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Michell structure for a uniform load over multiple spans

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Abstract

A new half-plane Michell structure capable of carrying a uniformly distributed load of infinite horizontal extent over a series of equally-spaced pinned supports is presented. For the case of equal allowable stresses in tension and compression a full kinematic description of the structure is provided. Although formal proof of optimality of the solution presented is not yet available, the proposed analytical solution is supported by available numerical evidence. Numerical solutions for cases of unequal allowable stresses are also presented, and suggest the existence of a wider family of related, simple, and practically relevant structures, which range in form from an arch with vertical hangers to a cable-stayed bridge.

Keywords: Michell structure, uniformly distributed loading.

1. Introduction

In his groundbreaking contribution to the field of structural optimization [11], A. G. M. Michell formulated the criteria to be satisfied by all least-volume trusses with equal tensile and compressive yield stresses, see also [6, 12]. In trusses satisfying these criteria the magnitudes of the tensile and/or compressive stresses in load-carrying members must everywhere be at maximum allowable values and the virtual strains in such members must not exceed these limiting values. The displacement field must remain continuous throughout the design domain and satisfy the kinematic restrictions imposed on the solution.

Michell's criteria can be satisfied in several different ways, implying that every optimal structure can be split into one or several regions, distinguished by values of the member force components f' and f'' :

$$T : \quad f' < 0, \quad f'' > 0, \quad \varepsilon' = -\epsilon, \quad \varepsilon'' = \epsilon; \quad (1)$$

$$R^C : \quad f' = 0, \quad f'' < 0, \quad |\varepsilon'| \leq \epsilon, \quad \varepsilon'' = -\epsilon; \quad (2)$$

$$R^T : \quad f' > 0, \quad f'' = 0, \quad \varepsilon' = \epsilon, \quad |\varepsilon''| \leq \epsilon. \quad (3)$$

Within (1)–(3), ε' and ε'' denote principal strains and ϵ is the positive infinitesimal. Optimal trusses may also contain regions of uniform tension and/or uniform compression [15]. All trusses constructed by Michell [11], as well as the majority of optimal trusses identified in the early literature, only feature one or several regions of type T ; the term ‘Michell structure’ is therefore sometimes considered to be synonymous with (1). However in this paper we use this term to describe any structure that satisfies the Michell criteria, and any number of regions (1)–(3) can be present.

The deceptive simplicity of the specified criteria should not obscure the fact that there is no known procedure for verifying whether a Michell structure exists for a given problem definition, or for determining its form. Unsurprisingly, the number of Michell structures to have been identified to date is not large, see e.g. [1, 2, 6, 8–11, 14]. Furthermore, whilst some notable exceptions exist [3, 7, 16, 17], the majority of known Michell structures are designed to support only a single external point load.

In this paper, we present details of an apparently new Michell structure, for a problem which appears to have been hitherto overlooked. The problem involves a uniformly distributed vertical load applied along a horizontal line spanning across an infinite number of equally spaced pinned supports. The motivation for this configuration originates from the (still unsolved) classical problem of finding the optimal half-plane structure to transmit a uniformly distributed load along a line between two level pinned supports, to these supports [3, 7]. In the case of equal allowable stresses and an *infinite* number of supports, the resulting geometry of the Michell structure, and the mathematical solution for kinematic fields, all turn out to be comparatively simple. Importantly, the volume per single span of the resulting structure is approximately 11.0% lower than that of the parabolic arch with vertical hangers and 7.86% lower than that of the classical solution [7], which is known to be sub-optimal. We stop short of proving the optimality of the proposed structure for the half-plane; however, results from numerical simulations presented in the

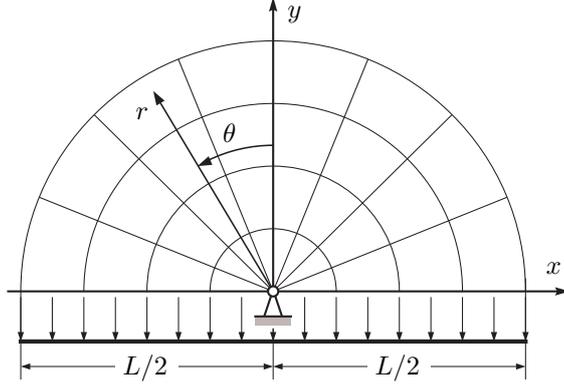


Figure 1: The Michell half-wheel subjected to a uniformly distributed load.

paper appear to support our claim. We also present a number of numerical solutions for similar problems with unequal allowable stresses, suggesting that a wider family of related, simple and practically relevant structures exists.

2. An auxiliary problem

Before analysing our main problem, featuring an infinite number of equally-spaced supports, it is instructive to examine a simpler set-up. Consider a uniformly distributed load (w per unit length) that is applied to a horizontal line segment of length L , and needs to be transmitted to a pinned support at the centre of the segment. It is not difficult to verify that the suitable optimal solution for the upper half-plane is a ‘half-wheel’, the structure comprising concentric semicircles and orthogonal radii, as shown on Fig. 1. Very similar structures for problems involving external point loads have been considered in [6, 11].

The structure is conveniently mapped by the orthogonal curvilinear system (α, β) , such that

$$\alpha = r, \quad \beta = \theta, \quad \phi = \beta + \frac{\pi}{2}, \quad A = 1, \quad B = \alpha, \quad (4)$$

where r , $0 \leq r \leq L/2$, is the linear distance from the support, θ , $-\pi/2 \leq \theta \leq \pi/2$, the polar angle measured counter-clockwise from the vertical symmetry axis and ϕ the angle measured from the horizontal line to the tangent of an α -line. Functions A and B are the components of the metric tensor. A suitable strain field is given by

$$u = -\epsilon\alpha, \quad v = 2\epsilon\alpha\beta, \quad \omega = 2\epsilon\beta, \quad (5)$$

in which u and v denote displacements along α and β , respectively, and ω denotes the rotation.

If T' and T'' denote the end loads per unit coordinate difference in the α and β directions, then they must satisfy the standard equilibrium equations in curvilinear coordinates:

$$\frac{\partial T'}{\partial \alpha} = T'' \frac{\partial \phi}{\partial \beta}, \quad \frac{\partial T''}{\partial \beta} = -T' \frac{\partial \phi}{\partial \alpha}, \quad (6)$$

see [6]. In our case $\partial T''/\partial \beta = 0$, and the equilibrium of vertical components of forces acting along the bottom of the structure requires that $T'' = w$. Equation (6)₁ can now be integrated, yielding $T' = w\alpha + t'(\beta)$. One needs to add another boundary condition to fully specify the force field within the structure. For example, if T' is required to vanish along the outer rim of the structure, then $t'(\beta) = -wL/2$ and $T' = w(\alpha - L/2)$, hence, completing the solution. The volume of the resulting structure is easily found from the virtual work of external forces, which in our particular case yields

$$W_{\text{aux}} = \frac{2}{\sigma\epsilon} \int_0^{L/2} -wv|_{\beta=-\pi/2} d\alpha = \frac{\pi wL^2}{4\sigma}. \quad (7)$$

3. The virtual displacement field

Strain field (4) may be trivially extended to cover the entire half-plane, thus signalling the global optimality of the solution obtained in Section 2. Perhaps unsurprisingly it cannot be as easily applied to problems featuring multiple supports. Indeed, if we were to consider two level supports, and attempt to match two

copies of field (5), expanding from each of the supports, then this would be found to be impossible due to the monotonic variation of each local u and v as functions of local α . Motivated by this observation, we consider an extension of the structure from Section 2, a half-span as shown in Fig. 2 (we assume that the other half is obtained by reflecting the structure about the vertical Oy). The original half-wheel, shaded in the new drawing, is expanded to the radius $L/\sqrt{2}$ and then cut along the vertical lines originating from points $x = \pm L/2$ in the global Cartesian coordinate system Oxy . Tensile circumferential members of the original half-wheel are then continued tangentially, as straight ties that connect the points along the cuts to horizontal compression members at $L/2 \leq |x| \leq L$. These concentrated members are needed to rotate the tie forces and equilibrate the portions of external load that are uniformly distributed along $L/2 \leq |x| \leq L$.

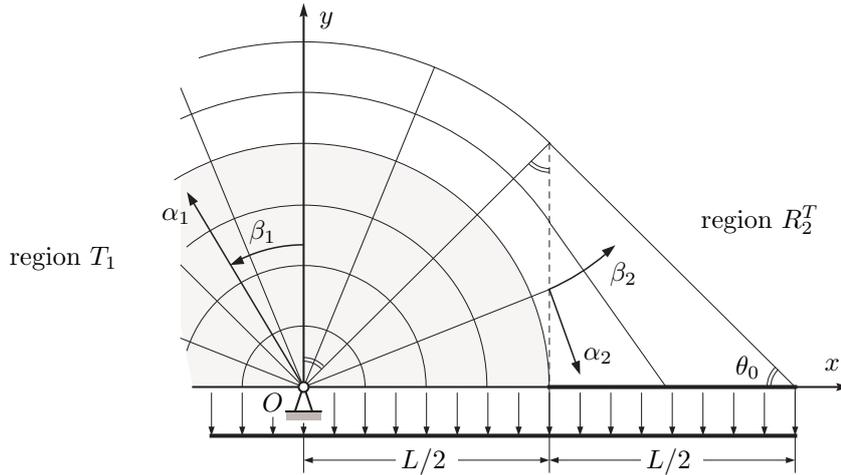


Figure 2: Half-span of the proposed structure.

The description of this structure requires the use of several curvilinear coordinate systems. The part of the structure for $-L/2 \leq x \leq L/2$, termed region T_1 , can be fully described using the same coordinate system as in Section 2. Thus, we assume that α_1 , β_1 , ϕ_1 , A_1 , B_1 , u_1 , v_1 and ω_1 are defined precisely as in (4) and (6)*. The only difference concerns the ranges of variation of the coordinates; since the verticals $x = \pm L/2$ are described within region T_1 by equation $\alpha_1 = L/2 |\sin \beta_1|$, therefore, $-\pi/2 \leq \beta_1 \leq \pi/2$ (as before) and $0 \leq \alpha_1 \leq \min\{L/\sqrt{2}, L/2 |\sin \beta_1|\}$. In particular, the curvilinear displacements and rotation are given along the boundary with region R_2^T by

$$u_1 = \frac{\epsilon L}{2 \sin \beta_1}, \quad v_1 = -\frac{\epsilon \beta_1 L}{\sin \beta_1}, \quad \omega_1 = 2\epsilon \beta_1, \quad \text{at } x = \frac{L}{2}. \quad (8)$$

The curvilinear coordinate system appropriate for describing the strain field within region R_2^T is harder to formulate. The systems of straight, non-intersecting ties are associated with regions described by (3); the mathematical formalism describing such regions is presented in [6, Sect. 4.2]. We begin by defining coordinate β_2 as the same polar angle as the one used within region T_1 . The bottom left corner of R_2^T corresponds to $\beta_2 = -\pi/2$, whereas the uppermost tie corresponds to $\beta_2 = -\pi/4$. More generally, all ties within R_2^T belong to the family of straight lines parametrised by β_2 :

$$2x - 2y \cot \beta_2 - L(1 + \cot^2 \beta_2) = 0. \quad (9)$$

It is possible to show that these lines envelop an evolute with the equation

$$y^2 + 2Lx - L^2 = 0. \quad (10)$$

In an orthogonal coordinate system with α_2 defined as the distance from a fixed involute, equation (10) may be alternatively written as $\alpha_2 + F(\beta_2) = 0$. Here $F(\beta_2)$ is an arc length measured along the evolute from the point where $\alpha_2 = 0$. Since evolute (10) touches the bottom left corner of R_2^T , it is convenient

* The numeric subscripts indicate which specific region a given quantity relates to.

to use the involute passing through this point as the coordinate axis. We can now integrate along the evolute to obtain the full description of our curvilinear coordinates in the form

$$\phi_2 = \beta_2 + \frac{\pi}{2}, \quad A_2 = 1, \quad B_2 = \alpha_2 + F(\beta_2), \quad \text{where } F(\beta_2) = \frac{L}{2}(\cot \beta_2 \csc \beta_2 - \ln[\cot \beta_2 - \csc \beta_2]), \quad (11)$$

see also [6]. The Cartesian description of coordinate lines in (α_2, β_2) is obtained by computing

$$x + iy = \frac{L}{2} + \alpha_2 e^{i\beta_2} + i \int_{-\pi/2}^{-\pi/4} e^{i\xi} F(\xi) d\xi, \quad (12)$$

which leads to the explicit formulae

$$x = (\alpha_2 - L \ln[\cot \beta_2 - \csc \beta_2]/2) \cos \beta_2 + L/2, \quad (13)$$

$$y = (\alpha_2 - L \ln[\cot \beta_2 - \csc \beta_2]/2) \sin \beta_2 - L \cot \beta_2/2. \quad (14)$$

An additional test of the validity of these equations may be performed by directly computing the metric tensor components from (13) and (14). The resulting expressions match equations (11) exactly. Table 1 presents some useful relationships between coordinates of various lines and points within the global Cartesian and the local curvilinear coordinate systems.

Table 1: Significant lines and points within the coordinate system (α_2, β_2) .

Cartesian	Curvilinear	Significance
$x = L/2$	$\alpha_2 = L \ln(\cot \beta_2 - \csc \beta_2)/2$	the boundary between T_1 and R_2^T
$y = 0$	$\alpha_2 = L (\cot \beta_2 \csc \beta_2 + \ln[\cot \beta_2 - \csc \beta_2]) / 2$	the bottom of R_2^T
$y = L - x$	$\beta_2 = -\pi/4$	the top tie of R_2^T
$(L/2, 0)$	$(0, -\pi/2)$	the bottom left corner of R_2^T
$(L/2, L/2)$	$(L \ln(\sqrt{2} - 1)/2, -\pi/4)$	the top left corner of R_2^T
$(L, 0)$	$(L[\sqrt{2} + \ln(\sqrt{2} - 1)]/2, -\pi/4)$	the right corner of R_2^T

Given orthogonal coordinates (11), we can formulate the system of partial differential equations describing principal and shear strains, as well as the rotation, in the form:

$$\frac{\partial u_2}{\partial \alpha_2} = \epsilon, \quad v_2 = \omega_2(\alpha_2 + F(\beta_2)) + \frac{\partial u_2}{\partial \beta_2}, \quad (15)$$

$$\omega_2 = \frac{\partial v_2}{\partial \alpha_2}, \quad \epsilon_2'' = (\alpha_2 + F(\beta_2))^{-1} \left(\frac{\partial v_2}{\partial \beta_2} + u_2 \right). \quad (16)$$

Equation (15)₁ implies that $u_2 = \epsilon(\alpha_2 + G(\beta_2))$, with $G(\beta_2)$ chosen to ensure the continuity along the line $x = L/2$. Since β_1 and β_2 denote the same angle, the continuity with circumferential displacements requires $u_2|_{\beta_2=\beta_1} = -v_1|_{\beta_1=\beta_2}$, so that a reference to (8)₂ leads to the full definition

$$u_2 = \epsilon(\alpha_2 + G(\beta_2)), \quad G(\beta_2) = F(\beta_2) + L(2\beta_2 - \cot \beta_2) \csc \beta_2/2. \quad (17)$$

The rotation is fixed along α_2 -lines within R_2^T ; therefore, the continuity of rotation along $x = L/2$ and equation (8)₃ give $\omega_2 = 2\epsilon\beta_2$. This enables us to compute u_2 directly from (15)₂, with the result

$$v_2 = \epsilon(2\beta_2[\alpha_2 + F(\beta_2)] - L[2\beta_2 \cot \beta_2 - 1] \csc \beta_2/2). \quad (18)$$

The substitution of displacement (18) into (16)₁ again gives $\omega_2 = 2\epsilon\beta_2$, as it should. By substituting the value for α_2 associated with $x = L/2$ from Table 1 into (18), it is also possible to verify that $v_2|_{\beta_2=\beta_1} = u_1|_{\beta_1=\beta_2}$. The only remaining equation (16)₂ provides a direct mean for computing the strain along β_2 -lines, which is found to be

$$\epsilon_2'' = 2\epsilon \left(2 - \frac{L \cot \beta_2 \csc \beta_2}{\alpha_2 + F(\beta_2)} \right) - \epsilon. \quad (19)$$

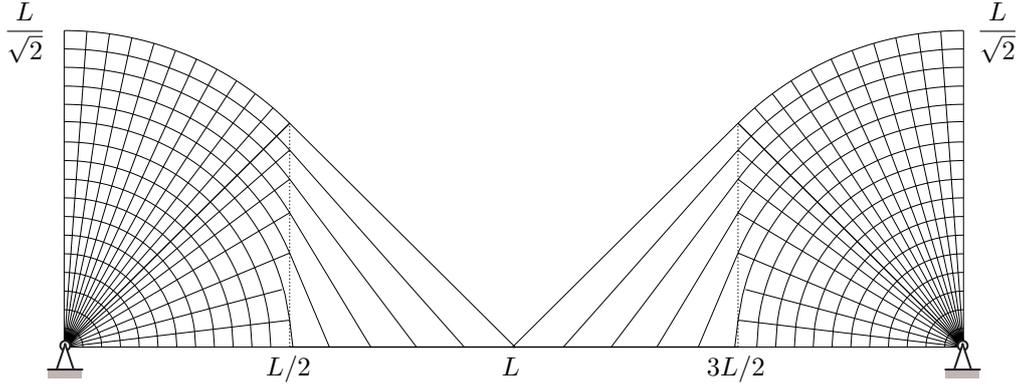


Figure 3: A single span of the proposed optimum structure.

For the field within R_2^T to satisfy the Michell criteria (3), we must ensure that $|\varepsilon_2''| \leq \epsilon$. It is worth reminding ourselves that the denominator within (19) is an equation of the evolute. It can only vanish in a single point of region R_2^T , where the evolute touches the bottom left corner, see (10). However, due to the cancellation of terms, one has everywhere along the bottom boundary of R_2^T :

$$\varepsilon_2''|_{y=0} = \epsilon, \quad (20)$$

see (19) and Table 1. Simultaneously, everywhere along the boundary between regions T_1 and R_2^T ,

$$\varepsilon_2''|_{x=L/2} = -\epsilon. \quad (21)$$

Keeping in mind that, for every fixed β_2 , ε_2'' is a monotonously increasing function of α_2 , see (19), that changes from $-\epsilon$ at $x = L/2$ to ϵ at $y = 0$, we come to the sought-for conclusion that R_2^T is a valid Michell region of type R^T , see (3).

Having constructed a consistent strain field for a single half-span does not yet solve the original problem, featuring infinite sequence of equally-spaced level supports. A full span of length $2L$ can be obtained by reflecting the constructed fields with respect to Oy . In addition, we can use equations (17), (18) and Table 1 to write u_2^x , the horizontal component of displacement along $y = 0$, in the form

$$u_2^x|_{y=0} \equiv u_2|_{y=0} \sin \phi_2 + v_2|_{y=0} \cos \phi_2 = -\epsilon \frac{L \cos 2\beta_2}{2 \sin^2 \beta_2}. \quad (22)$$

Clearly, this vanishes when $\beta_2 = -\pi/4$, i.e. when $x = L$. This means that we can also reflect the resulting structure with respect to the vertical line $x = L$. Therefore, it is now possible to produce a structure that, via a series of simple reflections, spans across an infinite sequence of level supports placed $2L$ apart along Ox . An illustration of a single span of such a structure is given in Fig. 3.

It has already been mentioned that regions of type T , i.e. the regions that satisfy the Michell criteria and conditions (1), are often perceived to be synonymous with all Michell structures. Since these regions feature systems of mutually orthogonal members, the requirement of member orthogonality often presumed for general Michell structures. This requirement is, evidently, violated at the bottom boundary of region R_2^T . Nevertheless, Rozvany [13] shows that the orthogonality requirement can be relaxed along boundaries between R^C and R^T regions. Thus, we can resolve the contradiction by interpreting the compression bar along the bottom of R_2^T as a degenerate region R_3^C , which satisfies all of the conditions (2).

The presence of a compressive concentrated member R_3^C also resolves what may appear to be a strain discontinuity at Cartesian point $(L/2, 0)$. Without R_3^C , the horizontal strain along the bottom of T_1 would be compressive, whereas the normal strain along the bottom of R_2^T , which becomes horizontal at $x = L/2$, would be tensile, see (20). Thus, the compressive member R_3^C ensures the continuity of the strain field at $y = 0$.

Although we have now obtained a continuous virtual displacement field that satisfies all of our kinematic requirements, this does not constitute a proof of global optimality for our solution. Such a proof would require constructing a continuous virtual displacement field that covers the entire half-space. The derivation of such a field is beyond the scope of the present paper.

4. The volume of the structure

The volume of a single span of the proposed structure can be computed by calculating the work done by the external forces and dividing it by $\epsilon\sigma$. The work \mathcal{W}_I done by the distributed load acting along $-L/2 \leq x \leq L/2$ has already been computed in (7): $\mathcal{W}_I \equiv \epsilon\sigma W_{\text{aux}} = \epsilon\pi wL^2/4$. In order to determine the work \mathcal{W}_{II} , done by the distributed load acting along $L/2 \leq |x| \leq L$, one needs to find the vertical displacement u_2^y along the bottom boundary of R_2^T . Using equations (17), (18) and Table 1, we obtain

$$u_2^y|_{y=0} \equiv -u_2|_{y=0} \cos \phi_2 + v_2|_{y=0} \sin \phi_2 = \epsilon L(\beta_2 \csc^2 \beta_2 + \cot \beta_2). \quad (23)$$

With the help of (23), the work integral is computed as

$$\mathcal{W}_{II} = 2 \int_{L/2}^L -wu_2^y|_{y=0} dx = 2wL \int_{-\pi/2}^{-\pi/4} u_2^y|_{y=0}(\cot^2 \beta_2 + 1) \cot \beta_2 d\beta_2 = \epsilon \left(\frac{4}{3} + \frac{\pi}{4} \right) wL^2. \quad (24)$$

It is now self-evident that

$$W_{\min} = \frac{\mathcal{W}_I + \mathcal{W}_{II}}{\epsilon\sigma} = \left(\frac{4}{3} + \frac{\pi}{2} \right) \frac{wL^2}{\sigma} \approx 2.90413 \frac{wL^2}{\sigma}. \quad (25)$$

Therefore, the volume of a single span of the described structure is 11.0% lower than the volume of a simple parabolic arch with vertical hangers, and 7.86% lower than that of the classical solution obtained by Hemp [7], which is known to be sub-optimal (see also [3]).

The solutions for force fields within regions T_1 , R_2^T and R_3^C can be computed without much difficulty and are omitted here for the sake of brevity. We used these solutions to compute the volume of the structure directly, and to verify formula (25). The volumes obtained via primal and dual formulations matched, therefore providing further confirmation of the correctness of the reported result.

5. Numerical solutions

In order to verify the optimality of the structure described in previous sections, a numerical solution has also been obtained using an efficient numerical layout optimisation procedure [5]. The same procedure was recently used to provide compelling numerical evidence that the parabolic arch is not an optimal structure to transfer a uniformly distributed transmissible load to two pinned supports [4]; see also subsequent formal proof of this [17].

The numerical solutions presented in this paper were computed for several combinations of allowable stresses, using numerical discretizations comprising 61 nodal points in the x direction and 41 nodal points in the y direction, therefore optimising over 3,126,250 potential members. The computations, in each of the cases, took around 30 seconds of CPU time on a modern PC. The plots of resulting solutions are grouped together in Fig. 4. Note that the solutions are plotted using a perspective projection, which makes the upper parts of structures appear narrower than they are.

Let σ_T and σ_C be the tensile and compressive yield stresses, respectively. The structure obtained in the case when $\sigma_C = \sigma_T$ and shown (twice) at the top of Fig. 4, displays a remarkable similarity to the analytical solution shown in Fig. 3. However, a slight mismatch in the positions of nodes at the top of Region 2 leads to the appearance of an additional (feint) fan region, comprising straight lines and concentric circles. Supplementary runs were performed to ensure that this vanishes as the numerical discretization is refined.

Interestingly, the numerical solutions for unequal allowable stresses indicate that our solution, although seemingly unusual, is closely related to two well known classes of structure, widely used in engineering practice. In particular, the left hand side of Fig. 4 presents structures dominated by compression ($\sigma_C > \sigma_T$). As σ_C/σ_T increases, the fans around the supports shrink in size, with the overall structure tending towards a simple arch with vertical hangers. In the case of structures dominated by tension ($\sigma_C < \sigma_T$), shown on the right hand side of Fig. 4, the solutions metamorphose into a cable stayed bridge structure, with the fans shrinking to become stocky, near-vertical, towers.

Closer inspection of the optimal structures dominated by tension enables fairly accurate determination of numerical values of the abscissas at which half-wheel fields are replaced by systems of straight tension members. This allows us to formulate a conjecture about the structures of this type. If X denotes an abscissa where the described transition takes place, it appears that

$$X = \frac{\sigma_C}{\sigma_C + \sigma_T} L. \quad (26)$$

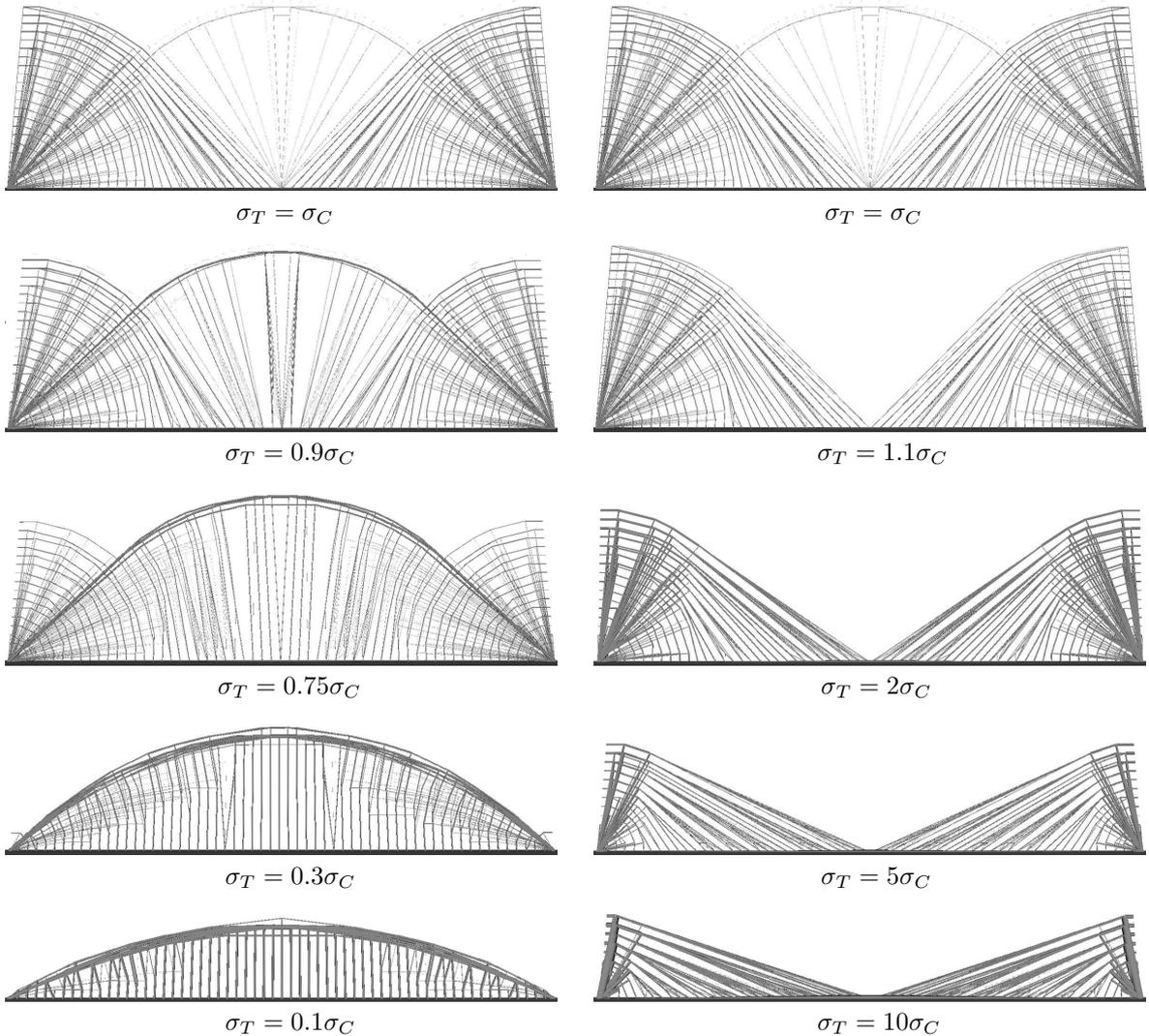


Figure 4: The numerical solutions that illustrate the effects of having unequal allowable stresses.

In particular, in the case when $\sigma_C = \sigma_T$, (26) yields $X = L/2$, precisely the same as assumed in our earlier derivations. Expression (26) can also be reformulated in terms of slope θ_0 of the top tie within region R_2^T , see Fig. 2. A simple geometric argument leads then to the following conclusion

$$\theta_0 = \arctan \sqrt{\frac{\sigma_C}{\sigma_T}}, \quad (27)$$

which is precisely the same condition as the one previously obtained for the parabolic funicular loaded by a transmissible, uniformly distributed load, see [4, 18].

6. Conclusions

Details of a new half-plane Michell structure capable of carrying a uniformly distributed load of infinite horizontal extent over a series of equally-spaced pinned supports have been presented. Although formal proof of optimality of the structure has not yet been demonstrated, the proposed analytical solution is supported by available numerical evidence. Numerical solutions also suggest the existence of a wider family of related, simple, and practically relevant structures, which range in form from an arch with vertical hangers to a cable-stayed bridge, depending on the specified ratio of limiting compressive to tensile stress.

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