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On the Global Bilinearization of Nonlinear Systems and the Existence of Volterra Series

by

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Abstract

A global Volterra series is developed for an analytic system defined on an analytic manifold M. Local bilinearizations are pieced togther using the theory of fibre bundles, giving rise to 'twisted' bilinear systems defined globally on M.

1. Introduction

The application of Volterra series to the study of nonlinear systems is now well known and has a long history; see, for example Volterra, 1958, Brockett, 1976, Lesiak and Krener, 1978, Crouch, 1981, Banks, 1985. The existence of bilinear representations of nonlinear systems has also been extensively studied since the application of Carleman linearization to linear analytic systems by Brockett, 1976. Generalizations of this idea have been given by Krener, 1975 and Lo, 1975 where the global bilinearization of systems of the form

$$\dot{x}(t) = f(x(t)) + G(x(t))u(t)$$

is discussed.

In this paper we shall treat a general nonlinear system

$$X(\cdot):U\rightarrow D(M) \tag{1.1}$$

where U is a control space and D(M) is the set of analytic vector fields on an analytic manifold M. In the first part we shall assume $U=\mathbb{R}^m$, $M=\mathbb{R}^n$ and that the system has the global representation

$$\dot{x} = f(x,u) \tag{1.2}$$

and obtain a bilinearization for this system.

In the second part of the paper, by replacing (1.1) by a system

$$Y(\cdot) : T(U)\rightarrow D(M\times U) \tag{1.3}$$

where T(U) is the tangent of U, we shall obtain a bilinearization for (1.1) which uses the local theory developed for (1.2). This will involve the introduction of a certain fible bundle and we shall associate a set of sections of this fibre bundle with (1.3) and regard this set of sections as a bilinear system. We can then associate an exponential mapping with this system which has the appropriate invariance properties, allowing us to obtain a global Volterra series for the system (1.2).

2. Notation and Terminology

In the case of differential equations defined on $\mathbb{R}^n$ we shall denote a generic equation by

$$\dot{x} = f(x,u)$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. The function f will be assumed to be (globally) analytic and we shall denote the homogeneous monomials in the Taylor expansion of f by

$$x^i u^j = x_1^{i_1} \dots x_n^{i_n} u_1^{j_1} \dots u_m^{j_m} ,$$

where

$$i = (i_1, \dots, i_n) \in \mathbb{N}^n \quad , \quad j = (j_1, \dots, j_m) \in \mathbb{N}^m .$$

In particular, $1(k)$ will denote the multi-index in $\mathbb{N}^\ell$ (some ℓ) in which the k^{th} entry is 1 and the others are zero. Note, however, that we shall not explicitly indicate the dimension ℓ of $1(k)$ so that if we write $x^{1(k)}$ or $u^{1(k)}$ then the dimension of $1(k)$ will be assumed to be that which is appropriate, i.e. n or m, respectively, in this case. I will denote the set of indices $(i,j) \in \mathbb{N}^n \times \mathbb{N}^m$.

ℓ^1 will denote the usual Banach space defined by

$$\{(\alpha_j)_{0 \le j < \infty} : \sum_{j=0}^{\infty} |\alpha_j| < \infty\}.$$

For systems defined on an analytic manifold M we shall use D(M) to denote the set of analytic vector field on M. T(M) will denote the tangent bundle of M while the tangent space at a point $p \in M$ is written $T_p(M)$. The main technical device in studying the global bilinearization of a nonlinear analytic system will be that of fibre bundle. Hence, we shall briefly recall the definition of a fibre bundle; its properties are given in detail by Kobayashi and Nomizu, 1963.

Let G be a topological group and X a topological space. X is called a right G-space if there is a map from $X \times G \to X$, written $(x,g) \mapsto xg$, such that

$$x(gh) = (xg)h$$

and $x1_G = x$ for all $x\varepsilon X, g, h\varepsilon X$. A map $f:X\to Y$ between G-spaces is called a <u>G-morphism</u> if $f(xg) = f(x)g$ for all $x\varepsilon X$, $g\varepsilon G$.

Given a G space X, we can define an equivalence relation on X as follows. Write $x_1 \sim x_2$ if there exists $g\varepsilon G$ such that $x_1 g = x_2$. This is an equivalence relation and we let $xG=\{xg : g\varepsilon G\}$ and $X/G=X/\sim$ with the quotient topology. Hence, if $\pi:X\to X/G$ is the canonical projection, then $(X,\pi,X/G)$ is a bundle. Since $f(xG)\subseteq f(x)G$ for any G morphism $f:X\to Y$, there is an induced map $f_\sim : X/G\to Y/G$ with $f_\sim(xG)=f(x)G$. Then $(f,f_\sim)$ is a bundle morphism.

A bundle (X,p,B) is called a <u>G-bundle</u> if $(1,f):(X,p,B)\to(X,\pi,X/G)$ is a bundle isomorphism, for some G-structure on X, where $f:B\to X/G$ is a homeomorphism. If G also has a differentiable structure then a <u>principal fibre bundle</u> over a manifold M with group G consists of a manifold P and an action of G on P such that

(1) G acts freely on P.

(2) $M=P/G$ and $\pi:P\to M$ is differentiable

(3) P is locally trivial; i.e. if $x\varepsilon M$ then there is a neighbourhood U of x in M such that $(\pi,\phi):\pi^{-1}(U)\to U\text{x}G$ is a diffeomorphism, for some G-morphism ϕ.

If U_α, U_β are open sets in M and ϕ_α, ϕ_β are associated with them as in (3) then if $u\varepsilon\pi^{-1}(U_\alpha \cap U_\beta)$ we have $\phi_\beta(ug)(\phi_\alpha(ug))^{-1} = \phi_\beta(u)(\phi_\alpha(u))^{-1}$. Hence we can define a mapping

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$$

by $g_{\alpha\beta}(\pi(u))=\phi_\beta(u)(\phi_\alpha(u))^{-1}$. These are the <u>transition functions</u> of the bundle and can be used to construct a bundle as shown in Kobayashi and Nomizu (1963). This is the approach we shall use to construct a fibre bundle in section 4.

Now let P be a principal fibre bundle and F a manifold on which G acts on the left; $(g,\zeta)\varepsilon G\text{x}F\to g\zeta\varepsilon F$. Then let G act on the right on PxF by $((u,\zeta),g)\varepsilon (P\text{x}F)\text{x}G\to(ug,g^{-1}\zeta)\varepsilon P\text{x}F$ and define

$$E = P\text{x}F/G$$

as a set. Define the projection $\pi_E : E \to M$ by $\pi_E((u,\zeta)G) = \pi(u)$ and call $\pi_E^{-1}(x)$ the fibre over $x \varepsilon M$. Since each point $x \varepsilon M$ has a neighbourhood U such that $\pi^{-1}(U) \cong U \times G$, it follows that the action of G on the right of $\pi^{-1}(U) \times F \cong U \times G \times F$ is given by $(x,g,\xi)h = (x, gh, h^{-1}\xi)$, $x \varepsilon U$, $g, h \varepsilon G$, $\xi \varepsilon F$ and so $\pi_E^{-1}(U) \cong U \times F$. We can therefore make E into a differentiable manifold in which $\pi_E^{-1}(u)$ is an open submanifold of E diffeomorphic with $U \times F$ under the isomorphism above. (E, π_E, M) is called the fibre bundle associated with the principal fibre bundle P.

3. Tensor-valued Differential Equations

Consider the controlled differential equation

$$\dot{x} = f(x,u) \tag{3.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$. We shall use standard index notation here, so that

$$(i,j) = (i_1, \dots, i_n, j_1, \dots, j_m)$$

will denote an $(n+m)$-tuple of integers and

$$x^i u^j = x_1^{i_1} \dots x_n^{i_n} u_1^{j_1} \dots u_m^{j_m}$$

Moreover, $1(k)$ will denote the multi-index with a 1 in the k^{th} place and zero elsewhere. We shall not specify explicitly the dimension of the multi-index $1(k)$, assuming simply that it has the appropriate dimension for the context.

In many cases, when studying control problems related to the system (1), it turns out that u is differentiable (although it does not follow, of course, for switching controls). When u is differentiable we may write

$$\begin{aligned} \dot{x} &= f(x,u) \\ \dot{u} &= v \end{aligned} \tag{3.2}$$

and regard v as the control. Now define the functions

$$\phi_{(i,j)} = x^i u^j \quad ,$$

and differentiate $\phi_{(i,j)}$ along the trajectories of (3.2). Then we have

$$\dot{\phi}_{(i,j)} = \sum_{k=1}^{n} i_k x^{(i-1(k))} u^j \dot{x}_k + \sum_{k=1}^{m} j_k x^i u^{(j-1(k))} \dot{u}_k$$

(where the first occurrence of $1(k)$ has dimension n and the second dimension m).

We shall assume, in this section, that f is globally analytic so that we may write

$$f_k(x,u) = \sum_{(i,j)\varepsilon I} \alpha^k_{(i,j)} x^i u^j \quad , \quad 1 \le k \le n \tag{3.3}$$

for some constants $\alpha^k_{(i,j)}$, where I is the set of all multi-indices with non-negative terms. Hence we have

$$\dot{\phi}_{(i,j)} = \sum_{k=1}^{n} i_k x^{(i-1(k))} u^j \sum_{(i',j')\varepsilon I} \alpha^k_{(i',j')} x^{i'} u^{j'} + \sum_{k=1}^{m} j_k x^i u^{(j-1(k))} v_k$$

$$= \sum_{(i',j')\varepsilon I} \sum_{k=1}^{n} i_k \alpha^k_{(i',j')} x^{i+i'-1(k)} u^{j+j'} + \sum_{k=1}^{m} j_k x^i u^{(j-1(k))} v_k$$

$$= \sum_{(i'',j'')\varepsilon I} \sum_{k=1}^{n} i_k \alpha^k_{(i''-i+1(k),j''-j')} x^{i''} u^{j''} + \sum_{k=1}^{m} j_k x^i u^{(j-1(k))} v_k \quad ,$$

where we set $\alpha^k_{(i,j)} = 0$ for each k if $(i,j)<(o,o)$. Hence

$$\dot{\phi}_{(i,j)} = \sum_{(i'',j'')\varepsilon I} a^{i''j''}_{ij} \phi_{(i'',j'')} + \sum_{(i'',j'')\varepsilon I} b^{i''j''}_{ij,k} \phi_{(i'',j'')} v_k \quad ,$$

where

$$a^{i''j''}_{ij} = \sum_{k=1}^{n} i_k \alpha^k_{(i''-i+1(k),j''-j')} \quad , \quad b^{i''j''}_{ij,k} = \sum_{k=1}^{m} j_k \delta^{i''j''}_{i\ j-1(k)} \quad .$$

Here, $\delta^{ij}_{k\ell}$ is the tensor defined by

$$\delta_{k\ell}^{ij} = \delta_{k_1}^{i_1}\delta_{k_2}^{i_2}\cdots\delta_{k_m}^{i_n}\delta_{\ell_1}^{j_1}\cdots\delta_{\ell_m}^{j_m}$$

Hence, defining the tensor operators A and B_μ by

$$(A\Phi)_{(i,j)} = \sum_{(k,\ell)\varepsilon I} a_{ij}^{k\ell}\phi_{(k,\ell)}$$

$$(B_\mu\Phi)_{(i,j)} = \sum_{(k,\ell)\varepsilon I} b_{ij,\mu}^{k\ell}\phi_{(k,\ell)} \quad , \quad 1\le\mu\le m .$$

where $(\Phi)_{(i,j)} = \phi_{(i,j)}$, we have

$$\dot{\Phi} = A\Phi + \sum_{\mu=1}^{m} v_\mu B_\mu \Phi = A\Phi + vB\Phi \tag{3.4}$$

where $vB \triangleq \sum_{i=1}^{m} v_i B_i$.

Now let ℓ^1 denote the standard Banach space of absolutely summable sequences and let ℓ_e^1 denote the Banach space of sequences $(\alpha_o,\alpha_1,\ldots)$ such that the sequence $(\alpha_o,\alpha_1/1!,\alpha_2/2!,\ldots)$ belongs to ℓ^1. Define a norm on ℓ_e^1 by

$$||\alpha||_e = \sum_{n=o}^{\infty} \frac{|\alpha_n|}{n!} \quad , \quad (\alpha)\varepsilon\ell_e^1$$

It is clear that ℓ_e^1 is , in fact, a Banach space and the map

$$E : \ell_e^1 \to \ell^1$$

defined by

$$E(\alpha) = (\alpha_n/n!) \quad , \ \alpha = (\alpha_n)\varepsilon\ell_e^1$$

is an isometric isomorphism. We can extend this definition to the space of infinite dimensional tensors of rank n in the following way. Let

$$\mathcal{L}_n = \otimes_n \ell^1_e$$

denote the algebraic tensor product of n copies of ℓ^1_e and let $\|\cdot\|$ be any cross norm on $\mathcal{L}_n$ (cf. Takesaki, 1979); i.e. for any tensor $\Phi \varepsilon \mathcal{L}_n$ of the form (called <u>simple tensors</u>)

$$\Phi = (\phi_{i_1 \ldots i_n}) = (\alpha_{1i_1}\alpha_{2i_2}\ldots\alpha_{ni_n}) = \alpha_1 \otimes \ldots \otimes \alpha_n \ ,$$

where $\alpha_k = (\alpha_{ko}, \alpha_{k1}, \alpha_{k2}, \ldots) \varepsilon \ell^1_e$, we have

$$\|\Phi\| = \prod_{k=1}^{n} \|\alpha_k\|_e \ .$$

<u>Lemma 3.1</u> For any n-vector $x = (x_1, \ldots, x_n)$ the tensor $(x_1^{i_1} \ldots x_n^{i_n})$ belongs to $\otimes_n \ell^1_e$ and we have

$$\| (x_1^{i_1} \ldots x_n^{i_n}) \| = \exp(\sum_{k=1}^{n} |x_k|) \ .$$

<u>Proof</u>

$$\| (x_1^{i_1} \ldots x_n^{i_n}) \| = \prod_{k=1}^{n} \|(x_k^{i_k})\|_e$$

$$= \prod_{k=1}^{n} \left\{ \sum_{\ell=o}^{\infty} \frac{|x_k|^{\ell}}{\ell!} \right\}$$

$$= \prod_{k=1}^{n} e^{|x_k|}$$

$$= \exp(\sum_{k=1}^{n} |x_k|) \ . \ \square$$

Returning to the system (3.4) we have

<u>Theorem 3.2</u> On the space $\mathcal{L}^T_{n+m}$ of temsors of the form $\Phi = (x^i u^j)$ we have

$$\| A\Phi \| \leq \sum_{k=1}^{n} |f_k(x,u)| \ \| \Phi \| \tag{3.5}$$

and

$$\| B\Phi \| \triangleq \sum_{i=1}^{m} \| B_i \Phi \| \leq m \| \Phi \| \qquad (3.6)$$

Proof

To prove (3.5) note that

$$(A\Phi)_{(i,j)} = \sum_{k=1}^{n} i_k\, x^{(i-1(k))} u^j f_k(x,u)$$

Consider the term

$$(A_1\Phi)_{(i,j)} \triangleq i_1 x_1^{i_1-1} x_2^{i_2} \dots x_n^{i_n} u_i^{j_1} \dots u_m^{j_m} f_1(x,u)$$

We have

$$\| A_1\Phi \| = \| i_1 x_1^{i_1-1} \|_e \prod_{k=2}^{n} \| (x_k^{i_n}) \|_e \prod_{k=1}^{m} \| (u_k^{j_k}) \|_e |f_1(x,u)|$$

$$= \sum_{\ell=1}^{\infty} \frac{i_1 |x_1|^{i_1-1}}{i_1!} \exp\left(\sum_{k=2}^{n} |x_k| + \sum_{k=1}^{m} |u_k|\right) |f_1(x,u)|$$

$$= \exp\left(\sum_{k=1}^{n} |x_k| + \sum_{k=1}^{m} |u_k|\right) |f_1(x,u)|$$

$$= \| \Phi \| \; |f_1(x,u)| ,$$

by lemma 5.1. The result (3.5) now follows. Since

$$(B\Phi)_{(i,j)} = \sum_{k=1}^{m} j_k\, x^i u^{(j-1(k))} ,$$

the result (3.6) follows similarly. □

Note that A can be extended to the space of simple tensors by defining

$$A\Phi = \sum_{k=1}^{n} i_k \alpha_{1i_1} \dots \alpha_{ki_k-1} \dots \alpha_{(n+m)i_{(n+m)}} f_k(x,u)$$

where $\Phi = (\alpha_{1i_1} \dots \alpha_{(n+m)i_{(n+m)}}) = \alpha_1 \otimes \dots \otimes \alpha_{n+m}$, and similarly for B.

The same inequalities (3.5),(3.6), then hold for A and B in the larger space. Hence we can define $A^k\Phi$, $B^k\Phi$ inductively on this space.

In particular, B is bounded on $\mathcal{L}^T_{n+m}$ by m. (Where we define the norm of B on $\mathcal{L}^T_{n+m}$ by $\sup_{\Phi\neq o\varepsilon\mathcal{L}^T_{n+m}} \|B\Phi\| / \|\Phi\|$. Of course, $\mathcal{L}^T_{n+m}$ is not a linear subspace of $\mathcal{L}_{n+m}$.)

Corollary 3.3 $\quad \|A^\ell\Phi\| \leq (\sum_{k=1}^{n} |f_k(x,u)|)^\ell \|\Phi\|$. □

Corollary 3.4 $\quad e^{At}\Phi$ exists for all t and for all $\Phi\varepsilon\mathcal{L}^T_{n+m}$, and we have

$$\|e^{At}\Phi\| \leq \exp\{(\sum_{k=1}^{n} |f_k(x,u)|)t\}\|\Phi\| . \square$$

In other words the system

$$\dot{\Phi} = A\Phi \tag{3.7}$$

is soluble in $\mathcal{L}^T_{n+m}$ and if $\Phi_o = (x_o^i u_o^j)$ for some fixed $x_o \varepsilon \mathbb{R}^n$, $u_o \varepsilon \mathbb{R}^m$, then the solution $\Phi(t)$ satisfies

$$\|\Phi(t)\| \leq \exp\{(\sum_{k=1}^{n} |f_k(x_o,u_o)|)t\}\|\Phi_o\|$$

Remark 3.5 (a) It should be noted that corollary 5.4 only holds on the non linear subspace $\mathcal{L}^T_{n+m}$ of tensors of the form $\Phi=(x^i u^j)$ and it does not follow that

$$\|e^{At}\Phi\| \leq \exp\{(\sum_{k=1}^{n} |f_k(x,u)|)t\}\|\Phi\|$$

for Φ in the closed linear span of tensors of this form. Hence, although A is a linear tensor operator, e^{At} cannot be extended to a linear semigroup on such a linear subspace of $\mathcal{L}_{n+m}$. e^{At} is a nonlinear semigroup, however, since strong continuity at t=o follows from corollary 3.4.

(b) It also follows from corollary 3.4 that e^{At} may be obtained from the usual series $\sum_{k=o}^{\infty} (At)^k/k!$. The $(i,j)^{th}$ component of $e^{At}\Phi$, where $\Phi=(x_o^i u_o^j)$ is then just the $(i,j)^{th}$ power of the Taylor series of the solution $(x(t),u(t))$ of equation (3.7) with initial condition (x_o,u_o).

We are now in a position to define the Volterra series solution for equation (3.4) in the usual way (Brockett (1976), Banks (1985)):

$$\Phi(t) = w_o(t) + \sum_{\nu=1}^{\infty} \int_o^t \ldots \int_o^t w_n(t,\sigma_1,\ldots,\sigma_\nu) v(\sigma_1)\otimes\ldots\otimes v(\sigma_\nu)\, d\sigma_1 \ldots d\sigma_\nu \tag{3.8}$$

where

$$w_n(t,\sigma_1,\ldots,\sigma_\nu) = e^{A(t-\sigma_1)} B e^{A(\sigma_1-\sigma_2)} B \ldots B e^{A\sigma_\nu} \Phi_o \quad \text{for } t \geq \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_\nu$$

and $w_\nu = 0$ otherwise.

Here, we define

$$e^{A(t-\sigma_1)} B e^{A(\sigma_1-\sigma_2)} B \ldots B e^{A\sigma_\nu} \Phi_o (v(\sigma_1) \ \ldots \ v(\sigma_2))$$

as

$$e^{A(t-\sigma_1)} v(\sigma_1) B e^{A(\sigma_1-\sigma_2)} v(\sigma_2) B \ldots v(\sigma_\nu) B e^{A\sigma_\nu} \Phi_o \ .$$

Then we have

Theorem 3.6 The Volterra series in (3.8) converges (in $\mathcal{L}_{n+m}$) and is the unique solution of equation (3.4), provided $v(t) \varepsilon L^\infty(o,\infty)$.

Proof The only nontrivial part is to prove convergence of the Volterra series. Let

$$\| v \|_\infty = \max_{i=1,..,m} \ \operatorname*{ess.\,sup}_{s\varepsilon[o,\infty)} |v_i(s)| .$$

Then,

$$\| \Phi(t) \| \leq \| w_o(t) \| + \sum_{\nu=1}^{\infty} \| v \|_\infty^\nu \int_o^t \ldots \int_o^{\sigma_{\nu-2}} \int_o^{\sigma_{\nu-1}} \| w_n(t,\sigma_1,\ldots,\sigma_\nu) \| \, d\sigma_1 \ldots d\sigma_\nu$$

However, by corollary 3.4 and (3.6) we have

$$\| w_\nu(t,\sigma_1,\ldots,\sigma_\nu) \| \leq m^\nu \exp\{(\sum_{k=1}^{n} |f_k(x_o,u_o)|)t\} \| \Phi_o \| \ ,$$

where $\Phi_o = (x_o^i u_o^j)$. Hence,

$$\|\Phi(t)\| \leq \|w_o(t)\| + \sum_{\nu=1}^{\infty} \|v\|_\infty^\nu m^\nu \int_o^t \cdots \int_o^{\sigma_{\nu-1}} \exp\{(\sum_{k=1}^{n} |f_k(x_o,u_o)|)t\} \|\Phi_o\| d\sigma_1 \cdots d\sigma_\nu$$

$$= \|w_o(t)\| + \sum_{\nu=1}^{\infty} \|v\|_\infty^\nu m^\nu \exp\{(\sum_{k=1}^{n} |f_k(x_o,u_o)|)t\} \|\Phi_o\| t^\nu/\nu!$$

$$\leq \exp\{(\sum_{k=1}^{n} |f_k(x_o,u_o)|)t + \|v\|_\infty mt\} \|\Phi_o\| . \square$$

4. Equations Defined on Manifolds

Now we shall consider the generalization of the above results to the case of a system defined by a set of vector fields on a (real) analytic manifold M. Hence let M and U be real analytic manifolds of dimensions n and m, respectively, and let

$$X(\cdot) : U \to D(M) \tag{4.1}$$

be an analytic map from U to the set D(M) of analytic vector fields on M. Thus, for each u in the control space U we are assigning a vector field X(u) on M. Locally we may express the vector field in the form of a differential equation

$$\dot{x} = f_p(x,u)$$

where $p \varepsilon M$ and x is a local coordinate system near p.

As in the previous section we shall reformulate the problem in the following way. Let MU denote the product manifold MxU which has the tangent bundle T(MxU), where we have the vector space isomorphism $T_{(p,u)}(MxU) \cong T_pM \oplus T_uU$

for each $(p,u)\varepsilon MxU$. If $D(MxU)$ denotes the set of analytic vector fields on MxU, then we consider, instead of (4.1), an analytic map

$$Y(\cdot) \, : \, T(U) \rightarrow D(MxU)$$

($T(U)$ is the tangent bumdle of U) such that, for each $Z\varepsilon T(U)$,

$$Y(Z)_{(p,u)} = (X(u)_p \, , \, v)$$

where $v\varepsilon T_u U$. Hence in a neighbourhood of (p,u) we can write Y in the form

$$\dot{x} = f_p(x,u) \tag{4.2}$$

$$\dot{u} = v$$

The control space is now the tangent bundle of U rather than U. Using the theory of section (3), we can replace the local system (4.2) by the tensor valued system

$$\dot{\Phi}_p = A_p \Phi_p + vB_p \Phi_p \tag{4.3}$$

where $vB_p = \sum_{i=1}^{m} v_i B_{i,p}$. Similar equations hold at each point $p\varepsilon M$, and to relate the systems arising from two intersecting coordinate neighbourhoods we must consider the effect of a coordinate transformation on the tensor space $\mathcal{L}^T_{n+m}$. Thus, let $(y,v) = g(x,u)$ be a coordinate transformation from (x,u)-coordinates to (y,v)- coordinates, where g is analytic with analytic inverse. Then we can write

$$g(x,u) = \sum_{i=o}^{\infty} \sum_{j=o}^{\infty} g^{ij} x^i u^j$$

and we have

$$y^{\alpha} v^{\beta} = \left(\sum_{i=o}^{\infty} \sum_{j=o}^{\infty} g^{ij} \, x^i \, u^j \right)^{(\alpha,\beta)}$$

$$= \sum_{i=o}^{\infty} \sum_{j=o}^{\infty} g^{ij}_{\alpha\beta} \, x^i \, u^j \tag{4.4}$$

for some numbers $g^{ij}_{\sigma\beta}$. Let G be the tensor operator with the representation $(g^{ij}_{\sigma\beta})$.

Then we have

$$\Psi = G\,\Phi,$$

where

$$\Psi = (y^{\alpha}v^{\beta}), \quad \Phi = (x^{i}u^{j}),$$

and

$$\Phi = G^{-1}\Psi,$$

where G^{-1} is the inverse tensor operator, which is defined in the same way from g^{-1} as G is from g.

If

$$(y,v) = g(x,u) \quad , \ (z,w) = h(y,v) \ ,$$

then

$$(z,w) = h\circ g\ (x,u)$$

and

$$\Xi = HG\Phi \ ,$$

where

$$\Xi = (z^{\alpha}w^{\beta}) \ , \ \Phi = (x^{i}u^{j})$$

and H, G are defined as above. Hence the set of tensor operators of the type defined by (4.4) is a group and operates as a transformation group on $\mathcal{L}^{T}_{n+m}$. Thus, if we assign a space of tensors $\mathcal{L}^{T}_{n+m,p}$ of type $(x^{i}u^{j})$ at each point p of M (with local coordinates (x,u)) then we can make $\bigcup_{p\in M}\mathcal{L}^{T}_{n+m,p}$ with projection

$$\pi : \bigcup_{p\in M}\mathcal{L}^{T}_{n+m,p} \rightarrow M$$

into a fibre bundle over M. We shall denote the bundle $\bigcup_{p\in M}\mathcal{L}^{T}_{n+m,p}$ by Γ_{n+m}.

<u>Definition</u>. Let $X(\cdot) : U \rightarrow D(M)$ be a system as defined above. We shall say that m+1 sections $\mathcal{A}_1, \mathcal{B}_1, \ldots, \mathcal{B}_m$ of the fibre bundle Γ_{n+m} form a bilinear system on M if the local representation of $X(\cdot)$ given by

$$\dot{x} = f_p(x,u)$$
$$\dot{u} = v$$

at p is related to the bilinear system

$$\dot{\Phi}_p = A_p\Phi_p + v_1 B_{1p}\Phi_p + \ldots + v_m B_{mp}\Phi_p \qquad (4.5)$$

as above, where $\mathcal{A}_p = A_p\Phi_p$, $\mathcal{B}_{ip} = B_{ip}\Phi_p$.

Note that the action of the group of transformations of type (4.4) imply that local representations of the form (4.5) are related by

$$\dot{\Psi}_q = G\, A_p G^{-1}\Psi_q + v_1 GB_{ip} G^{-1}\Psi_q + \ldots + v_m GB_{mp} G^{-1}\Psi_q$$

where $\Psi_q = G\Phi_p$ and G is a transformation of type (4.4) between the coordinates (y,v) at q and (x,u) at p where

$$y(q) = x(p) = 0,\ v(q) = u(p) = 0.$$

For a given section $\mathcal{A}$ of Γ_{n+m}, which belongs to a bilinear system, we can define an exponential map for this 'tensor field' by defining locally,

$$(e^{\mathcal{A}t})_p = e^{A_p t}\,\Phi_p$$

This is well defined since A_p is just a linear tensor operator and so $e^{A_p t}$ can be defined by the usual series locally and we have, under a change of coordinates G,

$$G(e^{A_p t})\, G^{-1} = e^{G\, A_p\, G^{-1} t}\ .$$

Then we have

<u>Theorem 4.1</u>. Given a nonlinear system $X(\cdot) : U \rightarrow D(M)$ on an analytic manifold M, we can associate with this system a Volterra series

$$\Phi(t) = w_o(t) + \sum_{\nu=1}^{\infty} \int_o^t \ldots \int_o^t w_n(t,\sigma_1,\ldots,\sigma_\nu)v(\sigma_1) \otimes \ldots \otimes v(\sigma_\nu)\, d\sigma_1 \ldots d\sigma_\nu)$$

where the kernels are given locally by

$$w_n(t,\ \sigma_1,\ \ldots\ ,\ \sigma_\nu;\ p) = e^{A_p(t-\sigma_1)}B_p e^{A_p(\sigma_1-\sigma_2)}B_p \ldots B_p\, e^{A_p\sigma_\nu}\,\Phi_{op}$$

where $B_p = (B_{1p},\ \ldots\ ,\ B_{mp})$. Moreover the kernels transform according to

$$\begin{aligned} w_n(t,\sigma_1,\ldots,\sigma_\nu;\ q) &= e^{A_q(t-\sigma_1)}B_q e^{A_q(\sigma_1-\sigma_2)}\ B_q \ldots B_q e^{A_q\,\sigma_\nu}\,\Psi_{oq} \\ &= Ge^{Ap(t-\sigma_1)}\ G^{-1}GB_pG^{-1}G\ e^{A_p(\sigma_1-\sigma_2)} \ldots G e^{A_p\sigma_\nu} G^{-1} G\Phi_{op} \\ &= G\, w_n(t,\ \sigma_1,\ \ldots,\ \sigma_\nu;\ p) \end{aligned}$$

where G is a matrix representation of the coordinate transformation from p to q. □

We can therefore compute the Volterra series locally just as in section 3 as if it were globally defined. When $\Phi(t)$ is 'near the boundary' of a given