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Successive elimination of shear layers by a hierarchy of constraints in inviscid spherical-shell flows

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In a rotating spherical shell, an inviscid inertia-free flow driven by an arbitrary body force will have cylindrical components that are either discontinuous across, or singular on, the tangent cylinder, the cylinder tangent to the inner core and parallel to the rotation axis. We investigate this problem analytically, and show that there is an infinite hierarchy of constraints on this body force which, if satisfied, sequentially remove discontinuities or singularities in flow derivatives of progressively higher order. By splitting the solution into its equatorial symmetry classes, we are able to provide analytic expressions for the constraints and demonstrate certain inter-relations between them. We show numerically that viscosity smoothes any singularity in the azimuthal flow component into a shear layer, comprising inner and outer layers, either side of the tangent cylinder, of width $O(E^{2/7})$ and $O(E^{1/4})$, respectively, where $E$ is the Ekman number. The shear appears to scale as $O(E^{-1/3})$ in the equatorially symmetric case, although in a more complex fashion when considering equatorial antisymmetry, and attains a maximum value in either the inner or outer sublayers depending on equatorial symmetry. In the low-viscosity magnetohydrodynamic system of the Earth’s core, magnetic tension within the fluid resists discontinuities in the flow and may dynamically adjust the body force in order that a moderate number of the constraints are satisfied. We speculate that it is violations of these constraints that excites torsional oscillations, magnetohydrodynamic waves that are observed to emanate from the tangent cylinder. © 2012 American Institute of Physics.

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I. INTRODUCTION

The study of shear layers in hydrodynamic flows has a long history. In a rotating, spherical-shell geometry, analytic progress dates back to the seminal work of Proudman and Stewartson, in which they considered a low-viscosity system which was driven through a small relative difference, $\epsilon \ll 1$, in the angular velocity of the boundary surfaces. The primary flow driven by the motion of the boundaries is that of a simple solid body rotation, although of considerably more interest is the secondary flow of magnitude $O(\epsilon)$ which is induced by viscous stresses at the boundaries. This flow is axisymmetric and, due to the strong influence of rotation, essentially geostrophic (i.e., azimuthal and independent of the axial coordinate) except in the vicinity of certain boundary layers. Of paramount importance is the internal boundary layer on the tangent cylinder (denoted $C$), the axial cylinder tangent to the inner sphere, which divides the spherical shell into two subdomains. Outside $C$ the geostrophic cylinders (axial cylinders on which the flow is constant) intersect only the outer boundary and the flow rotates as a solid body, oblivious to the differential boundary rotation of the inner sphere. Inside $C$ each geostrophic cylinder intersects both spherical boundaries and a formal asymptotic treatment for small viscosity at fixed $\epsilon$ shows that the geostrophic velocity is nonzero and intermediate between that of the outer and inner spherical boundaries, attaining that of the inner sphere on the inner-side of $C$. Thus across $C$ the azimuthal component of the flow is $\ldots$
formally discontinuous in the inviscid limit. However, by reinstating a weak viscosity, the solution is rendered continuous by a nested sequence of free-shear (so-called Stewartson) layers.

One important application of these ideas is to the Earth’s liquid core, for which there is evidence that the solid inner core may rotate at a different rate to the exterior mantle. So-called spherical Couette flows as described above, along with their associated shear layers, may therefore play a role in the dynamics of the Earth’s core. However, more detailed studies bound the differential rotation by about 0.2°/year, in many cases within an error tolerance of zero. Even if nonzero, such a super-rotation of the inner core is very small when compared with the daily rotation of the planet, corresponding to a value of $\epsilon = O(10^{-6})$. Due to their small magnitude, such shear layers driven by this process would not likely be of significant dynamical importance.

Motivated by the strong magnetic field in Earth’s core, we consider instead a flow $u$, driven not by boundary effects but by an internal body force. The body force $f$ comprises not only the Lorentz force due to the magnetic field, but also buoyancy which ultimately powers the geodynamo (for recent reviews). Both direct observations and geodynamo models indicate that the magnitude of the body force $f$ (for instance, as measured by the Elsasser number) is comparable to that of rotational (Coriolis) forces and, being of $O(1)$ rather than $O(\epsilon)$, can drive strong flows and associated dynamically important shear layers. The focus of this paper is to address what properties of $f$ lead to the formation of shear layers; or, expressed in another way, if the system dynamically favours a smooth flow, the manner in which $f$ must change in order to avoid discontinuities.

Although similar in many respects, the boundary-driven case has some significant differences from the internally forced system. From a mathematical point of view, of fundamental importance is that inviscid solutions may be written down directly in the latter case. In the boundary-driven system, the geostrophic flow is unspecified by the inviscid equations and is determined only by the low-viscosity limit of an involved viscous boundary-layer solution. Furthermore, since the forced flow depends everywhere on (the spatially dependent) $f$, it too may be dependent on the axial coordinate rather than simply geostrophic. However, in both cases the flow depends on the boundary conditions which, due to rotation, are only communicated parallel to the rotation axis (when ignoring viscosity). Thus, even in the forced system, only inside $C$ does the inner core have any influence and, in general, on $C$ any inviscid solution will have components that are each either discontinuous or singular; indeed, by restricting the possible choices of $f$ to a subclass which satisfies a certain homogeneous constraint, it is possible to render continuous the cylindrically radial component of flow, $u_r$, on $C$, following the work of Hollerbach and Proctor. This may be viewed as encoding into $f$ the existence of the inner core, making the solution outside $C$ “aware” of the matching condition that it should satisfy. However, this constraint guarantees continuity only of $u_r$; other components, for instance, $u_{\phi}$ (the azimuthal component of flow) may be not only discontinuous but singular on $C$.

The purpose of this paper is to investigate, analytically, the structure of an inviscid, inertia-free fluid subject to a body force $f$, and to determine the constraints on $f$ that are both necessary and sufficient in order to ensure any given continuity condition on the (non-axisymmetric) flow. Indeed, we will show that there is a hierarchy of such conditions, of which the Hollerbach-Proctor constraint is the first, which progressively smooth the flow on $C$. If a sufficient number of these conditions are satisfied then the inviscid flow will be smooth enough on $C$ that, if the weakly viscous problem were to be considered, any Stewartson-like shear layers would be entirely eliminated. As may be anticipated, this ordered list of constraints progressively include information about ever higher order derivatives of the body force and become inherently more mathematically complex statements.

For instance, although the Hollerbach-Proctor constraint involves only a certain simple average of $\nabla \times f$ over $C$, subsequent constraints involve a mixture of much more complex averages combined with pointwise values of $f$. Although we provide a general methodology to produce constraints to arbitrary order, in view of the complexity and typographic limitations, we provide only the first five members of the constraint hierarchy, these being sufficient to ensure that not only the flow but all of its first order derivatives are continuous on $C$. It is worth remarking that, even if the constraints are not satisfied, the (possibly only piecewise continuous) solution for the non-axisymmetric flow can always be found. This is quite different from the axisymmetric case in which there is no solution unless Taylor’s constraint is satisfied. Even if it is, the geostrophic flow is arbitrary and requires a further equation to close the system.
In the MHD system, the resistance to shear of magnetic field lines embedded in the fluid has a significant impact on the dynamics. In the Stewartson boundary-driven case, several studies have shown that the Lorentz force, associated with induced magnetic fields in the vicinity of strong velocity gradients, act to suppress the tangent-cylinder shear layers. In systems driven by a body force, similar adjustments occur and indeed there is numerical evidence that the magnetic field arranges itself to satisfy the Hollerbach-Proctor constraint, required for continuity of $u_\theta$ on $C$. In order for the magnetic field to have such a pronounced effect, not only must it be sufficiently strong, i.e., comparable to the Coriolis force, but it must be of the correct geometry. This latter qualification is important as field lines must cross $C$, coupling the two regions together in order to eliminate discontinuities in the flow. In particular, owing only to symmetry, dipolar fields do not cross $C$ on the equator and are therefore inefficient at removing the boundary-layer stump attached to the inner sphere. In this study, we discount the possibility of dynamical adjustment and simply consider $f$ as prescribed.

The remainder of the paper is laid out as follows. In Sec. II, we describe the model setup and give a complete solution of the flow in the forced, inviscid, non-axisymmetric problem. In Sec. III we derive a hierarchy of constraints on $f$ that ensure continuity across $C$ of any given component of the flow, and also confirm that the solution is finite on the rotation axis. In Sec. IV we provide some analytic examples illustrating the key ideas, and in Sec. V we explore how the singularities and discontinuities that may arise are smoothed into shear layers of finite thickness by viscosity. We end with a discussion in Sec. VI.

II. A COMPLETE DESCRIPTION OF THE INVISCID FORCED FLOW

We consider the forced, nondimensional Navier-Stokes equations in a frame that rotates about the $z$ axis:

\[
2k \times u = -\nabla p + f + \left[ E \nabla^2 u - R_o \frac{Du}{Dt} \right],
\]

where $k$ is the unit vector in the $z$-direction, $u$ is the incompressible velocity, $p$ is the pressure, and $f$ is a body force. Owing to the magnitude of the Rossby number (measuring the ratio of inertial to rotational effects), $R_o = O(10^{-9})$, and the Ekman number (measuring the ratio of viscosity to rotational effects), $E = O(10^{-15})$, in the Earth’s core, we neglect both viscosity and the inertial terms in the square parenthesis. In later sections, we will discuss how the system may be altered when these terms are added back in. In the specific application of the geodynamo, $f$ is simply the sum of the Lorentz and buoyancy forces, although for the present purposes it may be considered arbitrary but smooth (that is, it has continuous derivatives of all orders). We will solve (1) in a spherical-shell geometry which is most efficiently described in spherical polar coordinates $(r, \theta, \phi)$ as the domain $r_i \leq r \leq r_o$, where $r_i$ and $r_o$ are the inner and outer boundary radii. However, owing to the importance of the tangent cylinder, cylindrical polar coordinates $(s, \phi, z)$ will also be used throughout. In our rotating frame, the physical system of interest is modelled by considering both (stationary) boundaries as impenetrable and nonslip. However, in the inviscid case, due to the removal of the highest derivative in (1), only impenetrable conditions may be employed and the flow may have nonzero slip on $r = r_i$, $r_o$. It is assumed that, on reinstating weak viscosity, the solution will be modified by thin Ekman layers on both spherical boundaries, which form only in order to bring the fluid to rest and do not influence the free-stream solution (this will be justified by example in Sec. IV). Note that this stands in stark contrast to the boundary-forced spherical Couette case in which the secondary flow is driven entirely by Ekman pumping.

In the inviscid inertia-free limit of interest, $R_o = E = 0$, an explicit solution (exploiting incompressibility) may be immediately written down by first taking the curl,

\[
\frac{du}{dz} = -L,
\]

where $L = \frac{1}{2} \nabla \times f$ and then integrating with respect to $z$. In what follows, some of the ensuing algebraic expressions will be reduced in length by the introduction of the vector $L$. 
Equation (2) is not only linear but in fact its solutions separate into symmetry classes depending on the equatorial symmetry of $\mathbf{f}$. Recall that a vector $\mathbf{v}$ is equatorially symmetric ($E^5$) if both $v_z$ and $v_\phi$ are even functions of $z$ and $v_\phi$ is odd; conversely, $\mathbf{v}$ is equatorially antisymmetric ($E^3$) if both $v_z$ and $v_\phi$ are odd and $v_\phi$ is even in $z$. Further, it is elementary to show that (i) $\nabla \times \mathbf{v}$ is $E^3$ ($E^5$) if and only if $\mathbf{v}$ is $E^5$ ($E^3$) and (ii) $d\mathbf{v}/dz$ is $E^3$ ($E^5$) if and only if $\mathbf{v}$ is $E^3$ ($E^5$). It follows that $\mathbf{u}$ is $E^4$ ($E^5$) and $\mathbf{L}$ is $E^5$ ($E^4$) if $\mathbf{f}$ is $E^4$ ($E^5$). Therefore, rather than solving (2) in the entire spherical shell, it is expedient to solve for each symmetry separately, in each case restricting attention to the upper hemisphere. The geometry under consideration is shown in Figure 1.

At any cylindrical radius $0 \leq s \leq r_o$, let us define the vector $\mathbf{I}$ as the following componentwise integral of $\mathbf{L}$:

$$ I_{s,\phi,z}(s,\phi,z) = \int_z^{z_o} L_{s,\phi,z}(s,\phi,z') dz' $$

with $z_o = \sqrt{r_o^2 - s^2}$. The solution to (2) may then be written succinctly as

$$ \mathbf{u} = \{I_s, I_\phi, I_z\}(s,\phi,z) + [c_s, c_\phi, c_z](s,\phi), $$

where $c_s$, $c_\phi$, and $c_z$ are constants of the integration with respect to $z$, which depend on the remaining variables ($s, \phi$). Since $z_o$ marks the top of the outer boundary in both regions I and II, $\mathbf{I}$ is smooth everywhere in the upper hemisphere. Note that since $\mathbf{c}$ is independent of $z$, the boundary conditions are communicated in the $z$-direction and so $\mathbf{c}$ must be found in regions I and II separately. Thus, the solution in region I is oblivious to the inner core, leading to possible discontinuities on $\mathcal{C}$. Furthermore, due to the vertical slope of the inner boundary surface on $\mathcal{C}$, the solution in II may even admit singular behaviour.

In region II, in which $s \leq r_i$, the non-penetration condition $\mathbf{u} \cdot \mathbf{r} = \mathbf{u} \cdot [s, 0, z] = 0$, where $\mathbf{r}$ is the position vector, at both $r = r_o$ and $r = r_i$ supply the conditions

$$ s \ c_s + z_o \ c_z = 0, $$

$$ s \ (I_s(z_i) + c_s) + z_i(I_z(z_i) + c_z) = 0, $$

where $z_i = \sqrt{r_i^2 - s^2}$ and has solution

$$ c_s = \frac{z_o(I_s(z_o) + z_o I_z(z_o))}{s(z_i - z_o)}, \quad c_z = -\frac{s I_s(z_i) + z_i I_z(z_i)}{(z_i - z_o)}, $$

where we suppress the dependence of $\mathbf{I}$ on ($s, \phi$) for typographic purposes. Note that $c_s$ is everywhere finite in region II (even at $s = r_i$), except possibly at $s = 0$.

The solution in region I, in which $s > r_o$, is found slightly differently. Since it has no boundary on the equator, in addition to the non-penetration condition at $r = r_o$ we consider a symmetry condition on $\mathbf{u}$: if $\mathbf{u} \in E^5$ then $u_z = 0$ on the equator; conversely if $\mathbf{u} \in E^3$ then $u_z = 0$ on the equator. In region
I, the solution is then

\[ \begin{align*}
  c_s = \left\{ \begin{array}{ll}
    \frac{i \omega}{s} \int_0^\infty L_z \, dz & \text{if } u \in E^s \\
    - \int_0^\infty L_z \, dz & \text{if } u \in E^a
  \end{array} \right.
\]

(7)

with \( c_s = -s c_f/\omega \). The meridional flow is now completely specified in both regions; the remaining \( u_\phi \) component is determined in the axially non-symmetric case by using the divergence-free property:

\[ \frac{\partial u_\phi}{\partial \phi} = -\frac{\partial (s u_z)}{\partial s} - s \frac{\partial u_z}{\partial s} = -\frac{\partial (s c_s)}{\partial s} - s \frac{\partial (s I_s)}{\partial z} - s \frac{\partial I_s}{\partial z} \]

(8)

since \( c_s \) is independent of \( z \). By expanding all quantities into a complex Fourier series in azimuthal wavenumbers \( m > 0 \), \( u_\phi \) can easily be recovered using \( \partial u_\phi / \partial \phi = im u_\phi \) for each mode.

In both regions I and II, if \( f \) is smooth then so is \( I_{s,\phi} \). It follows immediately from (3) and (4) that \( u_t \) has the same continuity properties as \( c_s, c_t, \) and \( u_c \). Furthermore, from (8), continuity of \( u_\phi \) depends only on that of \( u_t \) and \( u_c \). Thus we may view continuity of all three components of \( u \) (and \( c \)) as depending only on that of \( u_t \). Indeed, (4) and (8), which hold on both sides of \( C \), show that if \( u_t \) has \( n \) continuous derivatives (with respect to \( s \)) across \( C \), then \( u_c \) has \( n \) and \( u_\phi \) has \( n - 1 \) continuous derivatives across \( C \). This idea is significantly expanded upon in Sec. III.

It is worth remarking that the above construction of the meridional flow does not depend on axial symmetry. That is, all longitudinal wavenumber components of \( u_t \) and \( u_c \) follow directly from (3), (6), and (7) and do not depend on (8). The azimuthal component of flow is quite different: using the machinery above, only the non-axisymmetric component of \( u_\phi \) can be found. Indeed, in order to find its axisymmetric component, although (8) provides no information, the contribution from \( I_\phi \) is known and it is only the geostrophic component of flow, \( c_\phi(s) \), that remains unspecified. In this case a further solvability condition known as Taylor’s constraint\(^{24} \) arises through taking the average of the azimuthal component of (1) over geostrophic cylinders,

\[ \int_{C(s)} f_\phi d\phi dz = 0, \]

(9)

where \( C(s) \) is any geostrophic cylinder. Unless (9) is satisfied no axisymmetric solution exists.

III. A HIERARCHY OF CONSTRAINTS

A. A series representation

In view of the central role played by \( u_t \) in the determination of continuity of all three components of \( u \), we may define the quantity

\[ \bar{Q}_t(\phi) = \lim_{s \to r_t^-} u_t'(s, \phi, z) - \lim_{s \to r_t^+} u_t''(s, \phi, z) = \lim_{s \to r_t^-} c_s'(s, \phi) - \lim_{s \to r_t^-} c_s''(s, \phi) \]

(10)

in which the limits approach \( r_t \) in \( s \) from above and below; this form may be extended in the obvious way to define step changes in derivatives of \( u_t \). The superscripts indicate the region of definition of each \( u_t \); by continuity of \( I_s \) across \( C \), the right-hand quantity is independent of \( z \). Since \( \bar{Q}_t \) must depend only on \( f \) (the only free parameter in the problem), considerations of continuity of \( u_t \) then translate via \( \bar{Q}_t \) into conditions on \( f \). In fact, our case is somewhat simpler than this difference of two distinct limits. Since \( u_t' \) is independent of \( r_t \) (from (7)), it is continuous across \( C \) and we may consider the quantity

\[ \bar{Q}_t(s, \phi) = u_t' - u_t'' \]

(11)

within region II, in which the one-sided limit of \( s \to r_t \) from below is of interest; \( \bar{Q}_t(\phi) \) is simply \( \lim_{s \to r_t^-} \bar{Q}_t(s, \phi) \). Furthermore, any limiting dependence of \( \bar{Q}_t \) close to \( C \) enters only through the variable \( \sqrt{r_t^2 - s^2} \), appearing in (6). Exploiting the above two observations permits the expansion...
of \( Q_t \) in the Taylor series near \( C \)

\[
Q_t(s, \phi) = \sum_{n=0}^{\infty} A_n(\phi) (r_i^2 - s^2)^n / 2.
\]

The lowest term in this series is given by \( n = 0 \) as, inspecting (6) and (7), \( u_t \) is finite in both regions I and II (except possibly at \( s = 0 \)); thus \( n < 0 \) which would render \( Q_t \) singular at \( s = r_i \) is not permitted. The first two derivatives of \( Q_t \) (providing continuity conditions on \( u_t \) and \( u_c \)) and \( s Q_t \) (providing continuity conditions on \( u_\phi \) by (8)) are given below,

\[
\frac{\partial Q_t}{\partial s} = -s A_1 (r_i^2 - s^2)^{1/2} - 2s A_2 + O \left( \sqrt{r_i^2 - s^2} \right),
\]

\[
\frac{\partial^2 Q_t}{\partial s^2} = -s^2 A_1 (r_i^2 - s^2)^{3/2} + \frac{3s^2 A_3 - A_1}{(r_i^2 - s^2)^{1/2}} + 2 \left[ (6s^2 - 2r_i^2) A_4 - A_2 \right] + O \left( \sqrt{r_i^2 - s^2} \right).
\]

\[
\frac{\partial(s Q_t)}{\partial s} = -s^2 A_1 (r_i^2 - s^2)^{1/2} + \left( A_0 - 3s^2 - r_i^2 \right) A_2 + O \left( \sqrt{r_i^2 - s^2} \right).
\]

\[
\frac{\partial^2(s Q_t)}{\partial s^2} = -s^3 A_1 (r_i^2 - s^2)^{3/2} + \frac{3s (s^2 A_3 - A_1)}{(r_i^2 - s^2)^{1/2}} + 2s \left[ (10s^2 - 6r_i^2) A_4 - 3A_2 \right] + O \left( \sqrt{r_i^2 - s^2} \right).
\]

If the structure of \( f \) is such that \( Q_t(r_i, \phi) = 0 \), occurring if and only if \( A_0(\phi) = 0 \), both \( u_t \) and \( u_c \) are continuous. Higher order zeros of \( Q_t \) (stemming from a more constrained structure of \( f \)) bestow more continuity on \( u \). For example, if the further condition \( \partial (s Q_t) / \partial s = 0 \) is imposed on \( C \), requiring that \( A_1 = 0 \) (annulling the singularity in (14)) and \( A_0 = 2r_i^2 A_2 \), then \( u_\phi \) is additionally continuous. Note that \( Q_t = \partial (s Q_t) / \partial s = 0 \) also gives \( \partial Q_t / \partial s = 0 \) which then provides continuity of \( \partial u_t / \partial s \) and \( \partial u_c / \partial s \). Overall then, continuity of all three components of \( u \) requires \( A_0 = A_1 = A_2 = 0 \).

We define a hierarchy of conditions that leads to \( u_t \) becoming progressively more smooth on \( C \), to be the ordered list \( A_i(\phi) = 0, i = 0, 1, 2, \ldots \infty \); increasing smoothness of \( u_t \) and \( u_\phi \) follows immediately as discussed above. The upper section of Table I summarises the continuity of \( u \) on \( C \) with both necessary and sufficient conditions that must be satisfied by the \( A_n \); in each row, we assume

**TABLE I.** Hierarchy of continuity conditions satisfied by \( u \) on the tangent cylinder \( C \) with the associated constraints on the expansion coefficients \( A_n \) (each being dependent on the forcing \( f \)). Each constraint is only active for one equatorial symmetry of \( f \) (discussed in Secs. III B and III C), as shown in column three. In the upper part of the table, the constraints on the meridional flow \( u_t \) and \( u_c \) must hold for all wavenumbers \( m \geq 0 \) individually; those for the azimuthal flow \( u_\phi \) only need hold for the non-axisymmetric components \( m > 0 \), \( u_\phi \) being undetermined in the axisymmetric case. In the lower part of the table, other continuity conditions are summarised.

<table>
<thead>
<tr>
<th>Smoothness condition on ( C )</th>
<th>Constraints</th>
<th>Symmetry</th>
</tr>
</thead>
<tbody>
<tr>
<td>Finiteness of ( u_t, u_c )</td>
<td>Automatically satisfied</td>
<td></td>
</tr>
<tr>
<td>Continuity of ( u_t, u_c )^a</td>
<td>( A_0 = 0 )</td>
<td>( E^8 )</td>
</tr>
<tr>
<td>Finiteness of ( \frac{\partial u_t}{\partial r}, \frac{\partial u_t}{\partial \phi} ), and ( u_\phi )</td>
<td>( A_1 = 0 )</td>
<td>( E^4 )</td>
</tr>
<tr>
<td>Continuity of ( \frac{\partial u_t}{\partial r}, \frac{\partial u_c}{\partial r} ), and ( u_\phi )</td>
<td>( A_2 = 0 )</td>
<td>( E^8 )</td>
</tr>
<tr>
<td>Finiteness of ( \frac{\partial^2 u_t}{\partial r^2}, \frac{\partial^2 u_t}{\partial \phi^2} ), and ( \frac{\partial u_\phi}{\partial r} )</td>
<td>( A_3 = 0 )</td>
<td>( E^4 )</td>
</tr>
<tr>
<td>Continuity of ( \frac{\partial u_t}{\partial r}, \frac{\partial u_c}{\partial r} ), and ( \frac{\partial u_\phi}{\partial \phi} )</td>
<td>( A_4 = 0 )</td>
<td>( E^8 )</td>
</tr>
<tr>
<td>Axisymmetric solution exists</td>
<td>Taylor’s constraint</td>
<td>( m = 0, E^8 )</td>
</tr>
<tr>
<td>Finiteness of ( u ) at ( s = 0 )</td>
<td>Automatically satisfied</td>
<td></td>
</tr>
</tbody>
</table>

^aIf \( A_1 \neq 0 \), then \( u_\phi \) is sufficiently singular on \( C \) to render the kinetic energy of the flow infinite.
that all lower order constraints are satisfied. That is, if \( A_n = 0 \) then \( A_j = 0, j = 0, 1, \ldots n - 1 \) are implicitly assumed to hold. This makes intuitive sense: in order to discuss (say) continuity of \( \partial u / \partial s \), it is reasonable to require that not only is this quantity finite on \( \mathcal{C} \), but that all lower order derivatives, in this case only \( u_s \), are themselves finite and continuous. As will be shown in Subsections III B and III C, each constraint in this hierarchy involves only one “active” equatorial symmetry class, shown in column three of Table I. The lower section of the table provides a statement of finiteness of \( u \) on the axis \( s = 0 \); this is proved in Sec. III E.

To ensure that all three components of \( u \) are continuous on \( \mathcal{C} \), it is required that \( A_0 = A_1 = A_2 = 0 \); continuity of \( \partial u / \partial s \) on \( \mathcal{C} \) further requires \( A_3 = A_4 = 0 \). Interestingly, if \( A_1 \neq 0 \) then the singularity of \( u_\phi \) on \( \mathcal{C} \) is sufficiently strong to render the kinetic energy of the flow infinite. This can be easily seen by noting that if \( u_\phi \sim (r_i^2 - s^2)^{1/2} \), then \( u_\phi \sim (r_i^2 - s^2)^{-1/2} \), and \( |u|^2 \sim (r_i^2 - s^2)^{-1} \) which has a divergent integral over the spherical shell.

Within our hierarchical structure, we note explicitly that any increment in the level of smoothness imposed on \( u \) introduces only a single further homogeneous constraint \( A_n = 0 \) (say); this is because the relevant derivative of \( Q_T \) involves terms which are homogeneous linear functions of \( A_j, j = 0, 1, \ldots n - 1 \), and if all lower order constraints are assumed satisfied then \( A_n \) must vanish. Further, we remark that as ever higher derivatives of \( Q_T \) are constrained to vanish, any given derivative of a component of flow progresses from being singular, to finite but discontinuous, to continuous on \( \mathcal{C} \). Finally, we note that it is possible (although non-intuitive) to construct examples in which this hierarchy is not followed. Consider, for instance, a case where \( 4r_i^2 A_4 - 3A_2 = 0, A_1 = A_3 = 0 \), but \( A_4 \neq 0 \neq A_2 \). This corresponds to an example in which \( \partial u_\phi / \partial s \) is continuous but \( u_\phi \) may be itself discontinuous.

The determination of the coefficients \( A_n \), which depend on which equatorial symmetry is under consideration, is greatly facilitated by computer algebra. To find \( A_n \), we simply need to evaluate \( Q_T \) at \( s = r_i \). We find a general \( A_n \) by an inductive argument, which is simply the limit of the quotient

\[
Q_T = \frac{\sum_{j=0}^{n-1} A_j (r_i^2 - s^2)^{j/2}}{(r_i^2 - s^2)^{n/2}}
\]

as \( s \to r_i \) from below, assuming that the \( A_j, j < n \), are already known. For \( n > 0 \) both the numerator and denominator of the above quotient vanish and the limit is undefined, although may be evaluated by using L’Hôpital’s rule to evaluate the quotient of the derivatives. However, for \( n > 1 \), a single application of L’Hôpital’s rule is insufficient and the limit remains undefined; empirically we find that the rule must be used \( n \) times to find \( A_n \) (assisted by re-expressing intermediate expressions in terms of a new numerator and denominator at each stage); see the Appendix for more details.

### B. Equatorially symmetric coefficients

In this subsection and in III C we supply the first five terms of the series expansion for \( Q_T \) in terms of \( A_n \) for each equatorial symmetry class. For typographic purposes, we suppress not only the \((s, \phi, z)\) dependence of all quantities appearing in the integrands, but also the dependency on \( \phi \) of the boundary terms. For each symmetry class we give the full representation of the series coefficients \( A_n \), not including any simplifications that may arise if all lower order constraints are simultaneously satisfied.

We begin by considering the equatorially symmetric class, this being the symmetry discussed by Hollerbach and Proctor.\(^6\) If \( \mathbf{f} \) is \( E^5 \), then \( \mathbf{L} \) is \( E^4 \) and \( \mathbf{u} \) is \( E^5 \). From (6), (7), and (11), the quantity \( Q_T \) can be written

\[
Q_T = \frac{z_\alpha (z_i + z_\alpha)}{s (r_i^2 - r_\alpha^2)} \left( \int_{z_i}^{z_\alpha} [z_1 L_z(s, z) + s L_z(s, z)] \, dz + (z_\alpha - z_i) \int_0^{z_i} L_z(s, z) \, dz \right).
\]
The first five expansion coefficients are given by

\[ r_i A_0 = \int_0^\zeta (\zeta L_z + r_i L_s) \, dz, \]

\[ \zeta A_1 = A_0, \]

\[ 2 \zeta^2 r_i^3 A_2 = \int_0^\zeta \left[ 2 r_i^3 L_s + \zeta \left( 3 r_i^2 + \zeta^2 \right) L_z - r_i^2 \zeta^2 \frac{\partial L_z}{\partial s} - r_i \zeta \frac{\partial L_z}{\partial s} \right] \, dz + \]

\[ \zeta r_i^2 \left( r_i L_s(r_i, \zeta) + \zeta L_z(r_i, \zeta) \right) - r_i^3 \zeta^2 \frac{\partial L_z}{\partial z}(r_i, 0) - 2 r_i^2 \zeta^2 L_z(r_i, 0), \]

\[ 2 \zeta^2 A_3 = 2 \zeta^2 A_2 - A_0, \]

\[ 24 \zeta^3 r_i^5 A_4 = 3 \int_0^\zeta \left[ 3 r_i^4 L_z - 3 \zeta^2 r_i \left( 2 r_i^2 + \zeta^2 \right) \frac{\partial L_z}{\partial s} + r_i^2 \zeta^4 \frac{\partial^2 L_z}{\partial s^2} - \right. \]

\[ \left. (4 \zeta r_i^4 + \zeta^2 r_i^2) \frac{\partial L_z}{\partial s} + r_i^3 \zeta^3 \frac{\partial^2 L_z}{\partial s^2} \right] \, dz + \]

\[ + 12 \zeta^3 r_i^3 \frac{\partial L_z}{\partial s}(r_i, 0) - 12 r_i^2 \zeta \left( 2 r_i^2 + \zeta^2 \right) L_z(r_i, 0) - 12 \zeta r_i^4 \frac{\partial L_z}{\partial z}(r_i, 0) \]

\[ + 6 \zeta^3 r_i^2 \frac{\partial^2 L_z}{\partial z\partial s}(r_i, 0) - 3 \zeta^3 r_i \frac{\partial^3 L_z}{\partial z^3}(r_i, 0) - 4 r_i^4 \zeta^3 \frac{\partial^2 L_z}{\partial z^2}(r_i, 0) \]

\[ + 9 r_i^5 L_s(r_i, \zeta) - 6 \zeta^2 r_i \frac{\partial L_z}{\partial s}(r_i, \zeta) + 3 \zeta^2 r_i^4 \frac{\partial L_z}{\partial z}(r_i, \zeta) \]

\[ + 3 \zeta^2 r_i^5 \frac{\partial L_z}{\partial z}(r_i, \zeta) - 6 \zeta^3 r_i^2 \frac{\partial L_z}{\partial s}(r_i, \zeta) + 3 \zeta^2 r_i \frac{\partial^2 L_z}{\partial z^2}(r_i, \zeta) \]

\[ \left( 5 r_i^2 + 2 \zeta^2 \right) L_z(r_i, \zeta). \]  \hspace{1cm} (17)

where \( \zeta = \sqrt{r_o^2 - r_i^2} \). In the above, we have used the equatorial symmetry imposing that \( L_s(s, \phi, z) \) is an odd function of \( z \) and \( L_z(s, \phi, z) \) is even to simplify the above expressions. Note that \( A_3 \) is a homogeneous function of \( (A_2, A_0) \) and \( A_1 \) a homogeneous function of \( A_0 \).

In this symmetry, \( u_\phi \) (and therefore) \( u_z \) are discontinuous unless the constraint of Hollerbach and Proctor\(^{10}\) is satisfied:

\[ \int_0^\zeta (\zeta L_z + r_i L_s) \, dz = 0. \]  \hspace{1cm} (18)

They conjectured that satisfaction of (18) leads not only to continuity of \( u_\phi \) but removal of any singular behaviour in \( u_\phi \) on \( C \). Here, we confirm this by noting that since \( A_1 = \zeta A_0 \), it is clear that if \( A_0 = 0 \) (giving continuity of \( u_z \)) then also \( A_1 = 0 \) giving finiteness of \( u_\phi \) automatically.

Furthermore, since \( A_3 \) depends homogeneously on \( A_2 \) and \( A_0 \), it follows that if \( A_0 = A_1 = A_2 = 0 \) (granting continuity of \( u_z \)), this automatically renders \( A_3 = 0 \) and \( \partial u_\phi/\partial s \) finite (but not necessarily continuous). We speculate that \( A_{2n-1} \) is linearly and homogeneously dependent on \( A_{2k}, \ k < n \), so that \( A_{2n-1} = 0 \) automatically if all lower order constraints are satisfied; that is, within this symmetry class, only \( A_{2n} = 0 \) are active constraints.

\section*{C. Equatorially antisymmetric coefficients}

If \( f \) is \( E^A \), then \( L \) is \( E^S \) and \( u \) is \( E^A \); the quantity \( Q_f \) can then be written

\[ Q_f = \frac{z_i + z_o}{s (r_o^2 - r_i^2)} \left( z_o \int_{z_i}^{z_o} [z_i L_z(s, z) + s L_z(s, z)] \, dz + s (z_i - z_o) \int_0^{z_o} L_z(s, z) \, dz \right). \]  \hspace{1cm} (19)
The first five series coefficients \( A_n \) are then given by

\[
A_0 = 0, \\
\sum_{i=0}^{\infty} \left( \xi L_z + r_i \right) dz - r_i \xi L_z(r_i, 0), \\
\xi A_2 = A_1, \\
6 \xi^3 r_i^3 A_3 = 3 \int_0^\xi \left[ r_i^3 L_z + \xi \left( 2 r_i^2 + \xi^2 \right) L_z - r_i^2 \frac{dL_z}{ds} - r_i \xi^3 \frac{dL_z}{dz} \right] dz + \\
3 \xi r_i^3 \left( r_i L_z(r_i, \xi) + \xi L_z(r_i, \xi) \right) - \\
6 \xi r_i^3 L_z(r_i, 0) + 3 \xi^3 r_i^2 \frac{dL_z}{dz}(r_i, 0) - 3 \xi^3 r_i^2 \frac{dL_z}{dz}(r_i, 0) - 3 \xi^3 r_i^3 \frac{d^2L_z}{dz^2}(r_i, 0), \\
2 \xi^3 A_4 = 2 \xi^2 A_3 - A_1. \tag{20}
\]

As before, we have used the equatorial symmetry of \( L \) to simplify the expressions, exploiting the fact that \( L_z(s, \phi, z) \) is an even function of \( z \) and \( L_z(s, \phi, z) \) is odd. Note that \( A_4 \) is a homogeneous function of \( (A_3, A_1) \) and \( A_2 \) a homogeneous function of \( A_1 \). Thus it would appear that the conditions \( A_{2n} \) are automatically satisfied (assuming that all lower order constraints are satisfied) and so only the constraints \( A_{2n+1} = 0 \) are active within this symmetry class.

In this symmetry, \( A_0 \equiv 0 \) and so both \( u_0 \) and \( u_t \) are automatically continuous on \( C \). This occurs because \( u_t = 0 \) at \( z = 0 \) on either side of \( C \); in region II, stemming from the non-penetration condition there, and in region I from the symmetry imposed. The behaviour of \( u_\phi \) is tied to \( \partial(s \Theta)/\partial s \) and requires \( A_1 = 0 \) to avoid singular behaviour. In fact, since \( A_1 = \xi A_2 \), this condition is also sufficient to ensure that \( A_2 = 0 \) and so renders \( u_\phi \) continuous. Thus, in this symmetry class, finiteness of both \( u_\phi \) and \( \partial u_t / \partial s \) implies also continuity of these quantities.

D. Comparison of the symmetry classes

Having derived the first few of the hierarchy of constraints for each equatorial symmetry class individually, we briefly remark on some general characteristics. First, although we only listed the coefficients \( A_n \) for \( n \leq 4 \), it is striking that if \( u \) is \( E^S \) \( (E^A) \) and \( n \) is an odd (even) integer, then \( A_n \) is linearly and homogeneously related to the \( A_1 \) for \( j < n \) and \( j \) is even (odd). In this sense, the hierarchy of constraints \( \{A_n = 0\} \) exactly partitions between the two symmetry classes: the \( E^S \) symmetry only requiring consideration of \( A_{2n} \), and the \( E^A \) symmetry consideration only of \( A_{2n+1} \) (see column three of Table I). Additionally, it cannot be the case that a threshold \( K \) is reached such that \( A_k \) is homogeneously dependent on \( A_j \) for any \( k > K, j \leq K \) (that is, progressing further down the series of \( A_j \) fails to produce any more constraints), since as \( n \) is increased ever higher derivatives are introduced which cannot be expressed in terms of those lower degrees (as can be seen, for example, in (17)).

Second, there appear to be strong links between the coefficients \( A_n \) from different symmetry classes. Perhaps most apparent is the homogeneous relations satisfied between coefficients being almost independent of symmetry class (but offset by one in the index \( n \)). For instance, it is striking to compare the relation for \( A_1 \) and \( A_3 \) in the \( E^S \) class with those for \( A_2 \) and \( A_4 \) in the \( E^A \) class. Further, it is interesting to compare the expressions for coefficients associated with the active constraints, for example, \( A_3 \) in the \( E^S \) class and \( A_3 \) in \( E^A \). Aside from the boundary terms evaluated on the equator (whose form is different, owing to the different behaviour that the two symmetry classes impart), all remaining terms are analytically identical if the preceding constraints are satisfied. For example, in the \( E^S \) symmetry class, it is easy to show that if \( A_0 = 0 \) then the integral terms in \( A_2 \) coincide (within a constant factor) with those with \( A_1 \) from the \( E^A \) class, as do the outer-sphere boundary terms. We speculate that the relations described above, motivated by the very restricted set of coefficients \( A_n \) presented, continue to hold for any integer \( n \).
E. Finiteness on the rotation axis

The rotation axis is a singular location in cylindrical coordinates and it is of interest to clarify whether or not $\mathbf{u}$ is guaranteed to be finite on the axis if $\mathbf{f}$ is everywhere smooth. Inspecting (1), it would seem likely that a singularity of $\mathbf{u}$ is inconsistent with the other (non singular terms) of the equation; here, we briefly prove that this is indeed the case.

We will assume that $\mathbf{f}$ is smooth everywhere, which means that $I_s, \phi, z$ is also smooth and so the behaviour of $\mathbf{u}$ yet again hinges on that of $\mathbf{c}$.

From (6), $c_z$ is continuous as $s \to 0$ and attains the finite value

$$\frac{r_i}{r_o - r_i} \int_{r_i}^{r_o} L_z(0, z) \, dz$$

at $s = 0$. Furthermore, as shown by (8), finiteness of $u_\phi$ as $s \to 0$ depends only on the behaviour of $\partial(s c_z)/\partial s$ and from (6) attains the finite value

$$\frac{r_o}{r_j - r_o} \int_{r_o}^{r_j} \left[ \frac{\partial L_z}{\partial s}(0, z) + L_z(0, z) \right] \, dz$$

at $s = 0$. It remains to show that $c_z$ is finite on the axis.

Finiteness of $c_z$ is not immediately assured as, from (6), as $s \to 0$:

$$c_z(s, \phi) = \frac{r_o r_i}{(r_j - r_o)} \int_{r_i}^{r_o} \frac{L_z(s, z)}{s} \, dz + O(1)$$

(21)

and we require that the integral above remains finite.

However, $\mathbf{L}$ is a smooth vector field and therefore $L_z$ is a smooth scalar. Standard regularity conditions\(^1\) then permit the expansion

$$L_z = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{m,n}(z) s^{m+2n} e^{im\phi}$$

(22)

and therefore (21) is guaranteed to be finite when $m > 0$. In the axisymmetric case, we may write $2 L_z = (\nabla \times \mathbf{f})_z = \frac{1}{s} \frac{d}{ds} (s f_\phi)$ and thus need to show that

$$\int_{z_i}^{\infty} \frac{1}{s^2} \frac{\partial}{\partial s} (s f_\phi) \, dz = \frac{1}{s^2} \int_{z_i}^{\infty} f_\phi \, dz + \frac{1}{s} \int_{z_i}^{\infty} \frac{\partial f_\phi}{\partial s} \, dz$$

(23)

has a finite limit as $s \to 0$. By (9), the first integral on the right hand side is zero for all $s$ and thus the first term on the right is identically zero. Since (9) is valid for all $s$, we may differentiate it to show that

$$0 = \frac{d}{ds} \int_{z_i}^{\infty} f_\phi \, dz = \int_{r_i}^{r_o} \frac{\partial f_\phi}{\partial s}(0, z) \, d\phi \, dz + O(s)$$

(24)

as $s \to 0$ where the derivative of the end points contribute the $O(s)$ term. It then follows that the second term on the right hand side of (23) is $O(1)$ and therefore finite as $s \to 0$.

IV. EXAMPLES

In this section we present some simple examples of forcings that satisfy a small number of the derived constraints, illustrating the different levels of continuity bestowed on the flow. We fix the inner and outer radii to be $1/2$ and $3/2$, respectively. We first remark that, in addition to the equatorial symmetry already discussed, any vector of azimuthal wavenumber $m > 0$ that possesses cylindrical components behaving as

$$[s^{m+1} f(s^2, z), s^{m+1} g(s^2, z), s^m h(s^2, z)],$$

where $f, g,$ and $h$ are arbitrary smooth functions, is guaranteed to be everywhere smooth, i.e., infinitely differentiable (including on the $z$ axis).\(^1\) This will constrain our forcings $\mathbf{f}$ when choosing examples.
Before embarking on the more general case, it is worth remarking that if \( f \) is both \( E^5 \) and independent of \( z \), then \( L_\varphi = 0 \) since it is simultaneously both odd in \( z \) and independent of \( z \). It follows immediately from (19) that \( Q_t \equiv 0 \) and all derivatives of \( u \) are automatically continuous across \( C \). We have not identified any such simple form of \( f \) that produces smooth flows in the \( E^5 \) symmetry.

We now present a family of non-trivial low-degree examples in each equatorial symmetry class. Consider the forcing

\[
\begin{align*}
f &= \left[ s^2(1 + s^2) + s^4 z^2 \sum_{k=0}^{K} a_k s^{2k} \right] \cos \phi \hat{s} + s z \cos \phi \hat{z} 
\end{align*}
\]

which drives an \( E^5 \) flow of the form

\[
(u_s, u_\phi, u_z) = (\tilde{u}_s(s, z) \sin \phi, \tilde{u}_\phi(s, z) \cos \phi, \tilde{u}_z(s, z) \sin \phi).
\]

Note the azimuthal phase shift between \( f \) and \( u \). We may choose the coefficients \( a_k \) to satisfy the constraints \( A_\varphi \). Table II shows how by adding in extra degrees of freedom, more constraints can be satisfied. For instance, choosing \( K = -1 \) (so that there are no degrees of freedom in \( f \)) violates all constraints \( A_\varphi \) and thus \( u_s, u_\phi \) are discontinuous, and \( u_z \) singular on \( C \). If \( K = 0 \), then if \( a_0 = -7, A_0 = A_1 = 0 \) and so \( u_s, u_\phi \) become continuous and \( u_z \) finite (but discontinuous) on \( C \). Contour plots of \( \tilde{u}_s, \tilde{u}_\phi \), and \( \tilde{u}_z \) for these two cases (rows one and two of Table II) are shown in the first two rows of Figure 2. Note that, following the observations in Sec. III D, each additional free parameter allows two more constraints to be satisfied as only half the constraints are active for any equatorial symmetry; we therefore anticipate that \( A_5 \) is also rendered zero in the last row.

As a second example, consider the forcing

\[
\begin{align*}
f &= s^2 z \cos \phi \hat{s} + \left[ s(1 + s^2) + s^3 z^2 \sum_{k=0}^{K} a_k s^{2k} \right] \cos \phi \hat{z}
\end{align*}
\]

which drives an \( E^4 \) flow of the form (26) and is, within some factors of \( s \), the \( E^5 \) forcing with the \( s \) and \( z \) components interchanged. This time, \( A_0 = 0 \) irrespective of \( f \) so that the first constraint is satisfied with no degrees of freedom. When \( K = 0 \) we may satisfy not only \( A_1 = 0 \) but also \( A_2 = 0 \) (since \( A_2 \) is proportional to \( A_1 \)). Table III summarises how an increasing number of constraints may become satisfied when admitting more degrees of freedom. Contour plots corresponding to the first two rows of this table are shown as rows three and four of Figure 2.

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<th>( a_2 )</th>
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<td>( A_0 = A_1 = A_2 = A_3 = 0 )</td>
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<tr>
<td>-315/16</td>
<td>153/2</td>
<td>-103</td>
<td>( A_0 = A_1 = A_2 = A_3 = A_4 (=A_5) = 0 )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( a_0 )</th>
<th>( a_1 )</th>
<th>Constraints satisfied</th>
</tr>
</thead>
<tbody>
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<td>0</td>
<td>( A_0 = 0 )</td>
</tr>
<tr>
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<td>( A_0 = A_1 = A_2 = 0 )</td>
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<td>( A_0 = A_1 = A_2 = A_3 = A_4 = 0 )</td>
</tr>
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V. THE EFFECT OF VISCOSITY

In this section we consider in what manner viscosity smooths out any singularity or discontinuity in the flow into shear layers of finite thickness. We solve

\[ 2k \times \mathbf{u} - E \nabla^2 \mathbf{u} = -\nabla p + \mathbf{f} \]  

(27)

numerically in the spherical-shell subject to nonslip boundary conditions, using a fully spectral method based on spherical harmonics and Chebyshev polynomials. Although spectral methods are not in general the method of choice for representing functions which are not smooth, the results will show that the viscous calculations converge to the inviscid case provided they are adequately...
resolved. We write

\[ \mathbf{u} = \nabla \times \left[ t(r, \theta) e^{i\phi} \hat{\mathbf{r}} \right] + \nabla \times \left[ s(r, \theta) e^{i\phi} \hat{\mathbf{r}} \right], \]

where each of the toroidal and poloidal scalar functions are further decomposed as, for instance,

\[ t(r, \theta) = \sum_{k=1}^{K_{\max}} \sum_{n=0}^{N_{\max}} t_{k,n} P_{2k-1}^1(\cos \theta) T_n(x), \quad s(r, \theta) = \sum_{k=1}^{K_{\max}} \sum_{n=0}^{N_{\max}} s_{k,n} P_{2k}^1(\cos \theta) T_n(x), \]

where \( P_n^m \) is an associated Legendre function and \( T_n(x) \) is a Chebyshev polynomial. The radial coordinate is stretched so that \( x = 2r - 2 \) satisfies \(-1 \leq x \leq 1\) over the spherical-shell domain \( 1/2 \leq r \leq 3/2 \). The representation above exploits the fact that (27) separates both in azimuthal mode and equatorial symmetry. The expression written above, involving \( P_n^m \) with \( n - m \) even for the toroidal scalar and \( n - m \) odd for the poloidal scalar is equatorially antisymmetric. Similar representations exist for the equatorially symmetric solutions. After projecting each term of the (linear) Eq. (27) back onto the spectral basis,\(^9\) the matrix-vector system, for the coefficients \( (t_{k,n}; s_{k,n}) \), may be found using standard Numerical Algorithms Group library routines on a shared-memory machine. Note that the solution is found directly, rather than as a perturbation to a given (e.g., the inviscid) background state. We found that for values of Ekman number \( E = 10^{-7}, E = 10^{-8}, \) and \( E = 10^{-9} \), sufficient resolutions were given by \( K_{\max} = N_{\max} \) of 400, 600, and 800, respectively, for converged solutions. By equatorial symmetry, the largest spherical harmonic degree used is \( L_{\max} = 2K_{\max} \), taking a maximum value of 1600. Although the matrix required to be inverted is of banded structure, at this highest truncation the inversion routines took 19 hours to run and required in total around 92 Gb of memory (including all overhead), close to the upper limit of our available computational resources. We therefore did not attempt to find solutions at higher truncations than this.

It is worth remarking that although the operator \( \nabla^2 \mathbf{u} \) leaves invariant not only the azimuthal wavenumber but the azimuthal phase of \( \mathbf{u} \), the viscous solutions do not remain of the simple form (26). This is because (27) combines the Coriolis operator \( \hat{k} \times \) that swaps the azimuthal phase (compare (26) to (25)) and \( \nabla^2 \) which does not, so the solution must be a mixture of phases. As \( E \to 0 \) however, any components of \( \mathbf{u} \) out of phase with solutions of the form (26) will reduce to zero.

Figure 3 compares cross sections of the flow components \( (\tilde{u}_i, \tilde{u}_\phi, \tilde{u}_z) \), derived using low-viscosity, and the formally inviscid solution. We show only the component of the viscous solutions of the form (26). The systems are driven by the equatorially symmetric body force of Sec. IV that satisfies no constraints. As viscosity is decreased from \( E = 10^{-7} \) (top row), \( E = 10^{-8} \) (middle row) to \( E = 10^{-9} \) (bottom row), all three components of flow converge to the formally inviscid solution shown in the top row of Figure 2. This plot also confirms that the boundary layers on either spherical boundary (owing to the nonslip conditions imposed) have no influence on the free-stream solution.

A more detailed view of the shear layers is afforded by cut-through sections at \( z = 1 \), midway between \( r_i \) and \( r_o \). Figure 4 examines \( \tilde{u}_\phi \) (leftmost plots) for our most singular examples (i.e., driven by forces that satisfy no constraints) for both the equatorially symmetric (top row) and antisymmetric (bottom row) symmetries. It is apparent that, as \( E \) decreases, the viscous solutions (colour) converge to the formally inviscid solution (black) which is singular at \( s = r_i = 1/2 \). Interestingly, viscosity smooths the singularity in \( u_\phi \) into a boundary layer centred on the tangent cylinder in which the peak shear is located either outside (inside) the tangent cylinder in the equatorially symmetric (antisymmetric) symmetry class.

Motivated by some of the known asymptotic scalings in both the boundary-driven\(^{22}\) and force-driven\(^{21}\) problems, we plot suitably rescaled solutions in the panels on the right. In both rows, the coincidence of the horizontal location of features \( a' \) to \( f' \), rescaled from features \( a \) to \( f \), supplies strong numerical evidence of inner and outer boundary layers of respective thickness \( E^{2/7} \) and \( E^{1/4} \). For the \( E^5 \) symmetry, \( u_\phi \) apparently scales as \( E^{-1/3} \) within both boundary layers (as indicated by the \( z \)-coordinate of features \( a' \) to \( c' \)). In the \( E^4 \) symmetry, the leftmost plot shows that the solution in the outer boundary-layer scales independently of \( E \) (feature \( f \)). The scaling in the \( z \)-direction suggests...
that the inner solution scales as $O(E^{1/20})$ (feature $d'$), although the peak values $e'$ do not then coincide. The inset shows a $z$-scaling of $E^{1/10}$ in which feature $e''$ appears to become independent of $E$. This more complex profile suggests a nested structure of boundary layers, in which the solution takes magnitudes $E^{-1/20}, E^{-1/10}$, and $O(1)$ as $s$ increases across $C$. Lastly, we remark that within boundary-layer theory as a whole, the scalings of $E^{-1/20}$ and $E^{-1/10}$ are not particularly commonplace and, in view of their empirical determination, the reader may wonder about the accuracy of the exponents we provide. Although they are the best fit to our data within our range $E \geq 10^{-9}$, any scalings of the form $E^{-1/k}, 16 \leq k \leq 24$ or $E^{-1/k}, 9 \leq k \leq 12$, respectively, are (broadly speaking) not inconsistent with the numerical calculations. It is likely that an accurate determination of the scalings will only emerge through a rigorous boundary-layer analysis.

VI. DISCUSSION

In this paper, we have described the analytic solution to a steady fluid driven by a body force $\mathbf{f}$ in a rotating spherical shell, in the absence of viscosity and inertia. Essentially, in what may be viewed as a generalisation of the Taylor-Proudman theorem, the impenetrable boundary conditions on the inner sphere are communicated only in the direction parallel to the rotation axis and this leads to singularities or discontinuities in all components of the driven flow on the tangent cylinder, $C$. We have set out a hierarchy of conditions, imposed on the body force $\mathbf{f}$, that progressively smooths the driven flow on $C$. The first condition, $A_0 = 0$, ensures that $u_z$ is continuous, the second, $A_1 = 0$, removes the singularity in $u_\phi$ and if $A_0 = A_1 = A_2 = 0$ then all three components of the flow are continuous. This hierarchy of conditions on $\mathbf{f}$ is analogous to the nested sequence of boundary layers found in similar problems that retain a small viscosity,\cite{12,22} each additional layer removing a higher...
order discontinuity from the flow. In the inviscid case considered, a smooth flow (i.e., infinitely differentiable) on $C$ must be driven by a force satisfying infinitely many constraints. Analogously, in the viscously adjusted problem, a smooth flow would require an infinite sequence of boundary layers, each ensuring the continuity of ever higher derivatives.

In this paper we have taken $f$ to be prescribed; but in general, it will be time-dependent, comprising buoyancy and the magnetic Lorentz force. It is this latter ingredient that, through the resistance to shearing motion by the magnetic field lines, we anticipate will adjust to satisfy as many constraints as is dynamically demanded. In the equatorially symmetric case, Hollerbach has already shown that indeed $f$ is perturbed to ensure that $A_0 = A_1 = 0$, providing continuity of $u$, and removing the discontinuity in $u_\theta$. How many more smoothness constraints are implicitly satisfied by the action of the magnetic field remains an open question although this is currently being investigated.

The influence of magnetic fields and their smoothing effect on shear layers may have significant consequences for the dynamics of the Earth’s core. So-called torsional waves, motions of axially symmetric cylinders of fluid connected by magnetic field lines, have been predicted both theoretically and are supported by mounting observational evidence. One interesting aspect of the latter study, corroborated by recent low-viscosity geodynamo simulations, is that these waves appear to emanate from the tangent cylinder and propagate both radially inwards and outwards. To explain this, we consider a background state of the core in which the body force satisfies enough of the constraints to ensure that the flow is (sufficiently) smooth on the tangent cylinder. Fluctuations in this state will cause violations of the constraints; hydrodynamically the system would respond by forming shear layers, but we speculate that the resisting influence of the magnetic field will cause (MHD) waves to propagate away from the tangent cylinder, carrying with them the...
problematic disturbance. Within our (principally non-axisymmetric) framework, we must then appeal to nonlinear interactions to produce (axisymmetric) torsional waves. Interestingly, there is evidence that persistent localised shear layers could spawn other observed instabilities such as geomagnetic jerks.\(^7\)

Even without the influence of a magnetic field, by reinstating linearised inertia, \(R_o \partial u/\partial t\), it is likely still possible to obtain hydrodynamic waves propagating from the tangent cylinder. Consider the time-dependent problem in which the initial flow is quiescent (in the rotating frame) and is at all times driven by a steady body force \(f\) that satisfies no constraints. Assuming that the set of inertial modes (solutions of the homogeneous equation given by \(f = 0\)) are complete, then the full-solution is simply the discontinuous/singular flow along with an infinite linear sum of inertial modes required to match the initial condition. The shortest length-scale modes required on the tangent cylinder, to remove any initial discontinuity or singularity, are associated with the fastest timescales and may effect a wavelike divergence from the tangent cylinder.

It is of interest to compare the constraints that we have derived to that of Taylor,\(^24\) particularly in terms of the number of conditions to which each constraint translates within a discretised system. Taylor’s constraint states that the azimuthal component of the body force (which reduces to the Lorentz force assuming that buoyancy is purely radial), must vanish when averaged on each geostrophic cylinder. Within a model discretised by spherical harmonics (of maximum degree and order \(L_{\text{max}}\)) in solid angle and a certain set of polynomials in radius of size \(L_{\text{max}}\), the number of conditions enumerates to \(O(5L_{\text{max}})^{15,16}\) In this study, our constraints involve certain averages of the body force which, although only applying on the tangent cylinder, apply separately to each azimuthal wavenumber \(1 \leq m \leq 2L_{\text{max}}\) (since \(f\) depends quadratically on the magnetic field). Accounting for both real and imaginary components of each complex exponential mode, these enumerate to a total of \(O(4L_{\text{max}})\) conditions. Although a very crude comparison, the number of discretised conditions corresponding to both types of constraint scales similarly within such a discretised scheme.

We finish by describing the outstanding issues that remain unresolved. First is the puzzle of why the equatorial symmetry classes are staggered between the constraints, each symmetry class being “active” only in every other constraint in our hierarchy. This mathematical property may well be tied to the remarkable similarity between the algebraic forms of the constraints between the two symmetry classes, whose origin might be revealed by addressing the problem in a different manner. Second is an understanding of precisely how viscosity smoothes out discontinuities or singularities into a shear layer and its scaling with the Ekman number, \(E\). In our study, we investigated numerically the most singular examples of flows, of both equatorial symmetries, driven by a body force satisfying no constraints. In both cases, the azimuthal component of flow has the same (inviscid) structure \(u_\phi \sim (r^2 - \hat{s}^2)^{-1/2}\), although, under equatorial symmetry, the singularity is arguably more severe than in the equatorially antisymmetric case, since \(u_s\) and \(u_\phi\) are also discontinuous on \(C\). Viscosity smoothes this singularity into a shear layer comprising an inner and outer layer on either side of the tangent cylinder. For both symmetries, these inner and outer layers have a thickness that appears to scale as \(O(E^{3/7})\) and \(O(E^{1/4})\), respectively, scalings identical to the boundary-forced case\(^22\) and with numerical studies of inertial waves with weak viscosity.\(^19\) Interestingly, the layer in which the shear takes its peak value depends on the symmetry class: the \(E^3\) \((E^4)\) example taking its peak value in the outer (inner) boundary layer. Further, the magnitude of the shear appears to scale differently between the symmetry classes, being of \(O(E^{-1/3})\) for \(E^3\) and of multiple scalings \(O(E^{-1/20})\), \(O(E^{-1/10})\) [or thereabouts, see the comments at the end of Sec. V] and \(O(1)\) for \(E^4\) depending on the position in the boundary-layer system. We have no explanation for this strong dependence on structure with equatorial symmetry, although it is likely to be revealed by a formal asymptotic treatment of the problem. It would further be of interest to investigate the scaling of the boundary layers in which derivatives of the flow change rapidly, when the system is driven by a body force satisfying some non-zero number of constraints.

Lastly, our results have focussed primarily on the non-axisymmetric constraints as a complete discussion of the axisymmetric flow is much more difficult. This is because the geostrophic flow of the form \(c_\phi(s)\hat{\phi}\) is formally undetermined by the governing reduced Navier-Stokes equation (1), essentially because \(k \times c_\phi(s)\hat{\phi} = -c_\phi(s)\hat{s}\) which can be absorbed into the pressure gradient. However, by considering the inviscid magnetohydrodynamical system, it is possible to show that
$c_\phi(s)$, both inside and outside the tangent cylinder, must be the solution of certain ordinary differential equations. It is not immediately clear what conditions of continuity must be required to match the two solutions on $C$, although a resolution of this may lead to a similar hierarchy of constraints for the axisymmetric flow.

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**APPENDIX: EVALUATION OF EXPANSION COEFFICIENTS**

We briefly outline the procedure for calculating the expansion coefficients, $A_n$, $n \geq 0$, for the $E^5$ symmetry class; the $E^A$ coefficients are derived in an identical fashion. The first term in the expansion, $A_0$, is simply $Q_\ell$ evaluated at $s = r$, which gives

$$A_0 = \frac{1}{r} \int_0^r (\zeta L_\ell + r; L_\ell) \, dz.$$ 

The next coefficient is given by

$$A_1 = \lim_{s \to r} \frac{Q_\ell - A_0}{(r^2 - s^2)^{1/2}}.$$ 

Since both numerator and denominator are zero in the required limit, we use L’Hôpital’s rule,

$$A_1 = \lim_{s \to r} \frac{d(Q_\ell - A_0)/ds}{(r^2 - s^2)^{-1/2}} = \frac{A_0}{\zeta} - L_\ell(r, 0),$$

reducing to $\zeta A_1 = A_0$ since, owing to symmetry, $L_\ell$ is zero at $z = 0$. Hidden in the above equation is some elementary but tedious differentiation, taking account of the fact that not only the integrands but the limits of the integrals appearing in $Q_\ell$ depend on $s$; finding the derivative required above is a task greatly facilitated by computer algebra. A similar procedure is followed to find $A_2$,

$$A_2 = \lim_{s \to r} \frac{Q_\ell - A_0 - (r^2 - s^2)^{1/2}A_1}{(r^2 - s^2)}.$$ 

Since both numerator and denominator are again zero, L’Hôpital’s rule gives

$$A_2 = \lim_{s \to r} \frac{d(Q_\ell - A_0 - (r^2 - s^2)^{1/2}A_1)/ds}{(r^2 - s^2)}.$$ 

However, this limit is also undefined and in order to make progress the above quantity can be re-expressed as a new fraction in which both numerator and denominator are zero at $s = r$,

$$H = \frac{2 r \zeta s \zeta_0 (z_0 - \zeta)}{r^3 \zeta^2},$$

where $H$ is a somewhat lengthy expression involving integrals and boundary terms. Using L’Hôpital’s rule again turns out to give a finite limit at $s = r$; the intermediate quantities in the calculation are again lengthy. The coefficient $A_1$ follows similarly, although we find that we need to apply L’Hôpital’s rule three times, re-expressing the intermediate terms twice. Further coefficients in the expansion follow similarly, and are found by applying L’Hôpital’s rule $n$-times and re-expressing the intermediate terms $n - 1$ times.