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Representing multipliers of the Fourier algebra on non-commutative L^p spaces

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Abstract

We show that the multiplier algebra of the Fourier algebra on a locally compact group G can be isometrically represented on a direct sum on non-commutative L^p spaces associated to the right von Neumann algebra of G. If these spaces are given their canonical Operator space structure, then we get a completely isometric representation of the completely bounded multiplier algebra. We make a careful study of the non-commutative L^p spaces we construct, and show that they are completely isometric to those considered recently by Forrest, Lee and Samei; we improve a result of theirs about module homomorphisms. We suggest a definition of a Figa-Talamanca–Herz algebra built out of these non-commutative L^p spaces, say $A_p(\hat{G})$. It is shown that $A_2(\hat{G})$ is isometric to $L^1(G)$, generalising the abelian situation.

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1 Introduction

The Fourier algebra $A(G)$ is, for a locally compact group G, the space of coefficient functionals $s \mapsto (\lambda(s)\xi|\eta)$ for $s \in G$, where $\xi, \eta \in L^2(G)$. Here λ denotes the left-regular representation of G on $L^2(G)$. For an abelian group, $A(G)$ is nothing but the Fourier transform of $L^1(\hat{G})$, where $\hat{G}$ is the Pontryagin dual of G. Eymard defined $A(G)$ for general G in [7]. We can also identify $A(G)$ as the predual of the group von Neumann algebra $VN(G)$, see [34, Chapter VII, Section 3].

In this paper we shall be interested in the multiplier algebra of $A(G)$. This can either be thought of abstractly as the double centraliser algebra (see [17]) of $A(G)$, or, as $A(G)$ is a regular algebra of functions on G, as the space of continuous functions f such that $fa \in A(G)$ for each $a \in A(G)$, see [31] for example. There is now much evidence that $A(G)$ is often best viewed as an *operator space*, when given the standard operator space structure as the predual of $VN(G)$. Then it is natural to consider only the *completely bounded* multipliers, leading to $M_{cb}A(G)$ (see [31] or [4]). In [24] a representation of $M_{cb}A(G)$ on $\mathcal{CB}(B(L^2(G)))$ was defined, extending a representation of $M(G)$ defined in [9]. It was shown that these representations are commutants of each other, hence in some sense extending Pontryagin duality. Similar ideas were considered for Kac algebras in [21] and have been extended (along with the commutation ideas) to Locally Compact Quantum Groups in [18].

In is shown in [4] that both $MA(G)$ and $M_{cb}A(G)$ are dual spaces, in such a way that the algebra products are separately weak*-continuous (so these are *dual Banach algebras*); see also [31, Section 6.2]. Now, $\mathcal{CB}(B(L^2(G)))$ is also a dual Banach algebra, and the representation of $M_{cb}A(G)$ constructed in [24] is weak*-weak*-continuous. However, it was shown in [3, Corollary 3.8] (and extended in [36] to the completely bounded case) that a dual Banach algebra $\mathcal{A}$ admits an isometric, weak*-weak*-continuous representation on $\mathcal{B}(E)$ for some *reflexive* Banach space E. The space E is built as the large direct sum of real interpolation spaces, and is rather abstract.

In this paper, we shall show that we can represent $MA(G)$ on a direct sum of non-commutative L^p spaces associated to $VN(G)$; we can also represent $M_{cb}A(G)$ on the same space, if it is equipped with the canonical operator space structure. Indeed, our construction is motivated by that of Young in [39]; as Young didn't consider multipliers, we sketch his ideas in Section 2 below.

Once we have motivated looking at (non-commutative) L^p spaces, we discuss weights on $VN(G)$ and non-commutative L^p for (possibly) non-semifinite von Neumann algebras in Section 3. This will involve introducing the complex interpolation method. In Section 4 we apply these ideas to the Fourier algebra, leading to a scale of spaces $L^p(\hat{G})$, for $1 < p < \infty$, which are $A(G)$-modules. We make a careful study of these spaces, and prove some approximation results which allow us to work with functions instead of abstract operators in the von Neumann algebra. With this perspective, the $A(G)$-module actions are just point-wise multiplication of functions. We show that our spaces are (completely) isometrically isomorphic to the two families of spaces constructed in [8, Section 6]. We think that our construction is easier and more natural than that of [8], although we have to worry more about the details of the complex interpolation method. The payoff is that, for example, we can easily extend a cohomological result from [8], which we can show to hold for all values of p (and not just $p \geq 2$).

In Section 5 we prove our representation result. Let $p_n \to 1$ in $(1, \infty)$, and let E be the ℓ^2 direct sum of the spaces $L^{p_n}(\hat{G})$. Then $MA(G)$ is weak*-weak*-continuously isometric to the idealiser of $A(G)$ in $\mathcal{B}(E)$. If we equip E with the canonical operator space structure, then $M_{cb}A(G)$ is weak*-weak*-continuously completely isometric to the idealiser of $A(G)$ in $\mathcal{CB}(E)$. As arguments involving multipliers often using bounded approximate identities, it's worth stressing that our results hold for all locally compact groups G. As hinted at in Section 2, Figa-Talamanca–Herz algebras make a natural appearance, and with our new tools, we define a notion of what $A_p(\hat{G})$ should be for a non-abelian group G. We show that $A_2(\hat{G})$ is canonically isometric to $L^1(G)$, but we have been unable to decide if $A_p(\hat{G})$ is always an algebra.

For Banach algebra notions, we follow [2] and [25]; we always write E^* for the dual of a Banach or Operator space E, reserving the notation A' for the commutatant. We shall only use standard facts about Operator spaces, for which we refer the reader to [5] and [28]. In the few places where we use matrix calculations, we shall simply write $\|\cdot\|$ for the norm on $\mathbb{M}_n(E)$, for any n.

2 Group convolution algebras

In this section we quickly review Young's construction in [39, Theorem 4], as applied to multipliers. Let G be a locally compact group, and consider the group convolution algebra $L^1(G)$. The multiplier algebra of $L^1(G)$ can be isometrically isomorphically identified with $M(G)$, the measure algebra of G. This is Wendel's theorem, [37] or [2, Theorem 3.3.40].

Let (p_n) be some sequence in $(1, \infty)$ converging to 1. Let E be the direct sum, in an ℓ^2 sense, of the spaces $L^{p_n}(G)$. To be exact, E consists of sequences (ξ_n) where, for each n, $\xi_n \in L^{p_n}(G)$, with

$$\|(\xi_n)\| := \Big(\sum_n \|\xi_n\|_{p_n}^2 \Big)^{1/2} < \infty.$$

Thus E is reflexive. Then $M(G)$ acts contractively on each $L^{p_n}(G)$ space by convolution, and hence also on E, leading to a contractive homomorphism $\theta : M(G) \to \mathcal{B}(E)$.

Theorem 2.1. *With notation as above, θ is isometric and weak*-weak*-continuous.*

We first introduce some further concepts. We write $\widehat{\otimes}$ for the (completed) projective tensor product (see [2, Appendix A3] for example). For any reflexive Banach space F, we thus have that $\mathcal{B}(F) = (F\widehat{\otimes}F^*)^*$. Let $\lambda_p : L^1(G) \to \mathcal{B}(L^p(G))$ be the left-regular representation, and let $(\lambda_p)_* : L^p(G)\widehat{\otimes}L^{p'}(G) \to L^\infty(G)$ be the adjoint. Here p' is the conjugate index to p, so that $L^p(G)^* = L^{p'}(G)$. For $a \in L^1(G), \xi \in L^p(G)$ and $\eta \in L^{p'}(G)$, we see that

$$\langle (\lambda_p)_*(\xi \otimes \eta), a\rangle = \langle \eta, \lambda_p(a)(\xi)\rangle = \int_G \int_G \eta(t)a(s)\xi(s^{-1}t)\ ds\ dt = \langle \omega_{\xi,\eta}, a\rangle.$$

Here $\omega_{\xi,\eta}$ denotes the function $s \mapsto \int_G \xi(s^{-1}t)\eta(t)\ dt$. Thus $\omega_{\xi,\eta}$ is a member of the Figa-Talamanca–Herz algebra $A_p(G)$, identified as a subalgebra of $C_0(G) \subseteq L^\infty(G)$. For further details see [12, 13].

This then suggests an abstract way to define $\tilde{\theta} : M(G) \to \mathcal{B}(L^p(G))$, namely

$$\langle \eta, \tilde{\theta}(\mu)(\xi)\rangle = \langle \mu, \omega_{\xi,\eta}\rangle \qquad (\mu \in M(G), \xi \in L^p(G), \eta \in L^{p'}(G)).$$

By the above calculation, this extends θ. Furthermore, if $\xi, \eta \in C_{00}(G)$, the space of compactly support continuous functions, then $\xi \in L^p(G)$, $\eta \in L^{p'}(G)$, and for $\mu \in M(G)$ we see that

$$\langle \eta, \tilde{\theta}(\mu)(\xi)\rangle = \int_G \int_G \xi(s^{-1}t)\eta(t)\ dt\ d\mu(s) = \langle \eta, \mu * \xi\rangle,$$

where $\mu * \xi$ has the unambiguous meaning of μ convolved with ξ. As such ξ and η are dense, we are justified in saying that $\tilde{\theta}$ is simply the convolution action of $M(G)$ on $L^p(G)$.

Proof of Theorem 2.1. Consider the adjoint map $\theta_* : E\widehat{\otimes}E^* \to M(G)^*$ given by

$$\langle \theta_*(\xi \otimes \eta), \mu\rangle = \langle \eta, \theta(\mu)(\xi)\rangle = \sum_n \langle \eta_n, \tilde{\theta}(\mu)(\xi_n)\rangle = \sum_n \langle \mu, \omega_{\xi_n,\eta_n}\rangle,$$

where $\xi = (\xi_n) \in E, \eta = (\eta_n) \in E^*$ and $\mu \in M(G)$. In particular, θ_* maps into $C_0(G)$, the predual of $M(G)$, so that θ is weak*-weak*-continuous.

For $f, g \in C_{00}(G)$, we have that $\omega_{f,g} = g * \check{f}$ as functions, where $\check{f}(s) = f(s^{-1})$ for $s \in G$. Furthermore, we have that

$$\lim_{p\to 1} \|f\|_p = \|f\|_1, \quad \lim_{p'\to\infty} \|g\|_{p'} = \|g\|_\infty.$$

For any $g \in C_{00}(G)$ and $\epsilon > 0$, we can find some $f \in C_{00}(G)$ with $\|f\|_1 = 1$ and $\|g*\check{f}-g\|_\infty < \epsilon$ (for example, see the proof of [2, Lemma 3.3.22]). As $p_n \to 1$, we can find n with $\|g\|_{p'_n} < (1+\epsilon)\|g\|_\infty$ and $\|f\|_{p_n} < 1+\epsilon$. It follows that

$$|\langle \mu, g\rangle| \geq |\langle \mu, \omega_{f,g}\rangle| - \epsilon\|\mu\|,$$

and that

$$\|\omega_{f,g}\|_{A_{p_n}(G)} \leq \|f\|_{p_n}\|g\|_{p'_n} < (1+\epsilon)^2\|g\|_\infty.$$

By taking suitable supremums, it now follows easily that θ is an isometry. ☐

For a Banach algebra $\mathcal{A}$, we say that $\mathcal{A}$ is *faithful* if for $a \in \mathcal{A}$, when $bac = 0$ for all $b, c \in \mathcal{A}$, then $a = 0$. We shall always assume that our algebras are faithful: notice that if $\mathcal{A}$ is unital, or has an approximate identity, then $\mathcal{A}$ is faithful. A pair (L, R) of linear maps $A \to A$ is a *multiplier* (or *centraliser*) if

$$L(ab) = L(a)b, \quad R(ab) = aR(b), \quad aL(b) = R(a)b \qquad (a, b \in \mathcal{A}).$$

The Closed Graph Theorem then shows that $L, R \in \mathcal{B}(\mathcal{A})$. For further details see [17], [2] or [25, Section 1.2]. Indeed, [25, Theorem 1.2.4] shows that if $L, R : \mathcal{A} \to \mathcal{A}$ are any maps with

$aL(b) = R(a)b$ for $a, b \in \mathcal{A}$, then (L, R) is already a multiplier. Let $M(\mathcal{A})$ be the space of multipliers, normed by embedding into $\mathcal{B}(\mathcal{A}) \times \mathcal{B}(\mathcal{A})$, and made into an algebra for the product $(L, R)(L', R') = (LL', R'R)$. Notice that $\mathcal{A}$ embeds (as $\mathcal{A}$ faithful) into $M(\mathcal{A})$ by $a \mapsto (L_a, R_a)$ where $L_a(b) = ab, R_a(b) = ba$ for $a, b \in \mathcal{A}$.

Then Wendel's Theorem tells us that for $(L, R) \in M(L^1(G))$ there exists a unique $\mu \in M(G)$ such that $L(a) = \mu a$ and $R(a) = a\mu$ for $a \in L^1(G)$. Indeed, from the proof of [2, Theorem 3.3.40], we have that μ is the weak*-limit of $(L(e_\alpha))$ in $M(G)$, where (e_α) is a bounded approximate identity for $L^1(G)$. It is then easy to show that $L(a) = \mu a$ for $a \in L^1(G)$. Notice then that $R(a)b = aL(b) = a(\mu b) = (a\mu)b$ for $a, b \in L^1(G)$, so as $L^1(G)$ is faithful, $R(a) = a\mu$ as required.

Theorem 2.2. *With notation as above, the image of $\tilde{\theta} : M(G) \to \mathcal{B}(E)$ is exactly the* idealiser *of $\theta(L^1(G))$, namely*

$$\mathcal{I} = \{T \in \mathcal{B}(E) : T\theta(a), \theta(a)T \in \theta(L^1(G)) \ (a \in L^1(G))\}.$$

Proof. For $\mu \in M(G)$, we have $\tilde{\theta}(\mu)\theta(a) = \theta(\mu a)$ and $\theta(a)\tilde{\theta}(\mu) = \theta(a\mu)$ for $a \in L^1(G)$, so that $\tilde{\theta}(M(G)) \subseteq \mathcal{I}$.

Conversely, let $T \in \mathcal{I}$ and define $L, R : L^1(G) \to L^1(G)$ by

$$L(a) = \theta^{-1}\big(T\theta(a)\big), \quad R(a) = \theta^{-1}\big(\theta(a)T\big) \qquad (a \in L^1(G)),$$

which makes sense, as θ is injective onto its range. For $a, b \in L^1(G)$ we see that $\theta(a)\theta(L(b)) = \theta(a)T\theta(b) = \theta(R(a))\theta(b)$, so that $aL(b) = R(a)b$. Thus $(L, R) \in M(L^1(G))$. Hence there exists $\mu \in M(G)$ with $L(a) = \mu a$ for $a \in L^1(G)$, so that $\tilde{\theta}(\mu)\theta(a) = T\theta(a)$ for $a \in L^1(G)$.

By the construction of E, we see that $\{\theta(a)\xi : a \in L^1(G), \xi \in E\}$ is linearly dense in E, from which it follows that $T = \tilde{\theta}(\mu)$, completing the proof. $\square$

Notice that we implicitly used the Closed Graph Theorem, in invoking [25, Theorem 1.2.4]. In the completely bounded setting, this would not be available to us, and indeed, it is unclear to the author if a direct analogue of this result would be true. However, if $\mathcal{A}$ is commutative (or has a bounded approximate identity) that L and R are closely related, allowing a modification of the proof to work, see Theorem 5.4 below. In relation to this, it is interesting to note that [18] works with *one-sided* multipliers (or centralisers).

It is classical that $L^p(G)$ can be described as a complex interpolation space between $L^1(G)$ and $L^\infty(G)$; see below for definitions, or [1, Chapter 4]. We can recover the action of $L^1(G)$ on $L^p(G)$ by interpolation, but some care is needed. Indeed, obviously $L^1(G)$ is an $L^1(G)$-bimodule over itself, and so by duality, $L^\infty(G)$ is an $L^1(G)$-bimodule. However, notice that the resulting left action of $L^1(G)$ on $L^\infty(G)$ is *not* the usual convolution action. With this in mind, the constructions in Section 4 below should appear less artificial.

3 Non-commutative L^p spaces

In this section we sketch the complex interpolation approach to non-commutative L^p spaces, see [35] and [15].

3.1 Weights on group von Neumann algebras

For a locally compact group G, let λ and ρ be, respectively, the left- and right-regular representations, defined by

$$\big(\lambda(s)\xi\big)(t) = \xi(s^{-1}t), \quad \big(\rho(s)\xi\big)(t) = \rho(ts)\nabla(s)^{1/2} \qquad (\xi \in L^2(G), s, t \in G).$$

Here ∇ is the modular function on G. For $f \in L^1(G)$, we shall write $\lambda(f)$ and $\rho(f)$ for the operators induced by integration, for example

$$\big(\rho(f)\xi\big)(s) = \int_G f(t)\xi(st)\nabla(t)^{1/2}\,dt \qquad (\xi \in L^2(G)).$$

Then the group von Neumann algebra $VN(G)$ is the von Neumann algebra generated by λ, so $VN(G) = \lambda(G)''$. Similarly, the right group von Neumann algebra, denoted here by $VN_r(G)$, is generated by ρ. We have that $VN(G)' = VN_r(G)$ and $VN_r(G)' = VN(G)$, see [34, Chapter VII, Section 3].

An alternative way to construct $VN(G)$ is to start with $C_{00}(G)$, considered as a left Hilbert algebra. The inner-product is inherited from $L^2(G)$, the product is convolution, and the involution is $f^\sharp(s) = \overline{f(s^{-1})}\nabla(s)^{-1}$ for $f \in C_{00}(G), s \in G$. See [34] or [33] for further details on left Hilbert algebras. One word of caution: for $f \in C_{00}(G)$ (or more generally, for *right bounded* elements of $L^2(G)$) we can define $\pi_r(f) \in VN_r(G)$ (using the notation of [34]). This is *not* equal to $\rho(f)$; we have $\pi_r(f) = \rho(K(f))$ for K defined below.

At this point, we shall stress that henceforth, for functions a, b on G, we denote the convolution product by ab (when this makes sense) and the point-wise product by $a \cdot b$. An exception is that ∇ always acts by point-wise multiplication.

The left Hilbert algebra leads naturally to a weight φ on $VN(G)$. This weight is explored in detail by Haagerup in [11, Section 2]. We let $\mathfrak{n}_\varphi = \{x \in VN(G) : \varphi(x^*x) < \infty\}$ and $\mathfrak{m}_\varphi = \operatorname{lin} \mathfrak{n}_\varphi^* \mathfrak{n}_\varphi$, and extend φ to $\mathfrak{m}_\varphi$ in the usual way. Let us just note that

$$\varphi(\lambda(f)) = f(e_G) \qquad (f \in C_{00}(G)^2),$$

where e_G is the unit of G, and $C_{00}(G)^2 = \operatorname{lin}\{fg : f, g \in C_{00}(G)\}$.

Let (π, H, Λ) be the GNS construction for $(VN(G), \varphi)$. We may hence identify H with $L^2(G)$ by $\Lambda(\lambda(f)) = f$ for $f \in C_{00}(G)$ (or more generally for left bounded $f \in L^2(G)$). Henceforth we shall drop π and always regard $VN(G)$ as acting on $L^2(G)$. The *modular conjugation* is the map

$$J : L^2(G) \to L^2(G), \quad J\xi(s) = \overline{\xi(s^{-1})}\nabla(s)^{-1/2} \qquad (\xi \in L^2(G), s \in G).$$

We define a linear version of J to be K, where $K(\xi) = J(\overline{\xi})$ for $\xi \in L^2(G)$. We define the "check map" by $\check{\xi}(s) = \xi(s^{-1})$, so $K\xi = \check{\xi}\nabla^{-1/2}$. We have that $VN_r(G) = VN(G)' = JVN(G)J$, and

$$\lambda(f) = J\rho(\overline{f})J = K\rho(f)K \qquad (f \in L^1(G)).$$

The *modular operator* is given by point-wise multiplication by ∇, and this leads to the *modular automorphism group* $(\sigma_t)_{t\in\mathbb{R}}$ given by $\sigma_t(\cdot) = \nabla^{it}(\cdot)\nabla^{-it}$.

We shall pick a canonical choice of weight φ' on $VN_r(G)$ by

$$\varphi'(x) = \varphi(Jx^*J) \qquad (x \in VN_r(G)^+).$$

Then $\mathfrak{m}_{\varphi'} = J\mathfrak{m}_\varphi J$ and the formula above defines φ' on $\mathfrak{m}_{\varphi'}$. Let (π', H', Λ') be the GNS construction for φ'. We can identify H' with H by $\Lambda'(x) = J\Lambda(JxJ)$ for $x \in \mathfrak{n}_{\varphi'} = J\mathfrak{n}_\varphi J$. Hence we identify H' with $L^2(G)$ by

$$\Lambda'(\rho(f)) = J\Lambda(\lambda(\overline{f})) = K(f) \qquad (f \in C_{00}(G)).$$

Again, we suppress π' and regard $VN_r(G)$ as acting on $L^2(G)$. Then the modular conjugation for φ' is simply J. The modular automorphism group for φ' is $(\sigma'_t)_{t\in\mathbb{R}}$, and this is given by $\sigma'_t(x) = J\sigma_t(JxJ)J$ for $x \in VN_r(G)$. Some care is required when analytically extending this to complex values; indeed, we have $\sigma'_z(x) = J\sigma_{\overline{z}}(JxJ)J$ for analytic x and $z \in \mathbb{C}$. Consequently

$$\sigma'_z(\rho(f)) = \rho\big(\nabla^{-i\overline{z}}f\big) \qquad (f \in C_{00}(G), z \in \mathbb{C}).$$

3.2 Non-commutative L^p spaces

There is a long history to non-commutative L^p spaces, for which we refer the reader to [29]. For a von Neumann algebra $\mathcal{M}$ with a finite normal trace τ, we can simply let $L^p(\mathcal{M}, \tau)$ be the completion of $\mathcal{M}$ with respect to the norm $\|x\|_p = \tau((x^*x)^{p/2})$, for $1 \leq p < \infty$. Similar remarks apply to semi-finite traces, although the framework of "measurable operators" gives a realisation of the completed space. See [34, Chapter IX, Section 2] for further details.

For a general von Neumann algebra which might only admit a weight, Haagerup introduced a crossed-product construction of a non-commutative L^p space in [10]. Building on work of Connes, Hilsum provided a spatial definition of a non-commutative L^p space in [14], and showed that the resulting space was isometrically isomorphic to Haagerup's. By analogy with the commutative case, we might expect the complex interpolation method to play a role. In [20], Kosaki provided a construction of a non-commutative L^p space associated to a von Neumann algebra with a finite weight (that is, a normal state) using the complex interpolation method. He showed that his space is isometrically isomorphic to Haagerup's. In [35], Terp extended a special case of Kosaki's construction to the semi-finite case, and she showed that her L^p space is isometrically isomorphic to Hilsum's (and hence to Haagerup's).

We shall instead follow Izumi's construction in [15], which simultaneously generalises Kosaki's and Terp's constructions. Of particular interest is that in [16], Izumi makes a detailed study of his spaces, introducing bilinear and sesquilinear products, and showing that his L^2 spaces are canonically isometrically isomorphic to the standard Hilbert space constructed from the underlying weight. As such, Izumi's constructions are self-contained (although we note that, technically, he relies upon Terp's work in a proof in [15]).

First let us define the complex interpolation method. See [1], [28, Section 2.7] for further details. A *compatible couple* of Banach spaces is a pair (E_0, E_1) continuously embedded into a Hausdorff topological vector space X. We can then make sense of the spaces $E_0 \cap E_1$ and $E_0 + E_1$, and define norms on them by

$$\|x\| = \max(\|x\|_{E_0}, \|x\|_{E_1}) \qquad (x \in E_0 \cap E_1),$$
$$\|x\| = \inf\{\|a\|_{E_0} + \|b\|_{E_1} : x = a + b, a \in E_0, b \in E_1\} \qquad (x \in E_0 + E_1).$$

We need X to be Hausdorff to ensure that we get a norm on $E_0 + E_1$. However, once we can form $E_0 + E_1$, in what follows, we can always just replace X by $E_0 + E_1$.

Let $\mathcal{S} = \{z = x + iy \in \mathbb{C} : 0 \leq x \leq 1, y \in \mathbb{R}\}$ and $\mathcal{S}_0 = \{z = x + iy \in \mathbb{C} : 0 < x < 1, y \in \mathbb{R}\}$. We let $\mathcal{F}$ be the space of functions $f : \mathcal{S} \to E_0 + E_1$ such that:

1. f is continuous and bounded, and analytic on $\mathcal{S}_0$;
2. for $j = 0, 1$, we have that $\mathbb{R} \mapsto E_j; t \mapsto f(j + it)$ is continuous, bounded, and tends to 0 as $|t| \to \infty$.

For more details on vector-valued analytic functions, see [34, Appendix] for example. We give $\mathcal{F}$ a norm by setting

$$\|f\| = \max_{j=0,1} \sup_{t \in \mathbb{R}} \|f(j + it)\|_{E_0} \qquad (f \in \mathcal{F}).$$

This is a norm, and then $\mathcal{F}$ becomes a Banach space.

For $0 \leq \theta \leq 1$, we define $(E_0, E_1)_{[\theta]}$ to be the subspace of $E_0 + E_1$ consisting of those x such that $x = f(\theta)$ for some $f \in \mathcal{F}$, together with the quotient norm

$$\|x\|_{[\theta]} = \inf\{\|f\| : f \in \mathcal{F}, f(\theta) = x\}.$$

The following is proved in [1, Theorem 4.1.2].

Theorem 3.1. *With notation as above, we have norm decreasing inclusions* $E_0 \cap E_1 \to (E_0, E_1)_{[\theta]} \to E_0 + E_1$. *Let* (F_0, F_1) *be another pair of compatible Banach spaces, and let* $T : E_0 + E_1 \to F_0 + F_1$ *be a linear map such that for* $j = 0, 1$, $T(E_j) \subseteq F_j$ *and the restriction* $T : E_j \to F_j$ *is bounded. Then*

$$T\big((E_0, E_1)_{[\theta]}\big) \subseteq (F_0, F_1)_{[\theta]}, \quad \|T\| \leq \|T : E_0 \to F_0\|^{1-\theta} \|T : E_1 \to F_1\|^{\theta}.$$

Lemma 3.2. *With notation as above, for* $j = 0, 1$ *let* $T_j \in \mathcal{B}(E_j, F_j)$. *There exists* $T : E_0 + E_1 \to F_0 + F_1$ *with* $T|_{E_j} = T_j$ *for* $j = 0, 1$ *if, and only if,* T_0 *and* T_1 *map* $E_0 \cap E_1$ *into* $F_0 \cap F_1$ *and agree on* $E_0 \cap E_1$.

Proof. If T_0 and T_1 agree on $E_0 \cap E_1$ and map into $F_0 \cap F_1$, then we try to define T by $T(x_0 + x_1) = T_0(x_0) + T_1(x_1)$ for $x_0 \in E_0, x_1 \in E_1$. This is well-defined, for if $x_0 + x_1 = x_0' + x_1'$ then $x_0 - x_0' = x_1' - x_1 \in E_0 \cap E_1$ and so $T_0(x_0) - T_0(x_0') = T_1(x_1') - T_1(x_1) \in F_0 \cap F_1$. The converse is clear. $\square$

There is also a bilinear version, see [1, Theorem 4.4.1].

Theorem 3.3. *Let* (E_0, E_1), (F_0, F_1) *and* (G_0, G_1) *be compatible couples, and let* $T : E_0 \cap E_1 \times F_0 \cap F_1 \to G_0 \cap G_1$ *be a bilinear map such that for some constants* M_0, M_1, *we have*

$$\|T(x_j, y_j)\|_{G_j} \leq M_j \|x_j\|_{E_j} \|y_j\|_{F_j} \qquad (j = 0, 1, x_j \in E_j, y_j \in F_j).$$

For $0 < \theta < 1$, *there is a bilinear map*

$$T_\theta : (E_0, E_1)_{[\theta]} \times (F_0, F_1)_{[\theta]} \to (G_0, G_1)_{[\theta]},$$

which extends T, *and which is bounded by* $M_0^{1-\theta} M_1^{\theta}$.

Now let $\mathcal{M}$ be a von Neumann algebra with normal semi-finite weight φ. Let (H, Λ) be the GNS construction, where we identify $\mathcal{M}$ with a subalgebra of $\mathcal{B}(H)$. Let J be the modular conjugation, and ∇ the modular operator. We shall now sketch Izumi's approach to non-commutative L^p spaces. The idea is to turn $(\mathcal{M}, \mathcal{M}_*)$ into a compatible couple; then $L^p(\varphi)$ will be defined as $(\mathcal{M}, \mathcal{M}_*)_{[1/p]}$, for $1 < p < \infty$.

Let (H, Λ) be a GNS construction for φ, so that $\mathfrak{A} = \Lambda(\mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*)$ is a full left Hilbert algebra in H, which generates $\mathcal{M}$ and induces φ. Let $\mathfrak{A}_0$ be the maximal Tomita algebra associated to $\mathfrak{A}$, see [34, Chapter VI, Section 2], and let $\mathfrak{a}_0 = \Lambda^{-1}(\mathfrak{A}_0)$. In particular, each $x \in \mathfrak{a}_0$ is analytic for (σ_t), and $\Lambda(x)$ is in the domain of ∇^α for each $\alpha \in \mathbb{C}$.

For $\alpha \in \mathbb{C}$, we let $L_{(\alpha)}$ be the collection of those $x \in \mathcal{M}$ such that there exists $\varphi_x^{(\alpha)} \in \mathcal{M}_*$ with $\langle y^* z, \varphi_x^{(\alpha)} \rangle = (x J \nabla^{\overline{\alpha}} \Lambda(y) | J \nabla^{-\alpha} \Lambda(z))$ for $y, z \in \mathfrak{a}_0$. Then $L_{(\alpha)}$ is a subspace of $\mathcal{M}$ which contains $\mathfrak{a}_0^2$, and is hence σ-weakly dense. We norm $L_{(\alpha)}$ by setting $\|x\|_{L_{(\alpha)}} = \max(\|x\|_{\mathcal{M}}, \|\varphi_x^{(\alpha)}\|_{\mathcal{M}_*})$ for $x \in L_{(\alpha)}$. Let $i_{(\alpha)} : L_{(\alpha)} \to \mathcal{M}$ be the inclusion map, and let $j_{(\alpha)} : L_{(\alpha)} \to \mathcal{M}_*$ be the map $x \mapsto \varphi_x^{(\alpha)}$. These are contractive injections, and $j_{(\alpha)}$ has norm dense range. Izumi proves that we have the following commuting diagram

$$\begin{tikzcd}
& \mathcal{M} \arrow[dr, "j^*_{(-\alpha)}"] & \\
L_{(\alpha)} \arrow[ur, "i_{(\alpha)}"] \arrow[dr, "j_{(\alpha)}"'] & & L^*_{(-\alpha)} \\
& \mathcal{M}_* \arrow[ur, "i^*_{(-\alpha)}"'] &
\end{tikzcd}$$

In particular, we have that $\langle y, \varphi_x^{(\alpha)}\rangle = \langle x, \varphi_y^{(-\alpha)}\rangle$ for $x \in L_{(\alpha)}$ and $y \in L_{(-\alpha)}$. By density we have that $i^*_{(-\alpha)}$ and $j^*_{(-\alpha)}$ are injective, and so we can view $(\mathcal{M}, \mathcal{M}_*)$ as a compatible couple. Izumi shows that under this identification, $\mathcal{M} \cap \mathcal{M}_*$ is precisely $L_{(\alpha)}$. We finally set

$$L^p_{(\alpha)}(\varphi) = \big(\mathcal{M}, \mathcal{M}_*\big)_{[1/p]} \qquad (1 < p < \infty).$$

We shall always view $L^p_{(\alpha)}(\varphi)$ as a subspace of $\mathcal{M} + \mathcal{M}_*$; consequently, by the commuting diagram and Theorem 3.1, we have that $j^*_{(-\alpha)}(x) \in L^p_{(\alpha)}(\varphi)$ for all $x \in L$ and all p.

For most of this paper, we shall actually work just with the case $\alpha = 0$, which is exactly the case which Terp considers in [35]. Set $L = L_{(0)}$, so we actually have the stronger property that $x \in L$ when there exists $\varphi_x \in \mathcal{M}_*$ with

$$\langle y^* z, \varphi_x\rangle = (Jx^*J\Lambda(z)|\Lambda(y)) = (xJ\Lambda(y)|J\Lambda(z)) \qquad (y, z \in \mathfrak{n}_\varphi).$$

As shown in [16], there are bilinear maps which satisfy

$$\langle\cdot,\cdot\rangle_{p,(\alpha)} : L^p_{(\alpha)}(\varphi) \times L^{p'}_{(-\alpha)}(\varphi) \to \mathbb{C}; \quad \langle j^*_{(-\alpha)}(x), j^*_{(\alpha)}(y)\rangle = \langle y, \varphi_x^{(\alpha)}\rangle = \langle x, \varphi_y^{(-\alpha)}\rangle,$$

where $1/p + 1/p' = 1$. There are sesquilinear maps which satisfy

$$(\cdot|\cdot)_{p,(\alpha)} : L^p_{(\alpha)}(\varphi) \times L^{p'}_{(\overline{\alpha})}(\varphi) \to \mathbb{C}; \quad (j^*_{(-\alpha)}(x)|j^*_{(-\overline{\alpha})}(y))_{p,(\alpha)} = \langle y^*, \varphi_x^{(\alpha)}\rangle = \overline{\langle x^*, \varphi_x^{(\overline{\alpha})}\rangle}.$$

Furthermore, these maps implement dualities between $L^p_{(\alpha)}(\varphi)$ and $L^{p'}_{(-\alpha)}(\varphi)$, and between $L^p_{(\alpha)}(\varphi)$ and $L^{p'}_{(\overline{\alpha})}(\varphi)$, respectively. As such, the dual of $L^p_{(0)}(\varphi)$ can be identified with $L^{p'}_{(0)}(\varphi)$, both linearly and anti-linearly.

We can identify $L^2_{(-1/2)}(\varphi)$ with H_φ, the standard GNS space for φ. Indeed, there is an isometric isomorphism

$$h : H_\varphi \to L^2_{(-1/2)}(\varphi); \quad h(\Lambda(x)) = j^*_{(-1/2)}(x) \qquad (x \in \mathfrak{n}_\varphi).$$

Furthermore, h respects the relevant inner-products, that is

$$(\xi|\eta) = (h(\xi)|h(\eta))_{2,(-1/2)} \qquad (\xi, \eta \in H_\varphi).$$

We can translate this to other values of α by using the fact that there are isometric isomorphisms

$$U_{p,(\beta,\alpha)} : L^p_{(\alpha)}(\varphi) \to L^p_{(\beta)}(\varphi); \quad U_{p,(\beta,\alpha)}(j^*_{(-\alpha)}(x)) = j^*_{(-\beta)}(\sigma_{i(\beta-\alpha)/p}(x)) \qquad (x \in \mathfrak{a}_0^2, \alpha, \beta \in \mathbb{R}).$$

Then, again for $\alpha, \beta \in \mathbb{R}$, we have that

$$(U_{p,(\beta,\alpha)}(\xi)|U_{p,(\beta,\alpha)}(\eta))_{p,(\beta)} = (\xi|\eta)_{p,(\alpha)} \qquad (\xi, \eta \in L^p_{(\alpha)}(\varphi)).$$

In particular, there is an isometric isomorphism $k : H_\varphi \to L^2_{(0)}(\varphi)$ with

$$k(\Lambda(x)) = j^*_{(0)}(\sigma_{i/4}(x)) \quad (x \in \mathfrak{a}_0^2), \qquad (\xi|\eta) = (k(\xi)|k(\eta))_{2,(0)} \quad (\xi, \eta \in H_\varphi).$$

Using convergence theorems for integration, it is easy to show that if (X, μ) is a measure space, and $f \in L^1(\mu) \cap L^\infty(\mu)$, then $f \in L^p(\mu)$ for all $p \in (1, \infty)$, and $\lim_{p\to 1} \|f\|_p = \|f\|_1$. The following is a non-commutative version of this.

Proposition 3.4. *With notation as above, let $x \in L$. Then $\lim_{p\to 1} \|j^*_{(0)}(x)\|_p = \|\varphi_x\|$, where $\|\cdot\|_p$ denotes the norm on $L^p_{(0)}(\varphi)$.*

Proof. Firstly, we show that

$$\|j^*_{(0)}(x)\|_p \leq \|x\|^{1/p'}\|\varphi_x\|^{1/p} \qquad (x \in L).$$

This is [32, Corollary 2.8], but we give a quick proof. Pick $\epsilon > 0$ and define $F : \mathcal{S} \to L$ by $F(z) = \exp(\epsilon(z^2 - \theta^2))\|\varphi_x\|^{\theta - z}\|x\|^{z-\theta}x$. Then $F \in \mathcal{F}$, $F(\theta) = x$, and we can check that

$$\|F\|_{\mathcal{F}} \leq \|\varphi_x\|^\theta \|x\|^{1-\theta} \exp\big(\epsilon(1-\theta^2)\big).$$

As $\epsilon > 0$ was arbitrary, we conclude that, as $\theta = 1/p$,

$$\|j^*_{(0)}(x)\|_p \leq \|x\|^{1-\theta}\|\varphi_x\|^\theta = \|x\|^{1/p'}\|\varphi_x\|^{1/p}.$$

We now use duality. For $\epsilon > 0$, there exists $p_0 > 1$ such that, if $1 < p \leq p_0$, then $\|j^*_{(0)}(x)\|_p \leq (1+\epsilon)\|\varphi_x\|$. As L is σ-weakly dense in $\mathcal{M}$, by Kaplansky density, we can find $y \in L$ with $\|y\| = 1$ and $|\langle y, \varphi_x\rangle| \geq (1-\epsilon)\|\varphi_x\|$. Then there exists $p_1 > 1$ such that, if $1 < p \leq p_1$, then $\|j^*_{(0)}(y)\|_{p'} \leq (1+\epsilon)\|y\| = 1+\epsilon$. Thus, if $1 < p < \min(p_0, p_1)$, then

$$\begin{aligned}(1+\epsilon)\|\varphi_x\| \geq \|j^*_{(0)}(x)\|_p &\geq \big|\langle j^*_{(0)}(x), j^*_{(0)}(y)\rangle_{p,(0)}\big| \|j^*_{(0)}(y)\|^{-1}_{p'} \\ &\geq |\langle y, \varphi_x\rangle|(1+\epsilon)^{-1} \geq (1-\epsilon)(1+\epsilon)^{-1}\|\varphi_x\|.\end{aligned}$$

As $\epsilon > 0$ was arbitrary, this completes the proof. □

3.3 Operator spaces

As noted by Pisier in [26], [28, Section 2.6], the complex interpolation method interacts very nicely with operator spaces. If E_0 and E_1 are operator spaces which, as Banach spaces, form a compatible couple, then, say, identifying $\mathbb{M}_n(E_0 + F_0)$ with $(E_0 + F_0)^{n^2}$, we turn $(\mathbb{M}_n(E_0), \mathbb{M}_n(E_1))$ into a compatible couple. We then define

$$\mathbb{M}_n\big((E_0, E_1)_{[\theta]}\big) = (\mathbb{M}_n(E_0), \mathbb{M}_n(E_1))_{[\theta]}.$$

It is an easy check that these matrix norms satisfy the axioms for an (abstract) operator space. Then the obvious completely bounded version of Theorem 3.1 holds.

Suppose that E and F are Banach spaces which form a sesquilinear dual pair. A typical example would be $E = L^\infty(\mu)$ and $F = L^1(\mu)$ for a probability measure μ, together with the pairing

$$(f|g) = \int f\overline{g}\ d\mu \qquad (f \in E, g \in F).$$

Then we can show that $(E, F)_{[1/2]}$ is a Hilbert space, if (E, F) is made a compatible couple in the correct way, see [28, Theorem 7.10] for example. In our example, we recover $L^2(\mu)$ for the canonical compatibility. Intrinsic in the proof is that a Hilbert space H can be canonically identified, in an anti-linear way, with its own dual, by way of the inner-product.

If E and F are also operator spaces, then we recover a Hilbert space with some operator space structure. There is a unique operator Hilbert space which is anti-linearly completely isometric to its dual: Pisier's *operator Hilbert space*. We write H_{oh} to denote this structure on H. As explained carefully in [28, Page 139], at least when $\mathcal{M}$ is semifinite, we should consider the compatible couple $(\mathcal{M}, \mathcal{M}_*^{\mathrm{op}})$. Here, for an operator space E, E^{op} denotes the space E with the *opposite* structure, namely $\|(x_{ij})\|_{\mathrm{op}} = \|(x_{ji})\|$ for $(x_{ij}) \in \mathbb{M}_n(E)$. If $\mathcal{A}$ is a C*-algebra, then $\mathcal{A}^{\mathrm{op}}$ can be identified with $\mathcal{A}$, but with the product reversed. See also [19, Section 4] for a slightly different perspective.

Indeed, as noted in [19], if $\mathcal{M}$ is in standard position on H with modular conjugation J, then we have a canonical $*$-isomorphism $\phi : \mathcal{M}^{\mathrm{op}} \to \mathcal{M}', x \mapsto Jx^*J$ and so $\phi_* : \mathcal{M}'_* \to \mathcal{M}_*^{\mathrm{op}}$ is a completely isometric isomorphism of operator spaces. We conclude that the natural operator space structure on $L^p(\mathcal{M})$ will arise from studying the compatible couple $(\mathcal{M}, \mathcal{M}'_*)$. Alternatively, if we privilege $\mathcal{M}_*$, then we should look at $(\mathcal{M}', \mathcal{M}_*)$. When $\mathcal{M}_* = A(G)$, it turns out that this simple observation will guide us as to how to give the resulting non-commutative L^p spaces an $A(G)$-module action.

Let us finish by showing the operator space version of Proposition 3.4.

Proposition 3.5. *With notation as above, let $x \in \mathbb{M}_n(L)$ for some $n \in \mathbb{N}$. Then $\lim_{p\to 1} \|j_{(0)}^*(x)\|_p = \|\varphi_x\|$.*

Proof. The norm on $\mathbb{M}_n(L^p_{(0)}(\varphi))$ is given by interpolating $\mathbb{M}_n(\mathcal{M})$ and $\mathbb{M}_n(\mathcal{M}_*^{\mathrm{op}})$, and so we can follow the first part of the proof of Proposition 3.4 to find $p_0 > 1$ such that, if $1 < p < p_0$, then $\|j_{(0)}^*(x)\|_p \leq (1+\epsilon)\|\varphi_x\|$.

By Smith's Lemma, [5, Proposition 2.2.2], and as $\mathbb{M}_n(L)$ is σ-weakly dense in $\mathbb{M}_n(\mathcal{M})$, there exists $y \in \mathbb{M}_n(L)$ with $\|y\| = 1$ and $|\langle\langle y, \varphi_x\rangle\rangle| \geq (1-\epsilon)\|\varphi_x\|$. We can now proceed as in the end of the proof of Proposition 3.4. ∎

4 Non-commutative L^p spaces associated to the Fourier algebra

Let G be a locally compact group G. We have that $VN(G)$ is a Hopf-von Neumann algebra; indeed, a Kac algebra, [6]; indeed, is a locally compact quantum group, [22, 23]. We have a normal $*$-homomorphism

$$\Delta : VN(G) \to VN(G)\overline{\otimes}VN(G) = VN(G\times G); \quad \Delta(\lambda(s)) = \lambda(s)\otimes\lambda(s) \qquad (s \in G).$$

It is not obvious that such a map exists, but if we define $W \in \mathcal{B}(L^2(G\times G))$ by $W\xi(s,t) = \xi(ts,t)$ for $s,t \in G$, then W is a unitary, and we can define $\Delta(x) = W^*(1\otimes x)W$ for $x \in VN(G)$. Then Δ is coassociative, namely $(\Delta\otimes\mathrm{id})\Delta = (\mathrm{id}\otimes\Delta)\Delta$. Thus Δ induces an associative product on $VN(G)_*$, leading to the Fourier algebra, [7]. For $\xi,\eta \in L^2(G)$ we write $\omega_{\xi,\eta}$ for the normal functional on $VN(G)$ given by $\langle x, \omega_{\xi,\eta}\rangle = (x\xi|\eta)$. As $VN(G)$ is in standard position, [34, Chapter IX, Section 1], every member of $A(G)$ arises in this way. We define a map, the *Eymard embedding*, $\Phi : VN(G)_* \to C_0(G)$ by

$$\Phi(\omega_{\xi,\eta})(s) = \langle\lambda(s), \omega_{\xi,\eta}\rangle = \int_G \xi(s^{-1}t)\overline{\eta(t)}\, dt \qquad (s \in G, \omega_{\xi,\eta} \in A(G)).$$

Then Φ is an algebra homomorphism. This follows [7], but we warn the reader that [34, Chapter VII, Section 3] uses a different map (with s^{-1} replacing s).

Then $VN_r(G) = VN(G)'$ carries a coassociative map Δ' given by $\Delta'(x) = (J\otimes J)\Delta(JxJ)(J\otimes J)$ for $x \in VN_r(G)$. We have that $\Delta'(\rho(s)) = \rho(s)\otimes\rho(s)$ for $s \in G$. Similarly $A_r(G) = VN_r(G)_*$ becomes an algebra. We write $\omega'_{\xi,\eta}$ for the functional on $VN_r(G)$ given by $\langle x, \omega'_{\xi,\eta}\rangle = (x\xi|\eta)$ for $x \in VN_r(G)$. We similarly define $\Phi' : A_r(G) \to C_0(G)$ by

$$\Phi'(\omega'_{\xi,\eta})(s) = \langle\rho(s), \omega_{\xi,\eta}\rangle = \int_G \xi(ts)\nabla(s)^{1/2}\overline{\eta(t)}\, dt \qquad (s \in G, \omega'_{\xi,\eta} \in A_r(G)).$$

Guided by the arguments in the previous section, we shall turn $(VN_r(G), A_r(G))$ into a compatible couple in the sense of Terp. As $VN_r(G) = VN(G)'$, we have a canonical $*$-isomorphism

$$\phi : VN(G)^{\mathrm{op}} \to VN_r(G); \quad x \mapsto Jx^*J \qquad (x \in VN(G)).$$

Then we have

$$\phi_* : A_r(G) \to A(G)^{\text{op}}; \quad \omega'_{\xi,\eta} \mapsto \omega_{J\eta,J\xi} \qquad (\xi,\eta \in L^2(G)).$$

This allows us to regard $(VN_r(G), A(G))$ as a compatible couple, and we shall often suppress the implicit ϕ_* involved. We then define

$$L^p(\hat{G}) = (VN_r(G), A(G))_{[1/p]} \qquad (1 < p < \infty).$$

Here we use the "dual group" notation which is common when studying the Fourier algebra. The motivation is that when G is abelian, we have that $VN_r(G) = L^\infty(\hat{G})$ and $A(G) = L^1(\hat{G})$ by the Fourier transform, where $\hat{G}$ is the Pontryagin dual of G, and so $L^p(\hat{G})$ agrees with the usual meaning. We keep the same notation in the non-abelian case, although now it is purely formal. We give $L^p(\hat{G})$ the canonical operator space structure

$$\mathbb{M}_n(L^p(\hat{G})) = \big(\mathbb{M}_n(VN_r(G)), \mathbb{M}_n(A(G))\big)_{[1/p]}.$$

We have that

$$\Phi(\phi_*(\omega'_{\xi,\eta}))(s) = (J\lambda(s)^* J\xi|\eta) = (\rho(s^{-1})\xi|\eta) = \Phi'(\omega'_{\xi,\eta})(s^{-1}) \qquad (s \in G, \xi,\eta \in L^2(G)).$$

Hence, under the maps Φ and Φ', ϕ_* induces the "check map". We also have the map K available, which allows us to define a $*$-homomorphism

$$\hat{\phi} : VN(G) \to VN_r(G); \quad x \mapsto KxK \qquad (x \in VN(G)).$$

The predual of this map is then

$$\hat{\phi}_* : A_r(G) \to A(G); \quad \omega'_{\xi,\eta} \mapsto \omega_{K\xi,K\eta} \qquad (\xi,\eta \in L^2(G)),$$

so that

$$\Phi(\hat{\phi}_*(\omega'_{\xi,\eta}))(s) = (K\lambda(s)K\xi|\eta) = (\rho(s)\xi|\eta) = \Phi'(\omega'_{\xi,\eta})(s) \qquad (s \in G, \xi,\eta \in L^2(G)).$$

Thus, under the maps Φ and Φ', we see that $\hat{\phi}_*$ is the formal identity.

Lemma 4.1. *For $f,g \in C_{00}(G)$, let $a = f^*g$. Then $\rho(a) \in VN_r(G)$ agrees with $\nabla^{1/2}a \in A(G)$ in $VN_r(G) \cap A(G) = L$.*

Proof. We have that $\rho(f),\rho(g) \in \mathfrak{n}_{\varphi'}$, and so by [35, Proposition 4], we have that $\rho(f^*g) \in L = VN_r(G) \cap A(G)$, with

$$\varphi_{\rho(f^*g)} = \omega'_{J\Lambda'\rho(f),J\Lambda'\rho(g)} = \omega'_{\overline{f},\overline{g}} = \phi_*^{-1}(\omega_{\Lambda'\rho(g),\Lambda'\rho(f)}) = \phi_*^{-1}(\omega_{Kg,Kf}) = \phi_*^{-1}\hat{\phi}_*(\omega'_{g,f}).$$

Now, for $s \in G$,

$$\begin{aligned}\omega_{Kg,Kf}(s) &= \int_G Kg(s^{-1}t)\overline{Kf(t)}\ dt = \int_G g(t^{-1}s)\nabla(t^{-1}s)^{1/2}\overline{f(t^{-1})}\nabla(t^{-1})^{1/2}\ dt \\ &= \nabla(s)^{1/2}\int_G f^*(t)g(t^{-1}s)\ dt = (\nabla^{1/2}a)(s),\end{aligned}$$

which completes the proof. $\square$

We wish to turn $L^p(\hat{G})$ into a (completely contractive) left $A(G)$-module. For $p = 1$, we obviously have a natural action of $A(G)$ on itself, and so the previous lemma suggests the following action.

Lemma 4.2. *There is a completely contractive action of $A(G)$ on $VN_r(G)$ such that $a \cdot \rho(f) = \rho(a \cdot f)$ for $a \in A(G)$ and $f \in C_{00}(G)$, where $a \cdot f$ denotes the point-wise product.*

Proof. We have that $VN_r(G)$ is a completely contractive $A_r(G)$-module (which is commutative, so we shall not distinguish between left and right actions) such that $a \cdot \rho(f) = \rho(a \cdot f)$ for $a \in A_r(G)$ and $f \in C_{00}(G)$. As above, we have that $\hat{\phi}_* : A_r(G) \to A(G)$ is a completely isometric homomorphism. So our required action is simply $a \cdot x = \hat{\phi}_*^{-1}(a) \cdot x$ for $a \in A(G), x \in VN_r(G)$. $\square$

The following is a useful approximation result, which allows us to work with concrete functions, rather than operators in $VN_r(G)$.

Proposition 4.3. *For $x \in VN_r(G)$, we have that $x \in L$ when there exists $\varphi_x \in A_r(G)$ with $(x(\overline{a})|\overline{b}) = \langle \rho(a^*b), \varphi_x \rangle$ for $a, b \in C_{00}(G)$.*

Proof. Let $\mathfrak{A} = \Lambda'(\rho(C_{00}(G))) = C_{00}(G)$, which is a Tomita algebra (but *not* the maximal Tomita algebra). We claim that $\mathfrak{A}$ generates the full left Hilbert algebra $\Lambda'(\mathfrak{n}_{\varphi'} \cap \mathfrak{n}_{\varphi'})$. This will follow from [33, Lemma 3, Section 10.5] if we can show that $C_{00}(G)$ is a core for the operator S, which is the closure of $\Lambda'(x) \mapsto \Lambda'(x^*)$ for $x \in \mathfrak{n}_{\varphi'} \cap \mathfrak{n}_{\varphi'}$ (meaning that the closure of the S operator associated to $\mathfrak{A}$ agrees with the canonical one associated to $\mathfrak{n}_{\varphi'}$).

Indeed, for us, S is the map $D(S) \to L^2(G), \xi \mapsto \overline{\check{\xi}}$ where $D(S) = \{\xi \in L^2(G) : \check{\xi} \in L^2(G)\}$. Then $D(S)$ is a Hilbert space for the inner-product $(\xi|\eta)_\sharp = (\xi|\eta) + (S\eta|S\xi)$ for $\xi, \eta \in D(S)$. We claim that $C_{00}(G)$ is dense in $D(S)$, from which it will follow that $C_{00}(G)$ is a core for S. Suppose that $\eta \in D(S)$ is such that $(\xi|\eta)_\sharp = 0$ for $\xi \in C_{00}(G)$. Then

$$0 = \int_G \xi(s)\overline{\eta(s)} \; ds + \int_G \xi(s^{-1})\overline{\eta(s^{-1})} \; ds = \int_G \overline{\eta(s)}\xi(s)(1 + \nabla(s)^{-1}) \; ds,$$

for all $\xi \in C_{00}(G)$. As the set $\{\xi \cdot (1 + \nabla^{-1}) : \xi \in C_{00}(G)\}$ is dense in $L^2(G)$, it follows that $\eta = 0$. So $C_{00}(G)$ is dense in $D(S)$, as required.

As $\mathfrak{A}$ generates $\Lambda'(\mathfrak{n}_{\varphi'} \cap \mathfrak{n}_{\varphi'})$, we can apply the approximation result [34, Theorem 1.26, Chapter VI]. This shows that for $x \in \mathfrak{n}_{\varphi'}$, we can find a sequence (f_n) in $C_{00}(G)$ such that

$$\lim_n \|\Lambda'(x) - \Lambda'\rho(f_n)\| = \lim_n \|\Lambda'(x) - Kf_n\| = 0, \qquad \|\rho(f_n)\| \leq \|x\| \qquad (n \in \mathbb{N}),$$

and that $\rho(f_n) \to x$ strongly.

Finally, suppose that $x \in VN_r(G)$ and $\varphi_x \in A_r(G)$ are such that $(x(\overline{a})|\overline{b}) = \langle \rho(a^*b), \varphi_x \rangle$ for $a, b \in C_{00}(G)$. Choose $\xi, \eta \in L^2(G)$ with $\varphi_x = \omega'_{\xi,\eta}$. Let $y, z \in \mathfrak{n}_{\varphi'}$, so we can find sequences $(a_n), (b_n)$ in $C_{00}(G)$, as above, associated to y and z respectively. Thus

$$\begin{aligned}\langle y^*z, \varphi_x \rangle &= (z\xi|y\eta) = \lim_n (\rho(b_n)\xi|\rho(a_n)\eta) = \lim_n \langle \rho(a_n^*b_n), \varphi_x \rangle \\ &= \lim_n (xJKa_n|JKb_n) = (xJ\Lambda'(y)|J\Lambda'(z)).\end{aligned}$$

We conclude that $x \in L$ as required. $\square$

We can immediately improve Lemma 4.1.

Proposition 4.4. *Let $a \in A(G)$. Then $a \in VN_r(G) \cap A(G)$ if and only if $\check{a}$ is* right bounded*, that is, there exists $K > 0$ such that $\|f\check{a}\|_2 \leq K\|f\|_2$ for $f \in C_{00}(G)$. In this case, the map $f \mapsto f\check{a}$ extends to an operator $x \in VN_r(G)$, and then $x \in L$ with $a = \phi_*(\varphi_x)$.*

Proof. Let $a = \omega_{\xi,\eta}$, so that $\check{a} = \Phi'\phi_*^{-1}(\omega_{\xi,\eta})$. Suppose that $\check{a}$ is right bounded. As convolutions on the right commutes with the action of $VN(G)$, we see that $x \in VN_r(G)$. For $f, g \in C_{00}(G)$, we see that

$$(x(\overline{f})|\overline{g}) = (\overline{f}\check{a}|\overline{g}) = \int \overline{f(s)}\check{a}(s^{-1}t)g(t)\ ds\ dt = \int \overline{f(s)}\check{a}(t)g(st)\ dt\ ds$$
$$= \int f^*(s)g(s^{-1}t)\check{a}(t)\ ds\ dt = \langle \rho(f^*g), \phi_*^{-1}(\omega_{\xi,\eta})\rangle.$$

So by the previous proposition, $x \in L$ and $a = \phi_*(\varphi_x)$, as claimed.

Conversely, if $a \in VN_r(G) \cap A(G)$ then there exists $x \in L$ with $a = \phi_*(\varphi_x)$. As $f\check{a}$ always exists for $f \in C_{00}(G)$, we can reverse the argument above to conclude that $x(f) = f\check{a}$ for $f \in C_{00}(G)$, so that $\check{a}$ is right bounded. $\square$

We can also apply our approximation idea to improve an approximation result of Terp, [35, Theorem 8].

Proposition 4.5. *For $x \in L$, we can find a net (f_i) in $C_{00}(G)^2$ such that $\sup_i \|\rho(f_i)\| < \infty$, $\rho(f_i) \to x$ σ-weakly, and $\varphi_{\rho(f_i)} \to \varphi_x$ in norm.*

Proof. By Terp's result [35, Theorem 8] we can a net bounded (x_i) in $\mathfrak{m}_{\varphi'}$ with $x_i \to x$ σ-weakly and $\varphi_{x_i} \to \varphi_x$ in norm. Indeed, from the proof, we can choose $x_i = y_i^* z_i$ for some $y_i, z_i \in \mathfrak{n}_{\varphi'}$ with (y_i) and (z_i) bounded nets.

For each i, choose a sequence $(a_{i,n})$ in $C_{00}(G)$ with $\rho(a_{i,n}) \to y_i$ strongly, $Ka_{i,n} \to \Lambda'(y_i)$ in norm, and with $\|\rho(a_{i,n})\| \leq \|y_i\|$. Similarly choose $(b_{i,n})$ associated to z_i. It follows (compare with the proof above) that $\rho((a_{i,n})^* b_{i,n}) \to y_i^* z_i = x_i$ σ-weakly, and that $\varphi_{\rho((a_{i,n})^* b_{i,n})} \to \varphi_{x_i}$ in norm. With the diagonal ordering, we see that $((a_{i,n})^* b_{i,n})$ is the required net. $\square$

Theorem 4.6. *There is a completely contractive left action of $A(G)$ on $L^p(\hat{G})$, for $1 < p < \infty$, such that $a \cdot j_{(0)}^* \rho(b) = j_{(0)}^* \rho(a \cdot b)$ for $a \in A(G)$ and $b \in C_{00}(G)^2$.*

Proof. Let $a \in A(G)$ and consider the bounded maps

$$T : A(G) \to A(G); b \mapsto a \cdot b, \qquad S : VN_r(G) \to VN_r(G); x \mapsto \hat{\phi}_*^{-1}(a) \cdot x.$$

By Lemma 3.2, we wish to show that T and S map L to L and agree on L. If this is so, then we get a map $R \in \mathcal{B}(L^p(\hat{G}))$ which extends T and S, and is bounded by $\|T\|^{1/p}\|S\|^{1/p'} \leq \|a\|$. Clearly $a \mapsto R$ is a homomorphism, and the resulting action of $A(G)$ on $L^p(\hat{G})$ is the one stated, by Lemma 4.2.

So, for $x \in L$, we need to show that $y = \hat{\phi}_*^{-1}(a) \cdot x \in L$ and that furthermore $a \cdot \phi_*(\varphi_x) = \phi_*(\varphi_y)$. Suppose that $x = \rho(f^*g)$ for $f, g \in C_{00}(G)$, so that from Lemma 4.1, $\Phi\phi_*(\varphi_x) = \nabla^{1/2} f^* g$. By Proposition 4.3, we have that $y \in L$ if

$$(y(\overline{c})|\overline{d}) = \langle \rho(c^*d), \phi_*^{-1}(a \cdot \phi_*(\varphi_x))\rangle \qquad (c, d \in C_{00}(G)).$$

Now, we have that

$$\langle \rho(c^*d), \phi_*^{-1}(a \cdot \phi_*(\varphi_x))\rangle = \langle \rho(c^*d), \phi_*^{-1}\Phi^{-1}(a \cdot \nabla^{1/2} f^*g)\rangle$$
$$= \int_G c^*d(s)a(s^{-1})\nabla(s)^{-1/2}(f^*g)(s^{-1})\ ds = \int_G c^*d(s^{-1})a(s)\nabla(s)^{-1/2}(f^*g)(s)\ ds$$
$$= \langle \hat{\phi}_*^{-1}(a) \cdot \rho(f^*g), \varphi_{\rho(c^*d)}\rangle = \langle y, \omega'_{\overline{c},\overline{d}}\rangle = (y(\overline{c})|\overline{d}),$$

using that $\Phi'\phi_*^{-1}\Phi^{-1}$ is the check map. Hence we are done in the case that $x \in \rho(C_{00}(G)^2)$.

For general $x \in L$, choose an approximating net $(f_i) \subseteq C_{00}(G)^2$ as in Proposition 4.5. Then, by the previous paragraph, for $a, b \in C_{00}(G)$,

$$\langle y, \omega'_{\overline{c},\overline{d}} \rangle = \lim_i \langle \rho(f_i), \omega'_{\overline{c},\overline{d}} \rangle = \lim_i \langle \rho(c^*d), \phi_*^{-1}(a \cdot \phi_*(\varphi_{\rho(f_i)})) \rangle = \langle \rho(c^*d), \phi_*^{-1}(a \cdot \phi_*(\varphi_x)) \rangle,$$

which completes the proof of the claim, by another application of Lemma 4.1.

In the completely bounded case, notice that T and S are completely bounded, and hence also R is, so we get a homomorphism $A(G) \to \mathcal{CB}(L^p(\hat{G}))$. To see that this is completely bounded, it is easier to prove the equivalent statement that $A(G) \times L^p(\hat{G}) \to L^p(\hat{G}); (a, \xi) \mapsto R(\xi)$ is jointly completely contractive, [5, Chapter 7]. That is, for $n \in \mathbb{N}$, the map $\mathbb{M}_n(A(G)) \times \mathbb{M}_n(L^p(\hat{G})) \to \mathbb{M}_{n^2}(L^p(\hat{G})); (a_{ij}) \times (\xi_{kl}) \mapsto (R_{ij}(\xi_{kl}))_{(i,k),(j,l)}$ is contractive. This follows immediately from Theorem 3.3, as the analogous statements hold for T and S. ☐

A slightly curious corollary of this proof is that L is an $A(G)$-submodule of $VN_r(G)$, and hence the image of L in $A(G)$ is a dense ideal. As a final application of our approximation ideas, we have the following.

Proposition 4.7. *For $1 < p < \infty$, we have that $j_{(0)}^*\rho(C_{00}(G)^2)$ is norm dense in $L^p(\hat{G})$.*

Proof. Following the proof of [16, Proposition 6.22], it suffices to show that $\rho(C_{00}(G)^2) \subseteq L$ separates the points of $VN_r(G) + A_r(G) \subseteq L^*$. Indeed, suppose that $x \in VN_r(G)$ and $\omega \in A_r(G)$ are such that

$$\langle x, \varphi_{\rho(f)} \rangle + \langle \rho(f), \omega \rangle = 0 \qquad (f \in C_{00}(G)^2).$$

For $y \in L$, use Proposition 4.5 to pick an approximating net (f_i), so that

$$0 = \lim_i \langle x, \varphi_{\rho(f_i)} \rangle + \langle \rho(f_i), \omega \rangle = \langle x, \varphi_y \rangle + \langle y, \omega \rangle.$$

In particular, this holds for any $y \in \mathfrak{m}_\varphi$, so by [35, Proposition 7] (or, essentially by definition) it follows that $x \in L$ with $\varphi_x = -\omega$. Hence $x + \omega = 0$ in $VN_r(G) + A_r(G)$, as required. ☐

Proposition 4.8. *There is an isometric isomorphism $\theta : L^2(G) \to L^2(\hat{G})$ satisfying $\theta(f) = j_{(0)}^*\rho(\nabla^{-3/4}\check{f})$ for $f \in C_{00}(G)^2$. Furthermore, θ intertwines the inner products on $L^2(G)$ and $L^2(\hat{G})$.*

Proof. In Section 3, we discussed the isometric isomorphism $k : H_{\varphi'} \to L^2(\hat{G})$ which is such that $k(\Lambda'\rho(f)) = j_{(0)}^*\sigma_{i/4}\rho(f)$ for $f \in C_{00}(G)^2$. If we identify $H_{\varphi'}$ with $L^2(G)$, then $\Lambda'\rho(f) = Kf$, and so we find a map θ which satisfies $\theta(f) = j_{(0)}^*\sigma_{i/4}\rho(Kf) = j_{(0)}^*\rho(\nabla^{-3/4}\check{f})$ for $f \in C_{00}(G)^2$. ☐

Notice that $L^2(G)$ carries a natural *bilinear* product, $\langle f, g \rangle = \int_G fg$ for $f, g \in L^2(G)$. Similarly $L^2(\hat{G})$ has the bilinear product $\langle \cdot, \cdot \rangle_{2,(0)}$, but θ does not intertwine these products.

4.1 Comparison with Forrest, Lee and Samei

In [8, Section 6], a different construction of non-commutative L^p spaces associated to $A(G)$ is given. We shall compare their construction to ours.

Firstly, they form the non-commutative L^p space using $VN(G)$, using Izumi's work with $\alpha = -1/2$. Let $\mathcal{O}L^p_{(-1/2)}(VN(G))$ be the operator space version, given by interpolating $VN(G)$ and

$A(G)^{\mathrm{op}}$. Here we write $(-1/2)$ to indicate the choice of α. Then they define

$$L^p(VN(G)) = \begin{cases} \mathcal{O}L^p_{(-1/2)}(VN(G))^{\mathrm{op}} & : 1 < p \leq 2, \\ \mathcal{O}L^p_{(-1/2)}(VN(G)) & : 2 \leq p < \infty. \end{cases}$$

Recall that the Hilbert space $\mathcal{O}L^2_{(-1/2)}(VN(G))$ will carry the operator Hilbert space structure, so that $\mathcal{O}L^2_{(-1/2)}(VN(G)) = \mathcal{O}L^2_{(-1/2)}(VN(G))^{\mathrm{op}}$.

By [8, Theorem 6.3], for $1 < p \leq 2$, the $A(G)$ module structure on $L^p(VN(G))$ satisfies

$$a \cdot j^*_{(1/2)}(\lambda(\check{f})) = j^*_{(1/2)}(\lambda(\check{a} \cdot \check{f})) \qquad (a \in A(G), f \in C_{00}(G)).$$

Here we use a different, but equivalent, notation to that of [8]. Similarly, by [8, Theorem 6.4], for $2 \leq p < \infty$, the module action of $A(G)$ on $L^p(VN(G))$ is

$$a \cdot j^*_{(1/2)}(\lambda(f)) = j^*_{(1/2)}(\lambda(a \cdot f)) \qquad (a \in A(G), f \in C_{00}(G)).$$

Recall the isometric isomorphism $U_{p,(0,-1/2)} : L^p_{(-1/2)}(VN(G)) \to L^p_{(0)}(VN(G))$ which satisfies, in particular,

$$U_{p,(0,-1/2)}(j^*_{(1/2)}\lambda(f)) = j^*_{(0)}(\sigma_{i/2p}\lambda(f)) = j^*_{(0)}\lambda(\Delta^{-1/2p}f) \qquad (f \in C_{00}(G)^2).$$

For $1 < p \leq 2$, we can hence regard $L^p(VN(G))$ as $\mathcal{O}L^p_{(0)}(VN(G))^{\mathrm{op}}$ with the module action

$$\begin{aligned} a \cdot j^*_{(0)}\lambda(f) &= U_{p,(0,-1/2)}(a \cdot U^{-1}_{p,(0,-1/2)} j^*_{(0)}\lambda(f)) = U_{p,(0,-1/2)}(a \cdot j^*_{(0)}\lambda(\Delta^{1/2p}f)) \\ &= U_{p,(0,-1/2)} j^*_{(0)}\lambda(\check{a} \cdot \Delta^{1/2p}f) = j^*_{(0)}\lambda(\check{a} \cdot f). \end{aligned}$$

for $a \in A(G)$ and $f \in C_{00}(G)^2$. Similarly, for $2 \leq p < \infty$, we regard $L^p(VN(G))$ as $\mathcal{O}L^p_{(0)}(VN(G))$ with the module action

$$a \cdot j^*_{(0)}\lambda(f) = j^*_{(0)}\lambda(a \cdot f) \qquad (a \in A(G), f \in C_{00}(G)^2).$$

Proposition 4.9. *For $2 \leq p < \infty$, there exists a completely isometric isomorphism $\hat{\phi}_p : L^p(VN(G)) \to L^p(\hat{G})$ which is also an $A(G)$-module homomorphism, with*

$$\hat{\phi}_p\big(j^*_{(0)}\lambda(f)\big) = j^*_{(0)}\rho(f) \qquad (f \in C_{00}(G)^2).$$

Proof. Our $L^p(\hat{G})$ spaces are formed by interpolating $VN_r(G)$ and $A_r(G)^{\mathrm{op}}$ (identified with $A(G)$). Consider again the maps $\hat{\phi} : VN(G) \to VN_r(G)$ and $\hat{\phi}_*^{-1} : A(G) \to A_r(G)$. We claim that these are compatible, that is, map $L_{(0)}$, for $(VN(G), A(G))$, into $L_{(0)}$, for $(VN_r(G), A_r(G))$. Indeed, let $x \in L_{(0)} \subseteq VN(G)$ with associated $\varphi_x \in A(G)$. Let $a, b \in C_{00}(G)$, so that

$$\begin{aligned} (\hat{\phi}(x)\overline{a}|\overline{b}) &= (xK(\overline{a})|K(\overline{b})) = (xJ(a)|J(b)) = (xJ\Lambda(\lambda(a))|J\Lambda(\lambda(b)) \\ &= \langle \lambda(a^*b), \varphi_x \rangle = \langle K\rho(a^*b)K, \varphi_x \rangle = \langle \rho(a^*b), \hat{\phi}_*^{-1}(\varphi_x) \rangle, \end{aligned}$$

so by Proposition 4.3 we see that $\hat{\phi}(x) \in L_{(0)}$, for $(VN_r(G), A_r(G)$, with $\varphi_{\hat{\phi}(x)} = \hat{\phi}_*^{-1}(\varphi_x)$. Consequently, by Lemma 3.2, we can interpolate these maps, leading to a contraction

$$\hat{\phi}_p : L^p_{(0)}(VN(G)) \to L^p_{(0)}(VN_r(G)) = L^p(\hat{G}).$$

As $\hat{\phi}_*^{-1}$ is also a complete isometry $A(G)^{\mathrm{op}} \to A_r(G)^{\mathrm{op}}$, we see that $\hat{\phi}_p$ is even a complete contraction. By symmetry, we also have a complete contraction in the other direction, showing that $\hat{\phi}_p$ is actually a completely isometric isomorphism. In particular,

$$\hat{\phi}_p j_{(0)}^* \lambda(f) = j_{(0)}^*\big(K\lambda(f)K\big) = j_{(0)}^*\rho(f) \qquad (f \in C_{00}(G)^2).$$

It is now clear from Proposition 4.7 that this map is an $A(G)$-module homomorphism. ☐

Proposition 4.10. *For $1 < p \leq 2$, there exists a completely isometric isomorphism $\phi_p : L^p(VN(G)) \to L^p(\hat{G})$ which is also an $A(G)$-module homomorphism, with*

$$\phi_p\big(j_{(0)}^*\lambda(f)\big) = j_{(0)}^*\rho(\check{f}\nabla^{-1}) \qquad (f \in C_{00}(G)^2).$$

Proof. For $1 < p \leq 2$, it is clear that $L^p(VN(G)) = \mathcal{O}L^p_{(0)}(VN(G))^{\mathrm{op}} = (VN(G)^{\mathrm{op}}, A(G))_{[1/p]}$. The idea now is to replicate the proof above, but using instead the maps $\phi : VN(G)^{\mathrm{op}} \to VN_r(G)$ and $\phi_*^{-1} : A(G) \to A_r(G)^{\mathrm{op}}$. For $x \in L \subseteq VN(G)$, let $\varphi_x = \omega_{\xi,\eta}$ for some $\xi, \eta \in L^2(G)$. Let $y', z' \in \mathfrak{n}_{\varphi'}$ so that $y = Jy'J, z = Jz'J \in \mathfrak{n}_\varphi$ and

$$\begin{aligned}(\phi(x)J\Lambda'(y')|J\Lambda'(z')) &= (Jx^*J\Lambda(y)|\Lambda(z)) = \langle z^*y, \varphi_x\rangle = (z^*y\xi|\eta) = (J(z')^*y'J\xi|\eta) \\ &= ((y')^*z'J\eta|J\xi) = \langle (y')^*z', \phi_*^{-1}(\varphi_x)\rangle.\end{aligned}$$

Hence $\phi(x) \in L \subseteq VN_r(G)$ with $\varphi_{\phi(x)} = \phi_*^{-1}(\varphi_x)$. Again, we interpolate to find a completely isometric isomorphism

$$\phi_p : L^p(VN(G)) \to L^p(\hat{G}).$$

We then see that for $f \in C_{00}(G)^2$,

$$\phi_p j_{(0)}^*\lambda(f) = j_{(0)}^*\phi(\lambda(f)) = j_{(0)}^*\big(J\lambda(f^*)J\big) = j_{(0)}^*\rho(\check{f}\nabla^{-1}).$$

It is now clear from Proposition 4.7 that this map is an $A(G)$-module homomorphism. ☐

4.2 Application to homological questions

The following is an improvement of [8, Proposition 6.8], which only showed the result for $p \geq 2$.

Proposition 4.11. *Let G be a non-discrete group, and let $1 < p < \infty$. Then the only bounded left $A(G)$-module homomorphism $L^p(\hat{G}) \to A(G)$ is the zero map.*

Proof. Let $T : L^p(\hat{G}) \to A(G)$ be a bounded left $A(G)$-module homomorphism, and suppose towards a contradiction that T is not zero. By density, we can find $x \in L$, such that setting $\xi = j_{(0)}^*(x)$, we have that $T(\xi) \neq 0$. Let $a = \phi_*(\varphi_x) \in A(G)$. For $y \in L$, let $\eta = j_{(0)}^*(y)$ and $b = \phi_*(\varphi_y)$. Then, with reference to Theorem 4.6, $z = \hat{\phi}_*^{-1}(a) \cdot y \in L$ with $\phi_*(\varphi_z) = a \cdot \chi_*(\varphi_y) = ab = ba = b \cdot \phi_*(\varphi_x) = \phi_*(\hat{\phi}_*^{-1}(b) \cdot x)$. Thus

$$a \cdot T(\eta) = Tj_{(0)}^*(z) = Tj_{(0)}^*(\hat{\phi}_*^{-1}(b) \cdot x) = b \cdot T(\xi).$$

Let V be a compact neighbourhood of the identity in G, so that $0 < |V| < \infty$. Let K be a compact neighbourhood of the identity with $KK^{-1} \subseteq V$, let $r \in G$, let $\alpha = |K|^{-1/2}\chi_{r^{-1}K} \in L^2(G)$ and $\beta = |K|^{-1/2}\chi_K \in L^2(G)$. Then $\|\alpha\|_2 = \|\beta\|_2 = 1$, and so $b = \omega_{\alpha,\beta} \in A(G)$ with $\|b\|_{A(G)} \leq 1$. We see that

$$b(s) = \frac{1}{|K|}\int \chi_{r^{-1}K}(s^{-1}t)\chi_K(t)\,dt = \frac{|sr^{-1}K \cap K|}{|K|} \qquad (s \in G).$$

So $b(r) = 1$ and $b(s) \neq 0$ implies that $s \in KK^{-1}r \subseteq Vr$. So b has compact support and is bounded, and hence $b \in L^1(G)$ with $\|b\|_1 \leq |Vr|$. By Proposition 4.4, $b = \phi_*(\varphi_y)$ where $y \in L$ with $y(f) = f\check{b}$ for $f \in C_{00}(G)$. We can check that actually $y = \rho(\nabla^{-1/2}b)$, so that $\|y\| \leq \|\nabla^{-1}b\|_1 \leq |Vr|\|\nabla^{-1}|_{Vr}\|_\infty = K(V)$ say. By the estimate in Proposition 3.4 we see that

$$\|j^*_{(0)}(y)\|_p \leq \|y\|^{1/p'}\|\varphi_y\|^{1/p} \leq K(V)^{1/p'}.$$

With $\eta = j^*_{(0)}(y)$, we hence see that

$$|T(\xi)(r)| \leq \|b \cdot T(\xi)\|_{A(G)} = \|a \cdot T(\eta)\|_{A(G)} \leq \|a\|_{A(G)}\|T\|K(V)^{1/p'}.$$

In particular, we can make $K(V)$ as small as we like by choosing V small (as G is not discrete). As r was arbitrary, we conclude that $T(\xi) = 0$, giving our contradiction. $\square$

We can now follow the proof of [8, Theorem 6.9] to show the following; we refer the reader to [8] for the definition of *operator projective*.

Theorem 4.12. *Let G be a non-discrete group and $1 < p < \infty$. Then $L^p(\hat{G})$ is not operator projective as a left $A(G)$-module.*

5 Representing the multiplier algebra

Let G be a locally compact group, let (p_n) be a sequence in $(1, \infty)$ tending to 1, and let

$$E = \ell^2 - \bigoplus_n L^{p_n}(\hat{G}).$$

In the Banach space case, this is the direct sum in the ℓ^2 sense, defined in Section 2. In the operator space case, we regard this as a discrete vector-valued commutative ℓ^2 space, which carries a natural operator space structure, see [38, Section 1] and [27]. Indeed, $E_\infty = \ell^\infty - \oplus L^{p_n}(\hat{G})$ carries an obvious operator space structure. We give $E_1 = \ell^1 - \oplus L^{p_n}(\hat{G})$ the operator-space structure arising as a subspace of the dual of $\ell^\infty - \oplus L^{p_n}(\hat{G})^*$. Then (E_∞, E_1) is a compatible couple, and E is simply $(E_\infty, E_1)_{[1/2]}$. Notice that the underlying Banach space is the same as the usual definition.

Then $A(G)$ acts co-ordinate wise on E, so that E becomes a (completely) contractive $A(G)$-module. In the operator space case, notice that this is clear for E_1 and E_∞, and hence also for E by bilinear interpolation. In this section, we shall show that $MA(G)$, respectively $M_{cb}A(G)$, have actions on E extending those of $A(G)$, and that the resulting homomorphisms $MA(G) \to \mathcal{B}(E)$ and $M_{cb}A(G) \to \mathcal{CB}(E)$ are weak*-weak*-continuous (complete) isometries.

Proposition 5.1. *For $1 < p < \infty$, there is a natural action of $MA(G)$ on $L^p(\hat{G})$ extending the action of $A(G)$, such that $a \cdot j^*_{(0)}\rho(f) = j^*_{(0)}\rho(a \cdot f)$ for $a \in MA(G)$ and $f \in C_{00}(G)^2$. Furthermore, this action of $MA(G)$ restricts to give a completely contractive action of $M_{cb}A(G)$ on $L^p(\hat{G})$.*

Proof. We let $MA(G)$ act on $A(G)$ in the canonical way. As in the proof of Lemma 4.2, we note that $MA_r(G)$ acts on $A_r(G)$ and hence on $VN_r(G)$ by duality. This action satisfies $a \cdot \rho(f) = \rho(a \cdot f)$ for $a \in MA(G)$ and $f \in C_{00}(G)$. We then extend $\hat{\phi}_*^{-1}$ to an isometric homomorphism $\psi : MA(G) \to MA_r(G)$, which completes the argument as in Lemma 4.2. We define ψ by

$$\psi(a)(b) = \hat{\phi}_*^{-1}\big(a\hat{\phi}_*(b)\big) \qquad (a \in MA(G), b \in A_r(G)).$$

As $\hat{\phi}_*$ is a homomorphism, this does extend $\hat{\phi}_*^{-1}$ and is itself a homomorphism. Clearly ψ is contractive, and has an obvious contractive inverse, so that ψ is isometric as required. Notice that, if we view $a \in MA(G)$ and $\psi(a)$ as functions on G (using Φ and Φ') then these functions agree.

We now follow Theorem 4.6 and use interpolation to extend this $MA(G)$ action to $L^p(\hat{G})$. We hence need to show that if $x \in L$, then $y = \psi(a) \cdot x \in L$ with $a \cdot \phi_*(\varphi_x) = \phi_*(\varphi_y)$. As in the proof of Theorem 4.6, by our approximation result, it is enough to show this for $x = \rho(f^*g)$ for $f, g \in C_{00}(G)$. But then the proof of Theorem 4.6 follows *mutatis mutandis*.

The remark about $M_{cb}A(G)$ will follows if we can show that ψ restricts to a complete contraction $\psi : M_{cb}A(G) \to M_{cb}A_r(G)$. However, this follows immediately because $\hat{\psi}_*$ is a complete isometry. $\square$

By [4], $MA(G)$ is a dual Banach algebra with a predual Q, which is the completion of $L^1(G)$ for the norm

$$\|f\|_Q = \sup\Big\{ \Big| \int_G f(s)a(s)\ ds \Big| : a \in MA(G), \|a\| \leq 1 \Big\} \qquad (f \in L^1(G)).$$

Let $\lambda_Q : L^1(G) \to Q$ be the inclusion map. Similarly, $M_{cb}A(G)$ has a predual Q_{cb} which is defined in the same way, but taking the supremum over the unit ball of $M_{cb}A(G)$. Define similarly $\lambda_{Q_{cb}} : L^1(G) \to Q_{cb}$.

For $1 < p < \infty$, let $\pi^p : A(G) \to \mathcal{B}(L^p(\hat{G}))$ be the contractive homomorphism given by Theorem 4.6, and let $\hat{\pi}^p : MA(G) \to \mathcal{B}(L^p(\hat{G}))$ be the contractive homomorphism given by Proposition 5.1. Using Izumi's bilinear product, we have that $L^p(\hat{G})^* = L^{p'}(\hat{G})$, and so we can consider the map

$$\pi^p_* : L^p(\hat{G})\widehat{\otimes}L^{p'}(\hat{G}) \to A(G)^* = VN(G); \quad \langle \pi^p_*(\xi \otimes \eta), a\rangle = \langle a \cdot \xi, \eta\rangle_{p,(0)},$$

for $a \in MA(G), \xi \in L^p(\hat{G})$ and $\eta \in L^{p'}(\hat{G})$). Let $\hat{\pi}^p_* : MA(G) \to MA(G)^*$ be the analogous map.

Let $\pi^{p,cb} : A(G) \to \mathcal{CB}(L^p(\hat{G}))$ and $\hat{\pi}^{p,cb} : M_{cb}A(G) \to \mathcal{CB}(L^p(\hat{G}))$ be analogously given by Theorem 4.6 and Proposition 5.1. Similarly, define $\pi^{p,cb}_* : L^p(\hat{G})\widehat{\otimes}L^{p'}(\hat{G}) \to A(G)^*$ and $\hat{\pi}^{p,cb}_* : L^p(\hat{G})\widehat{\otimes}L^{p'}(\hat{G}) \to MA(G)^*$.

Proposition 5.2. *The maps π^p_* and $\pi^{p,cb}_*$ take values in $C^*_\lambda(G)$, the reduced group C^*-algebra. The map $\hat{\pi}^p_*$ takes values in the predual Q, and $\hat{\pi}^{p,cb}_*$ takes values in the predual Q_{cb}, so that both $\hat{\pi}^p$ and $\hat{\pi}^{p,cb}$ are weak*-weak*-continuous.*

Proof. Suppose that $\xi = j^*_{(0)}(f)$ and $\eta = j^*_{(0)}(g)$ for $f, g \in C_{00}(G)^2$. Then, for $a \in MA(G)$, using the calculations of Lemma 4.1,

$$\begin{aligned}\langle \hat{\pi}^p_*(\xi \otimes \eta), a\rangle &= \langle j^*_{(0)}\rho(a \cdot f), j^*_{(0)}\rho(g)\rangle_{p,(0)} = \langle \rho(a \cdot f), \varphi_{\rho(g)}\rangle \\ &= \int_G a(s)f(s)\nabla(s)^{-1/2}g(s^{-1})\ ds = \langle a, \lambda_Q(f \cdot Kg)\rangle.\end{aligned}$$

Hence $\hat{\pi}^p_*(\xi \otimes \eta) = \lambda_Q(f \cdot Kg) \in Q$. By Proposition 4.7, such ξ and η are norm dense, showing that $\hat{\pi}^p_*$ takes values in Q. It is now standard that $\hat{\pi}^p$ is weak*-weak*-continuous. The same calculation shows that $\hat{\pi}^{p,cb}_*(\xi \otimes \eta) = \lambda_{Q_{cb}}(f \cdot Kg) \in Q_{cb}$, so that $\hat{\pi}^{p,cb}_*$ takes values in Q_{cb} and hence also $\hat{\pi}^{p,cb}$ is weak*-weak*-continuous.

We have that $\hat{\pi}^p$, restricted to $A(G)$, is π^p. Similarly, and for $f \in L^1(G)$, we see that $\lambda_Q(f)$, restricted to $A(G)$, is simply $\lambda(f) \in C^*_\lambda(G) \subseteq VN(G)$. The above calculation hence also shows that π^p_* takes values in $C^*_\lambda(G)$, as claimed. The same argument applies in the completely bounded case. $\square$

If $\mathcal{A}$ is a commutative Banach algebra and $(L,R) \in M(\mathcal{A})$ then for $a,b \in \mathcal{A}$, $L(a)b = L(ab) = L(ba) = L(b)a = aL(b) = R(a)b$. If $\mathcal{A}$ is faithful, then $L = R$. We remark that $A(G)$ is faithful, as by [7, Lemme 3.2], for any compact $K \subseteq G$ there exists $a \in A(G)$ which is identically 1 on K.

The following is now the $A(G)$ version of the results in Section 2.

Theorem 5.3. *Let G and E be as above. Let $MA(G)$ act on E co-ordinate wise. Then the resulting homomorphism $\pi : MA(G) \to \mathcal{B}(E)$ is an isometry, and is weak*-weak*-continuous. Furthermore, the image of π is the idealiser of $\pi(A(G))$ in $\mathcal{B}(E)$.*

Proof. Clearly π is contractive. Let $a \in MA(G)$ and $\epsilon > 0$. As $j_{(0)}(L)$ is dense in $A_r(G)$, we can find $x \in L$ with $\|\varphi_x\| = 1$ and

$$\|a \cdot \phi_*(\varphi_x)\| \geq (1-\epsilon)\|a\|_{MA(G)}.$$

Then, using Proposition 3.4, we see that

$$\|\pi(a)\| \geq \lim_n \|a \cdot j_{(0)}^*(x)\|_{p_n} \|j_{(0)}^*(x)\|_{p_n}^{-1} = \|a \cdot \phi_*(\varphi_x)\| \|\varphi_x\| \geq (1-\epsilon)\|a\|_{MA(G)}.$$

As $\epsilon > 0$, we conclude that π is an isometry, as required.

Let $\xi = (\xi_n) \in E$ and $\eta = (\eta_n) \in E^*$ be sequences which are eventually zero. For $a \in MA(G)$, we see that

$$\langle \pi(a)\xi, \eta \rangle = \sum_n \langle a, \hat{\pi}_*^p(\xi_n \otimes \eta_n) \rangle,$$

so that $\pi_*(\xi \otimes \eta) \in Q$. As such ξ and η are dense, by continuity we see that $\pi_* : E \widehat{\otimes} E^* \to MA(G)^*$ takes values in Q. Again, this implies that π is weak*-weak*-continuous.

Clearly $\pi(MA(G))$ is contained in the idealiser of $\pi(A(G))$. Conversely, given T in the idealiser of $\pi(A(G))$, we can follow the proof of Theorem 2.2 to find $a \in MA(G)$ with $\pi(ab) = T\pi(b)$ and $\pi(ba) = \pi(b)T$ for $b \in A(G)$. For each $L^p(\hat{G})$, by Proposition 4.7 and again using [7, Lemme 3.2], it follows that $\{\pi(a)\xi : a \in A(G), \xi \in L^p(\hat{G})\}$ is linearly dense in $L^p(\hat{G})$. This is enough to show that then $T = \pi(a)$ as required to complete the proof. ☐

The completely bounded version of this result requires a subtly different proof.

Theorem 5.4. *Let G and E be as above, where we now regard E as an operator space. Let $M_{cb}A(G)$ act on E co-ordinate wise. Then the resulting homomorphism $\pi_{cb} : M_{cb}A(G) \to \mathcal{CB}(E)$ is a weak*-weak*-continuous complete isometry. Furthermore, the image of π_{cb} is the idealiser of $\pi_{cb}(A(G))$ in $\mathcal{CB}(E)$.*

Proof. Again, clearly π_{cb} is completely contractive. As the norm on $\mathbb{M}_n(L^p(\hat{G}))$ is given by interpolating $\mathbb{M}_n(VN_r(G))$ and $\mathbb{M}_n(A(G))$, we can simply apply the proof of the previous theorem, but working with matrices, and using Proposition 3.5, to show that π_{cb} is a complete isometry. Similarly, it follows that π_{cb} is weak*-weak*-continuous.

Clearly $\pi_{cb}(M_{cb}A(G))$ is contained in the idealiser of $\pi_{cb}(A(G))$. Conversely, given T in the idealiser of $\pi_{cb}(A(G))$, we can follow the proof of Theorem 2.2 to find $(L,R) \in M(A(G))$ with $\pi_{cb}(L(a)) = T\pi_{cb}(a)$ and $\pi_{cb}(R(a)) = \pi_{cb}(a)T$ for $a \in A(G)$.

For $n \in \mathbb{N}$, let $i_n : L^{p_n}(\hat{G}) \to E$ be the inclusion map, which is a completely contractive $A(G)$-bimodule homomorphism. Then

$$Ti_n(a \cdot \xi) = T\pi_{cb}(a)i_n(\xi) = \pi_{cb}(L(a))i_n(\xi) = i_n(L(a) \cdot \xi) \qquad (a \in A(G), \xi \in L^{p_n}(\hat{G})).$$

As $A(G)\cdot L^p(\hat G)$ is dense in $L^p(\hat G)$ for all p, we conclude that there exists $T_n\in\mathcal{CB}(L^{p_n}(\hat G))$ with $Ti_n = i_nT_n$ and $\|T_n\|_{cb}\leq\|T\|_{cb}$. It now follows that

$$T_n(a\cdot\xi) = L(a)\cdot\xi,\quad a\cdot T_n(\xi) = R(a)\cdot\xi\qquad (a\in A(G),\xi\in L^{p_n}(\hat G)).$$

Let $A_0 = A(G)\cap VN_r(G)$ regarded as a subspace of $A(G)$ (so that A_0 is $\phi_*j_{(0)}(L)$). Consider the map $i_{(0)}^*\phi_*^{-1}: A(G)\to L_{(0)}^*$, which maps A_0 into $L^p(\hat G)$ for all p. Let $\iota_n: A_0\to L^{p_n}(\hat G)$ be the resulting map. We have that $a\cdot\iota_n(b) = \iota_n(ab)$ for $a\in A(G)$ and $b\in A_0$. We hence see that

$$a\cdot T_n\iota_n(b) = R(a)\cdot\iota_n(b) = \iota_n(R(a)b) = \iota_n(aL(b)) = a\cdot\iota_n(L(b))\qquad (a\in A(G),b\in A_0).$$

It follows that $T_n\iota_n = \iota_nL$. By much the same argument as at the start of the proof, we see that for $a=(a_{ij})\in\mathbb{M}_m(A_0)$,

$$\|(L)_m(a)\| = \lim_n\|(\iota_nL(a_{ij}))\| = \lim_n\|(T_n\iota_n(a_{ij}))\|\leq\|T\|_{cb}\lim_n\|(\iota_n(a_{ij}))\|\leq\|T\|_{cb}\|(a_{ij})\|.$$

Thus L is completely bounded, with $\|L\|_{cb}\leq\|T\|_{cb}$, and so induces a member of $M_{cb}A(G)$. We can now follow the end of the previous proof to conclude that $T\in\hat\pi_{cb}(M_{cb}A(G))$. ☐

6 Analogues of the Figa-Talamanca–Herz algebras

In Section 2 we saw that the Figa-Talamanca–Herz algebras $A_p(G)$ naturally appeared. We have now developed enough theory to very easily suggest a definition for analogues of the Figa-Talamanca–Herz algebras, starting with $A(G)$ instead of $L^1(G)$. Indeed, consider the map $\pi^p: A(G)\to\mathcal{B}(L^p(\hat G))$ as in the previous section. We define $A_p(\hat G)$ to be the image of π^p_*, equipped with the quotient norm, so that $A_p(\hat G)$ is isometric to $(L^p(\hat G)\widehat{\otimes}L^{p'}(\hat G))/\ker\pi^p_*$. By Proposition 5.2 we see that $A_p(\hat G)$ is a subspace of $C^*_\lambda(G)$, which we would expect, as this is the "dual" statement to the fact that $A_p(G)\subseteq C_0(G)$.

The following says, informally, that $A_2(\hat G) = L^1(G)$.

Theorem 6.1. *For a locally compact group G, $A_2(\hat G)$ is equal to $\lambda(L^1(G))$ as a subset of $C^*_\lambda(G)$, and the norm on $A_2(\hat G)$ agrees with that on $L^1(G)$.*

Proof. We recall the isometric isomorphism $\theta: L^2(G)\to L^2(\hat G)$ given by Proposition 4.8, $\theta(f) = j_{(0)}^*\rho(\Delta^{-3/4}\check f)$ for $f\in C_{00}(G)^2$. Then, from above,

$$\pi^2_*(\theta(f)\otimes\theta(g)) = \lambda\big(\Delta^{-3/4}\check f\cdot K(\Delta^{-3/4}\check g)\big) = \lambda\big(\Delta^{-1/2}\check f\cdot g\big) = \lambda\big(Kf\cdot g\big)\qquad (f,g\in C_{00}(G)^2).$$

As $K: L^2(G)\to L^2(G)$ is unitary, by continuity, we have that $\pi^2_*(\theta(\xi)\otimes\theta(\eta)) = \lambda(K\xi\cdot\eta)$ for $\xi,\eta\in L^2(G)$. In particular, by Cauchy-Schwarz, we have that $K\xi\cdot\eta\in L^1(G)$ with $\|K\xi\cdot\eta\|_1\leq\|K\xi\|_2\|\eta\|_2$, for $\xi,\eta\in L^2(G)$.

For $\tau\in L^2(G)\widehat{\otimes}L^2(G)$ and $\epsilon>0$, we can find sequences (ξ_n) and (η_n) in $L^2(G)$ with

$$\tau = \sum_n\xi_n\otimes\tau_n,\quad\|\tau\|\leq\sum_n\|\xi_n\|_2\|\eta_n\|_2 < \|\tau\|+\epsilon.$$

Then let $f = \sum_n K\xi_n\cdot\eta_n\in L^1(G)$, the sum converging by Cauchy-Schwarz, with $\|f\|_1\leq\|\tau\|+\epsilon$. We see that

$$\pi^2_*(\theta\otimes\theta)\tau = \lambda\Big(\sum_n K\xi_n\cdot\eta_n\Big) = \lambda(f).$$

As $(\theta \otimes \theta)$ is an isometric isomorphism, it follows that $A_2(\hat{G}) \subseteq \lambda(L^1(G))$.

For $f \in L^1(G)$, let $\xi = K(|f|^{1/2}) \in L^2(G)$ and $\eta = f|f|^{-1/2} \in L^2(G)$, so that $\pi_*^2(\theta(\xi)\otimes\theta(\eta)) = f$, and $\|\xi\|_2 = \|\eta\|_2 = \|f\|_1^{1/2}$. We conclude that $A_2(\hat{G}) = \lambda(L^1(G))$, with the quotient norm on $A_2(\hat{G})$ agreeing with the L^1 norm on $\lambda(L^1(G))$. □

In particular, $A_2(\hat{G})$ is a subalgebra of $C^*_\lambda(G)$, and with the quotient norm, $A_2(\hat{G})$ is a Banach algebra. We have been unable to decide if the same is true for $A_p(\hat{G})$, for $p \neq 2$. However, we do have the following.

Proposition 6.2. *For $1 < p < \infty$, $A_p(\hat{G})$ contains a dense subset which is a subalgebra of $C^*_\lambda(G)$.*

Proof. Let $a, b, c, d \in C_{00}(G)^2$, let $\xi_1 = j^*_{(0)}(a), \xi_2 = j^*_{(0)}(c) \in L^p(\hat{G})$ and let $\eta_1 = j^*_{(0)}(b), \eta_2 = j^*_{(0)}(d) \in L^{p'}(\hat{G})$. Then, as above, $\pi_*^p(\xi_1 \otimes \eta_1) = \lambda(a \cdot Kb)$ and $\pi_*^p(\xi_2 \otimes \eta_2) = \lambda(c \cdot Kd)$. Let $f = (a \cdot Kb)(c \cdot Kd) \in C_{00}(G)^2$.

Pick $g_1 \in C_{00}(G)$ with $\int_G g_1(s)\, ds = 1$. Let $X \subseteq G$ be a compact set containing the support of f, and let $Y \subseteq G$ be a compact set containing the support of g_1. Let $e = |Y|^{-1}\chi_{(XY)^{-1}Y}$ and $f = \chi_Y$, so that $g_0 = e\check{f} \in C_{00}(G)$. Then, for $s \in G$,

$$g_0(s) = \int_G e(t)\check{f}(t^{-1}s)\ dt = \frac{1}{|Y|}\int_{(XY)^{-1}Y} \chi_Y(s^{-1}t)\ dt = \frac{|sY \cap (XY)^{-1}Y|}{|Y|},$$

so if $s \in (XY)^{-1}$, then $sY \subseteq (XY)^{-1}Y$ and so $g_0(s) = |sY||Y|^{-1} = 1$. Now let $g = (\nabla^{-1/2}g_1)(\nabla^{-1/2}g_0) \in C_{00}(G)^2$, so for $s \in X$,

$$\begin{aligned}(\nabla^{-1/2}g_1)(\nabla^{-1/2}g_0)(s^{-1}) &= \int_G \nabla(t)^{-1/2}g_1(t)\nabla(t^{-1}s^{-1})^{-1/2}g_0(t^{-1}s^{-1})\ dt \\ &= \nabla(s)^{1/2}\int_Y g_1(t)g_0(t^{-1}s^{-1})\ dt = \nabla(s)^{1/2}\int_Y g_1(t)\ dt = \nabla(s)^{1/2},\end{aligned}$$

as if $t \in Y$ then $t^{-1}s^{-1} \in (XY)^{-1}$. Hence, for $s \in X$, we see that $Kg(s) = g(s^{-1})\nabla(s)^{-1/2} = 1$. Thus $f \cdot Kg = f$, showing that

$$\pi_*^p(\xi_1 \otimes \eta_1)\pi_*^p(\xi_2 \otimes \eta_2) = \lambda(f) = \pi_*^p(j^*_{(0)}\rho(f) \otimes j^*_{(0)}\rho(g)).$$

We conclude that

$$\operatorname{lin}\left\{\pi_*^p(j^*_{(0)}\rho(f) \otimes j^*_{(0)}\rho(g)) : f, g \in C_{00}(G)^2\right\} \subseteq A_p(\hat{G})$$

is a dense subalgebra. □

One could instead work with $\pi_*^{p,cb}$, which would lead to an operator space version of $A_p(\hat{G})$, say $OA_p(\hat{G})$. However, as this would naturally use the operator space projective tensor product, in general we would only have that $A_p(\hat{G}) \subseteq OA_p(\hat{G})$. Indeed, in [30], Runde used the natural operator space structure on vector valued *commutative* L^p spaces to define algebras $OA_p(G)$, as an attempt to find an operator space structure on $A_p(G)$. If G is abelian, then by the Fourier transform, $OA_p(\hat{G})$ has an unambiguous meaning (either ours or Runde's). Let $PM_p(\hat{G})$ be the weak*-closure of $\pi^p(A(G))$ in $\mathcal{B}(L^p(\hat{G}))$. After [30, Proposition 2.1], in a remark attributed to G. Pisier, it is shown that there exist abelian G with $PM_p(\hat{G}) \not\subseteq \mathcal{CB}(L^p(\hat{G}))$. It follows that $OA_p(\hat{G}) \neq A_p(\hat{G})$. If we wish to view $OA_p(\hat{G})$ as a generalisation of $A_p(G)$, then this a problem!

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