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SOME NOTES ON OPTIMIZATION OF CONFIDENCE
BANDS IN PROCESS CONTROL BASED
ON APPROXIMATE MODELS

by

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1. Introduction

Let $E^P(t)$, $t \geq 0$, be the $m \times m$ matrix

$$E^P(t) = Y(t)K_p - Y_A(t) \quad (1)$$

where K_p is $m \times m$ and constant, $Y(t)$ is $m \times m$ and $Y_A(t) = \text{diag}\{h_j(t)\}_{1 \leq j \leq m}$.

The minimization of the widths of the confidence circles can be expressed in the form

$$\min_{K_p} \max_j \sum_{\ell=1}^m N_{\infty}(E_{j\ell}^P) \quad (2)$$

if 'row dominance' conditions are used or

$$\min_{K_p} \max_j \sum_{\ell=1}^m N_{\infty}(E_{\ell j}^P) \quad (3)$$

if 'column dominance' conditions are used. If Y_A is (as yet) unspecified then E_{jj}^P is unspecified, $1 \leq j \leq m$, and (2) and (3) can be replaced by

$$\min_{K_p} \max_j \sum_{\ell \neq j} N_{\infty}(E_{j\ell}^P) \quad (4)$$

and

$$\min_{K_p} \max_j \sum_{\ell \neq j} N_{\infty}(E_{\ell j}^P) \quad (5)$$

provided that one extra constraint (say)

$$\|K_p\| = 1 \quad (6)$$

is introduced to avoid the trivial solution $K_p = 0$. Here $\|K_p\|$ is any convenient matrix norm on K_p such as

$$\|K_p\| = \max_{ij} |(K_p)_{ij}| \quad (7)$$

Other norms or constraints are, of course, possible and may be desirable for numerical purposes or to avoid singularity of K_p . (Note: There is an advantage in using the 'column dominance' conditions as, if

$$K_p = [K_1, K_2, \dots, K_m] \quad (8)$$

then problems (3) and (5) can be represented as

$$\min_{K_j} \sum_{\ell=1}^m N_{\infty}(E_{\ell j}^P), \quad 1 \leq j \leq m, \quad (9)$$

and

$$\min_{K_j} \sum_{\ell \neq j} N_{\infty}(E_{\ell j}^P), \quad \|K_j\| = 1, \quad 1 \leq j \leq m \quad (10)$$

respectively)

An alternative approach is to solve

$$\min_r r(N_{\infty}^P(E^P)) \quad (11)$$

or, if G_A is unspecified,

$$\min_{K_P} r(N_{\infty}^P(E^P) - \text{diag}\{N_{\infty}(E_{jj}^P)\}_{1 \leq j \leq m})$$

$$\|K_P\| = 1 \quad (12)$$

All of these problems are non-quadratic and suffer from non-differentiability and hence need further work to produce computational algorithms. They can also be extended to cover minimization with respect to

- (1) K_P diagonal (i.e. the simplest case of input scaling only)
- (2) K_P structured to reflect the loop connections possible.
- (3) K_P free

The could also be extended to design post and precompensators

Another tentative approach is to solve the problems

$$\min_{K_P} N_{\infty}^P(E^P) \quad (13)$$

or

$$\min [N_{\infty}^P(E^P) - \text{diag}\{N_{\infty}(E_{jj}^P)\}_{1 \leq j \leq m}] , \quad \|K_P\| = 1 \quad (14)$$

where the minimization is with respect to the partial ordering on $L(R^m)$ defined by $A \geq B$ iff $A_{ij} \geq B_{ij}$, $1 \leq j \leq m$. It is not obvious however that this problem is well-posed.

2. Existence of Solutions and Duality

The basic structure of the problem is characterized by:

Theorem 1: Problems (2) -(6) and (13)-(14) are (possibly constrained) convex optimization problems over a finite dimensional subspace $\tilde{M}$ of products of the space $BV[0, \infty)$ of functions of bounded variation on $[0, +\infty)$. The dimension of the subspace is equal to m^2 if $|Y(t)| \neq 0$.

Proof: Problems (2), (3), (4), (5), (9) and (10) are minimum norm problems in products of $BV[0, \infty)$ as N_∞ is the norm in $BV[0, \infty)$. They take the form

$$\min \{ \|Z - Y_A\|_{BV} ; Z = YK_p, K_p \in L(R^m) \} \quad (15)$$

together with any other constraints on K_p . These problems are convex provided any extra constraints on K_p are convex. Problems (13) and (14) are similarly convex as, for example,

$$\begin{aligned} N_\infty^p(\lambda Z_1 + (1-\lambda)Z_2 - Y_A) &= N_\infty^p(\lambda(Z_1 - Y_A) + (1-\lambda)(Z_2 - Y_A)) \\ &\leq \lambda N_\infty^p(Z_1 - Y_A) + (1-\lambda) N_\infty^p(Z_2 - Y_A) \end{aligned} \quad (16)$$

wherever $0 \leq \lambda \leq 1$.

Finally

$$\tilde{M} = \{ Z ; Z = YK_p \}$$

which is a subspace of products of $BV[0, \infty)$. Writing K_p as

$$K_p = \sum_{i,j} \alpha_{ij} e_i e_j^T \quad (17)$$

$$Y(t) = [Y_1(t), \dots, Y_m(t)] \quad (18)$$

then

$$Z = \sum_{i,j} Y_i(t) e_j^T \alpha_{ij} \quad (19)$$

and $Z = 0$ iff $\alpha_{ij} = 0, 1 \leq i, j \leq m$, from the nonsingularity of $Y(t)$. $\tilde{M}$ therefore has dimension m^2 .

The existence of a solution K_p to many of the optimization problems can be proved by using a dual-space framework. Note that $BV[0,T]$ is the dual of $C[0,T]$ in this context and, in the following development, replace (for technical simplicity) the interval $[0,+\infty)$ by $[0,T]$. This reflects the practical situation where only a finite length of data is available. The following result is pinched from Luenberger p121:

Theorem 2: Let M be a subspace of a real normal space X . Let $x^* \in X^*$ be distance d from $M^\perp$. Then

$$d = \min_{m^* \in M^\perp} \|x^* - m^*\| = \sup_{\substack{x \in M \\ \|x\| \leq 1}} \langle x, x^* \rangle \quad (20)$$

where the minimum on the left is achieved by $m_o^* \in M^\perp$. If the supremum on the right is achieved for $x_o \in M$, then $x^* - m_o^*$ is aligned with x_o .

(Notes: (1) $\langle x, x^* \rangle$ denotes $x^*(x)$.

(2) $M^\perp = \{x^* \in X^* ; \langle x, x^* \rangle = 0\}$

(3) x is aligned with x^* if $\langle x, x^* \rangle = \|x\| \cdot \|x^*\|$.)

For application to our problems, let $X^* = (BV[0,T])^{m^2}$ and $X = (C[0,T])^{m^2}$.

If the elements are arranged in matrix form

$$x^* = [x_{ij}^*], \quad x = [x_{ij}] \quad (21)$$

denote

$$\langle x, x^* \rangle = \sum_{i,j} \langle x_{ij}, x_{ij}^* \rangle \quad (22)$$

The dual functional $\langle \cdot, \cdot \rangle$ can be represented as

$$\langle x_{ij}, x_{ij}^* \rangle = \int_0^T x_{ij}(t) dx_{ij}^*(t) \quad (23)$$

from the Riesz Representation Theorem on p113 of Luenberger.

The norms chosen in X^* and X will depend on the problem!

The existence of solutions to our problems can be verified by setting the problem in the form of theorem 2 where K_p is unconstrained. If K_p is also subjected to boundedness constraints this is unnecessary as the existence of a

solution follows from compactness arguments. A potential further use of the results is the possibilities that m_0^* can be found by solving for x^* and 'deducing' m_0^* from the alignment condition. The 'primal' problem of finding x^* could be simpler than trying to find m_0^* directly!

Our first job is to find M by identifying x^* with Y_A , m with YK_p and $M^\perp$ with $\tilde{M}$. Clearly $\tilde{M}$ is closed as it is finite dimensional. We contend that

$$M = \{z \in X; \langle z, z^* \rangle = 0, z^* \in \tilde{M}\} \quad (24)$$

which is closed as it is the intersection of a finite number of closed hyperplanes in X . The finite dimensionality of $\tilde{M}$ implies that $\tilde{M} = M^\perp$. Now write $\langle z, z^* \rangle = 0$ with $z^* = YK_p$ as

$$\sum_{i,j,k} \langle z_{ij}, Y_{ik} \rangle \alpha_{kj} = 0 \quad (25)$$

As this must hold for all K_p , we obtain the characterization

$$M = \{z \in X; \sum_i \langle z_{ij}, Y_{ik} \rangle = 0, 1 \leq j, k \leq m\} \quad (26)$$

No further reduction seems to be possible!

Case 1: Consider problem (2) written as the optimization problem in X^*

$$\min_{m^* \in M^\perp} \|Y_A - m^*\| \quad (27)$$

where the norm in X^* is taken as

$$\|m^*\| \triangleq \max_i \sum_{j=1}^m \|m_{ij}^*\| \quad (28)$$

which is induced by the norm in X

$$\|x\| \triangleq \sum_i \max_j \|x_{ij}\| \quad (29)$$

as is easily proved by writing

$$\begin{aligned}
 |\langle x, m^* \rangle| &= \left| \sum_{i,j} \langle x_{ij}, m_{ij}^* \rangle \right| \\
 &\leq \sum_{i,j} \|x_{ij}\| \|m_{ij}^*\|
 \end{aligned} \tag{30}$$

That is

$$\begin{aligned}
 |\langle x, m^* \rangle| &\leq \sum_i \max_j \|x_{ij}\| \sum_k \|m_{ik}^*\| \\
 &\leq \left(\max_i \sum_j \|m_{ij}^*\| \right) \sum_i \max_j \|x_{ij}\|
 \end{aligned} \tag{31}$$

equality being almost achieved as follows: let k be the index in (28) such that

$$\|m^*\| = \sum_{j=1}^m \|m_{kj}^*\| \tag{32}$$

and let $x_{ij} = 0$, $i \neq k$, with $\|x_{kj}\| = 1$, $1 \leq j \leq m$, such that, with ϵ arbitrary,

$$0 \leq (1-\epsilon) \|m_{kj}^*\| \leq \langle x_{kj}, m_{kj}^* \rangle \leq \|m_{kj}^*\| \tag{33}$$

Then (30) and (31) yield

$$(1-\epsilon) \sum_j \|m_{kj}^*\| \leq |\langle x, m^* \rangle| \leq \sum_j \|m_{kj}^*\|$$

or

$$(1-\epsilon) \|m^*\| \leq |\langle x, m^* \rangle| \leq \|m^*\| \tag{34}$$

as required.

The structure of theorem 2 is therefore satisfied and we conclude that a solution K_p^* exists to problem (2). The corresponding primal problem is given by the RHS of (20) ie

$$\sup \sum_{i,j} \langle x_{ij}, (Y_A)_{ij} \rangle \quad (35)$$

subject to the constraints (c.f.(26))

$$\sum_i \langle x_{ij}, Y_{ij} \rangle = 0, \quad 1 \leq j, k \leq m \quad (36)$$

and

$$\sum_i \max_j \|x_{ij}\| \leq 1 \quad (37)$$

Equivalently, we can write

$$\sup \sum_{i,j} \int_0^T x_{ij}(t) d(Y_A(t))_{ij} \quad (38)$$

subject to

$$\sum_i \int_0^T x_{ij}(t) dY_{ik}(t) = 0, \quad 1 \leq j, k \leq m \quad (39)$$

and

$$\sum_i \max_j \max_{0 \leq t \leq T} |x_{ij}(t)| \leq 1 \quad (40)$$

A further simplification is possible if $(Y_A(t))$ has impulse response $h_{ij}^A(t)$ when (38) reduces to

$$\sup \sum_{i,j} \int_0^T x_{ij}(t) h_{ij}^A(t) dt \quad (41)$$

This is also possible for (39) but, for process control studies the impulse response of Y_{ij} may not be known. We therefore leave it unchanged.

The primal problem is an infinite dimensional linear programming problem. If it has a solution x_o , then the alignment condition reduces to

$$\begin{aligned} & \sum_{i,j} \int_0^T (x_o(t))_{ij} d(Y_A(t) - m_o^*(t))_{ij} \\ &= \left(\sum_i \max_j \max_{0 \leq t \leq T} |(x_o(t))_{ij}| \right) \left(\max_i \sum_j N_{\infty}((Y_A(t) - m_o^*(t))_{ij}) \right) \end{aligned} \quad (42)$$

If this is solved for $m_o^*(t)$, then K_p^* satisfies

$$m_o^*(t) = Y(t) K_p^* \quad (43)$$

Alternatively, K_p^* could be obtained in principle by substituting (43) into (42) and 'solving' for K_p^* . In essence the optimization problem is replaced by a nonlinear algebraic problem.

It appears that both the 'dual' and 'primal' problems are rather complex. It may be best to attempt a direct numerical solution of the dual problem! Let $t_0 = 0 < t_1 < \dots < t_N = T$ be a partition of $[0, T]$ and consider the finite-dimensional approximating problem

$$\min_{K_p} \max_i \sum_j \sum_{k \geq 1} |E_{ij}^P(t_k) - E_{ij}^P(t_{k-1})| \quad (44)$$

subject to

$$E^P(t_k) = Y(t_k) K_p - Y_A(t_k), \quad 0 \leq k \leq N \quad (45)$$

Case 2: Consider problem (3) written as the optimization problem in X^*

$$\min_{m^* \in M} \|Y_A - m^*\| \quad (46)$$

or, noting that it can be written in the form of (9) regarded as the problem

$$\min_{m^* \in M_j} \|Y_A^{(j)} - m^*\| \quad (47)$$

in $X_j^* \triangleq (BV[0, T])^m$ with norm

$$\|m^*\| \triangleq \sum_j \|m_j^*\| \quad (48)$$

induced by the norm

$$\|x\| \triangleq \max_i \|x_i\| \quad (49)$$

in $X_j \triangleq (C(0, T))^m$. Here

$$M_j^\perp = \{ m^* ; m^* = YK_j \} \quad (50)$$

in the orthogonal complement of

$$\begin{aligned} M_j &\triangleq \{ x ; \langle x, m^* \rangle = 0, \quad m^* \in M_j^\perp \} \\ &= \{ x ; \sum_{i,k} \langle x_i, Y_{ij} \rangle K_{kj} = 0, \quad \forall K_{kj} \} \\ &= \{ x ; \sum_i \langle x_i, Y_{ik} \rangle = 0, \quad 1 \leq k \leq m \} \end{aligned} \quad (51)$$

The structure of theorem 2 is satisfied and hence a solution K_j^* exists. The primal problem is

$$\sup \sum_i \langle x_i, (Y_A)_{ij} \rangle \quad (52)$$

subject to the constraints

$$\sum_i \langle x_i, Y_{ik} \rangle = 0, \quad 1 \leq k \leq m \quad (53)$$

and

$$\max_i \|x_i\| \leq 1 \quad (54)$$

interpreted as

$$\sup_i \int_0^T x_i(t) d(Y_A(t))_{ij} \quad (55)$$

with

$$\int_0^T x_i(t) dY_{ik}(t) = 0, \quad 1 \leq k \leq m \quad (56)$$

and

$$\max_{0 \leq t \leq T} |x_i(t)| \leq 1, \quad 1 \leq i \leq m \quad (57)$$

If the primal problem is solved by x_0 , then the alignment condition reduces to

$$\begin{aligned} & \int_0^T (x_0(t))_i d((Y_A(t))_{ij} - (m_0^*(t))_i) \\ &= \max_i \max_{0 \leq t \leq T} |(x_0(t))_i| \left(\sum_{\ell=1}^m N_{\infty}((Y_A)_{\ell j} - (m_0^*)_{\ell}) \right) \end{aligned} \quad (58)$$

where

$$m_0^*(t) = Y(t)K_j \quad (59)$$

Again both primal and dual problems are relatively complex and it may be better to use a discrete approximation to the dual problem directly.

Case 3: Consider problem (13) written as the problem

$$\min_{m^* \in M} \|Y_A - m\| \quad (60)$$

with M as in case 1 and the norm in $X^* = (BV[0, T])^m$

$$\|m^*\| = \sum_{i,j} \|m_{ij}^*\| \quad (61)$$

induced by the norm

$$\|x\| = \max_{i,j} \|x_{ij}\| \quad (62)$$

in $X = (C(0, T))^m$.

The structure of theorem 2 is satisfied and hence K_p^* exists!

The primal problem is

$$\begin{aligned} \sup \sum_{i,j} \langle x_{ij}, (Y_A)_{ij} \rangle \\ \sum_i \langle x_{ij}, Y_{ik} \rangle &= 0, \quad 1 \leq j, k \leq m \\ \|x_{ij}\| &\leq 1, \quad 1 \leq i, j \leq m \end{aligned} \quad (63)$$

Again both primal and dual problems are complex unless discretized.