ROBUSTNESS AND SENSITIVITY OF SMITH PREDICTOR
CONTROLLERS FOR TIME-DELAY SYSTEMS

by

D. H. Owens, BSc, ARCS, PhD, AFIMA, CEng, MIEE

and

H. M. Wang*

and

A. Chotai, BSc, PhD

Department of Control Engineering
University of Sheffield
Mappin Street, Sheffield S1 3JD

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*Visitor on leave from Shaanxi Institute of Technology, China
1. **Introduction**

There is now an extensive literature on the subject of the control of time delay systems \(^1\). The well known Smith predictor\(^2\) and its extensions are often used in this case. Recent work pays particular attention to the robustness of the Smith predictor scheme\(^3\) to copy with the plant/model mismatch and to retain stability in the presence of the changes in plant dynamics. A general theory describing the robustness of the Smith scheme is given by \(^3\) and some problems, particularly the choice of approximate model, are investigated in \(^4\).

In this report, we will first extend the result of \(^3\) to the more general case - nonunity feedback case in both inner and outer loops. Then, by looking at a multivariable process control example, we will discuss the optimal parameter problems and the effects of nonunity feedback. In addition the sensitivity function in multivariable case will be derived and discussed.

2. **Robustness of the Extended Smith Schemes**

The plant is regarded as a linear operator mapping input linear vector space \(U^k\) into output linear vector space \(Y^m\) and suppose that it can be expressed into separable form (Fig.1)

\[
Y = TZ 
\]

\[
Z = GU 
\]

where the linear operator \(T\) maps \(Y^m\) into itself and represents the delays at the plant output and \(G\) maps \(U^k\) into \(Y^m\). Because of the generality of the theory which will be given it can be applied to plants expressed as above\(^4\). Where \(T\) is a pure delay matrix and \(G\) is a rational and strictly proper TFM but not necessarily a 'delay free' component (i.e. it can be a component with time delay).
The extended Smith scheme is illustrated in Fig.2 where the linear operator $G_A$ and $T_A$ represent models of the plant components $G$ and $T$ respectively and $K$ is a forward path controller regarded as a mapping of $Y^m$ into $U^l$. The only difference with the usual Smith scheme is that the feedback components are, instead of unity, $F_1(s)$ in the inner loop and $F_2(s)$ in the outer loop respectively. Here $F_1$ and $F_2$ are regarded as mapping of $Y^m$ into itself. As to be mentioned in section 3.2, $F_1$ and $F_2$ may lead to some benefits sometimes, e.g. improve stability or performance or increase permissible mismatch.

We now derive the general theory of robustness\(^3\) for the scheme of Fig.2. The signal $\gamma \in Y^m$ is the demand signal. If the initial conditions are zero, the dynamics can be expressed as follows

$$Y = TGU$$

$$U = K(\gamma-F_1G_AU-F_2(Y-T_A^lG_AU))$$

...(3)

...(4)

After a little manipulation, we get

$$U = \bar{K}^*(\gamma-F_2TGU)$$

...(5)

where

$$\bar{K}^* = (I+KF_1G_A-KF_2T_A^lG_A)^{-1}K$$

...(6)

is an uniquely defined linear mapping of $Y^m$ into $U^l$. It is trivially verified from equation (5) that scheme Fig.2 is equivalent to scheme Fig.3.

For convenience of comparison and use, we follow the method of reference (3) to give a theorem to characterize the stability of the extended Smith scheme. Let $U^l_0$ and $Y^m_0$ be linear vector subspace of $U^l$ and $Y^m$ respectively (regarded as spaces of 'stable' inputs and outputs respectively).
Theorem 1

The scheme of Fig. 2 is stable in the BIBO sense if

(i) the plant component G and its model $G_A$ map $U_o^L$ into $Y_o^m$

and that their restrictions to $U_o^L$ have finite induced norms,

(ii) the delay component T and its model $T_A$ map $Y_o^m$ into itself with restrictions to $Y_o^m$ of finite induced norms,

(iii) the restriction to $Y_o^m$ of the mapping

$\gamma \rightarrow U_A^L \triangleq (I+KF_1 G_A)^{-1} K$ has range in $U_o^L$ and finite induced norm,

(iv) $\lambda_1 \triangleq \| (I+KF_1 G_A)^{-1} KF_2 \Delta T_A \| < 1 \quad \ldots (7)$

(v) $\lambda_2 \triangleq \frac{1}{1-\lambda_1} \| (I+KF_1 G_A)^{-1} KF_2 \Delta T_A \| < 1 \quad \ldots (8)$

where $\Delta G = G-G_A \quad \ldots (9)$

$\Delta T = T-T_A \quad \ldots (10)$

represent the mismatch between plant and its model.

Proof: As G and T are stable and bounded by assumption it is sufficient to prove that $U \in U_o^L$ whenever $\gamma \in Y_o^m$.

From $U = K^* (\gamma-F_2 T G U)$

$= K^* (\gamma-F_2 (T G U+F_2 T G U)) \quad \ldots (11)$

we can get

$U = (I+K^*_2 F_2 T G A)^{-1} K^* (\gamma-F_2 (T G A U)) \quad \ldots (12)$

This is an equation in $U_o^L$ of form $U = W_r(U)$. Suppose that $W_r$ map $U_o^L$ into itself whenever the demand $\gamma \in Y_o^m$. Clearly the BIBO stability is ensured if $W_r$ is a contraction [5]. This is the case if

$\lambda_0 \triangleq \| (I+K^*_2 F_2 T G A)^{-1} K^* F_2 (T G A U) \| < 1 \quad \ldots (13)$
We then prove that the following equality is true

$$(I + \mathbf{K}^{*} \mathbf{F}_{2}^{T} \mathbf{G}_{A} A)^{-1} \mathbf{K}^{*} = (I + \mathbf{K} \mathbf{F}_{1} \mathbf{G}_{A})^{-1} \mathbf{K}$$  \hspace{1cm} \ldots(14)$$

Write

$$U_{A} = V_{1} - V_{2}$$

with

$$V_{1} = \mathbf{K}^{*}Y$$  \hspace{1cm} \ldots(15)$$

$$V_{2} = \mathbf{K}^{*} \mathbf{F}_{2}^{T} \mathbf{G}_{A} A U_{A}$$  \hspace{1cm} \ldots(16)$$

and express $U_{A}$ in the form

$$U_{A} = (I + \mathbf{K}^{*} \mathbf{F}_{2}^{T} \mathbf{G}_{A} A)^{-1} \mathbf{K}^{*}Y$$  \hspace{1cm} \ldots(17)$$

By definition (6) we get

$$V_{1} = (I + \mathbf{K} \mathbf{F}_{1} \mathbf{G}_{A} - \mathbf{K} \mathbf{F}_{2}^{T} \mathbf{G}_{A} A U_{A})^{-1} \mathbf{K}Y$$  \hspace{1cm} \ldots(18)$$

Rewriting it yields

$$V_{1} = \mathbf{K}(Y - (\mathbf{F}_{1} - \mathbf{F}_{2}^{T} \mathbf{G}_{A} A U_{A}) V_{1})$$  \hspace{1cm} \ldots(19)$$

By similar means, we can get

$$V_{2} = \mathbf{K}(\mathbf{F}_{2}^{T} \mathbf{G}_{A} A U_{A} - (\mathbf{F}_{1} - \mathbf{F}_{2}^{T} \mathbf{G}_{A} A U_{2}) V_{2})$$  \hspace{1cm} \ldots(20)$$

Subtracting (19) and (20), we can obtain

$$U_{A} = V_{1} - V_{2} = \mathbf{K}Y - \mathbf{K} \mathbf{F}_{1} \mathbf{G}_{A} U_{A}$$  \hspace{1cm} \ldots(21)$$

or its equivalent

$$U_{A} = (I + \mathbf{K} \mathbf{F}_{1} \mathbf{G}_{A})^{-1} \mathbf{K}Y$$  \hspace{1cm} \ldots(22)$$

and equality (14) is then proved.

Substitute (14) into (13), the result follows by noting
\[ \lambda_0 \triangleq \|(I+KF_1G_A)^{-1}KF_2(TG-T_A G_A)\| \]
\[ = \|(I+KF_1G_A)^{-1}KF_2(T_{\Delta G}+\Delta T_A)\| \]
\[ \leq \|(I+KF_1G_A)^{-1}KF_2T_{\Delta G}\| + \|(I+KF_1G_A)^{-1}KF_2\Delta T_A\| \]
\[ < (1-\lambda_1) + \lambda_1 = 1 \]

...(23)

The condition (i)-(iii) ensures that \( W_1 \) maps \( U_o \) into itself for all \( \gamma \in Y_o \). In physical viewpoint, condition (i) and (ii) is the requirement of open-loop stability of plant \( TG \) and its model \( T_A G_A \); conditions (iii) simply require that the feedback scheme of Fig.4 is stable in normal practical sense. Condition (iv) and (v) provide upper bounds on the mismatch \( \Delta G \) and \( \Delta T \) that guarantee the BIBO stability of Fig.2.

For simplicity of application, following [3], we suppose that \( G \) and \( G_A \) are rational and strictly proper TFMs, that \( K \) is rational and proper and that both \( T = \text{diag}(e^{-\tau_j s})_{1 \leq j \leq m} \) and \( T_A = \text{diag}(e^{-\tau_{\Delta j} s})_{1 \leq j \leq m} \) are \( m \times m \) diagonal matrices of pure delay. In this case, the theorem has following simple form [3]:

**Theorem 2**

If the plant component \( G \) and its model \( G_A \) are asymptotically stable and the feedback system of Fig.4 is input-output stable then the extended Smith scheme of Fig.2 is BIBO stable if

\[ \lambda_1 = \max_{1 \leq i \leq m} \sup_{s \in \Omega} \sum_{j=1}^{m} \|(I+KF_1G_A)^{-1}KF_2\Delta T_A\)_{ij}\| < 1 \]

...(24)

\[ \lambda_2 \triangleq \frac{1}{1-\lambda_1} \max_{1 \leq i \leq m} \sup_{s \in \Omega} \sum_{j=1}^{m} \|(I+KF_1G_A)^{-1}KF_2T_{\Delta G}\)_{ij}\| < 1 \]

...(25)
where \( s \) is complex variable and \( \Omega \) is the Nyquist contour.

Proof: The stability assumptions are equivalent to conditions (i)-(iii) of theorem 1 whilst (24) and (25) are identical to (7) and (8) according to reference [7].

This result is easily used to evaluate \( \lambda_1 \) and \( \lambda_2 \) by numerical calculation.

3. Parameter Optimal Control

In reference [4] we discussed the choice of approximate model \( G_A \) and forward path controller \( \mathbf{K} \) for a kind of process control. But the choice of time delay component \( T_A \) has not yet been discussed in detail. Hence we shall investigate this problem from a parameter optimal view-point. The parameters of the controller \( \mathbf{K} \) and the model \( G_A \) can also be analysed from this view-point. The effects of the components \( F_1 \) and \( F_2 \) in the feedback path will also be investigated in this section.

Throughout this report the performance indexes used are defined as integral square-errors:

\[
J_i = \int_0^\infty e_i^2 dt \tag{26}
\]

where

\[
e_i = y_i(t) - x_i(t - \tau_i) \tag{27}
\]

and \( \tau_i \) is the time delay in output \( y_i \).

3.1. Temporal Optimality

The term 'temporal optimality' is used to present the optimal choice \( T_A \) for a certain \( G, T, K \) and \( G_A \). We will indicate that the
presence of temporal mismatch $\Delta T$ can lead some benefits sometimes. In other words, the matched case is not necessarily the optimal case\cite{6}. The choice of temporal model $T_A$ is hence still a problem which should be handled carefully.

Let's consider the same example used in ref.\cite{4}, where the plant can be expressed into separate form $TG$:

$$G = \begin{bmatrix} \frac{119.3}{1+812.8s} & \frac{-62}{1+904s} \\ \frac{55.3}{1+776s} & \frac{-109.7}{1+1715s} \end{bmatrix}, \quad T = \begin{bmatrix} e^{-35 \times 22.8s} & 0 \\ 0 & e^{-35 \times 3s} \end{bmatrix}$$

For simplicity suppose that $F_1 = I$, $F_2 = I$, i.e. its a usual Smith predictor scheme.

We choose the first order model\cite{8} as an approximate model $G_A$, which is defined as

$$G_A = (A_0 + A_1)^{-1} \quad \ldots (28)$$

where

$$A_0^{-1} = \lim_{s \to \infty} sG(s) \quad \ldots (29)$$

$$A_1^{-1} = \lim_{s \to 0} G(s)$$

The controller $K$ is chosen as proportional plus integral control\cite{8}, which is

$$K = (k + c + \frac{kc}{s})A_0 - A_1 \quad \ldots (30)$$

where $k$ and $c$ are constant scalar.

Regarding $m$ as a parameter we choose temporal component $T_A$ as

$$T_A = \begin{bmatrix} e^{-mx \times 22.8s} & 0 \\ 0 & e^{-mx \times 3s} \end{bmatrix} \quad \ldots (31)$$
Fig. 5 gives the variation of $J - J_o$ with $m$ under the stability condition (24) and (25). Here $J_o$ is the performance index when $m = 1$, i.e. the matched performance index. It is evident from Fig. 5 that the mismatch optimal is lower than matched case. The similar conclusion was indicated in (6) for single input/single output. In other words, the performance can be improved by 'correct' mismatch. So the minimization of the performance index can be, in the authors' opinion, one of the criterions of choosing temporal component.

3.2. The Effects of Feedback Components

We investigate the effects of $F_1$ and $F_2$ from the following view-point:

(i) do they improve the stability characteristics?
(ii) do they improve the performance?
(iii) do they increase the permissible mismatch?

The effects of $F_1$ is first investigated by looking at the same example with previous subsection.

Let

$$T_A = \begin{bmatrix} e^{-30x22.8s} & 0 \\ 0 & e^{-30x3s} \end{bmatrix}$$

...(32)

and choose both $F_1$ and $F_2$ to be of diagonal form and regard $n$ as a parameter

$$F_1 = \begin{bmatrix} 1+30xnxs & 0 \\ 0 & 1+20xnxs \end{bmatrix}$$

...(33)

$$F_2 = I$$
The norms $\lambda_1$ and $\lambda_2$ can be evaluated by (24) and (25) and are illustrated in Fig. 6 as a function of frequency for various fixed value of $n$. The performance index is shown in Table 1.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$J$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>138.66</td>
</tr>
<tr>
<td>0</td>
<td>151.45</td>
</tr>
<tr>
<td>1</td>
<td>164.53</td>
</tr>
</tbody>
</table>

Table 1

It is evident from this example that when $n > 0$, $F_1$ can reduce the max value of the norms but increase the performance index. In other words, it can increase the permissible mismatch and hence robustness but deteriorate the performance. Conversely, when $n < 0$, $F_1$ can improve the performance but reduce the permissible mismatch. So $F_1$ provides a margin of choice between the robustness and performance. When the main purpose is to increase the robustness, designers should put $n > 0$ (proportional plus differential) in inner loop. And the performance can be improved by putting $n < 0$ in inner loop.

We then investigate the effects of $F_2$ using the same example but supposing

$$
F_2 = \begin{pmatrix}
\frac{1}{1+30\pi n x s} & 0 \\
0 & \frac{1}{1+20\pi n x s}
\end{pmatrix}
$$

$$
F_1 = I
$$

The norms $\lambda_1$ and $\lambda_2$ are shown in Fig. 7 for various fixed values of $n$ and the performance index are shown in Table 2.
<table>
<thead>
<tr>
<th>n</th>
<th>J</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>151.45</td>
</tr>
<tr>
<td>1</td>
<td>154.62</td>
</tr>
<tr>
<td>2</td>
<td>153.07</td>
</tr>
</tbody>
</table>

Table 2

We can see from Fig.7 that when $n > 0$, the presence of $F_2$ can increase the permissible mismatch and hence increase the robustness. The effect of $F_2$ on performance is very small and can be neglected. The reason of this small effect on performance is that because this example is a small mismatch case, the outer-loop feedback is of no importance in practice. In extreme cases, when it is exactly matched, $F_2$ has no effect on the performance. However, it is expected that in serious mismatched cases, $F_2$ might produce more obvious effect on the performance.

As for the effect of $F_1$ and $F_2$ on stability characteristics, we only indicate that $F_1$ is in fact a parallel compensator for the scheme of Fig.4. We know very well that in single-input/single-output case the addition of a zero to a open-loop transfer function has the effects of tending to make the system more stable and to speed up the settling of the response. Hence a $F_1$ formed as (33) can lead the scheme of Fig.4 to be more easily stabilized and it is this stability that is a necessary condition for stabilizing the scheme of Fig.2.

In short, the presence of $F_1$ and $F_2$ may yield some benefits and provide a margin of choice to the designer. Even though the general analysis has not yet been achieved, we can expect that the extended Smith scheme may have some advantage over the usual Smith scheme.
3.3. The Steady State Error

We here note the steady state error in $F_1 \neq I$, $F_2 \neq I$ case.

From Fig. 3 the closed loop TFM can be obtained as

$$H_C(s) = T(1 + GK^*F_2T)^{-1}GK^*$$  \hspace{1cm} \text{(35)}$$

where

$$K^* = (I + KF_1G_A - KF_2G_A^{-1}K)^{-1}K$$ as before (see formula (6)).

According to final value theorem, steady state value for unit step demand is

$$Y(\infty) = T(0)(I + G(0)K^*(0)F_2(0)T(0))^{-1}G(0)K^*(0)$$  \hspace{1cm} \text{(36)}$$

When both $T$ and $T_A$ are diagonal matrix of pure time delays, the final value is

$$Y(\infty) = (I + G(0)K^*(0)F_2(0))^{-1}G(0)K^*(0)$$  \hspace{1cm} \text{(37)}$$

where

$$K^*(0) = (I + K(0)(F_1(0) - F_2(0))G_A(0))^{-1}K(0)$$  \hspace{1cm} \text{(38)}$$

We then look at some special case:

1. When $F_1(0) = F_2(0)$, it is clear that

$$K^*(0) = K(0)$$

and

$$Y(\infty) = (I + G(0)K(0)F_2(0))^{-1}G(0)K(0)$$  \hspace{1cm} \text{(39)}$$

In this case if integral action is included in the controller, $K(0) \to \infty$ then

$$Y(\infty) = F_2^{-1}(0)$$  \hspace{1cm} \text{(40)}$$
That means steady state error will exist if \( F_2(o) \neq I \) even though integral action is included in the controller.

(2) \( F_1(o) \neq F_2(o) \) and controller includes integral action, in this case

\[
K^*(o) = G_A^{-1}(o) (F_1(o) - F_2(o))^{-1}
\]

...(41)

the steady state value of \( y \) is

\[
Y(o) = \left[ I + G(o) G_A^{-1}(o) (F_1(o) - F_2(o))^{-1} F_2(o) \right]^{-1} G(o) G_A^{-1}(o) (F_1(o) - F_2(o))^{-1}
\]

...(42)

If the approximate model is chosen such that

\[
G_A(o) = G(o)
\]

then \( Y(o) \) is of more simple form,

\[
Y(o) = \left[ I + (F_1(o) - F_2(o))^{-1} F_2(o) \right]^{-1} (F_1(o) - F_2(o))^{-1} = F_1^{-1}(o)
\]

...(43)

The steady state error will again exist if \( F_1(o) \neq I \).

Summarising the analysis above, we can conclude that the steady state error exists in general even though integral action is included in the controller. This is perhaps the expense for obtaining the increase in permissible mismatch or the improvement of the performance. The designer should hence check the steady state error when either inner loop or outer loop feedback is not unity.

4. **Sensitivity Function for Multivariable Smith Scheme**

The sensitivity problem for time delay systems in single input/single output case has been studied in [6]. We will here investigate this problem in multivariable case. Following [6], the term 'temporal sensitivity' is used to describe the sensitivity of system performance
to the time delay parameter and 'parameter sensitivity' for the other parameters. Throughout this report we suppose that the perturbation of the parameter is time invariant. For simplicity assume both $F_1$ and $F_2$ in scheme Fig.2 are unity.

4.1. General Sensitivity Function

From Fig.3 we obtain

$$Z = GK^*(\gamma - TZ) \quad \ldots (44)$$
or

$$Z = (1 + MT)^{-1}M\gamma \quad \ldots (45)$$

where $M = GK^*$, includes all parameters except temporal parameter.

The output $y$ can then be expressed as

$$Y = TZ \quad \ldots (46)$$
$$= T(1 + MT)^{-1}M\gamma \quad \ldots (47)$$

Differentiate both sides of (46) with respect to a general parameter $\alpha$

$$\dot{Y} = \dot{T}Z + T\dot{Z} \quad \ldots (48)$$

where the notation '.' means $\frac{\partial}{\partial \alpha}$.

It is clear from (45) that

$$Z + MTZ = M\gamma \quad \ldots (49)$$

and by differentiating (49) we get

$$\dot{Z} + MT + MTZ = M\gamma \quad \ldots (50)$$

Noting (45), after a little manipulation, we obtain

$$\dot{Z} = (I + MT)^{-1}M(I - T(I + MT)^{-1}M)\gamma - (I + MT)^{-1}MT(I + MT)^{-1}M\gamma \quad \ldots (51)$$

The general sensitivity function is

$$\dot{Y} = T(I + MT)^{-1}M(I - T(I + MT)^{-1}M)\gamma + (I - T(I + MT)^{-1}M)^T(I + MT)^{-1}M\gamma \quad \ldots (52)$$

The parameter sensitivity function $\dot{Y}_\alpha$ can be obtained by taking $\dot{T} = 0$ in (52)
\[ \ddot{Y}_\alpha = T(I+MT)^{-1}M(I-T(I+MT)^{-1}M)Y \]  
\[ \ldots(53) \]

The corresponding temporal sensitivity \( \dot{Y}_T \) is
\[ \dot{Y}_T = (I-T(I+MT)^{-1}M)\dot{\dot{Y}}(I+MT)^{-1}MY \]  
\[ \ldots(54) \]

4.2. Sensitivity Function for Matched Case

In the matched case the outer loop disappears in fact and the Smith scheme is simplified to be Fig.8. The output \( y \) is
\[ Y = TZ \]  
\[ \ldots(55) \]
where
\[ Z = (I+M)^{-1}MY \]  
\[ \ldots(56) \]
and
\[ M = GK \]

Differentiate both sides of (55) with respect to a parameter \( \alpha \)
\[ \dot{Y} = T\dot{Z} + TZ \]  
\[ \ldots(57) \]
\( \dot{Z} \) can be obtained by the similar means with 4.1
\[ \dot{Z} = (I+M)^{-1}\dot{\dot{Y}}(I-(I+M)^{-1}M)Y \]  
\[ \ldots(58) \]

It is simpler than (51) because the disappearance of the outer-loop. The sensitivity function in matched case is then
\[ \dot{Y} = T(I+M)^{-1}M(I-(I+M)^{-1}M)\dot{Y} + T(I+M)^{-1}MY \]  
\[ \ldots(59) \]
because
\[ (I-(I+M)^{-1}M)\dot{Y} = (I+M)^{-1}\dot{Y} \]  
\[ \ldots(60) \]
substitute (60) into (59) we then get
\[ \dot{Y} = T(I+M)^{-1}M(I+M)^{-1}\dot{Y} + \dot{\dot{Y}}(I+M)^{-1}MY \]  
\[ \ldots(61) \]
The parameter sensitivity function can be easily obtained from (61)
\[ \dot{Y}_\alpha = T(I+M)^{-1}M(I+M)^{-1}\dot{\dot{Y}} \]  
\[ \ldots(62) \]
The corresponding temporal sensitivity is

\[ \dot{Y}_T = \dot{T}(I+M)^{-1}M \gamma \] \hspace{1cm} \text{...(63)}

The parameter sensitivity can be further analysed as follows. If only a parameter \( \alpha \) in \( G \) changes, then

\[ \dot{M} = \dot{G}K \] \hspace{1cm} \text{...(64)}

We use the term 'plant sensitivity function' and notation \( \dot{Y}_G \) to describe the sensitivity of system performance to this parameter. Which is

\[ \dot{Y}_G = T(I+GK)^{-1} \frac{3G}{3\alpha} K(I+GK)^{-1} \gamma \] \hspace{1cm} \text{...(65)}

When only a parameter \( \alpha \) in the controller \( K \) changes, so called 'controller sensitivity function' \( \dot{Y}_K \) can be defined by similar way:

\[ \dot{Y}_K = T(I+GK)^{-1} G \frac{3K}{3\alpha} (I+GK)^{-1} \gamma \] \hspace{1cm} \text{...(66)}

It describes the sensitivity of system performance to a parameter of controller.

4.3. Sensitivity Function for Mismatched Case

We rewrite the parameter sensitivity function (53) and make further observations to it.

\[ \dot{Y}_\alpha = T(I+MT)^{-1}M(I-T(I+MT)^{-1}M) \gamma \] \hspace{1cm} \text{...(67)}

where \( M = GK^* \)

\[ = G(I+KG_A - KT_A G_A)^{-1}K \] \hspace{1cm} \text{...(68)}
(1) When only a parameter $\alpha$ in $G$ changes,

$$\dot{M} = \dot{G}(I+KG_A-KT_A G_A)^{-1}K$$

...(69)

then the plant sensitivity function $\dot{Y}_G$ is

$$\dot{Y}_G = T(I+MT)^{-1} \frac{\partial G}{\partial \alpha} (I+KG_A-KT_A G_A)^{-1}K(I-T(I+MT)^{-1}M)\gamma$$

...(70)

(2) When only a parameter $\alpha$ in $G_A$ changes, define 'model sensitivity function' and use notation $\dot{Y}_{GA}$ to describe the sensitivity of system performance to this parameter.

From (68), we have

$$(I+KG_A-KT_A G_A)G^{-1}M = K$$

...(71)

Differentiate both sides of (71) with respect to $\alpha$ included in $G_A$, after a little manipulation, obtain

$$\dot{M} = GM(T_A-I)\dot{G}_A G^{-1}M$$

...(72)

The model sensitivity function is obtained by substituting (72) into (67)

$$\dot{Y}_{GA} = T(I+MT)^{-1}GM(T_A-I)\frac{\partial G_A}{\partial \alpha} G^{-1}M(I-T(I+MT)^{-1}M)\gamma$$

...(73)

We can see from (72) that the nearer the $T_A$ to $I$, the smaller the model sensitivity. This is easy to be understood - if $T_A = I$, the feedbacks of $G_A$ in inner loop and outer loop will offset each other (see Fig.2 when $F_1 = F_2 = I$) and the model sensitivity will be zero.

(3) Similarly, when only a parameter in $K$ changes, we can find $\dot{M}$ from (71),
\[
\dot{M} = G(I + KG_A^{-1}K_A G_A)^{-1} \frac{\partial K}{\partial \alpha} (I - (I - T_A)G_A G_A^{-1} M) \quad \ldots (74)
\]

The controller sensitivity function \( \dot{Y}_K \) can be easily found by substituting (74) into (67) but will be omitted here for brevity.

4.4. The Application of Sensitivity Functions

It is well known that sensitivity plays an important part in system synthesis. The sensitivity functions have been found in both matched and mismatched case without difficulty but we suffer from their complex structure. The structure of the sensitivity function in the multivariable case is much more complex than it is in single-input/single-output case. Because of this the application of the sensitivity function has to be carried forward by numerical method rather than analysis.

If the topic is the investigation of the sensitivity of performance to one parameter \( \alpha \) which is included in plant or model (or controller, temporal component), regard the corresponding sensitivity function as a function of \( s \) and parameter \( \alpha \): \( \dot{Y}(s, \alpha) \). Where \( s \) is the complex variable. We then can obtain the induced norm of the sensitivity function for a fixed \( \alpha \). The induced norm is of course different for different \( \alpha \). The best \( \alpha \), from the sensitivity view-point, is such a \( \alpha \) which makes norm reach their smallest value under the stability condition. In other words the proper \( \alpha \) should tend to minimize the induced norm of sensitivity function.

Like in theorem 2, suppose that \( G \) and \( G_A \) are rational and strictly proper TFMs, that \( K \) is rational and proper and that both \( T \) and \( T_A \) are diagonal matrices of pure time delay, and suppose that \( \frac{\partial G}{\partial \alpha} \), \( \frac{\partial G_A}{\partial \alpha} \),
\[ \frac{\partial^2 K}{\partial \alpha^2}, \frac{\partial T}{\partial \alpha} \] are all proper and rational, then the induced norm is as follows

\[ \eta \triangleq \max_{1 \leq i \leq m} \sup_{s \in \Omega} |(\hat{Y}(s, \alpha))_i| \] ... (75)

where \( \Omega \) is usual Nyquist contour.

The proper \( \alpha \) should tend to minimize the \( \eta \), i.e. for a series of \( \alpha \), the best \( \alpha \) is the \( \alpha \) which make \( \eta \) get their smallest value under the stability condition.

We here give an example to illustrate the application of the sensitivity function. The example is to investigate the controller sensitivity in the matched case. Suppose that the plant components \( T \) and \( G \) are the same as in section 3.1 and that the controller is of form

\[ K = (k + c + \frac{kc}{s})A_0 - A_1 \] ... (76)

where \( A_0 \) and \( A_1 \) are as in (29).

The parameter which changes is \( k \), i.e. a 'multiple gain'. It is clear that

\[ \frac{\partial^2 K}{\partial k} = (1 + \frac{c}{s})A_0 \] ... (77)

The controller sensitivity function can be found by substituting (77) into (66), which is

\[ \hat{Y}_K = (1 + \frac{c}{s})T(1+GK)^{-1}GA_0(I+GK)^{-1}Y \] ... (78)

Choose a unit-impulse function as the input, which Laplace transformed is \( \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] \). The norm can be calculated according (75) and is illustrated in Fig.9 as a function of \( s \) for a series fixed \( k \).
It is evident from this graph that the larger the \( k \) the smaller the norm \( \eta \). So designers should choose \( k \) as large as possible under the stability condition for reducing the sensitivity of performance to parameter \( k \).

The sensitivity to other parameters or temporal can be evaluated by the same means.

5. **Conclusions**

This report has derived the robustness theorem for the extended Smith scheme. Because the nonunity feedback in the inner loop or outer loop may lead to some benefits, so the extension of the robustness theorem is helpful in general.

Some parameter optimality problems are also discussed. The results indicate that the performance may be improved by temporal mismatch\(^{[6]}\) or proper chosen feedback component. Minimization of the performance index may be one of the criterions for choosing temporal model \( T_A \).

When the feedback in either inner or outer loop is not unity, the steady state error should be carefully checked even though an integral action is included in the controller.

The sensitivity functions for Smith scheme in the multivariable case have been established in this report. In spite of their complex structure they can be checked numerically.
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Fig. 1. Plant decomposition

Fig. 2. Extended Smith Control Scheme
Fig. 3. Equivalent Smith Scheme

Fig. 4. Delay-Free Control Scheme
Fig. 5. Variation of $(J-J_0)$ with $M$
Fig. 7. Induced Norm
Fig. 8. Matched Case

Fig. 9. Controller Sensitivity