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# On the Equivalent Bilinearization of Nonlinear Controlled Delay Systems

by

S.P.Banks

Department of Control Engineering
University of Sheffield
Mappin Street
SHEFFIELD S1 3JD

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## Abstract

The Taylor polynomials are used to obtain an equivalent bilinear system for a nonlinear analytic delay equation with control and the optimal control of the resulting bilinear system is considered.



Key Words: Delay Systems, Equivalent Bilinearization, Optimal Control.

## 1. Introduction

The theory and control of delay systems has been widely studied (Hale, 1971, Curtain and Pritchard, 1978, Banks, 1983). In particular, the semigroup approach of Delfour and Mitter (1972) has enabled the linear control theory of delay equations to be subsumed under the general Hilbert-statespace theory of linear systems. Nonlinear delay systems are more difficult to deal with and in this paper we propose an equivalent linearization technique for analytic systems. We shall see that we can replace such a system by a sequence of nonautonomous linear infinite-dimensional systems or by a similar sequence of bilinear systems if the delay equation contains a control.

This technique has been used previously for ordinary differential equations by Takata (1979), Banks and Ashtiani (1985) and for certain infinite-dimensional nonlinear systems by Banks (1985). The reader may also consult Hunt et al (1983) for a different method of equivalent linearization using a differential geometric approach.

In section 2 we shall introduce some notation and terminology from the theory of tensors on $\ell^2$ and then in section 3 we shall consider a general nonlinear analytic delay equation without control, obtaining an equivalent system of linear equations. In section 4 a control input will be added and we show how to obtain an infinite-dimensional bilinear system for which we study the optimal control problem in section 5. A predator-prey example is finally considered in section 6.

## 2. Notation and Terminology

In this paper we shall use the theory of tensors on the Hilbert space $\ell^2$ (see Greub, 1978 and Banks and Yew, 1985). In particular $\otimes_n(\ell^2)$ will denote the tensor product of n copies of $\ell^2$ and this is a Hilbert space under the obvious norm. It therefore makes sense to consider the space $\mathcal{L}(\otimes_n \ell^2)$ of bounded operators on $\otimes_n \ell^2$. It will be convenient, occasionally, to use the isomorphism $\otimes_{2n}\ell^2 \cong \otimes_n \ell^2 \otimes_n \ell^2$ and to consider an element $\underline{\alpha}$ of $\otimes_{2n}\ell^2$ as having n 'covariant' and n 'contravariant' components. We then write (in the standard basis of $\otimes_{2n}\ell^2$)

$$(\underline{\alpha})_{i_1\ldots i_n, j_1\ldots j_n} = \alpha^{j_1\ldots j_n}_{i_1\ldots i_n} .$$

If $\Phi = (\phi_{i_1\ldots i_n}) \in \otimes_n \ell^2$ we then define the contraction of $\underline{\alpha}$ and $\Phi$ by

$$\sum_{j_1,\ldots,j_n} \alpha^{j_1\ldots j_n}_{i_1\ldots i_n} \phi_{j_1\ldots j_n}$$

and we write this as $C(\underline{\alpha} \otimes \Phi)$.

## 3. Delay Equations

Consider the delay equation

$$\dot{x}(t) = f(x(t),x(t-\delta)) \quad , \quad x(t) \in \mathbb{R}^n \tag{3.1}$$

with initial conditions

$$x(o) = x_o \; , \quad x(\theta) = \xi(\theta) \; , \quad \theta \in [-\delta, o) \; ,$$

where $x_o$ and $\xi \in C[-\delta, o]$ (say) are given. We shall assume that f is analytic and if $x=(x_1,\ldots,x_n)$ we introduce the functions

$$\phi_{i_1\ldots i_n}(t) = x_1^{i_1}(t)\ldots x_n^{i_n}(t) \; , \quad i_1,\ldots,i_n \geq 0 \tag{3.2}$$

Then,

$$\dot{\phi}_{i_1\ldots i_n}(t) = \sum_{k=1}^{n} i_k x_1^{i_1}(t)\ldots x_k^{i_k-1}(t)\ldots x_n^{i_n}(t)\; f_k(x(t),x(t-\delta))$$

(We shall interpret the kth term as 0 if $i_k = 0$).

By Taylor's theorems, we can write

$$f_k(x(t),x(t-\delta)) = \sum_{\substack{i_1,\dots,i_n\\ j_1,\dots,j_n}} \alpha^{j_1\dots j_n}_{i_1\dots i_n}(k)x_1^{i_1}(t)\dots x_n^{i_n}(t)x_1^{j_1}(t-\delta)\dots x_n^{j_n}(t-\delta)$$

for some tensor $\underline{\alpha}(k) = (\alpha^{j_1\dots j_n}_{i_1\dots i_n}(k))$. Hence

$$\dot{\phi}_{i_1\dots i_n}(t)= \sum_{k=1}^{n} \sum_{\substack{i'_1,\dots,i'_n\\ j'_1,\dots,j'_n}} i_k x_1^{i_1}(t)\dots x_n^{i_n}(t)\alpha^{j'_1\dots j'_n}_{i'_1\dots i'_n}(k)x_1^{i'_1}(t)\dots x_n^{i'_n}(t)\cdot x_1^{j'_1}(t-\delta)\dots x_n^{j'_n}(t-\delta)$$

$$= \sum_{k=1}^{n} \sum_{\substack{i'_1,\dots,i'_n\\ j'_1,\dots,j'_n}} i_k\alpha^{j'_1\dots j'_n}_{i'_1\dots i'_n}(k)\phi_{i_1+i'_1\dots i_k+i'_k-1\dots i_n+i'_n}(t)\phi_{j'_1\dots j'_n}(t-\delta)$$

$$= \sum_{k=1}^{n} \sum_{i'_1,\dots,i'_n} i_k\phi_{i_1+i'_1\dots i_k+i'_{k-1}\dots i_n+i'_n}(t) \sum_{j'_1,\dots,j'_n} \alpha^{j'_1\dots j'_n}_{i'_1\dots i'_n}(k)\phi_{j'_1\dots j'_n}(t-\delta) \qquad (3.3)$$

Now consider the tensor valued function $\Phi(\cdot) : \mathbb{R} \to \bigotimes_n(\ell^2)$ with components $\phi_{i_1,,,i_n}$ in the natural basis (see section 2). Then we can write

$$\sum_{j'_1,\dots,j'_n} \alpha^{j'_1\dots j'_n}_{i'_1\dots i'_n}(k)\phi_{j'_1\dots j'_n}(t-\delta) = C(\underline{\alpha}(k)\otimes\Phi(t-\delta))$$

where $\otimes$ is the tensor product and C is the complete contraction operator. Moreover for any tensor $\Psi$ with components $\psi_{i_1\dots i_n}$ we define the tensor $\Psi^{(k_1,\dots,k_n)}$, for any fixed index set $(k_1,\dots k_n)$, which has components

$$\Psi_{i_1+k_1 \ldots i_n+k_n}.$$

Then we can write (3.3) in the form

$$\dot{\phi}_{i_1 \ldots i_n}(t) = \sum_{k=1}^{n} i_k C\{\Phi^{(i_1,\ldots,i_k-1,\ldots,i_n)}(t)\left[C(\alpha(k)\otimes\Phi(t-\delta))\right]\} \qquad (3.4)$$

Now define the function $A^{\Psi}(\cdot):\mathbb{R}\to\mathcal{L}(\otimes_n \ell^2)$, for any tensor-valued function $\Psi(\cdot):\mathbb{R}\to\otimes_n \ell^2$, by

$$(A^{\Psi(\cdot)}(t)\Phi)_{i_1,\ldots,i_n} = \sum_{k=1}^{n} i_k C\{\Phi^{(i_1,\ldots,i_k-1,\ldots,i_n)}\left[C(\alpha(k)\otimes\Psi(t))\right]\}. \qquad (3.5)$$

and then (3.4) can be written

$$\dot{\Phi}(t) = A^{\Phi(\cdot-\delta)}(t)\Phi(t) \qquad (3.6)$$

Returning to the original equation (3.1) we suppose that sufficient conditions for the existence of a unique solution are placed on f. Then (3.6) has a unique solution with initial conditions

$$(\Phi(o))_{i_1\ldots i_n} = x_{o1}^{i_1} \ldots x_{on}^{i_n}$$

where $x_o = (x_{o1}, \ldots, x_{on})$ and

$$(\Phi(\theta))_{i_1\ldots i_n} = \xi_1^{i_1}(\theta)\ldots\xi_n^{i_n}(\theta) \quad , \quad \theta\varepsilon[-\delta,o).$$

$$\triangleq (\Xi(\theta))_{i_1\ldots i_n} \quad , \quad \theta\varepsilon[-\delta,o)$$

We can then solve (3.6) inductively in the following way:

Set

$$\dot{\Phi}_o(t) = A^{\Xi(\cdot-\delta)}(t)\Phi_o(t) \qquad t\varepsilon[o,\delta)$$

and

$$\dot{\Phi}_m(t) = A^{\Phi_{m-1}(\cdot-\delta)}(t)\Phi_m(t)\ , \quad m\geq 1, \quad t\varepsilon[m\delta,(m+1)\delta) \tag{3.7}$$

with initial conditions

$$\Phi_o(0) = \Phi(0)$$

$$\Phi_m(m\delta) = \Phi_{m-1}(m\delta) \tag{3.8}$$

We shall now assume that the system (3.1) has bounded global solutions and that if $\| x_o\| \leq\varepsilon$, $\|\xi\|_{C[-\delta,o]}\leq\varepsilon$ for some $\varepsilon<1$, then $\| x(t)\| <1$, for all $t\geq o$. Then we introduce the space

$$[\otimes_n(\ell^2)]_T$$

which is the closed linear span of the set $P_T$ of all elements $\Phi$ in $\otimes_n(\ell^2)$ of the form $\Phi = (x_1^{i_1}\ldots x_n^{i_n})$ , with $\| x\| \leq\varepsilon$.

<u>Definition 3.1</u> If $E:S\to X$ is a function defined on a subset S (not generally a linear manifold) of a Banach space X, then we say that E is <u>bounded</u> on S if

$$\| Ex\| \leq \mathfrak{m}\| x\|$$

for all $x\varepsilon S$ and some constant $\mathfrak{m}$. We then write

$$\| E\| = \sup_{\substack{x\varepsilon S\\ x\neq o}} \frac{\| Ex\|}{\| x\|}$$

<u>Remark 3.2</u> Note that if E is 'linear' in the sense that

$$E(\alpha x+\beta y) = \alpha E(x)+\beta E(y)$$

whenever $\alpha x+\beta y$, $x,y\varepsilon S$, and if E also denotes the extension of E to the linear manifold $\ell(S)$ generated by S, then E is not necessarily bounded on $\ell(S)$.

<u>Lemma 3.3</u> The operator $B:P_T\to\otimes_n\ell^2$ defined by

$$(B\Phi)_{i_1,\ldots,i_n} = (i_1x_1^{i_1-1}x_2^{i_2}\ldots x_n^{i_n})\ , \quad \Phi = (x_1^{i_1}\ldots x_2^{i_n})$$

is bounded on $P_T$.

Proof If $\Phi = (x_1^{i_1} \dots x_n^{i_n}) \in P_T$ , then

$$\| B\Phi \|^2 = \left( \sum_{i_1=1}^{\infty} \sum_{i_2=o}^{\infty} \dots \sum_{i_n=o}^{\infty} i_1^2 x_1^{2i_1-2} x_2^{2i_2} \dots x_n^{2i_n} \right) .$$

$$= \frac{1}{1-x_2^2} \frac{1}{1-x_3^2} \dots \frac{1}{1-x_n^2} \sum_{i_1=o}^{\infty} ((i_1+1)^{2/i_1} x_1^2)^{i_1}$$

Now, $(j+1)^{1/j} \to 1$ as $j \to \infty$ and so $\exists\, m$ such that $(i_1+1)^{1/i_1} \le 2/(1+\varepsilon)$ $(\varepsilon < 1)$ for $i_1 \ge m$. Then,

$$\sum_{i_1=o}^{\infty} ((i_1+1)^{2/i_1} x_1^2)^{i_1} \le m^2 \sum_{i_1=o}^{\infty} \left( \frac{2}{1+\varepsilon} x_1 \right)^{2i_1} = m^2 \frac{1}{1-\left(\frac{2}{1+\varepsilon} x_1\right)^2}$$

Hence, if

$$\eta = \max_{|x_1| \le \varepsilon} \frac{1-|x_1|^2}{1-\left(\frac{2}{1+\varepsilon} x_1\right)^2}$$

then

$$\| B\Phi \|^2 \le \frac{1}{\prod_{j=2}^{n} (1-x_j^2)} m^2 \eta \frac{1}{1-x_1^2} = m^2 \eta \| \Phi \|^2 . \ \square$$

Since the equation (3.1) has a unique solution it follows that the system (3.7) has a unique solution if

$$\| \Phi(0) \| \le \varepsilon , \quad \| \xi \|_{C[o,1]} \le \varepsilon$$

Moreover, each operator $A^{\Phi_{m-1}(\cdot - \delta)}(t)$ generates an evolution operator on $[m\delta, (m+1)\delta)$ which we denote by $U_m(t,s)$ . Using lemma 3.3 it follows that the operator $B_m$ given by†

$$B_m \Phi = (A^{\Phi_{m-1}(\cdot - \delta)}(t) - A^{\Phi_{m-1}((m-1)\delta)}) \Phi .$$

is bounded on $P_T$ and so $U_m(t,s)$ is given by the solution of the equation

† Here we regard $\Phi_{m-1}((m-1)\delta)$ as the constant function with value $\Phi_{m-1}((m-1)\delta)$ on $[m\delta, (m+1)\delta)$

$$U_m(t,s) = \exp\{[A^{\Phi_{m-1}((m-1)\delta)}](t-s)\} + \int_s^t \exp\{[A^{\Phi_{m-1}((m-1)\delta)}](t-\tau) B_m U_m(t,s)\,ds$$

where $\exp\{[A^{\Phi_{m-1}((m-1)\delta)}](t)\}$ is the semigroup (which clearly exists) defined by $A^{\Phi_{m-1}((m-1)\delta)}$ on the closed linear span of $P_T$. Writing

$$T_m(t) = \exp\{[A^{\Phi_{m-1}((m-1)\delta)}](t)\}$$

we have

$$U_m(t,s) = T_m(t-s) + \int_s^t T_m(t-\tau)\, B_m U_m(\tau,s)\,ds.$$

Since $B_m$ is bounded on $P_T$ and the solutions of (3.7) are all in $P_T$ for the assumed initial conditions we can obtain $U_m(t,s)$ inductively, in the usual way, as follows:

Put

$$U_{mo}(t,s) = T_m(t-s)$$

$$U_{mk}(t,s) = T_m(t-s) + \int_s^t T_m(t-\tau) B_m U_{mk-1}(\tau,s)\,ds$$

and then

$$U_m = \sum_{k=o}^{\infty} U_{mk} .$$

## 4. Nonlinear Delay Equations with Control

In this section we shall show that the nonlinear delay system

$$\dot{x}(t) = f(x(t),x(t-\delta),\, u(t)) \;,\; x(t)\in\mathbb{R}^n,\; u(t)\in\mathbb{R} \quad (4.1)$$

is reducible to an infinite dimensional bilinear system on $\ell^2$. (We have chosen a scalar control merely for ease of exposition. The general case where $u(t)\in\mathbb{R}^m$ follows in much the same way.) We shall restrict attention to differentiable controls u(t) and consider the 'augmented' system.

$$\dot{x}(t)=f(x(t),x(t-\delta),u(t))$$

$$\dot{u}(t)=v(t)$$

making u effectively a state. Then generalising (3.2) we define

$$\Phi_{i_1 \dots i_{n+1}}(t) = x_1^{i_1}(t) \dots x_n^{i_n}(t)\, u^{i_{n+1}}(t) .$$

Then,

$$\dot{\phi}_{i_1 \cdots i_{n+1}}(t) = \sum_{k=1}^{n} i_k x_1^{i_1}(t) \cdots x_k^{i_k - 1}(t) \cdots x_n^{i_n}(t) u^{i_{n+1}}(t) \cdot$$

$$f_k(x(t), x(t-\delta), u(t)) + i_{n+1} x_1^{i_1}(t) \cdots x_n^{i_n}(t)\, u^{i_{n+1}-1}(t) v(t)$$

Again, by Taylor's theorem, we can write

$$f_k(x(t), x(t-\delta), u(t)) = \sum_{\substack{i_1, \ldots, i_{n+1} \\ j_1, \ldots, j_n}} \alpha^{j_1 \cdots j_n}_{i_1 \cdots i_{n+1}}(k) x_1^{i_1}(t) \cdots$$

$$\cdots x_n^{i_n}(t) x_1^{j_1}(t-\delta) \cdots x_n^{j_n}(t-\delta) u^{i_{n+1}}(t).$$

for some tensor $\underline{\alpha}(k) = (\alpha^{j_1 \cdots j_n}_{i_1 \cdots i_{n+1}}(k))$. Hence,

$$\dot{\phi}_{i_1 \cdots i_{n+1}}(t) = \sum_{k=1}^{n} \sum_{\substack{i'_1, \ldots, i'_{n+1} \\ j'_1, \ldots, j'_n}} i_k x_1^{i_1}(t) \cdots x_n^{i_n}(t)\, u^{i_{n+1}}(t)\, \alpha^{j'_1 \cdots j'_n}_{i'_1 \cdots i'_{n+1}}(k) \cdot$$

$$x_1^{i'_1}(t) \cdots x_n^{i'_n}(t)\, x_1^{j'_1}(t-\delta) \cdots x_n^{j'_n}(t-\delta)\, u^{i'_{n+1}}(t)$$

$$+ i_{n+1} x_1^{i_1}(t) \cdots x_n^{i_n}(t) u^{i_{n+1}-1}(t) v(t)$$

$$= \sum_{k=1}^{n} \sum_{\substack{i'_1, \ldots, i'_{n+1} \\ j'_1, \ldots, j'_n}} i_k \alpha^{j'_1 \cdots j'_n}_{i'_1 \cdots i'_{n+1}}(k)\, \phi_{i_1+i'_1 \cdots i_k+i'_k-1 \cdots i_n+i'_n\, i_{n+1}+i'_{n+1}}(t).$$

$$\phi_{j'_1 \cdots j'_n o}(t-\delta) + i_{n+1} \phi_{i_1 \cdots i_n\, i_{n+1}-1}(t) v(t)$$

$$= \sum_{k=1}^{n} \sum_{i_1',\dots,i_{n+1}'} i_k \phi_{i_1+i_1' \dots i_k+i_{k-1}' \dots i_n+i_n' i_{n+1}+i_{n+1}'}(t).$$

$$\cdot \sum_{j_1',\dots,j_n'} \alpha^{j_1' \dots j_n'}_{i_1',\dots,i_{n+1}'}(k) \phi_{j_1' \dots j_n' o}(t-\delta)$$

$$+ \; i_{n+1} \phi_{i_1 \dots i_n i_{n+1}-1}(t) v(t)$$

With a similar notation to that in section 3, we now have

$$\dot{\phi}_{i_1 \dots i_n}(t) = \sum_{k=1}^{n} i_k \; C\{\Phi^{(i_1,\dots,i_k-1,\dots,i_n,i_{n+1})}(t) [C(\alpha(k) \otimes \Phi^o(t-\delta))]\}$$

$$+ \; i_{n+1} \phi_{i_1 \dots i_n i_{n+1}-1}(t) v(t)$$

where $\Phi^o$ is the tensor with components $x_1^{i_1} \dots x_n^{i_n} u^o$. Hence the equation takes the form

$$\dot{\Phi}(t) = A^{\Phi^o(\cdot-\delta)}(t) \Phi(t) + v(t) B \Phi(t) \tag{4.2}$$

where

$B : \otimes_n(\ell^2) \to \otimes_n(\ell^2)$ is defined by

$$(B\Phi)_{i_1 \dots i_{n+1}} = i_{n+1} \phi_{i_1 \dots i_n i_{n+1}-1}$$

when $(\Phi)_{i_1 \dots i_{n+1}} = \phi_{i_1 \dots i_{n+1}}$. Using the same initial conditions as in section 3, we therefore obtain from (4.2) the following system of equations:

$$\dot{\Phi}_o(t) = A^{\Xi^o(\cdot-\delta)}(t) \Phi_o(t) + v(t) B \Phi_o(t) \;, \quad t \in [o, \delta)$$

$$\tag{4.3}$$

$$\dot{\Phi}_m(t) = A^{\Phi^o_{m-1}(\cdot-\delta)}(t) \Phi_m(t) + v(t) B \Phi_m(t) \;, \quad m \geq 1, \; t \in [m\delta, (m+1)\delta)$$

It has therefore been shown that a general delay equation with control of the form (4.1), which has an analytic right hand side, can be 'reduced' to a set of nonautonomous bilinear systems.

## 5. Optimal Control

The (sub) optimal control of infinite dimensional bilinear systems has recently been considered by Banks and Yew (1985) on the filtered tensor product space $\bigoplus_{i=1}^{\infty} \bigotimes_i (\ell^2)$. This method can be applied to the equations (4.3) to obtain a suboptimal control for equation (4.1).

To illustrate a slightly different approach we recall the following result of Tzafestas et al (1984):

Theorem 5.1 Consider the bilinear system

$$\dot{x}(t) = A(t)x(t) + v(t)Bx(t)$$

and a cost function of the form

$$J(t_f) = \int_{t_o}^{t_f} L(x, \ ,t)dt + \frac{1}{2} x_f^T \Lambda_f x_f$$

where

$$L(x,u,t) = \frac{1}{2} \{x^T K(x,t)x + R(t)v^2\} \quad , \quad x_f = x(t_f)$$

and $R(t)$ is a positive scalar function, $K(x,t)$ is a symmetric semipositive definite matrix function for $t\varepsilon[t_o,t_f]$ and $\Lambda_f$ is a given symmetric positive definite matrix. If $\Lambda(t)$ satisfies

$$-\dot{\Lambda}(t) = Q(t) + A^T\Lambda(t) + \Lambda(t)A, \quad \Lambda(t_f) = \Lambda_f \tag{5.1}$$

with

$$0<G\leq\Lambda(t)\leq H<\infty \quad \text{for all } t \ ,$$

and $K(x,t)$ is chosen to satisfy

$$K(x,t) = Q(t) + \Lambda(t)Bx(t)R^{-1}(t)x^T(t)B^T\Lambda(t)$$

where $Q(t)$ is chosen to satisfy

$$-Q(t) \le -F < 0 \qquad (F<\infty)$$

then

$$v(t) = -R^{-1}(t)x^T(t)B^T\Lambda(t)x(t)$$

is the optimal control for the above problem. $\square$

Remark 5.2 This theorem was proved by Tzafestas et al (1984) in the finite dimensional case, but it is clearly still true for bounded operators on an infinite dimensional space or even for an unbounded operator A(t) which generates an evolution operator, provided (5.1) is interpreted in the weak sense (see Curtain and Pritchard, 1978, Banks, 1983). We shall apply this result to the system (4.3); since the operators $A^{\Phi_m(\cdot-\delta)}(t)$ are bounded on $P_T$ we may regard (5.1) as being true in the strong sense since the control depends only on states $\Phi$ in $P_T$. We then have

Theorem 5.3 Consider the sequence of bilinear systems (4.3), which are equivalent on $P_T$ to the delay equation (4.1), and associate the sequence of cost functionals

$$J_m = \int_{m\delta}^{(m+1)\delta} L_m(\Phi_m,v,t)dt + \frac{1}{2}\Phi_m((m+1)\delta)\Lambda_m^f\Phi_m((m+1)\delta)$$

where

$$L_m(\Phi_m,v,t) = \frac{1}{2}\{\Phi_m^T K_m(\Phi_m,t)\Phi_m + R_m(t)v^2\}$$

with $\Lambda_m^f$ a given symmetric positive definite (infinite) tensor. If $\Lambda_m(t)$ satisfies

$$-\dot{\Lambda}_m(t) = Q_m(t) + (A^{\Phi^o_{m-1}(\cdot-\delta)}(t))^T\Lambda_m(t) + \Lambda_m(t)A^{\Phi^o_{m-1}(\cdot-\delta)}(t), \qquad (5.2)^{\dagger}$$

with $\Lambda_m((m+1)\delta) = \Lambda_m^f$ and

$$0<G_m\le\Lambda_m(t)\le H_m<\infty, \quad \forall t,$$

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† Dual tensors are defined with respect to the usual inner product on $\otimes_n \ell^2$; see Banks and Yew, 1985.

and $K_m$ satisfies

$$K_m(\Phi_m,t) = Q_m(t) + \Lambda_m(t)B\Phi_m(t)R_m^{-1}(t)\Phi_m^T(t)B^T\Lambda_m(t)$$

where $$-Q_m(t) \leq -F_m < o ,$$

then

$$v(t) = -R_m^{-1}(t)\Phi_m^T(t)B^T\Lambda_m(t)\Phi_m(t) \tag{5.3}$$

is the optimal control for equation (4.3) subject to the functionals $J_m$. □

## 6. Example

In order to illustrate the above theory we shall consider the predator-prey equations of Wangersky and Cunningham (1957)

$$\begin{aligned} \dot{x}(t) &= \alpha x(t)\left[\frac{m-x(t)}{m}\right] -bx(t)y(t) \\ \dot{y}(t) &= -\beta y(t) + cx(t-1)y(t-1) \end{aligned} \tag{6.1}$$

(cf. Hale, 1971, p2). Define

$$\phi_{ij}(t) = x^i(t)y^j(t).$$

Then

$$\dot{\phi}_{ij} = ix^{i-1}(t)\dot{x}(t)y^j(t) + jx^i(t)y^{j-1}(t)\dot{y}(t)$$

$$= (i\alpha-j\beta)\phi_{ij}(t)-\frac{i\alpha}{m}\phi_{i+1j}(t)-ib\phi_{ij+1}(t)+jc\phi_{ij-1}(t)\phi_{11}(t-1)$$

It is convenient here to define a different operator $A^{\Psi}$ than that given by (3.5). Put

$$\alpha^{oo}(t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \alpha^{10}(t) = \begin{pmatrix} 0 & \alpha & -b \\ 0 & -\alpha/m & 0 \end{pmatrix}, \quad \alpha^{01}(t) = \begin{pmatrix} c\phi_{11}(t-1) & -\beta & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\alpha^{ij}(t) = \begin{pmatrix} c_j\phi_{11}(t-\delta) & i\alpha-j\beta & -ib \\ 0 & -i\alpha/m & 0 \end{pmatrix}.$$

Then we define $A^{\phi_{11}(\cdot-1)}$ by

$$(A^{\phi_{11}(\cdot-1)}(t)\Phi)_{ij} = \sum (\alpha^{ij} * \Phi^{ij}) \tag{6.2}$$

where

$$\Phi^{ij} = \begin{pmatrix} \phi_{ij-1} & \phi_{ij} & \phi_{ij+1} \\ \phi_{i+1j-1} & \phi_{i+1j} & \phi_{i+1j+1} \end{pmatrix},$$

$*$ denotes Schur produc of matrices (i.e. $(x_{ij})(y_{ij}) = (x_{ij}y_{ij})$) ,

and

$$\sum (X) = \sum_{i=1}^{2} \sum_{j=1}^{3} x_{ij}$$

when $$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{pmatrix}.$$

We therefore obtain formally a system of the type (3.7) where

$$\Xi(\theta-1) = \phi_{11}(\theta-1) = x(\theta-1)y(\theta-1) \quad , \quad 0 \leq \theta < 1 .$$

Suppose now that $\alpha$ (the rate of prey population increase) is a control.

Then we have the equations

$$\dot{x}(t) = u(t)x(t)\left[\frac{m-x(t)}{m}\right] - bx(t)y(t)$$

$$\dot{y}(t) = -\beta y(t) + cx(t-1)y(t-1)$$

$$\dot{u}(t) = v(t)$$

Defining

$$\phi_{ijk}(t) = x^i(t)y^j(t)u^k(t)$$

we have

$$\dot{\phi}_{ijk}(t) = ix^{i-1}(t)y^j(t)u^k(t)\left\{u(t)x(t)\left(\frac{m-x(t)}{m}\right) - bx(t)y(t)\right\}$$

$$+ jx^i(t)y^{j-1}(t)u^k(t)(-\beta y(t)+cx(t-1)y(t-1))$$

$$+ kx^{i}(t)y^{j}(t)u^{k-1}(t)\ (t)$$

and it is clear that we obtain a system of the form (4.2), namely,

$$\dot{\Phi}(t) = A^{\phi_{11}(\cdot-1)}(t)\Phi(t) + v(t)B\Phi(t)$$

where B is defined in the obvious way and

$$(A^{\phi_{11}(\cdot-1)}(t)\Phi)_{ijk} = \sum (\alpha^{ijk} * \Phi^{ijk})$$

with $\alpha^{ijk}$ equal to the tensor

$$\alpha^{ijk} = \begin{array}{cc} & \begin{array}{ccc} j-1 & j & j+1 \end{array} \\ k \begin{array}{c} i \\ i+1 \end{array} & \begin{pmatrix} cj\phi_{11}(t-1) & -j\beta & -ib \\ 0 & 0 & 0 \end{pmatrix} \\ k+1 \begin{array}{c} i \\ i+1 \end{array} & \begin{pmatrix} 0 & i & 0 \\ 0 & -i/m & 0 \end{pmatrix} \end{array}$$

and $\Phi^{ijk}$ equal to the subtensor $(\phi_{\ell mn})_{i\leq\ell\leq i+1,\, j-1\leq m\leq j+1,\, k\leq n\leq k+1}$ of $\Phi$. Splitting the equation into a system of the form (4.3) and then solving (5.2) enables us to obtain the optimal control

$$u(t) = \int_{0}^{t} v(t)dt = -\int_{m}^{m+1} R_{m}^{-1}(t)\Phi_{m}^{T}(t)B^{T}\Lambda_{m}(t)\Phi_{m}(t)dt.$$

## 7. Conclusions

In this paper we have considered the equivalent linearization of nonlinear analytic delay systems, obtaining a sequence of bilinear systems which are equivalent to the original equation. This enables us to use one of the