s+1+se^{-s})}, \qquad k = 0, 1, 2, \dots. \tag{1}$$

This transfer function is asymptotically stable (i.e., no poles in the closed right-hand half-plane); it is known that it does not lie in H^∞ for $k = 0$, but it is H^∞ stable for $k \geq 1$ (see [14]). The question of BIBO stability is far more difficult: G_k is clearly not BIBO stable for $k = 0$, but following the results of [4, 14] it is known to be BIBO stable for $k \geq 4$. The remaining cases were open, but with new methods we are now able to resolve the cases $k = 2$ and $k = 3$.
Before stating a more general result, we shall analyse G_k for $k \geq 2$, as the method is easiest to explain with this example.

Note that in many cases of stability, its robustness will be subject to small perturbations in the gap metric, but not to changes in the delay. Notions such as w-stability may be used to shed further light on such delicate issues (see [7]).

Lemma 1. *For $k \geq 0$ let $h_k \in L^1(0,\infty)$ satisfy $\mathcal{L}h_k(s) = \dfrac{s^k}{(s+1)^{k+3}}$. Then $\|h_k\|_1 = O(k^{-5/4})$ as $k \to \infty$.*

Proof. Write $g_k(t) = e^{t/4}h_k(t)$. Note that $\mathcal{L}g_k(s) = \dfrac{(s-\frac{1}{4})^k}{(s+\frac{3}{4})^{k+3}}$. Then, by the Cauchy–Schwarz inequality we have

$$\|h_k\| \leq \|e^{-t/4}\|_{L^2}\|g_k\|_{L^2}.$$

Now $\|g_k\|_{L^2} = \frac{1}{\sqrt{2\pi}}\|\mathcal{L}g_k\|_{H^2}$, and

$$\begin{aligned}\|\mathcal{L}g_k\|^2_{H^2} &= 2\int_0^\infty \frac{|iy-\frac{1}{4}|^{2k}}{|iy+\frac{3}{4}|^{2k+6}}\,dy \\ &= 2\left(\int_0^{\sqrt{k}} + \int_{\sqrt{k}}^\infty\right)\frac{(y^2+\frac{1}{16})^k}{(y^2+\frac{9}{16})^{k+3}}\,dy.\end{aligned}$$

We may estimate the first integral as at most $\sqrt{k}$ times the maximum value of the integrand on $[0,\sqrt{k}]$, or $O(k^{1/2}k^{-3})$, since the integrand is an increasing function of y. The second integral is at most $\displaystyle\int_{\sqrt{k}}^\infty y^{-6}\,dy$, which is also $O(k^{-5/2})$. This gives the result. □

Theorem 2. *The system with transfer function $G_k(s) = \dfrac{1}{(s+1)^k(s+1+se^{-s})}$ is BIBO stable for $k \geq 2$.*

Proof. It is sufficient to consider the case $k = 2$, as higher-order G_k are simply cascades of G_2 with BIBO-stable finite-dimensional systems. Now

$$G_2(s) = \sum_{k=0}^\infty (-1)^k e^{-sk}\frac{s^k}{(s+1)^{k+3}},$$

converging pointwise in $\mathbb{C}_+$, and it is easy to see that the inverse Laplace transforms converge pointwise on $(0,\infty)$, since the kth term vanishes on $[0,k)$. Thus if $\mathcal{L}h = G_2$, we have

$$\|h\|_1 \leq \sum_{k=0}^\infty \left\|(-1)^k e^{-sk}\frac{s^k}{(s+1)^{k+3}}\right\|_{BIBO} = \sum_{k=0}^\infty \left\|\frac{s^k}{(s+1)^{k+3}}\right\|_{BIBO},$$

by Fatou's lemma (in the form that asserts that if $f_n \to f$ pointwise then $\|f\|_1 \leq \liminf \|f_n\|$). Using Lemma 1, we conclude that $h \in L^1$, and the system G_2 is BIBO stable. □

A more general result can be proved by the same method. Note that one necessary condition on p and q for a neutral system $\dfrac{1}{p(s) + q(s)e^{-s}}$ to be asymptotically stable is that

$$\lim_{|s|\to\infty} |q(s)/p(s)| \leq 1, \tag{2}$$

(see [14, Prop. 2.1]), as otherwise the poles are asymptotic to a vertical line strictly in $\mathbb{C}_+$.

Theorem 3. *Let $G(s) = \dfrac{f(s)}{p(s) + q(s)e^{-s}}$ be the transfer function of a neutral delay system, satisfying condition (2), and write $p = r\widetilde{p}$, $q = r\widetilde{q}$, where r is the greatest common divisor of p and q. Suppose that*

- $\deg f = N'$ *and* $\deg p = \deg q = N \geq 3 + N'$*;*
- *all roots of p lie in $\mathbb{C}_-$;*
- *there is a $c > 0$ such that $|\operatorname{Re}(s_1 + c)| < |\operatorname{Re}(s_2 + c)|$ for all s_1, s_2 with $\widetilde{q}(s_1) = \widetilde{p}(s_2) = 0$;*
- *there exists an index $\alpha > 2/5$ such that for sufficiently large k the function $y \mapsto \left| \dfrac{f(iy - c)\widetilde{q}(iy - c)^k}{r(iy - c)\widetilde{p}(iy - c)^{k+1}} \right|$ is increasing on an interval $[0, \delta_k]$ where $\delta_k \asymp k^\alpha$.*

Then G is BIBO stable, and hence H^∞ stable.

Proof. Let h_k be such that $\mathcal{L}h_k(s) = \dfrac{f(s)\widetilde{q}(s)^k}{r(s)\widetilde{p}(s)^{k+1}}$, and write $g_k(t) = e^{ct}h_k(t)$. Then by the Cauchy–Schwarz inequality

$$\|h_k\|_{L^1} \leq \|e^{-ct}\|_{L^2} \|g_k\|_{L_2}.$$

Moreover

$$\mathcal{L}g_k(s) = \mathcal{L}h_k(s - c) = \frac{f(s - c)\widetilde{q}(s - c)^k}{r(s - c)\widetilde{p}(s - c)^{k+1}}.$$

Hence

$$\begin{aligned}
\|g_k\|_{L^2}^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{f(s-c)\widetilde{q}(s-c)^k}{r(s-c)\widetilde{p}(s-c)^{k+1}}\right|^2 ds \\
&= \frac{1}{\pi}\left(\int_0^{\delta_k}+\int_{\delta_k}^{\infty}\right)\left|\frac{f(iy-c)^2\widetilde{q}(iy-c)^{2k}}{r(iy-c)^2\widetilde{p}(iy-c)^{2k+2}}\right| dy.
\end{aligned}$$

We may estimate the first integral using the maximum value of the integrand at δ_k, to obtain $O(\delta_k^{2N'-2N+1})$; for the second we use the fact that the integrand is asymptotic to $y^{2N'-2N}$ to obtain a similar quantity. That is,

$$\|g_k\|_2 = O(k^{(2N'-2N+1)\alpha/2}),$$

and hence $\sum \|h_k\|_1 < \infty$, so we may deduce BIBO stability using the series

$$G(s) = \sum_{k=0}^{\infty}\frac{(-1)^k f(s)e^{-sk}q(s)^k}{p(s)^{k+1}},$$

and Fatou's lemma, as in Theorem 2. $\square$

Example 4. The transfer function $G(s) = \dfrac{1}{(s+3)(s+2)^2+(s-\frac{1}{2})s^2e^{-s}}$, for which the poles are asymptotic to $i\mathbb{R}$, is BIBO stable: apply Theorem 3, with $c = \frac{1}{2}$ and $\alpha = \frac{1}{2}$.

3 Fractional systems

In this section we discuss the asymptotic stability of fractional systems with transfer functions of the form

$$G(s) = \frac{r(s)}{p(s)+q(s)\exp(-hs^\alpha)}, \tag{3}$$

where p and q are real quasi-polynomials, $h > 0$, and $0 < \alpha < 1$. For analysis in the complex plane, we make a branch cut from 0 to $-\infty$ along the negative real axis, so that G is a single-valued function. The change of variable $u = s^\alpha$ transforms the transfer function into a function of the form

$$\widetilde{G}(u) = \frac{R(u)}{P(u)+Q(u)e^{-hu}}, \tag{4}$$

the location of whose poles (in terms of u) can be determined from the general theory of delay systems.

Note that questions of BIBO stability and H^∞ stability reduce to this case since by the results in [2] all three notions of stability coincide for such systems. This is in contrast to the case $\alpha = 1$ (delay systems), as we saw in Section 2.

We start by discussing the asymptotic location of poles of G ("large poles"), and then the determination of the poles closest to the origin ("small poles").

Theorem 5. *For a system with transfer function $\widetilde{G}$, as given in (4), it holds that for every $\varepsilon > 0$ the poles of large modulus lie in a sector*

$$S_\varepsilon := \left\{ u \in \mathbb{C} : \frac{\pi}{2} - \varepsilon < |\arg u| < \frac{\pi}{2} + \varepsilon \right\}.$$

Hence if $\frac{1}{3} < \alpha < 1$ the poles of large modulus of G, as given in (3) lie in the left-hand half-plane.

Proof. Standard results (see, e.g. [1, 15]) indicate that in the case that $\widetilde{G}$ is of retarded type ($\deg P > \deg Q$), the large poles satisfy $s_n = x_n + iy_n$ where $x_n < 0$ and $|x_n| \asymp \log |n|$, while $y_n \asymp n$ (for $n \in \mathbb{Z}$). Likewise, for $\widetilde{G}$ of advanced type ($\deg P < \deg Q$), we have $x_n > 0$ and $x_n \asymp \log |n|$, while $y_n \asymp n$ (for $n \in \mathbb{Z}$). Finally, for $\widetilde{G}$ of neutral type ($\deg P = \deg Q$) we have $x_n = O(1)$ and $y_n \asymp n$. Thus in each case $|y_n/x_n| \to \infty$ as $n \to \pm\infty$. Thus the poles lie in the sector S_ε for n sufficiently large.
It follows that with $s = u^{1/\alpha}$ and $\alpha^{-1} < 3$, the corresponding values of s lie in $\mathbb{C}_-$. □

For stability analysis, it remains to consider the location of the small poles of G, and for this purpose we develop a new technique inspired by the Walton–Marshall method [16, 9, 15]. We illustrate it with a simple example, before giving the complete algorithm.

Example 6. *Let*

$$G_h(s) = \frac{1}{\sqrt{s} + e^{-h\sqrt{s}}},$$

where $h \geq 0$. Then G_h is stable for $0 \leq h < \frac{3\pi}{2\sqrt{2}} e^{3\pi/4}$. As h increases, the poles cross the axis from left to right.

Proof. We consider the variation in the zeroes of $\sqrt{s} + e^{-h\sqrt{s}}$ as h increases: in particular the values of h at which they cross the y-axis. Equivalently, we consider the values of $h > 0$ for which $G_h(u) = u + e^{-hu}$ has a zero on the line $\{u \in \mathbb{C} : \arg u = \pi/4\}$.

Accordingly, suppose that $e^{-hu} = -u$, and let $u = xe^{i\pi/4}$, where $x > 0$. We have

$$xe^{i\pi/4} + \exp(-hxe^{i\pi/4}) \quad = \quad 0,$$

and so

$$xe^{-i\pi/4} + \exp(-hxe^{-i\pi/4}) \quad = \quad 0.$$

Thus

$$e^{2hx\cos\pi/4} \quad = \quad x^2$$

and

$$e^{-2ihx\sin\pi/4} \quad = \quad e^{-i\pi/2}.$$

We now eliminate h and solve for x, so that

$$i\log x^2 = \frac{i\pi}{2} + 2in\pi \qquad (n \in \mathbb{Z}),$$

whence $x = \exp(\frac{\pi}{4} + n\pi)$, and

$$h = -\frac{\frac{\pi}{2} + 2n\pi}{\sqrt{2}\exp(\pi/4 + n\pi)}.$$

The smallest positive value of h occurs at $n = -1$, giving $h = \dfrac{3\pi}{2\sqrt{2}}e^{3\pi/4}$. Now, it is straightforward to check that for very small positive values of h the transfer function G_h is asymptotically stable, and so it remains stable until the first pole-crossing, which is at $h = \dfrac{3\pi}{2\sqrt{2}}e^{3\pi/4}$.

It is possible to show that the poles cross from left to right as h increases by calculating $\dfrac{\partial s}{\partial h}$ at a point where $\sqrt{s} + \exp(-h\sqrt{s}) = 0$. Similar calculations are done for delay systems in [16] and [15]. Indeed, we have

$$\frac{1}{2\sqrt{s}}\frac{\partial s}{\partial h} - \left[\sqrt{s}\exp(-h\sqrt{s}) + \frac{h}{2\sqrt{s}}\frac{\partial s}{\partial h}\right]\exp(-h\sqrt{s}) = 0,$$

from which it is easy to obtain a formula for $\dfrac{\partial s}{\partial h}$.

□

Remark 7. This leads to a general method for finding zero-crossings of a transfer function $p(s) + q(s)\exp(-hs^\alpha)$ with $0 < \alpha < 1$; it involves setting $u = s^\alpha = xe^{i\pi\alpha/2}$, with $x > 0$, and so

$$P(xe^{i\pi\alpha/2}) + Q(xe^{i\pi\alpha/2})\exp(-hxe^{i\pi\alpha/2}) = 0,$$

with a second equation obtained by complex conjugation, namely,

$$P(xe^{-i\pi\alpha/2}) + Q(xe^{-i\pi\alpha/2})\exp(-hxe^{-i\pi\alpha/2}) = 0.$$

From these two equations one can eliminate h and solve for x. Next, by solving for h one obtains the values of the delay for which the poles of (3) cross the axis.

4 Conclusions

The general question of BIBO stability of a linear system given in terms of a transfer function is difficult in general; although our methods now enable us to resolve the question for many systems, the case of G_1, as defined in (1) remains open. Recall also that BIBO stability is a necessary condition for the Hankel operator of a linear system to be *nuclear*, a property that has certain implications for model reduction [8], and several associated questions remain open.

For fractional systems, stability issues seem to be somewhat easier to decide; future research is expected to include the analysis of a more general class of systems that can be presented by means of a diffusive representation [13].

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