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NON-LINEAR CIRCUIT MODE ANALYSIS

by

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ABSTRACT

An exposition is given of recent advances in the theoretical analysis of circuits comprising sets of intercoupled non-linear oscillators. Motivation for this work has come from mathematical modelling of biomedical systems, particularly for the electrical activity of the gastro-intestinal digestive tract. Over the past few years a matrix linearisation method has been developed and applied to a wide range of structures comprising chains, arrays, rings and tubes. Also, different coupling components including pure time delay, and two types of van der Pol oscillator dynamics have been investigated. Numerous mode patterns have been predicted and experimentally observed, and stability criteria established for a wide range of conditions. The multi-oscillator mode analysis entails a two-stage process comprising matrix mode decoupling followed by equivalent linearisation similar to the Krylov-Bogolioubov approach.

1. Introduction

Extensive theoretical advances were made in the 1930's by both Western and Eastern researchers in the field of non-linear oscillators under both free and externally stimulated conditions. These studies were based largely upon the now-classic dynamics referred to as the van der Pol Equation. It is this equation which forms the basis of the mode analysis methods which will be described in this paper. In this early work the emphasis was on a single oscillator system with forcing in one direction only. In some respects, this parallels the early development of control systems theory based upon the single variable, single loop concept. Only in recent years has there been any strong attempt to extend non-linear mode analysis into the multi-oscillator condition with mutual interaction (i.e. bidirectional coupling). This also parallels the current interest in multivariable control systems theory.

Non-linear mode analysis has been developed around the concept of distributed arrays of coupled circuits. These circuits comprise a mixture of linear and non-linear components, where the basic mode structure can be determined from the linear components only. In this paper the emphasis is on coupled oscillatory circuit elements, although similar treatment can also be used for distributed filter structures. The analytical method involves two stages comprising determination of basic mode behaviour via the unperturbed system (i.e. linear components only) followed by an investigation of mode stability via a matrix linearisation method. The analysis is therefore concerned with weakly non-linear systems, and attention is not given to relaxation oscillations involving strong non-linearities.

The theoretical advances have been strongly motivated by parallel research into biomedical oscillations. In particular, the electrical activity in the gastro-intestinal digestive tract of humans and other mammals has relevance in this area. This activity, known as 'slow-waves', has been extensively modelled using non-linear oscillatory dynamics Such a mathematical model was first proposed by Nelsen and Becker in 1968[1] as an alternative to a linear cable model. Since that time the mathematical model has been extended to comprise a one-dimensional chain for small-intestinal studies[2] and a two-dimensional array for gastric modelling[3]. The classic van der Pol equation has often been used in these studies, and modifications to this equation to make the zero state stable have been used in modelling the human large-intestine (colon)[4]. For this latter work a small tubular structure was also considered. All of these structures and dynamics, together with different coupling conditions, have proven to be amenable to theoretical mode analysis surveyed in this paper.

After a brief historical survey of theoretical work in non-linear oscillator analysis, a verbal classification is given of the various types of mode behaviour which will be encountered. This is followed by a development of non-linear mode theory for a two-dimensional array. A ladder structure is then treated as a special case of this array. A tubular connection is next presented, followed by a ring as a special case of this structure. Coupling delay is then introduced in the ring connection. This is followed by a treatment of oscillator dynamics including fifth-power characteristics for a chain connection, both with and without coupling delay. An alternative approach which removes some of the restrictions which apply to the matrix mode analysis method is described in the next section on harmonic balancing, and which summarises recent advances using this technique. Finally, a brief resumé is given of experimental investigations which have verified the predictions arising from the theoretical studies. It should be noted that many other studies have been made on coupled oscillators using dynamics other than the van der Pol type.

These studies which involve both theoretical and simulation approaches are partially described in the area of biomedical modelling in[5].

2. Non-linear circuit mode analysis

2.1 Historical survey

Mention must first be made of the work by van der Pol himself whose interest in oscillations was life-long and spanned the disciplines of electronics, biology and music. In his early writings he considered mode behaviour in chains and rings of coupled circuits under both transient and resonant conditions[6,7]. In a series of papers which followed, van der Pol extended the knowledge of non-linear circuit oscillations using a number of methods. He considered free and forced oscillations in a triode, using methods based on the celestial mechanics perturbation approach and a technique based on energy averaging which was the precursor of the harmonic balancing approach[8]. He produced graphical procedures to explain hysteresis effects of switching between modes both for a single oscillator with fifth power conductance characteristic[9], and for an oscillator mutually coupled to a passive resonant circuit[10]. While these papers used methods based on almost-linear conditions, van der Pol also analysed relaxation oscillations in his own equation using a phase - plane approach for this highly nonlinear condition[11]. Subsequent work dealt with frequency 'pulling' in a forced oscillator, and the phenomenon of suppression of the free oscillation by the external oscillations(i.e. synchronisation)[12]. These theoretical circuit analysis studies were coupled with hypothesised application to heart rhythms[13] and many biological and social phenomena[14].

Although emphasis is placed in this review on the analytical advances made for weakly non-linear systems, much work has been done in both the West and the East on relaxation oscillations (eg. 15, 16 and 17). For weakly non-linear systems the method of perturbations had been introduced as early as 1772 by Euler[18] for consideration of celestial mechanics. This was generalised by Poisson in 1834[19], the problem of secular terms overcame by Lindstedt in 1882 [20]. The method was reduced to a systematic averaging procedure by Poincaré in 1886[21] and further generalised in the area of celestial mechanics by von Zeipel in 1911[22]. As stated by Giacaglia[23], "celestial mechanics had been worked down to the bones by means of available tools of classical analysis by the end of the last century".

At the same time as van der Pol was producing work on non-linear circuit analysis, similar studies were being made by Russian mathematicians. A series of averaging methods (called equivalent linearisation, harmonic balancing and energy balancing) was applied to non-linear circuits and mechanical systems by Krylov and Bogolioubov in the 1930's[24] and introduced to the West in 1947[25]. The connection between non-linear circuit theory and classical averaging methods of celestial mechanics was shown by Cesari in 1959[26], while the equivalence of the Krylov-Bogolioubov-Mitropolsky and von Zeipel's methods was demonstrated in 1961[27]. The Krylov-Bogolioubov method is a generalisation of that used by van der Pol, and is closely related to the describing function technique introduced in England by Tustin[28] and in America by Kochenberger[29].

Although the majority of literature on synchronisation has been on unidirectional forcing of single degree of freedom oscillators (e.g. Haag[30]), the first example of synchronisation observed appears to have been a mutually coupled system. It is well known that Huyghens (1629-1695) observed that if two clocks were slightly de-synchronised with one another when hung on a wall, they became synchronised when attached to a thin wooden board. The first analytical treatment of mutually coupled non-linear oscillatory circuits was due to Russian authors in the 1950's. Thus, synchronisation and power sharing were considered for mutual inductive coupling by Akalovsky[31], a bridge comprising R,L and C elements by Rapoport[32], and parallel RC coupling by Parygin[33]. The analysis was extended to the case of the action of an external force on two synchronised oscillators by Khoklov[34], and the mutual synchronisation of three RC coupled oscillators by Parygin[35]. It was not until the 1960's that equivalent literature began to appear in the West. Thus, in 1963 Nag analysed the case of two capacitvely coupled tuned anode oscillators[36], and showed the stability of two modes using Liapunov theory. Using the same equations Aggarwal and Ritchie[37] employed Liapunov theory to find the inner bound both for periodic and aperiodic oscillations.

In the Russian literature applications of the above analysis were frequently in the microwave devices field. This is also true in analytical treatment of noise performance in non-linear oscillatory systems. Kurokawa[38] considered noise in a single oscillator, while Schlosser[39] investigated noise in two mutually synchronised oscillators. The coupling was via a three-port, characterised by a scattering matrix, with an application in the combining of solid-state microwave devices to obtain larger output powers.

In the field of circuit theory, an important paper on both one and two dimensional arrays of non-linear circuits is that by Scott in 1970[40]. He considered a set of parallel capacitor and non-linear conductance circuits linked via inductive elements to form a low-pass filter structure. He analysed the lumped parameter system using a distributed approximation commonly employed in solid-state physics to determine the properties of crystal lattices. Scott showed that simultaneous existence of two modes could not occur in a line array but could occur in a two-dimensional array. In this work, and that which follows, only asynchronous modes are considered. Such modes, often termed non-resonant, have ratios which are non-rational and hence the averaging methods can be applied to their analysis. The same structure was analysed by Parmentier[41] using the method of continuum approximation[42], producing the same conclusions. Using the same method the equivalent high-pass filter structure was analysed by Aumann[43]. Similar work mostly dealing with distributed oscillatory systems was appearing in the Russian literature at this time, with the emphasis being on microwave and laser devices. Thus, analyses have appeared on the mutual locking of lasers[44], asynchronous modes in distributed oscillators[45], external forcing of a one-dimensional distributed oscillator[46], and 'ensemble' coupling of oscillators[47,48].

The results for ladder filters were extended to ladder oscillators by Endo and Mori[49] for both the low-pass case (inductive coupling) and the high-pass case (capacitive coupling). They used a matrix decoupling method which will be outlined in the next section, and which has now been applied to a number of related structures. Thus, mode analysis has been performed for non-linear oscillator structures comprising two-dimensional arrays[50], rings[51], fifth-power oscillators[52], coupling networks with pure time-delay[53,54], tubular structures with third power and fifth power dynamics[55]. Since intercoupled oscillators are the basis of gastro-intestinal modelling studies and other biomedical applications, it is this form of non-linear circuit analysis that is singled out for description in the following sections.

2.2 Non-linear mode analysis techniques

The technique pioneered by Endo and Mori for the determination of mode behaviour in coupled non-linear oscillatory circuits comprises a two stage process. The first stage entails a linear transformation applied to the multivariable interacting system to reduce it to a diagonalised uncoupled structure which is the matrix equivalent of a single oscillator van der Pol equation. Equivalent linearisation of the non-linear terms is then performed via the Krylov-Bogolioubov approach with mode stability being investigated via perturbations and resulting variational equations.

When considering a non-linear system with many degrees of freedom a number of distinctions arise in classifying the various modes. The first is the presence or absence of resonance. If the ratio of two mode frequencies (ω_i/ω_j) is a rational number, these two frequencies are called resonant; if it is an irrational number, they are called non-resonant . Another matter is that of degeneration. If two or more different modes have the same frequency, these modes are called degenerate; otherwise they are called non-degenerate. In degenerate modes there are resonant interactions between them, and their phase relationships should be considered. When there is a fixed phase relationship between degenerate modes the modes are called regular. When the phase relationship cannot be determined and is arbitrary, the modes are called irregular. All of these mode distinctions are necessary in the following analyses of different coupling structures, although it should be noted that resonant multimodes will not be considered since the normal method of averaging cannot be used in this case.

The most general structure which has been analysed comprises a tubular connection of oscillators, and this is of most relevance in the gastro-intestinal models mentioned earlier. The structure is shown in Fig. 1, where each node point contains an oscillator comprising parallel LC and conductance terms and is equivalent to a van der Pol type equation. Both third power and fifth power polynomial conductance terms will be considered in the following sections. In the initial section a two-dimensional array of oscillators will be considered since this introduces most of the mathematical and modal ideas.

2.2.1 Two-dimensional Arrays

The structure comprises Fig. 1 with a cut along the axis of the tube, and open-circuit boundary conditions along each dimension of the array. A conventional van der Pol oscillator is obtained by considering the conductance to have the form

$$I_{ij} = - g_1 V_{ij} + g_3 V_{ij}^3 \,, \; g_1, g_3 > 0 \tag{1}$$

Applying Kirchoff's laws to the ij^{th} node gives the following difference-differential equation

$$\frac{d^2 V_{ij}}{dt^2} - \frac{g_1}{c}\left(1 - \frac{3g_3}{g_1} V_{ij}^2\right) \frac{dV_{ij}}{dt} - \frac{1}{CL_o} V_{ij-1} + \left(\frac{1}{2CL} + \frac{1}{CL_o}\right) V_{ij}$$

$$- \frac{1}{CL_o} V_{i,j+1} - \frac{1}{CL_o} V_{i-1,j} + \left(\frac{1}{2CL} + \frac{2}{CL_o}\right) V_{ij} - \frac{1}{CL_o} V_{i+1,j} = 0 \tag{2}$$

$$i = 1,2,\ldots,m; \quad j = 1,2,\ldots,n$$

This equation can be put in normalised format by making the following substitutions

$$\tau = \sqrt{\left(\frac{1}{2CL} + \frac{1}{CL_o}\right)}\; t$$

$$V_{ij} = \sqrt{\left(\frac{g_1}{3g_3}\right)} x_{ij}$$

$$\varepsilon = \frac{g_1}{\sqrt{\left(\frac{C}{2L} + \frac{C}{L_o}\right)}}$$

$$\alpha = \frac{2L}{2L + L_o} \tag{3}$$

Thus (1) becomes

$$\ddot{x}_{ij} - \varepsilon(1 - x_{ij}^2)\dot{x}_{ij} - \alpha x_{i,j-1} + (1+\alpha) x_{ij} - \alpha x_{i,j+1}$$

$$- \alpha x_{i-1,j} + (1+\alpha) x_{ij} - \alpha x_{i+1,j} = 0$$

$$\dot{x} \equiv \frac{\alpha x}{\alpha \tau} \quad ; \quad \ddot{x} \equiv \frac{\alpha^2 x}{\alpha \tau^2} \tag{4}$$

Defining X and Z matrices of order mxn as

$$X \equiv [x_{ij}]$$

$$Z \equiv [x_{ij}^3]$$

(4) can be expressed as a matrix differential equation

$$\ddot{X} + B_m X + X B_n = \varepsilon \dot{X} - \frac{1}{3} \varepsilon \dot{Z} \tag{5}$$

with a symmetric tridiagonal matrix of order m(n)

$$B_{m(n)} = \begin{pmatrix} 1 & -\alpha & & & 0 \\ -\alpha & 1+\alpha & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & -\alpha & 1+\alpha & -\alpha \\ 0 & & & -\alpha & 1 \end{pmatrix} \tag{6}$$

The unperturbed form of (5)

$$\ddot{X} + B_m X + X B_n = 0$$

can be transformed to the canonical form

$$\ddot{Y} + (P^T B_m P) Y + Y(Q^T B_n Q) = 0 \tag{7}$$

by substituting

$$X = PYQ \tag{8}$$

The orthogonal transformation matrices P and Q which diagonalise matrix B to produce an uncoupled system can be determined as an eigenvalue problem of matrices[56] by solving the difference equation

$$-\alpha p_{i-1,j} + (1+\alpha) p_{ij} - \alpha p_{i+1,j} = \lambda_j p_{ij} \tag{9}$$

with boundary conditions

$$p_{oj} = p_{1j} \; ; \; p_{mj} = p_{m+1,j}$$

where p_{ij} is an element of P and λ_j is an eigenvalue of B_m, which are uniquely **determined** to be

$$\rho_{i1} = \frac{1}{\sqrt{m}}$$

$$p_{ij} = \sqrt{\frac{2}{m}} \cos \frac{(2i-1)(j-1)\pi}{2m} , \ j \neq 1$$

$$\lambda_i = 1 + \alpha - 2\alpha \cos \frac{(i-1)\pi}{m} , \ i=1,2,\ldots,m \qquad (10)$$

Similarly, the eigenvalues λ_i' and q elements can be determined for the B_n matrix, so that (7) can be rewritten as

$$\ddot{Y} + \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_m \end{pmatrix} Y + Y \begin{pmatrix} \lambda_1' & & 0 \\ & \ddots & \\ 0 & & \lambda_m' \end{pmatrix} = 0 \qquad (11)$$

whose solution gives the elements y_{ij} of Y as

$$y_{ij} = A_{ij} \sin(\omega_{ij}\tau + \phi_{ij})$$

$$\omega_{ij} = \sqrt{\lambda_i + \lambda_i'} \qquad (12)$$

The non-linear term of the fundamental equation (5) transformed into Y-space is now linearised using the Krylov-Bogolioubov technique. Ignoring resonant interaction between modes and retaining the fundamental component only in the Fourier series expansion of the non-linear term leads to

$$\ddot{y}_{ij} + \omega^2_{ij}\, y_{ij} = \varepsilon \dot{y}_{ij} - \frac{1}{3}\varepsilon \sum_{k=1}^{m} \sum_{\ell=1}^{n} S_{ij}(k,\ell)\, \dot{y}_{k\ell} \qquad (13)$$

where

$$S_{ij}(k,\ell) = \frac{3}{2} \sum_{k=1}^{m} \sum_{\ell=1}^{n} I_{mn}(i,j,k,\ell) A^2_{k\ell} - \frac{3}{4} I_{mn}(i,j,k,\ell) A^2_{ij} \qquad (14)$$

and

$$I_{mn}(i,j,k,\ell) \equiv \sum_{a=1}^{m} \sum_{b=1}^{n} p^2_{ai}\, q^2_{bj}\, p^2_{ak}\, q^2_{b\ell} \qquad (15)$$

In (13) we can ignore all the $\dot{y}_k$ terms on the right hand side except $\dot{y}_{ij}$, since the left-hand side has a resonance centred around ω_{ij}. Thus, (13) can be re-written simply as

$$\ddot{y}_{ij} + \omega_{ij}^2 \, y_{ij} = \varepsilon\dot{y}_{ij} - \frac{1}{3}\,\varepsilon\dot{u}_{ij}$$

$$u_{ij} = S_{ij} \, y_{ij} \tag{16}$$

To determine the stationary values of the amplitudes A_{ij} the unperturbed solution (12) is substituted into (16) with the assumption that A_{ij} is a slowly-varying function of time to give

$$\dot{\upsilon}_{ij} = \upsilon_{ij} \left(1 - \tfrac{1}{2} \sum_{k=1}^{m} \sum_{l=1}^{n} I_{mn}(i,j,k,l)\upsilon_{kl} + \tfrac{1}{4} \, I_{mn}(i,j,k,l)\upsilon_{ij} \right)$$

where

$$\upsilon_{ij} = A_{ij}^2 \tag{17}$$

To investigate the stability of the various modes it is necessary to classify the values of $I_{mn}(i,j,k,l)$. For the array structure it can be shown that there are five groups of points in both (i.k) space and (j,l) space giving the combinations shown in Table 1[(50)]. Using this information a table of values for all I_{mn} can be constructed and an example of this is shown in Table 2 for a 3x4 array. The stationary values can be found by putting

$$\dot{\upsilon}_{ij} = 0 \quad , \quad \text{for all } i \text{ and } j \tag{18}$$

Mode stability is determined by linearising the averaged equations around the stationary values and investigating the variational equation via the characteristic equation. If the real parts of all the roots of the characteristic equation are negative then the corresponding mode is stable. For single modes (i.e. only one mode is excited) stability of mode (i_o, j_o) is investigated by letting

$$\upsilon_{i_o j_o} \neq 0 \; , \; \upsilon_{ij} = 0 \text{ for all } i,j \text{ except } i_o, j_o \tag{19}$$

The resulting characteristic equation yields a stability criterion that

$$\frac{I_{mn}(i,j,i_o,j_o)}{I_{mn}(i_o,j_o,i_o,j_o)} > \tfrac{1}{2} \tag{20}$$

Thus the stability of a mode can be determined solely from the (i_o,j_o)th column in the I_{mn} table such as that of Table 2. In this example it can be seen that modes (1,1)(1,3) are the onlv stable single modes out of the 12 possibilities.

For non-resonant double modes, two equal amplitude modes are excited simultaneouslv, and thus

$$\upsilon_{i_oj_o} = \upsilon_{r_os_o} \neq 0, \ \upsilon_{ij} = 0 \text{ for all } ij \text{ except } i_oj_o \text{ and } r_os_o$$

where the mode amplitudes can be found to be

$$\upsilon_{i_oj_o} = \frac{4}{I_{mn}(i_o,j_o,i_o,j_o) + 2\,I_{mn}(i_o,j_o,r_o,s_o)} \tag{21}$$

Investigating stability via the variational eauation leads to the following two criteria for stability

$$\frac{I_{mn}(i_o,j_o,i_o,j_o)}{I_{mn}(i_o,j_o,i_o,j_o) + 2\,I_{mn}(i_o,j_o,r_o,s_o)} > \tfrac{1}{2}$$

$$\frac{I_{mn}(i,j,i_o,j_o) + I_{mn}(i,j,r_o,s_o)}{I_{mn}(i_o,j_o,i_o,j_o) + 2\,I_{mn}(i_o,j_o,r_o,s_o)} > \tfrac{1}{2} \tag{22}$$

These criteria can be obtained via the I_{mn} table such as Table 2. and in the 3x4 example it can be seen that the only stable double mode is the (1,2)(1,4) pair.

In the case of degenerate modes. phase must also be considered and additional terms are required in the linearised equations. Considering the two degenerate modes to be (α,β) and (γ,δ) the averaged equations reduce to

$$\mathring{\upsilon}_{\alpha\beta} = \varepsilon\upsilon_{\alpha\beta}\left(1 - \tfrac{1}{2}\sum_{k=1}^{m}\sum_{l=1}^{n} I_{mn}(\alpha,\beta,k,\ell)\upsilon_k + \right.$$

$$\tfrac{1}{4} I_{mn}(\alpha,\beta,\alpha,\beta)\upsilon_{\alpha\beta} + \tfrac{1}{4} I_{mn}(\alpha,\beta,\gamma,\delta)\upsilon_{\gamma\delta} -$$

$$\left. \tfrac{1}{2} I_{mn}(\alpha,\beta,\gamma,\delta)\upsilon_{\gamma\delta}\cos^2\phi \right) \tag{23}$$

$$\mathring{\phi} = \tfrac{1}{4}\,\varepsilon\upsilon_{\gamma\delta}\, I_{mn}(\alpha,\beta,\gamma,\delta)\cos\phi\,\sin\phi$$

plus a similar expression for $\mathring{\upsilon}_{\gamma\delta}$.

The stationary values for amplitudes and phase are determined by equating the first derivatives of (23) to zero to give

$$I_{mn}(\alpha,\beta,\alpha,\beta)\upsilon_{\alpha\beta} + I_{mn}(\alpha,\beta,\gamma,\delta)(1 + 2\cos^2\phi)\upsilon_{\gamma\delta} = 4$$

$$I_{mn}(\alpha,\beta,\gamma,\delta)(1 + 2\cos^2\phi)\upsilon_{\alpha\beta} + I_{mn}(\gamma,\delta,\gamma,\delta)\upsilon_{\gamma\delta} = 4$$

$$I_{mn}(\alpha,\beta,\gamma,\delta)\sin\phi\,\cos\phi\,\upsilon_{\gamma\delta} = 0 \tag{24}$$

Two solutions emerge from (24), one of which is given by

$$\upsilon_{\alpha\beta} = \upsilon_{\gamma\delta} = \frac{4}{I_{mn}(\alpha,\beta,\alpha,\beta) + 3I_{mn}(\alpha,\beta,\gamma,\delta)}$$

$$\phi = 0,\pi \tag{25}$$

This degenerate mode pair corresponds to a standing wave condition, and stability analysis reveals that this is an unstable condition for the two-dimensional array.

The other solution is given by

$$\upsilon_{\alpha\beta} = \upsilon_{\gamma\delta} = \frac{4}{I_{mn}(\alpha,\beta,\alpha,\beta) + I_{mn}(\alpha,\beta,\gamma,\delta)}$$

$$\phi = \pm\,\pi/2 \tag{26}$$

This mode pair corresponds to a travelling wave condition, and stability analysis yields the following criteria for the pair to be stable

$$I_{mn}(\alpha,\beta,\alpha,\beta) > I_{mn}(\alpha,\beta,\gamma,\delta)$$

$$\frac{I_{mn}(i,j,\alpha,\beta) + I_{mn}(i,j,\gamma,\delta)}{I_{mn}(\alpha,\beta,\alpha,\beta) + I_{mn}(\alpha,\beta,\gamma,\delta)} > \tfrac{1}{2} \tag{27}$$

for all i and j except $(i,j) = (\alpha,\beta),(\gamma,\delta)$. These criteria (27) can also be determined from the I_{mn} table such as Table 2 and in the 3x4 array example it can be seen that the degenerate mode pair (2,3)(3,1) is stable. Its spatial distribution can be determined using (10) and (26) to be

$$x_{ij} = \begin{bmatrix} 2(-30^\circ) & 2(-150^\circ) & 2(-150^\circ) & 2(-30^\circ) \\ 2(90^\circ) & 2(90^\circ) & 2(90^\circ) & 2(90^\circ) \\ 2(-150^\circ) & 2(-30^\circ) & 2(-30^\circ) & 2(-150^\circ) \end{bmatrix} \sin(\sqrt{(2+\alpha)}\,\tau)$$

and

$$x_{ij} = \begin{bmatrix} 2(-150^\circ) & 2(-30^\circ) & 2(-30^\circ) & 2(-150^\circ) \\ 2(90^\circ) & 2(90^\circ) & 2(90^\circ) & 2(90^\circ) \\ 2(-30^\circ) & 2(-150^\circ) & 2(-150^\circ) & 2(-30^\circ) \end{bmatrix} \cos(\sqrt{(2+\alpha)}\,\tau) \tag{28}$$

2.2.2 One-dimensional chain

A ladder structure is clearly a particular case of the two-dimensional array considered above, but has been more extensively treated by Endo and Mori[48]. Thus, they analysed both open-circuit and short-circuit end conditions, and also investigated the possibility of multimodes comprising more than two frequencies. They showed that under no condition could more than two modes be excited simultaneously. Non-resonant double modes can exist in a ladder structure with open boundary conditions, but cannot exist for a short-circuit boundary.

For a three oscillator ladder the stability table is given in Table 3. Using the particular form of the stability criterion for single modes given by (20), investigation of the columns of Table 3 shows that mode 1 is the only stable single mode. Similarly, using stability criteria given by (22) and Table 3 shows that the only stable non-resonant double mode is given by modes 2 and 3.

For a 4 oscillator ladder the stability table comprises the top left-hand corner of Table 2. From this and criterion (20) it can be seen that modes 1 and 3 are stable single modes. Similarly, from criteria (22) only one of the 6 possible non-resonant double modes is stable, being the mode pair 2 and 4. In these examples no degenerate modes can exist since the mode frequencies are all different.

2.2.3 Tubular Structure

This type of interconnection is shown in Fig. 1 and is relevant to modelling studies for both the small-intestine (particularly for the duodenum) and the large-intestine (colon). An extensive mode analysis has been performed on this structure by Alian[55].

The analysis follows the same procedure as that outlined for the two-dimensional array. The symmetrical tri-diagonal matrix for the peripheral direction around the tube changes to

$$B_m = \begin{bmatrix} 1+\alpha & -\alpha & & 0 \\ -\alpha & 1+\alpha & \ddots & \\ & \ddots & \ddots & -\alpha \\ 0 & & -\alpha & 1+\alpha \end{bmatrix} \tag{29}$$

The elements of the P matrix and the eigenvalues for the unperturbed solution are found in an analogous manner, and are given by

$$p_{i1} = \sqrt{\frac{1}{m}} \quad \text{for } i = 1,2,\ldots m$$

$$p_{i,\,\frac{m}{2}+1} = (-1)^i \sqrt{\frac{1}{m}} \quad \text{for } i = 1,2,\ldots m$$

when m is even

$$p_{ij} = \sqrt{\frac{2}{m}} \cos \frac{2\pi i(j-1)}{m} \quad \text{for } i=1,2,\ldots m \text{ and}$$

$$j=2,3,\ldots \left(\frac{m+1}{2}\right)$$

when m is odd

$$j = 2,3,\ldots \frac{m}{2} - 1, \frac{m}{2} + 1$$

when m is even

$$p_{ij} = \sqrt{\frac{2}{m}} \sin \frac{2\pi i(j-1)}{m} \quad \text{for } i=1,2,\ldots m \text{ and}$$

$$j = \left(\frac{m+1}{2}\right), \ldots, m-1, m$$

when m is odd

$$j = \left(\frac{m}{2} + 2\right), \ldots, m-1, m$$

when m is even

$$\lambda_j = 1 + \alpha - 2\alpha \frac{\cos 2\pi(j-1)}{m} \quad \text{for } j=1,2,\ldots,m \tag{30}$$

Similarly, for the axial direction of the tube the elements of the Q matrix and the eigenvalues are given by

$$q_{i1} = \frac{1}{\sqrt{n}} \quad \text{for } i = 1,2,\ldots,n$$

$$q_{ij} = \sqrt{\frac{2}{n}} \cos \frac{(2i-1)(j-1)\pi}{2n} \quad \text{for } i=1,2,\ldots, n$$

$$\text{and } j=2,3,\ldots,n$$

$$\lambda_i^* = 1 + \alpha - 2\alpha \cos \frac{(i-1)\pi}{n} \tag{31}$$

The equivalent linearistaion and stabilitv analysis follows a similar procedure as that outlined for the array, with the crucial part being the establishment of the equivalent stability table. The same criteria can then be used to determine the various modes which are stable. In the tubular case, two general stability tables are established depending on whether the number of oscillators around the periphery is odd or even.

Both of these conditions are illustrated in the following examples.

Consider firstly a 3x4 tube structure in which each ring has an odd number of oscillators. The mode frequencies are given in Table 4 from which it can be seen that single, non-resonant double, and degenerate modes are feasible. The relevant stability table for this example is provided by Table 5. Use of this table and stability criterion (20) shows that only modes (1,1) and (1,3) represent stable single modes out of the 12 possibilities. Similarly, using the table and stability criteria (22) only one non-resonant double mode is stable, being the mode pair (1,2) and (1,4). Also, regular degenerate modes transpire to be stable using Table 5 and stability criteria (27). These modes are the pairs (2,1) with (3,1) and (2,3) with (3,3). Thus a 3x4 tubular structure produces the following stable conditions

1) Two single ordinary modes

2) One non-resonant double mode

3) Two regular degenerate modes

Secondly, consider a 4x3 tubular structure in which each ring has an even number of oscillators. The resulting mode frequencies are given in Table 6 while the necessary stability table is provided by Table 7. Use of Tables 6 and 7 in conjunction with the stability criteria (20), (22) and (27) yield the following summarised stable mode behaviour:-

1) Two single ordinary modes, being mode (1,1) and mode (3,1)

2) One non-resonant double mode, being modes (1.2) and (3,3)

3) One regular degenerate double mode, being modes (2,1) and (4,1).

In all cases, the amplitude and spatial distribution of the voltages throughout the network can be determined by noting that for single modes

$$x_{k\ell} = p_{ki_o} q_{\ell j_o} A_{i_o j_o} \sin(\omega_{i_o j_o} \tau + \phi_{i_o j_o}) \;, \quad k=1,2,\ldots m \quad \ell=1,2,\ldots,n \qquad (32)$$

where the elements p and q are obtained from (30,31), ω is determined from (30,31,12) and $A_{i_o j_o}$ is given by

$$A_{i_o j_o} = \frac{2}{\sqrt{I_{mn}(i_o,j_o,i_o,j_o)}} \tag{33}$$

For non resonant double modes the spatial distribution is determined by

$$x_{k\ell} = p_{ki_o}\, q_{\ell j_o}\, A_{i_o j_o} \sin(\omega_{i_o j_o}\tau + \phi_{i_o j_o}) +$$

$$p_{kr_o}\, q_{\ell s_o}\, A_{r_o s_o} \sin(\omega_{r_o s_o} + \phi_{r_o s_o}) \tag{34}$$

where the amplitudes $A_{i_o j_o}$ and $A_{r_o s_o}$ can be found from (21) and the remaining quantities as before. Likewise, the degenerate modes spatial distribution is found from

$$x_{k\ell} = p_{k\alpha}\, q_{\ell\beta} A_{\alpha\beta} \sin(\omega\tau + \phi_{\alpha\beta}) + p_{k\gamma}\, q_{\ell\delta} A_{\gamma\delta} \sin(\omega\tau + \phi_{\gamma\delta}) \tag{35}$$

Again, these quantities can be found from the given equations plus (26) to determine the amplitudes $A_{\alpha\beta}$ and $A_{\gamma\delta}$.

2.2.4 Ring Structure

In an analogous manner to the ladder structure, a ring formulation can be considered as a special case of the tubular structure outlined above. Further analysis has, however, been carried out on this structure and will now be summarised. The unperturbed solution is readily shown to be

$$y_k = A_k \sin(\omega_k\tau + \phi_k)$$

$$w_k = \sqrt{(1+\alpha-\frac{2\alpha\cos 2\pi(k-1)}{n}} \tag{36}$$

From (36) it appears that the mode frequencies can be classified as follows

$$
\begin{array}{ll}
\text{class 1} & \omega_1 \ldots\ldots\ldots\ldots \text{a non-degenerate mode} \\
\text{class 2} & \left.\begin{array}{l} \omega_2 = \omega_m \\ \omega_3 = \omega_{m-1} \\ \vdots \\ \omega_{\left[(m+1)/2\right]} = \omega_{\left[(m/2)+2\right]} \end{array}\right\} \begin{array}{l}\text{degenerate} \\ \text{modes}\end{array} \\
& \omega_{(m/2)+1} \ldots\ldots\ldots \text{a non-degenerate mode only} \\
& \qquad\qquad\qquad\quad \text{for n even}
\end{array}
\qquad (37)
$$

where [ ] indicates the highest integer number.

Classifying these modes as 1 and 2 it has been shown that no non-resonant double modes comprising mode combinations 1 and 2, or 2 and 2 can exist as stable conditions. Also, it has been shown that no triple non-resonant modes can be stable. Because of the mode pair degeneration shown by (37), only one single mode occurs for an odd number of oscillators in the ring, and this is always stable. For an even number of oscillators the additional single mode can also be demonstrated to be stable. Stability of the degenerate modes has also been determined with the additional situation that an irregular degenerate condition arises when the number of oscillators is a multiple of 4. Such degeneracy is given by mode pairs $\omega_{(m/4)+1} = \omega_{(3m/4)+1}$. The phase between two components of this irregular degenerate mode cannot be determined, and hence no phase synchronisation can be observed.

These concepts can be clarified by considering two examples. For a ring of 4 oscillators the stable modes are illustrated in the vector diagram of Fig. 2. Two single nondegenerate modes occur since n is even, and one irregular degenerate mode pair exists since n is a multiple of 4. For this latter mode, although adjacent oscillators have an arbitrary phase relationship, opposite oscillators are locked in anti-phase (i.e. 1 with 3.

and 2 with 4). For a ring of 5 oscillators the equivalent vector diagrams are shown in Fig. 3. Since n is odd, only one single mode exists, while two regular degenerate mode pairs occur. In a ring connection the phase between adjacent oscillators under degeneracy is given by $\pm 360^{\circ} t/n$, where t is an integer, and n is the number of oscillators in the ring.

2.2.5 Coupling including Time Delays

In the treatment so far it has been assumed that coupling between oscillators was via an inductive component, i.e. low-pass conditions. Dual results can be obtained for the high-pass case, i.e. capacitive coupling under the assumed conditions of small non-linearity. The matrix Krylov-Bogolioubov mode analysis can be extended to the case of time delay (Δ) inserted in all the coupling paths[53]. In the case of a ring connection with coupling delay the unperturbed solution is identical to that given in the previous section. The stability of the various modes is, however, affected by the presence of the delay. The necessary stability table can be obtained as a special case from the tubular structure given by either Table 4 or 5 for rings of 3 or 4 oscillators. For non degenerate single modes, stability is found by investigation of the elements

$$b_i = \varepsilon(1 - \tfrac{1}{2}g_{i1}\ \upsilon_1) - \frac{\lambda_i}{\omega_i}\sin(\omega_i\Delta) \text{ for } i=1,2,\ldots m \qquad (38)$$

where $$\upsilon_1 = \left(\frac{4}{g_{11}}\right)\left\{1 - \frac{\lambda_1}{\omega_1\varepsilon}\sin(\omega_1\Delta)\right\} \qquad (39)$$

and g_{i1} are corresponding entries in the stability table.

If all b_i are negative then the mode is stable. The stability of the other single mode $\omega_{(N/2)+1}$ can be determined in a similar fashion. The frequency and spatial dependence for mode ω_1 is given by

$$x_k = 2\surd\left\{1 - \left(\frac{\lambda_1}{\varepsilon\omega_1}\right)\sin(\omega_1\Delta)\ \sin(\omega_1\tau)\right\} \text{ for } k = 1,2,\ldots m \qquad (40)$$

Also, the other mode $\omega_{(N/2)+1}$ (for even N) spatial dependence is given by

$$x_k = 2(-1)^k \surd\{1 - (\frac{\lambda_{(N/2)+1}}{\varepsilon\omega_{(N/2)+1}}) \sin(\omega_{(N/2)+1}\Delta)\sin(\omega_{(N/2)+1}\tau)$$

$$\text{for } k = 1,2,\ldots m \qquad (41)$$

The stability of regular degenerate modes is determined by the signs of b_k and b'_{m-i+1} being negative, where

$$b_k = \varepsilon(1 - \tfrac{1}{2}g_{k(i+1)}\ \upsilon_{i+1} - \tfrac{1}{2}g_{k(m-i+1)}\ \upsilon_{m-i+1}) - \frac{\lambda_k}{\omega_k}\sin(\omega_k\Delta)$$

$$b'_{m-i+1} = \varepsilon(1 - \tfrac{1}{2}g_{(m-i+1)(m-i+1)}\ \upsilon_{m-i+1}) - \frac{\lambda_{m-i+1}}{\omega_{m-i+1}} \sin(\omega_{m-i+1}\Delta) \qquad (42)$$

The spatial distribution of the regular degenerate modes is given by

$$x_k = 2\surd\{1 - \frac{\lambda_{i+1}}{\varepsilon\omega_{i+1}} \sin(\omega_{i+1}\Delta)\}\sin(\omega_{Di}\tau \pm \phi_{ki})$$

$$\phi_{ki} = \frac{2\pi k_i}{m}, \qquad k=1,2,\ldots m \text{ and}$$

$$i=1,2,\ldots \left[\frac{(m-1)}{2}\right]; i \neq N/4 \qquad (43)$$

The irregular degenerate modes occur when m is a multipe of 4 and their stability is determined by the signs of b''_k being negative, where

$$b''_k = -\varepsilon - \frac{\lambda_k}{\omega_k}\sin(\omega_k\Delta) \text{ for } k \neq (m/4)+1,$$

$$(3m/4+1) \qquad (44)$$

The corresponding spatial mode is given by

$$x_k = 2\cos\frac{\pi k}{2}\sin(\omega_{D(m/4)}\tau + \phi) +$$

$$2\sin\frac{\pi k}{2}\sin(\omega_{D(m/4)}\tau) \text{ for } k = 1,2,\ldots m \qquad (45)$$

$$\phi \text{ undetermined}$$

Examples of the variation in amplitude, frequency and stability of these various modes with increasing time delay are given for a 3 oscillator ring in Fig. 4 and a 4 oscillator ring in Fig. 5.

2.2.6 Oscillators with fifth power characteristic

Instead of the conventional third power conductance in the non-linear van der Pol oscillator, the negative conductance can be extended to be

$$I(V) = g_1 V - g_3 V^3 + g_5 V^5 \quad , \; g_1, g_3, g_5 > 0 \tag{46}$$

with this characteristic it has long been known that zero becomes a stable state, and that one other limit cycle condition is also stable. The zero stable state has been of considerable interest in modelling the electrical activity of the large intestine (colon) where periods of rhythmic electrical oscillations have appeared to be interspersed with sections of electrical 'silence'(57). Coupled oscillators of this type have been hypothesised as a model to explain the phenomena of dual rhythms and electrical silence in the human large-intestine (4). Two coupled fifth power van der Pol oscillators were analysed using the matrix Krylov-Bogolioubov method by Datardina and Linkens(52) using a differential equation in the form

$$\ddot{x} + \varepsilon(b - cx^2 + dx^4)\dot{x} + \omega^2 x = 0 \tag{47}$$

They showed that the zero state is stable for all conditions, and that single modes are stable provided that

$$c^2 - 8bd > 0 \tag{48}$$

Unlike the third power case for which a two oscillator case cannot produce a stable non-resonant double mode, it was also found that a stable double mode can exist provided that the following inequality is satisfied

$$|53.33bd - 6c^2| < c\sqrt{36c^2 - 320bd} < |160bd - 6c^2| \tag{49}$$

The analysis has been extended to consider a ladder connection of such oscillators(58) including also a parallel loss conductance across the coupling inductors. In this case the basic matrix equations become

$$\begin{aligned} &\ddot{X} + B_n X = -\varepsilon C_n \dot{X} + \frac{1}{3}\varepsilon\beta \dot{X}_c - \frac{1}{5}\varepsilon \dot{X}_f \\ &X = [x_1, x_2, \ldots, x_n]^T \\ &X_c = [x_1^3, x_2^3, \ldots, x_n^3]^T \\ &X_f = [x_1^5, x_2^5, \ldots, x_n^5]^T \end{aligned} \tag{50}$$

In (50) parameter $\varepsilon(=g_1 \sqrt{\{LL_o/C(L+L_o)\}} << 1)$ indicates the degree of non-linearity, and parameter $\beta(=3g_3/\sqrt{(5g_1g_5)}$ determines both amplitude and threshold conditions for the modes. The matrices

B_n and C_n are given by

$$B_n = \begin{bmatrix} 1 & -\alpha_L & & & 0 \\ -\alpha_L & 1+\alpha_L & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 1+\alpha_L & -\alpha_L \\ 0 & & & -\alpha_L & 1 \end{bmatrix} \quad C_n = \begin{bmatrix} 1+\alpha_G & -\alpha_G & & & 0 \\ -\alpha_G & 1+2\alpha_G & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 1+2\alpha_G & -\alpha_G \\ 0 & & & -\alpha_G & 1+\alpha_G \end{bmatrix}$$

The coupling parameters $\alpha_L(0 < \alpha_L < 1)$ and $\alpha_G(0 < \varepsilon\alpha_G << \alpha_L)$ are given as follows

$$\alpha_L = L/(L+L_o) \qquad \varepsilon\alpha_G = G\sqrt{\{LL_o/C(L+L_o)\}} \tag{52}$$

Using the linear transformation method the system can be decoupled into canonical form. The eigenvalues (λ_K) and eigenvectors (p_{ij}) for B_n are identical to those given for the ladder by equations (10). Similarly, matrix C_n can also be diagonalised by this transformation yielding eigenvalues given by

$$\gamma_k = 1+2(1 - \cos\frac{(k-1)\pi}{n}\ \alpha_G\ , \ k=1,2,\ldots,n \tag{53}$$

The resulting canonical matrix form can be written as

$$\ddot{Y} + \Lambda_n Y = -\varepsilon\Gamma_n\dot{Y} + \frac{1}{3}\varepsilon\beta\dot{U} - \frac{1}{5}\varepsilon\dot{V}$$

Λ_n: diagonal matrix with elements λ_k

Γ_n: diagonal matrix with elelemtns γ_k

$$U^n = [u_1,u_2,\ldots,u_n]^T = P^T X_c$$

$$V = [v_1,v_2,\ldots,v_n]^T = P^T X_f \tag{54}$$

The unperturbed solution is obtained as previously, while equivalent linearisation of the non-linear terms υ and V in (54) can be performed by ignoring harmonics which appear in the Fourier expansion of x_k^3 and x_k^5. The linearised elements of υ and V can be written as

$$u_i = \xi_i y_i \quad ; \xi_i = \frac{3}{4} g_{ii} \upsilon_i + \frac{3}{2} \sum_{t=1, t\neq i}^{n} g_{it} \upsilon_t ; \; i=1,2,\ldots,n$$

$$v_i = \zeta_i y_i \quad ; \zeta_i = \frac{5}{8} h_{ii} \upsilon_i^2 + \frac{15}{8} \sum_{t=1, t\neq i}^{n} h_{it} \upsilon_t^2 +$$

$$\frac{15}{4} \sum_{t=1, t\neq i}^{n} h_{ti} \upsilon_t \upsilon_i + \frac{15}{4} \sum_{\substack{s=1 \\ s\neq i, t\neq i, s\neq t}}^{n} \sum_{t=1}^{n} h'_{ist} \upsilon_s \upsilon_t \quad ; \; i=1,2,\ldots,n \qquad (55)$$

The linearised equation is thus given from (54) and (55) by

$$\ddot{y}_i + \lambda_i y_i = \varepsilon(- \gamma_i + \frac{1}{3} \beta\xi_i - \frac{1}{5} \zeta_i y_i) \qquad (56)$$

The averaged equation obtained by assuming the amplitude $A_k = \sqrt{\upsilon_k}$ and phase ϕ_k to be slowly varying functions of time are given by

$$\dot{\upsilon}_i = -\varepsilon \upsilon_i (\gamma_i - \frac{1}{3} \beta\xi_i + \frac{1}{5} \zeta_i)$$

$$\dot{\phi} = 0 \qquad , \quad i = 1,2,\ldots n \qquad (57)$$

In (55) the g,h and h' terms are defined by

$$g_{ij} = \sum_{k=1}^{n} P_{ki}^2 P_{kj}^2$$

$$h_{ij} = \sum_{k=1}^{n} P_{ki}^2 P_{kj}^4 \; ; \; h'_{ijk} = \sum_{s=1}^{n} P_{si}^2 P_{sj}^2 P_{sk}^2 \qquad (58)$$

These values can be calculated from (10) as before and are used in the stability determination. The values for n = 2,3,4 are given in Table 8.

The zero state is stable for all conditions of parameters. The stationary amplitudes of single modes are obtained by equating $\dot{\nu} = 0$ in (57) to give

$$\upsilon_p = (g_{pp}/h_{pp})(\beta + \sqrt{\{\beta^2 - 8\gamma_p h_{pp}/g_{pp}^2\}}) \qquad (59)$$

Stability analysis via the characteristic equation leads to the following criterion for stability of single ordinary modes

$$S_i = Q_i + R_i\,\beta^2 + R_i\beta\sqrt{(\beta^2 - S)} > 0\ ,\ i = 1,2,\ldots,n \tag{60}$$

where

$$Q_i = 8(\gamma_i - 3h_{ip}\gamma_p/h_{pp})$$
$$R_i = (2g_{pp}/h_{pp})(3h_{ip}g_{pp}/h_{pp} - 2g_{ip})$$
$$S = 8\gamma_p h_{pp}/g_{pp}^{\,2}$$

General conditions are more difficult to obtain for non-resonant double modes, but investigation of the linearised characteristic equation results in the following criteria for double-mode stability

$$I_{pp} + I_{qq} < 0\ ,\ I_{pp}I_{qq} - I_{pg}I_{qp} > 0$$

$$I_{ii} < 0\quad ,\ i = 1,2,..,n;\ i \neq p,q \tag{61}$$

where

$$I_{kk} = -\,\varepsilon(\gamma_k - \frac{1}{2}\,\beta g_{kp}\,\upsilon_p - \frac{1}{2}\,\beta g_{kq}\,\upsilon_q +$$

$$\frac{3}{8}h_{kp}\,\upsilon_p^{\,2} + \frac{3}{8}\,h_{kq}\,\upsilon_q^{\,2} + \frac{3}{2}h'_{kpq}\,\upsilon_p\,\upsilon_q),\ k = 1,2,\ldots,n$$

$$I_{pq} = -\,\varepsilon(-\,\frac{1}{2}\,\beta g_{pq}\,\upsilon_p + \frac{3}{4}h_{pq}\,\upsilon_p\,\upsilon_q + \frac{3}{4}h_{qp}\,\upsilon_p^{\,2})$$

Stationary amplitudes are calculated as usual by setting $\dot{\upsilon} = 0$ in (57), but in this case no general analytical expression is obtained, and numerical methods must be used to find the υ values. These values can then be substituted in (61) to determine mode stability.

For n = 2 one double mode occurs, whose stability depends on the values of β and α_G. This is shown in Fig. 6 where the shaded region indicates stability. For n = 3, there are no stable double modes, whereas for n=4 two of these modes are stable, being modes 1 and 3 or modes 2 and 4. The stability zones for these modes for varying β and α_G are portrayed in

Figures 7 and 8, which show that the modes 2 and 4 do not have an upper bound on β for stability.

Investigation of single mode stability for n = 2,3,4 yields the following summarised results

n = 2 a) mode 1 : stable region $\beta^2 > 8$

b) mode 2 : stable region $\beta^2 > 8(1 + 2\alpha_G)$

n = 3 a) mode 1 : stable region $\beta^2 > 8$

b) mode 2 : stable region $\beta^2 > 8(1 + \alpha_G)$

c) mode 3 : unstable for all β

n = 4 a) mode 1 : stable region $\beta^2 > 8$

b) mode 2 : stable region

$$(80/9)(1+(2 - \sqrt{2})\alpha_G) < \beta^2 < 16(1 + 2(2\sqrt{2}+1)\alpha_G)^2/(1 + (5\sqrt{2}+2)\alpha_G)$$

c) mode 3 : stable region $\beta^2 > 8(1 + 2\alpha_G)$

d) mode 4 : stable region

$$(80/9(1+(2 + \sqrt{2})\alpha_G) < \beta^2 < 16(1 - 2(2\sqrt{2}-1)\alpha_G)^2/(1-(5\sqrt{2}-2)\alpha_G)$$

2.2.7 Fifth power oscillator chain with coupling delays

Mode analysis has been performed on a structure which combines the coupling delays of section (2.2.5) with the oscillator dynamics of the previous section (59).

The two oscillator case has been fully analysed and the following results obtained.

The zero state is stable if the following inequalities are simultaneously satisfied

$$-\varepsilon - \frac{\alpha}{\omega_1}\sin(\omega_1\Delta) < 0$$

$$-\varepsilon + \frac{\alpha}{\omega_2}\sin(\omega_2\Delta) < 0 \qquad (62)$$

where the parameters refer to a governing differential equation structure given by

$$\ddot{x} + \varepsilon(1 - \beta x_1^2 + x_1^4)\dot{x}_1 + x_1 - \alpha x_{2\Delta} = 0 \tag{63}$$

For typical values of ε and α involving weak non-linearity and weak coupling, stability criteria (62) show that the zero state has alternate stable and unstable zones as the delay Δ is increased. Thus, the structure switches between 'hard' and 'soft' oscillator characteristics as the coupling delay varies.

There are two single modes representing in-phase and anti-phase synchronisation with stationary amplitudes A_1 and A_2. The in-phase mode amplitude A, is given by

$$-\varepsilon(1 - \frac{\beta}{8} A_1^2 + \frac{1}{32} A_1^4) - \frac{\alpha}{\omega_1} \sin(\omega_1\Delta) = 0 \tag{64}$$

This mode is stable under the inequalities

$$-\varepsilon(- \frac{\beta}{8} + \frac{A_1^2}{16}) < 0$$

$$-\varepsilon(1 - \frac{\beta}{4} A_1^2 + \frac{3}{32} A_1^4) + \frac{\alpha}{\omega_2} \sin(\omega_2\Delta) < 0 \tag{65}$$

The anti-phase mode amplitude is given by

$$-\varepsilon(1 - \frac{\beta}{8} A_2^2 + \frac{1}{32} A_2^4) + \frac{\alpha}{\omega_2} \sin(\omega_2\Delta) = 0 \tag{66}$$

Similarly, this mode is stable provided

$$-\varepsilon(1 - \frac{\beta}{4} A_2^2 + \frac{3}{32} A_2^4) - \frac{\alpha}{\omega_1} \sin(\omega_1\Delta) < 0$$

$$-\varepsilon(- \frac{\beta}{8} + \frac{1}{16} A_2^2) < 0 \tag{67}$$

One non-resonant double mode is possible for this structure, with stationary amplitude A_1 and A_2 determined from the relationships:-

$$-\varepsilon(1 - \frac{\beta}{8}(A_1^2+2A_2^2) + \frac{1}{32}(A_1^4+3A_2^4+6A_1^2A_2^2)) - \frac{\alpha}{\omega_1}\sin(\omega_1\Delta) = 0$$

$$-\varepsilon(1 - \frac{\beta}{8}(2A_1^2+A_2^2) + \frac{1}{32}(3A_1^4+A_2^4+6A_1^2A_2^2)) + \frac{\alpha}{\omega_2}\sin(\omega_2\Delta) = 0 \tag{68}$$

Investigation of the characteristic equation for this double mode leads to the following stability criteria

$$J_{11} + J_{22} < 0$$
$$J_{11}J_{22} - J_{12}J_{21} > 0 \qquad (69)$$

where the Jacobian elements are given by

$$J_{11} = -\varepsilon(1 - \frac{\beta}{4}(A_1^2+A_2^2) + \frac{3}{32}(A_1^4+A_2^4+4\ A_1^2A_2^2)) - \frac{\alpha}{\omega_1}\sin(\omega_1\Delta)$$

$$J_{12} = -\varepsilon(-\frac{\beta}{4}A_1^2 + \frac{3}{16}(A_1^4 + A_1^2A_2^2))$$

$$J_{21} = -\varepsilon(-\frac{\beta}{4}A_2^2 + \frac{3}{16}(A_1^2A_2^2 + A_2^4))$$

$$J_{22} = -\varepsilon(1 - \frac{\beta}{4}(A_1^2 + A_2^2) + \frac{3}{32}(A_1^4 + A_2^4 + 4A_1^2A_2^2)) + \frac{\alpha}{\omega_2}\sin(\omega_2\Delta) \qquad (70)$$

For typical small values of ε and α it can be seen that the double mode remains stable only for low values of coupling delay Δ. This is consistent with the concept of switching from a 'hard' to a 'soft' oscillator characteristic with increasing delay, since two coupled third power oscillators cannot support a stable non-resonant double mode.

2.2.8 Fifth Power Tubular Structure

The unperturbed solution for this case yields the same eigenvalues and eigenvectors as for the third power tubular structures and these are given by (30) and (31). The equivalent linearisation is lengthy in this case and is given in detail by Alian[55]. The averaged equation is given by

$$\dot{\upsilon}_{ij} = -\varepsilon\upsilon_{ij}(1 - \frac{1}{3}\beta\eta_{ij} + \frac{1}{5}\zeta_{ij}) \qquad (71)$$

where η_{ij} and ζ_{ij} are given by the following relationships

$$\eta_{ij} = \frac{3}{2}\sum_{k=1}^{m}\sum_{\ell=1}^{n} I_{mn}(i,j,k,\ell)A_{k\ell}^2 - \frac{3}{4}I_{mn}(i,j,i,j)A_{ij}^2 \qquad (72)$$

$$\zeta_{ij} = \frac{5}{8} I^*_{mn}(i,j,i,j)\upsilon^2_{ij} + \frac{15}{8} \sum_{k=1}^{m} \sum_{\ell=1}^{n} I^*_{mn}(i,j,k,\ell)\upsilon^2_{k\ell}$$

$$+ \frac{15}{4} \sum_{k=1}^{m} \sum_{\ell=1}^{n} I^*_{mn}(k,\ell,i,j)\, \upsilon_{ij}\upsilon_{k\ell}$$

$$+ \frac{15}{4} \sum_{k=1}^{m} \sum_{\ell=1}^{n} \sum_{r=1}^{m} \sum_{s=1}^{n} I^{**}_{mn}(i,j,k,\ell,r,s)\upsilon_{k\ell}\,\upsilon_{rs} \qquad (73)$$

In (72) the values of I_{mn} are identical to those for the third power tube, as illustrated for the 3x4 structure in Table 5. The value of I^*_{mn} is given by

$$I^*_{mn}(i,j,k,\ell) = \sum_{a=1}^{m} (p^2_{ai})(p^4_{ak}) \sum_{b=1}^{n} (q^2_{bj})(q^4_{b\ell})$$

$$= I^*_m(i,k)\,I^*_n(j,\ell) \qquad (74)$$

Values of I^*_m and I^*_n can be determined from the element of the P and Q matrices given in (30) and (31). Similarly, I^{**}_{mn} can be written as

$$I^{**}_{mn}(i,j,k,\ell,r,s) = I^{**}_m(i,k,r)\; I^{**}_n(j,\ell,s) \qquad (75)$$

where

$$I^{**}_m(i,k,r) = \sum_{a=1}^{m} p^2_{ai}p^2_{ak}p^2_{ar}$$

$$+ \quad I^{**}_n(j,\ell,s) = \sum_{b=1}^{n} q^2_{bj}q^2_{b\ell}\,q^2_{bs}$$

which can also be determined from the P and Q matrix elements. For a 3x4 example values for $I^*_3(i,k)$ and $I^*_4(j,\ell)$ are given in Table 9. Use of these tables results in the complete I^*_{mn} Table 10 which is used in determining the mode stability as well as the stationary amplitudes in (71).

Single mode stability has been investigated using the variational approach and produces the following stability criteria

$$F_{ij}(i_o,j_o) + H_{ij}(i_o,j_o)\beta^2 + H_{ij}(i_o,j_o)\beta\sqrt{\beta^2 - R_{i_oj_o}} > 0$$

$$\text{for } i = 1,2,\ldots,m_j \quad j=1,2,\ldots,n \qquad (76)$$

where the F,H and R terms are given by

$$F_{ij}(i_o,j_o) = 1 - 3\,\frac{I_{mn}(i,j,i_oj_o)}{I_{mn}(i_o,j_o,i_o,j_o)}$$

$$R_{i_oj_o} = \frac{8\,I^{*}_{mn}(i_o,j_o,i_o,j_o)}{I^{2}_{mn}(i_oj_o,i_o,j_o)}$$

$$H_{ij}(i_o,j_o) = \tfrac{1}{4}\,\frac{I_{mn}(i_o,j_o,i_o,j_o)}{I^{*}_{mn}(i_o,j_o,i_o,j_o)}\,\{3\,I^{*}_{mn}(i,j,i_o,j_o)$$

$$\frac{I_{mn}(i_o,j_o,i_o,j_o)}{I^{*}_{mn}(i_o,j_o,i_o,j_o)} - 2\,I_{mn}(i,j,i_o,j_o)\} \qquad (77)$$

Stability is thus determined by use of the two stability tables for I_{mn} and I^{*}_{mn} to find the F,H and R quantities in criterion (76). For a 3x4 tubular example the following single mode stability conditions can be determined

modes (1,1)(1,3)(3,1)(3,3) stable for $\beta^2 > 8$

modes (1,2)(1,4)(3,2)(3,4) stable for $8.8889 < \beta^2 < 16$.

The remaining modes are unstable, and thus this particular case supports 4 stable modes in the parameter range $8 < \beta^2 < 8.8889$, and 8 stable modes in the range $8.8889 < \beta^2 < 16$. The corresponding stationary amplitudes are given by

$$U_{11} = U_{13} = 12(\beta + \sqrt{\beta^2 - 8})$$

$$U_{12} = U_{14} = 7.2(\beta + \sqrt{\beta^2 - 8.8889})$$

$$U_{31} = U_{33} = 8(\beta + \sqrt{\beta^2 - 8})$$

$$U_{32} = U_{34} = 4.8(\beta + \sqrt{\beta^2 - 8.8889}) \qquad (78)$$

The manipulation of the characteristic equation for determination of non-resonant double mode stable is also lengthy and is given by Alian[55]. The necessary stability criteria are summarised as follows

$$J_{i_oj_o}(i_o,j_o) + J_{r_os_o}(r_o,s_o) < 0$$

$$J_{i_oj_o}(i_o,j_o)J_{r_os_o}(r_o,s_o) - J_{i_oj_o}(r_o,s_o)J_{r_o,s_o}(i_o,j_o) > 0$$

$$J_{ij}(i,j) < 0 \qquad (79)$$

where the various Jacobian elements are given by

$$J_{ij}(i,j) = -\,\varepsilon\{1 - \tfrac{1}{2}\beta I_{mn}(i,j,i_o,j_o)\upsilon_{i_oj_o} -$$

$$\tfrac{1}{2}\beta I_{mn}(i,j,r_o,s_o)\upsilon_{r_os_o} + \frac{3}{8}\,I^{*}_{mn}(i,j,i_o,j_o)\upsilon^2_{i_oj_o} +$$

$$\frac{3}{8}\,I^{*}_{mn}(i,j,r_o,s_o)\upsilon^2_{r_os_o} + \frac{3}{2}\,I^{**}_{mn}(i,j,i_o,j_o,r_o,s_o)\,\upsilon_{i_oj_o}\,\upsilon_{r_os_o}\} \qquad (80)$$

$$J_{i_oj_o}(r_o,s_o) = -\,\varepsilon\{-\,\tfrac{1}{2}\beta I_{mn}(i_o,j_o,r_o,\ {}_o)\upsilon_{i_oj_o} +$$

$$\tfrac{3}{4}\,I^{*}_{mn}\,(i_o,j_o,r_o,s_o)\,\upsilon_{i_oj_o}\,\upsilon_{r_os_o} + \tfrac{3}{4}\,I^{*}_{mn}(r_o,s_o,i_o,j_o)\upsilon^2_{i_oj_o}\} \qquad (81)$$

and $J_{r_os_o}(i_o,j_o)$ is obtained via interchange of suffixes in (81). As for the case of a fifth power chain there is no simple analytical expression for those stationary amplitudes, and hence numerical methods must be used to determine the stationary amplitude values before the stability criteria (79) can be used to investigate the non-resonant double modes.

3. Coupled Oscillator Analysis via Harmonic Balancing

In parallel with the matrix linearisation mode analysis pioneered by Endo and Mori and applied to many situations as summarised in the previous sections, a number of further concepts have been elucidated using classical harmonic balancing techniques . In many ways these studies have been complementary to the Endo and Mori approach since they are capable of investigating situations which violate the assumptions of the matrix linearisation method. Specifically, these assumptions include very small non-linearity, equal frequencies for all uncoupled oscillators, and complete synchronisation. It will now be shown that these restrictions can be relaxed using harmonic balancing.

Basically, harmonic balancing comprises the assumption of oscillatory solutions containing one or more spectral components which may or may not be harmonically related. The assumed solutions are substituted into the governing equations for the oscillators, and time is eliminated by equating terms of the same frequency. In this way, a number of non-linear algebraic equations are obtained whose solution gives all the required information about frequencies, amplitudes and phases. In all but the simplest cases these non-linear algebraic equations must be solved numerically using standard hill-climbing or equivalent methods.

An early paper by Linkens[60] using this technique was motivated by the requirement in gastro-intestinal modelling to have a chain (i.e. ladder) of mutually coupled non-linear oscillators. In particular, it is known that in the duodenum section of the human (and other mammalian (small-intestine synchronisation occurs. However, it is also known that the intrinsic (i.e. uncoupled) frequencies are not equal, but have a decreasing gradient. To analyse such a model using the matrix linearisation method is not possible. It was shown that harmonic balancing could deal with such a situation provided that care was taken in setting up the initial conditions for a stable mode under equal frequency conditions. Subsequently, parameter scans can be made varying the frequency gradient or the degree of non-linearity using the results of a previous run as the initial conditions for the next hill-climb. Using this approach, convergence to a correct minimum is usually obtained easily. With frequency mismatch between oscillators a non-zero phase shift occurs during synchronisation. This locked phase shift increases with either increasing frequency mismatch or decreasing coupling strength. In the limit it approaches the theoretical maximum of 90° when synchronisation is lost altogether, although for larger degrees of non-linearity maximum locked phase shift between oscillators is about 30°.

The majority of matrix linearisation mode analyses have involved a low-pass structure (i.e. inductive coupling alone). In physiological modelling, the equivalent circuits are usually based on coupling structures involving ionic conductances and membrance capacitances. The harmonic balance method has been applied to two coupled third power oscillators with a parallel RLC coupling network[61]. For purely resistive coupling only one mode is stable, whereas for inductive or capacitive coupling, both in-phase and anti-phase modes are stable. In the case of equal uncoupled frequencies and symmetrical coupling (assumed throughout the matrix linearisation mode analysis) the algebraic equations have an analytical solution; and stability can be determined via a perturbational approach. Using this approach a stability space was determined for the anti-phase mode for general RLC coupling. This is shown in Fig. 9 where the zone between the two contours represents instability.

A similar analysis was performed for the case of general RLC coupling together with a pure time delay. The physiological motivation for this is that interactions between biological oscillatory subsystems such as the respiratory cycle and the cardio-vascular blood pressure reflex are mediated via neural pathways which involve significant time delay. Also, the form of coupling may well include output and rate of change of output terms, which is equivalent to RC coupling in equivalent circuit notation. Again, in the case of two such coupled oscillators, mode stability can be determined via a perturbational approach. Since coupling delay affects the stability of both in-phase and anti-phase modes, two stability spaces were determined with axes comprising output, output rate and coupling delay parameters. These are shown in Fig. 10 (in-phase mode) and Fig. 11 (anti-phase mode). A series of rotating planes is obtained as delay is increased giving alternative mode stability, but with the areas decreasing for very large delays. This indicates decreasing likelihood of mode stability as delay increases.

It was observed that the synchronised frequency varied cyclically with increasing coupling delay with a sinusoidal type shape. As the degree of non-linearity increases (i.e. ε larger), however, the sinusoidal shape becomes increasingly skewed giving wider zones of delay for which both modes are stable for fixed RC coupling. To analyse this further requires the incorporation of odd harmonic spectral components in the assumed solution. This is easily feasible with the harmonic balancing approach, and has yielded at interesting 'jump' phenomena[62]. Variations in entrained frequency, amplitudes and phase shift with increasing delay are shown in Fig. 12 for $\varepsilon = 1$. A hysteresis characteristic occurs for delays near to 11.5 or 18.5. This causes a classic jump phenomenon whereby two possible anti-phase modes can exist at a given set of parameters. The particular mode excited will depend on the past time hisotry of the oscillators. Although not shown on this figure, the in-phase mode is also stable at these values of coupling delays. Thus, in the hysteresis region there are three possible modes which are stable. The jump phenomenon has been verified via digital simulation, together with the existence of three stable modes at one value of coupling delay. These results were obtained using only a third harmonic extension to the assumed solution, but the simulations verified jumps at the narrow hysteresis region at a delay of 11.5.

Since the matrix linearisation method assumes only a fundamental component in the oscillator outputs, symmetrical waveforms are considered. In biomedical oscillations it is commonplace to find waveforms which are asymmetrical. Again, harmonic balancing can easily be extended to include odd harmonics in the assumed oscillator solution to allow for asymmetry in the time waveforms[63]. A dc component was also allowed for in the oscillator output, and results obtained showing the effect on harmonic content and synchronised frequency as the degree of asymmetry increased. Digital simulation studies also showed that increasing asymmetry decreased the likelihood of obtaining two stable modes.

In gastro-intestinal rhythms and other biomedical systems conditions of complete synchronisation do not always exist. Thus, in the human small-intestine, only the top end of the duodenum is completely entrained. In the lower regions, conditions of almost-entrainment occur causing fluctuations in amplitude (and/or frequency) referred to in the medical literature as 'waxing and waning'. Even in the top section of the duodenum periods of regular amplitude fluctuations sometimes occur, and it has been hypothesized that these might correspond to simultaneous double modes often referred to in the previous sections[64]. Spectral analysis of these data certainly suggest the presence of two frequencies, rather than the three or more that would be expected from amplitude and/or frequency modulation. Further down the small intestine, however, 'waxing and waning' appears to be either a 'combination tone' or modulation phenomenon. In their early work on forced weakly non-linear oscillators, Appleton and van der Pol[9] had noted the experimental existence of combination tones and produced a subsequent analysis of the condition. In this analysis, two components are assumed to exist simultaneously in the oscillator output; one at the external forcing frequency, and the other at the self (i.e. intrinsic) frequency. A number of factors have encouraged the extension of this analysis to include other effects.

In a simulation study on mutually coupled oscillators under almost-synchronised conditions it was found that combination tones only occur for very small non-linearity and very weak coupling. For moderate non-linearity and coupling, amplitude and/or frequency modulation conditions exist depending on the values of the parameters [65]. It was observed that when the oscillator spectrum contained 3 major components, the frequencies were such that for two oscillators the assumed solutions could be taken to be

$$x_1 = A_1\cos(\omega_3 t + \alpha_1) + B_1\cos(\omega_1 t + \beta_1) + C_1\cos(\omega_2 t + \gamma_1)$$

$$x_2 = A_2\cos(\omega_1 t + \alpha_2) + B_2\cos(\omega_2 t + \beta_2) + C_2\cos(\omega_4 t + \gamma_2)$$

where $\omega_3 = 2\omega_1 - \omega_2$

and $\omega_4 = 2\omega_2 - \omega_1$ (82)

In (82) the frequencies ω_1 and ω_2 correspond to the predominant self-components which may not equal the intrinsic frequencies. This is because the oscillators are 'pulled' towards each other in frequency as the coupling increases. Thus, ω_1 and ω_2 must be unknowns in the harmonic balance equations, unlike the fixed values used by Appleton and van der Pol. The third additional spectral component in each oscillator is a sideband reflected about the major component. Using (82) as the assumed solutions for the two oscillator case produces a set of 12 non-linear algebraic equations whose solution gives all the amplitudes and phases of the spectral components. Using this approach, conditions of almost-synchronisation have been analysed for varying nonlinear parameter, and RLC coupling parameters [66]. An example of this is shown in Fig. 13 for varying resistive coupling. From this figure it can be seen how the oscillator frequencies are pulled together as coupling strength increases. In contrast, Fig. 14 which shows equivalent results for capacitive coupling, indicates that instead of being pulled together, the individual frequencies are diverging in this case. The frequency pulling or repelling correlates with one stable mode for resistive coupling and two stable modes for capacitive coupling.

A further extension to non-linear mode analysis has been made via harmonic balancing for the case of sub-harmonic synchronisation of mutually coupled oscillators. Although this condition had been analysed for a single forced oscillator[67], it is only recently that the more complex condition of bidirectionally coupled oscillators has been investgated. In this case several components must be assumed to exist in each oscillator, for which extended harmonic balancing is suitable. For example, in the case of 3:1 sub -harmonic synchronisation the lower frequency oscillator can be assumed to contain terms in ω and 3ω, whereas the higher frequency oscillator may contain terms in ω(injected from the neighbouring oscillator), 3ω and 9ω(third harmonic of its self-component). An example of the variations in entrained frequency, amplitudes and phases for increasing output coupling strength is shown in Fig. 15, which was obtained using hill-climbing on the non-linear algebraic harmonic balance equations. In the case of 3:1 sub-harmonic synchronisation it was found that the assumed solution mentioned above was insufficient to describe the spectrum in certain regions of the coupling space[63]. Under these conditions, further harmonic components would be required to obtain accurate results, leading to a larger number of equations for solution via hill-climbing.

4. Experimental investigations

A wide range of experiments has been conducted on coupled oscillator models for the structures considered in this paper. For gastro-intestinal modelling alone, ladders, arrays, rings and tubes have all been studied using a number of different individual oscillator dynamics. Mention will, however, only be made here of experimental studies which relate to, and support, the non-linear circuit mode analysis described in this paper. In this area, two approaches have been made. One is via computer simulation, either in analogue form for earlier studies, or in digital implementation for more recent work (particularly those involving time delays). The other method has been via electronic implementation, using diode circuitry and operational amplifiers to approximate the polynomial negative conductance terms in the van der Pol equivalent circuit. Fig. 16 shows the circuit diagram for one implementation which allows for the fifth-power conductance approximation. Fig. 17 shows a 16-oscillator experimental rig which has been used to investigate some of the mode behaviour previously considered analytically. Each oscillator can be tuned in frequency using a variable inductor, while the degree of non-linearity can be adjusted via a potentiometer. An 8-bank switch on each oscillator allows selection of 3rd power/5th power characteristic, resistive loading, and RLC coupling components in two dimensions. Use of external patch cords allows structuring of oscillators into ladders, arrays, rings and tubes. Oscillator read-out onto a multi-beam oscilloscope is facilitated via a 16-position switch which selects a read-out pattern of any oscillator plus its adjacent four neighbours.

Using an electronic emplementation, nonresonant double mode behaviour has been verified for a 3-oscillator chain[(49)]. Similarly, the degenerate mode condition was observed in a 3x4 array structure[(49)]. The irregular degenerate modes peculiar to a ring with a multiple of 4 oscillators have been observed in a 4-oscillator ring[(51)]. This condition had been observed previously in an analogue computer simulation in colonic electronic activity modelling[(4)]. These irregular degenerate modes were not observed in an electronic implementation when coupling delay was introduced via a dleay time[(53)]. It was suggested that this was due to approximations involved in the circuitry representing the mathematical equations.

Analogue and digital computer simulation studies have verified nearly all of the theoretical predictions reviewed. This includes the existence of double modes in a fifth power 2-oscillator structure[(52)], and good correlation found between analytical and experimental values for mode frequencies and spatial amplitude distribution for most of the structures considered. Stability zones under parameter variations have also been verified, giving good agreement with theory for non-linearities of $\varepsilon = 0.1$ in the normalised van der Pol equation. One aspect of mode behaviour not covered in the analysis is the relative ease with which certain modes can be excited. This is of particular interest in gastro-intestinal modelling where different rhythms occur in the colon for varying periods, which have been quantified as percentage activity (57). To investigate likelihood of mode occurence, analogue simulations have been performed to determine regions of attraction planes for a two-oscillator structure with the two variables being the initial conditions of the oscillator outputs (69). These planes were established for different types of coupling (i.e. R,L or C), different dynamics (i.e. 3rd or 5th power), and varying degrees of asymmetry. It was shown that the likelihood of particular mode incidence is strongly affected by these factors.

5. Conclusion

Knowledge about mutually interacting oscillatory systems has been growing steadily during the past few years. One strong motivation for this is the interest in such non-linear structures for a number of bio-medical applications. A particular case is the electrical activity of the gastro-intestinal digestive tract which can be modelled as a set of inter-coupled non-linear oscillators. Chains, arrays and tubes are all of interest in this particular field of application. It has been shown in this review paper that all of thes structures have yielded to theoretical studies via the method of non-linear mode analysis. Pioneered by Endo and Mori this approach uses a two-stage process comprising matrix decoupling followed by equivalent linearisation based on the Krylov-Bogolioubov classical approach for a single oscillator.

Non-linear mode analysis has revealed a complex pattern of mode behaviour involving numerous types of mode behaviour unquantified before. Thus, prediction and experimental verification has shown the existence of single modes, non-resonant double modes, regular degenerate modes and irregular degenerate modes. This review paper has briefly expounded all of these conditions for all the structures mentioned above, giving the necessary equations and criteria from which mode amplitudes, spatial distribution and stability can be determined for any order of system of coupled oscillators of this type.

The parallel development of an application (i.e. gastro-intestinal modelling) giving the motivation, and a theoretical technique (i.e. matrix non-linear mode analysis) illustrates the way in which scientific advances can be accelerated when both components are present. It is noteworthy that such advance has been characterised by a multi-disciplinary research effort involving different groups of workers in different countries working in the areas of electrical circuit theory, systems engineering and biomedicine.

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(i,k)	(j,l)	$\Psi_{mn}(i,j,k,l)$
A	A	1/mn
A	B	3/2mn
A	C	1/2mn
A	E	1/mn
A	D	1/mn
B	B	9/4mn
B	C	3/4mn
B	E	3/2mn
B	D	3/2mn
C	C	1/4mn
C	E	1/2mn
C	D	1/2mn
E	E	1/ mn
E	D	1/mn
D	D	1/mn

Table 1

Classification of Ψ_{mn}. The Ψ_{mn} is invariable for the change of (i,k) and (j,l).

(i,j) \ (k,l)	(1,1)	(1,2)	(1,3)	(1,4)	(2,1)	(2,2)	(2,3)	(2,4)	(3,1)	(3,2)	(3,3)	(3,4)
(1,1)	1	1	1	1	1	1	1	1	1	1	1	1
(1,2)	1	1.5	1	0.5	1	1.5	1	0.5	1	1.5	1	0.5
(1,3)	1	1	1	1	1	1	1	1	1	1	1	1
(1,4)	1	0.5	1	1.5	1	0.5	1	1.5	1	0.5	1	1.5
(2,1)	1	1	1	1	1.5	1	1.5	1.5	0.5	0.5	0.5	0.5
(2,2)	1	1.5	1	0.5	1	2.25	1.5	0.75	0.5	0.75	0.5	0.25
(2,3)	1	1	1	1	1.5	1.5	1.5	1.5	0.5	0.5	0.5	0.5
(2,4)	1	0.5	1	1.5	1.5	0.75	1.5	2.25	0.5	0.25	0.5	0.75
(3,1)	1	1	1	1	0.5	0.5	0.5	0.5	1.5	1.5	1.5	1.5
(3,2)	1	1.5	1	0.5	0.5	0.75	0.5	0.25	1.5	2.25	1.5	0.75
(3,3)	1	1	1	1	0.5	0.5	0.5	0.5	1.5	1.5	1.5	1.5
(3,4)	1	0.5	1	1.5	0.5	0.25	0.5	0.75	1.5	0.75	1.5	2.25

$\times \frac{1}{12}$

Table 2. Stability table $\varphi_{34}(i,j,k,l)$ for array of $m=3, n=4$

j \ k	1	2	3
1	1	1	1
2	1	1.5	0.5
3	1	0.5	1.5

Table 3. Stability table $\varphi_3(j,k)$ for 3 oscillator ladder

$i \backslash j$	1	2	3	4
1	$\sqrt{2-2\alpha}$	$\sqrt{2-\sqrt{2}\alpha}$	$\sqrt{2}$	$\sqrt{2+\sqrt{2}\alpha}$
2	$\sqrt{2+\alpha}$	$\sqrt{2+(3-\sqrt{2})\alpha}$	$\sqrt{2+3\alpha}$	$\sqrt{2+(3+\sqrt{2})\alpha}$
3	$\sqrt{2+\alpha}$	$\sqrt{2+(3-\sqrt{2\alpha})}$	$\sqrt{2+3\alpha}$	$\sqrt{2+(3+\sqrt{2})\alpha}$

Table 4 Angular frequencies ω_{ij} of the tube oscillator system with m=3 and n=4.

$i \backslash j$	1	2	3
1	$\sqrt{2-2\alpha}$	$\sqrt{2-\alpha}$	$\sqrt{2+\alpha}$
2	$\sqrt{2}$	$\sqrt{2+\alpha}$	$\sqrt{2+3\alpha}$
3	$\sqrt{2+2\alpha}$	$\sqrt{2+3\alpha}$	$\sqrt{2+5\alpha}$
4	$\sqrt{2}$	$\sqrt{2+\alpha}$	$\sqrt{2+3\alpha}$

Table 6 Angular frequencies ω_{ij} of the tube oscillator system with m=4 and n=3

(k,l) / (i,j)	(1,1)	(1,2)	(1,3)	(1,4)	(2,1)	(2,2)	(2,3)	(2,4)	(3,1)	(3,2)	(3,3)	(3,4)
(1,1)	1/12	1/12	1/12	1/12	1/12	1/12	1/12	1/12	1/12	1/12	1/12	1/12
(1,2)	1/12	1/8	1/12	1/24	1/12	1/8	1/12	1/24	1/12	1/8	1/12	1/24
(1,3)	1/12	1/12	1/12	1/12	1/12	1/12	1/12	1/12	1/12	1/12	1/12	1/12
(1,4)	1/12	1/24	1/12	1/8	1/12	1/24	1/12	1/8	1/12	1/24	1/12	1/8
(2,1)	1/12	1/12	1/12	1/12	1/8	1/8	1/8	1/8	1/24	1/24	1/24	1/24
(2,2)	1/12	1/8	1/12	1/24	1/8	3/16	1/8	1/16	1/24	1/16	1/24	1/48
(2,3)	1/12	1/12	1/12	1/12	1/8	1/8	1/8	1/8	1/24	1/24	1/24	1/24
(2,4)	1/12	1/24	1/12	1/8	1/8	1/16	1/8	3/16	1/24	1/48	1/24	1/16
(3,1)	1/12	1/12	1/12	1/24	1/24	1/24	1/24	1/24	1/8	1/8	1/8	1/8
(3,2)	1/12	1/8	1/12	1/24	1/24	1/16	1/24	1/48	1/8	1/16	1/8	1/16
(3,3)	1/12	1/12	1/12	1/12	1/24	1/24	1/24	1/24	1/8	1/8	1/8	1/8
(3,4)	1/12	1/24	1/12	1/8	1/24	1/48	1/24	1/16	1/8	1/16	1/8	3/16

Table 5 The values $\psi_{mn}(i,j,k,l)$ when m=3, n=4 for a tube

(i,j) \ (k,l)	(1,1)	(1,2)	(1,3)	(2,1)	(2,2)	(2,3)	(3,1)	(3,2)	(3,3)	(4,1)	(4,2)	(4,3)
(1,1)	$^1/_{12}$	$^1/_{12}$	$^1/_{12}$	$^1/_{12}$	$^1/_{12}$	$^1/_{12}$	$^1/_{12}$	$^1/_{12}$	$^1/_{12}$	$^1/_{12}$	$^1/_{12}$	$^1/_{12}$
(1,2)	$^1/_{12}$	$^1/_8$	$^1/_{24}$	$^1/_{12}$	$^1/_8$	$^1/_{24}$	$^1/_{12}$	$^1/_8$	$^1/_{24}$	$^1/_{12}$	$^1/_8$	$^1/_{24}$
(1,3)	$^1/_{12}$	$^1/_{24}$	$^1/_8$	$^1/_{12}$	$^1/_{24}$	$^1/_{18}$	$^1/_{12}$	$^1/_{24}$	$^1/_8$	$^1/_{12}$	$^1/_{24}$	$^1/_8$
(2,1)	$^1/_{12}$	$^1/_{12}$	$^1/_{12}$	$^1/_6$	$^1/_6$	$^1/_6$	$^1/_{12}$	$^1/_{12}$	$^1/_{12}$	0	0	0
(2,2)	$^1/_{12}$	$^1/_8$	$^1/_{24}$	$^1/_6$	$^1/_4$	$^1/_{12}$	$^1/_{12}$	$^1/_8$	$^1/_{24}$	0	0	0
(2,3)	$^1/_{12}$	$^1/_{24}$	$^1/_8$	$^1/_6$	$^1/_{12}$	$^1/_4$	$^1/_{12}$	$^1/_{24}$	$^1/_8$	0	0	0
(3,1)	$^1/_{12}$	$^1/_{12}$	$^1/_{12}$	$^1/_{12}$	$^1/_{12}$	$^1/_{12}$	$^1/_{12}$	$^1/_{12}$	$^1/_{12}$	$^1/_{12}$	$^1/_{12}$	$^1/_{12}$
(3,2)	$^1/_{12}$	$^1/_8$	$^1/_{24}$	$^1/_{12}$	$^1/_8$	$^1/_{24}$	$^1/_{12}$	$^1/_8$	$^1/_{24}$	$^1/_{12}$	$^1/_8$	$^1/_{24}$
(3,3)	$^1/_{12}$	$^1/_{24}$	$^1/_8$	$^1/_{12}$	$^1/_{24}$	$^1/_8$	$^1/_{12}$	$^1/_{24}$	$^1/_8$	$^1/_{12}$	$^1/_{24}$	$^1/_8$
(4,1)	$^1/_{12}$	$^1/_{12}$	$^1/_{12}$	0	0	0	$^1/_{12}$	$^1/_{12}$	$^1/_{12}$	$^1/_6$	$^1/_6$	$^1/_6$
(4,2)	$^1/_{12}$	$^1/_8$	$^1/_{24}$	0	0	0	$^1/_{12}$	$^1/_8$	$^1/_{24}$	$^1/_6$	$^1/_4$	$^1/_{12}$
(4,3)	$^1/_{12}$	$^1/_{24}$	$^1/_8$	0	0	0	$^1/_{12}$	$^1/_{24}$	$^1/_8$	$^1/_6$	$^1/_{12}$	$^1/_4$

Table 7 The values $\psi_{mn}(i,j,k,l)$ corresponding to tube structure with m=4, n=3

N=2

$$[g_{ij}] = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \qquad [h_{ij}] = \begin{bmatrix} 1/4 & 1/4 \\ 1/4 & 1/4 \end{bmatrix}$$

N=3

$$[g_{ij}] = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/2 & 1/6 \\ 1/3 & 1/6 & 1/2 \end{bmatrix} \qquad [h_{ij}] = \begin{bmatrix} 1/9 & 1/6 & 1/6 \\ 1/9 & 1/4 & 1/36 \\ 1/9 & 1/12 & 11/36 \end{bmatrix}$$

N=4

$$[g_{ij}] = \begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 3/8 & 1/4 & 1/8 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/8 & 1/4 & 3/8 \end{bmatrix} \qquad [h_{ij}] = \begin{bmatrix} 1/16 & 3/32 & 1/16 & 3/32 \\ 1/16 & 5/32 & 1/16 & 1/32 \\ 1/16 & 3/32 & 1/16 & 3/32 \\ 1/16 & 1/32 & 1/16 & 5/32 \end{bmatrix}$$

$$h'_{2,1,3} = h'_{4,1,3} = 1/16, \quad h'_{1,2,4} = h'_{3,2,4} = 1/32$$

Table 8. Fifth power stability terms for N=2,3,4

$$\begin{bmatrix} \frac{1}{9} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{9} & \frac{11}{36} & \frac{1}{6} \\ \frac{1}{9} & \frac{1}{36} & \frac{1}{4} \end{bmatrix} \qquad \begin{bmatrix} \frac{1}{9} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{9} & \frac{1}{4} & \frac{1}{36} \\ \frac{1}{9} & \frac{1}{12} & \frac{11}{36} \end{bmatrix}$$

$\psi_3^*(i,k)$ $\qquad$ $\psi_3^*(j,\ell)$

$$\begin{bmatrix} \frac{1}{16} & \frac{1}{8} & \frac{1}{16} & \frac{1}{8} \\ \frac{1}{16} & \frac{1}{4} & \frac{1}{16} & 0 \\ \frac{1}{16} & \frac{1}{8} & \frac{1}{16} & \frac{1}{8} \\ \frac{1}{16} & 0 & \frac{1}{16} & \frac{1}{4} \end{bmatrix} \qquad \begin{bmatrix} \frac{1}{16} & \frac{3}{32} & \frac{1}{16} & \frac{3}{32} \\ \frac{1}{16} & \frac{5}{32} & \frac{1}{16} & \frac{1}{32} \\ \frac{1}{16} & \frac{3}{32} & \frac{1}{16} & \frac{3}{32} \\ \frac{1}{16} & \frac{1}{32} & \frac{1}{16} & \frac{5}{32} \end{bmatrix}$$

$\psi_4^*(i,k)$ $\qquad$ $\psi_4^*(j,\ell)$

Table 9. Stability Table for $m, n = 3, 4$

(1,1)	(1,2)	(1,3)	(1,4)	(2,1)	(2,2)	(2,3)	(2,4)	(3,1)	(3,2)	(3,3)	(3,4)	
1/9	1/6	1/9	1/6	1/6	1/4	1/6	1/4	1/6	1/4	1/6	1/4	
1/9	5/18	1/9	1/18	1/6	5/12	1/6	1/12	1/6	5/12	1/6	1/12	
1/9	1/6	1/9	1/6	1/6	1/4	1/6	1/4	1/6	1/4	1/6	1/4	
1/9	1/18	1/9	5/18	1/6	1/12	1/6	5/12	1/6	1/12	1/6	5/12	
1/9	1/6	1/9	1/6	11/36	11/24	11/36	11/34	1/6	1/4	1/6	1/4	
1/9	5/18	1/9	1/18	11/36	55/72	11/36	11/72	1/6	5/12	1/6	1/12	$X\frac{1}{16}$
1/9	1/6	1/9	1/6	11/36	11/24	11/36	11/24	1/6	1/4	1/6	1/4	
1/9	1/18	1/9	5/18	11/36	11/72	11/36	55/72	1/6	1/12	1/6	5/12	
1/9	1/6	1/9	1/6	1/36	1/24	1/36	1/24	1/4	3/8	1/4	3/8	
1/9	5/18	1/9	1/18	1/36	5/72	1/36	1/72	1/4	5/8	1/4	1/8	
1/9	1/6	1/9	1/6	1/36	1/24	1/36	1/24	1/4	3/8	1/4	3/8	
1/9	1/18	1/9	5/18	1/36	1/72	1/36	5/72	1/4	1/8	1/4	5/8	

Table 10 The values of ψ^*_{mn} (i,j,k,l) when m=3 and n=4 for a tube.

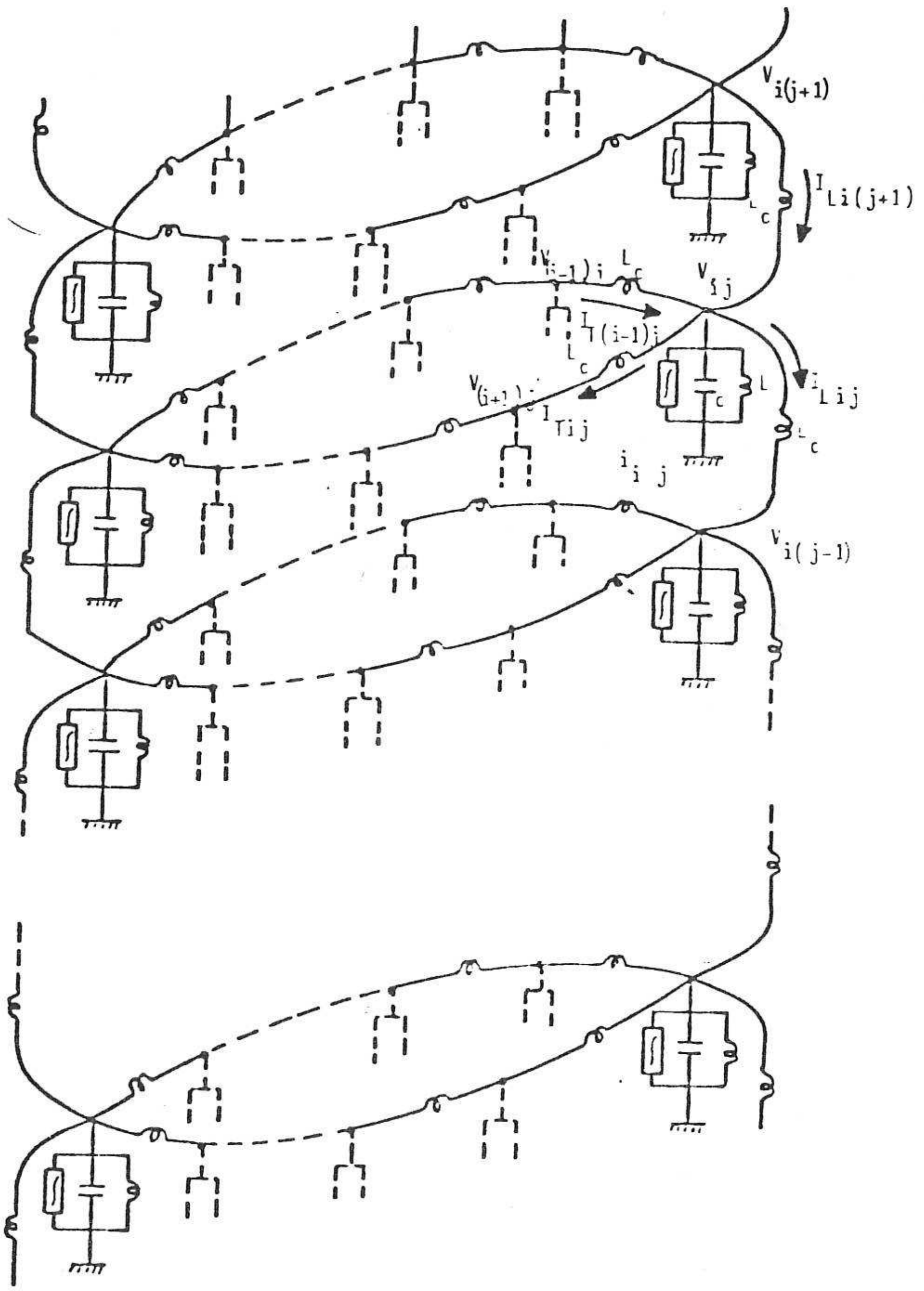


Fig. 1 Tube structure with an arbitrary number of oscillators in each ring and an arbitrary number of rings

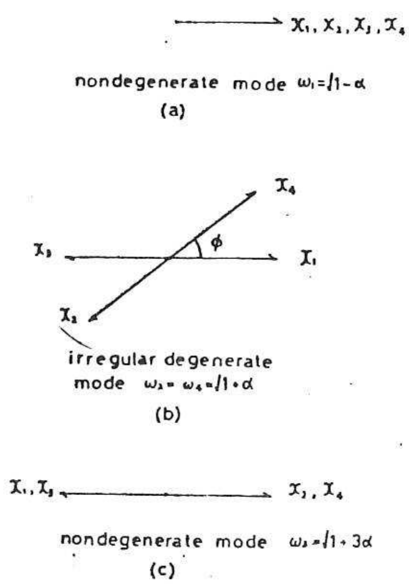


Fig.2. The phase diagrams of stable modes for a four-coupled oscillator ring.

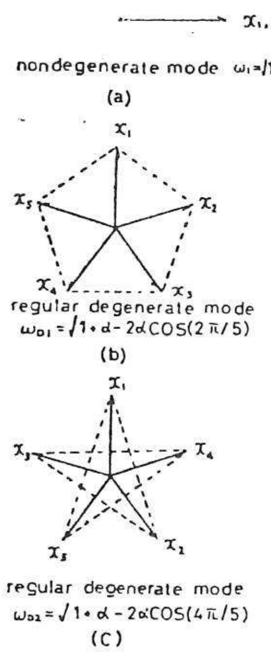
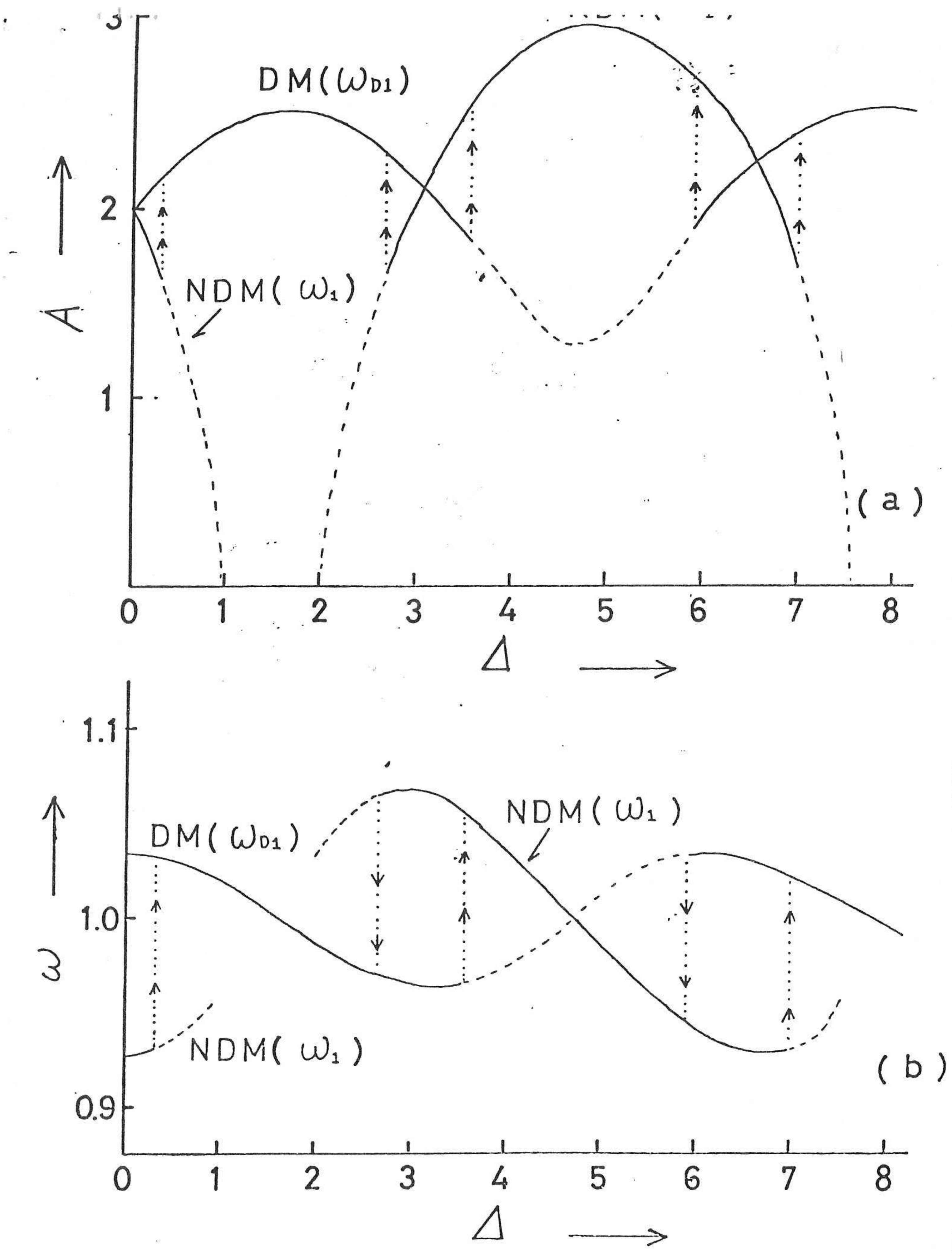


Fig3. The phase diagrams of stable modes for a five-coupled oscillator ring. (b), (c) A right-handed rotation. Of course, a left-handed rotation is also stable.



3 oscillators. a) Relationship between mode amplitude (A) and delay time (Δ)

Fig 4. b) " " " frequency (ω) and delay time (Δ)

NDM: Nondegenerate mode. DM: Degenerate mode. — stable. --- unstable

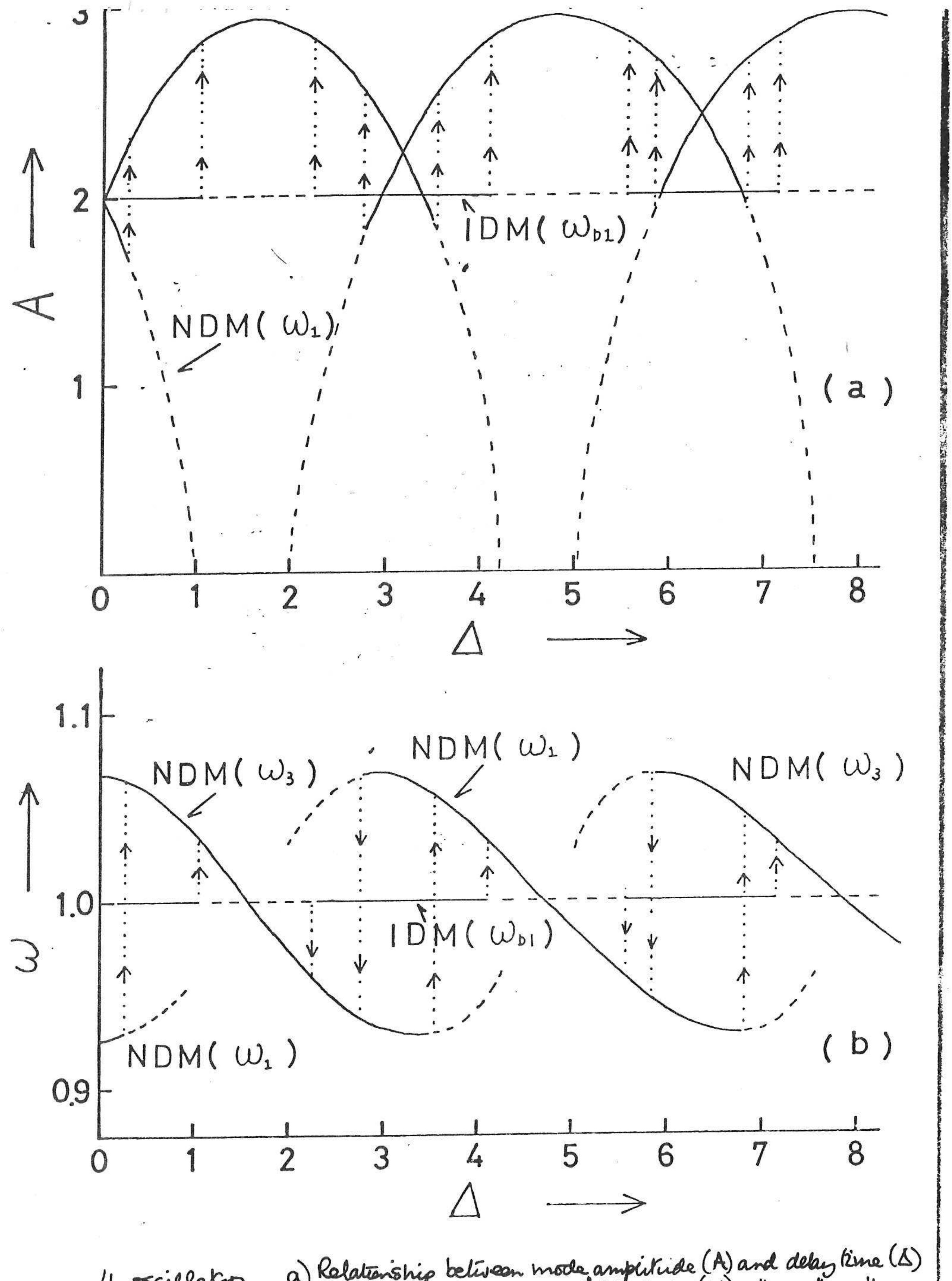


Fig 5. 4 oscillators. a) Relationship between mode amplitude (A) and delay time (Δ)
b) " " " frequency (ω) " " "

NDM: Nondegenerate mode. IDM: Irregular degenerate mode. — stable. --- unstable

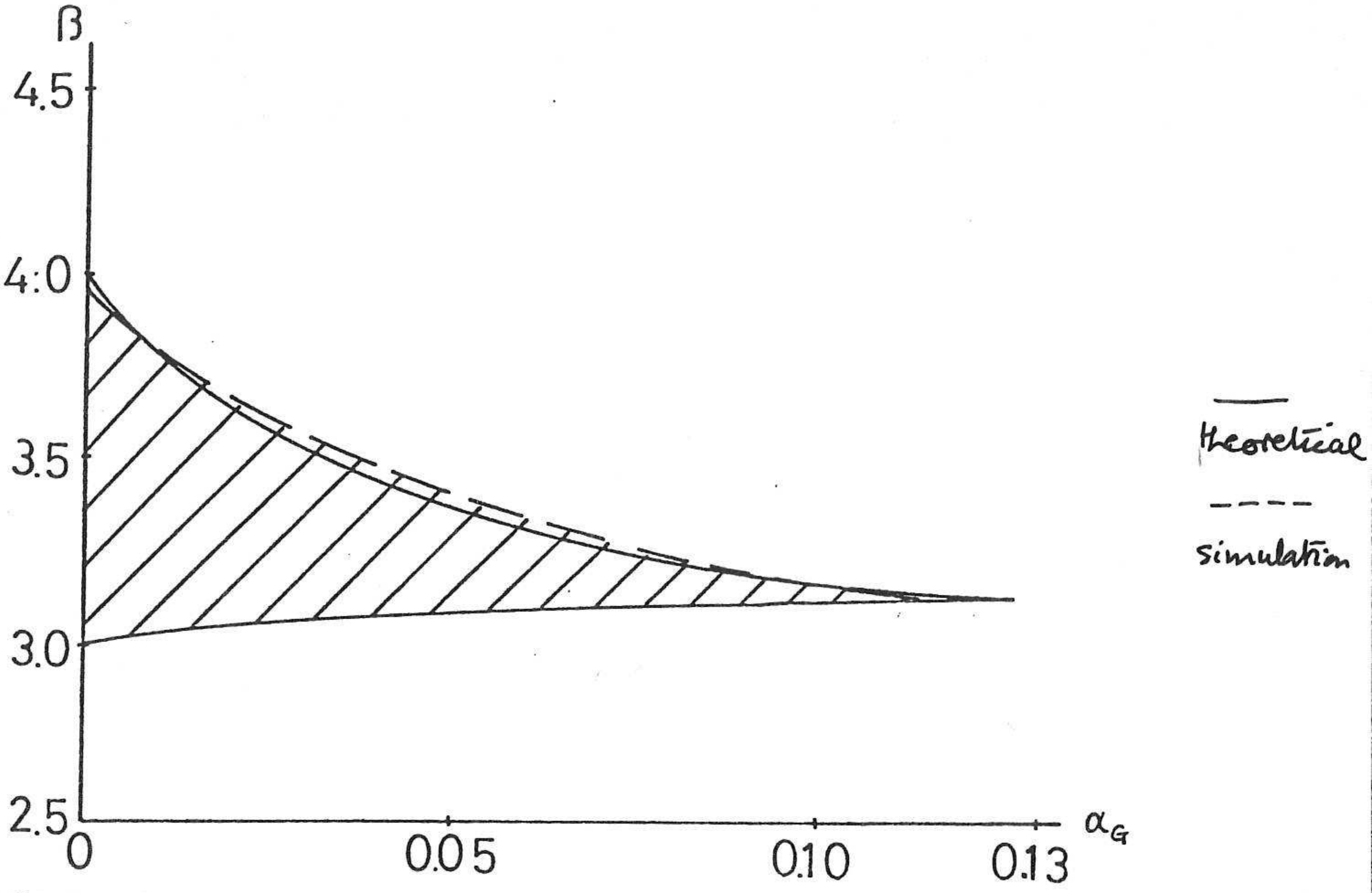


Fig 6. Stable region of β for nonresonant double mode y_1 & y_2 for n=2 oscillators ($\varepsilon = 0.1$) $\alpha_L = 0.2$

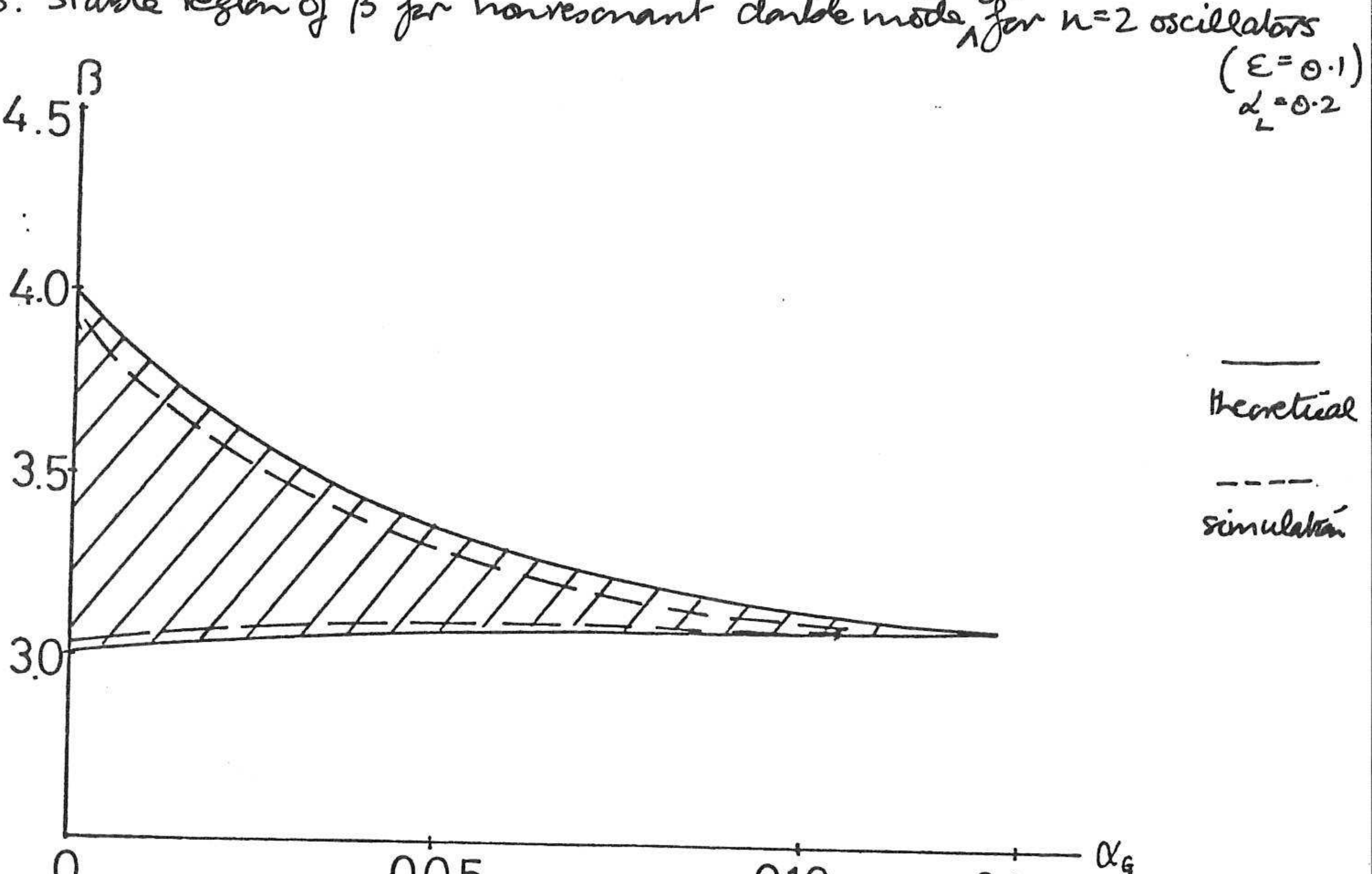


Fig. 7 Stable region of β for nonresonant double mode y_1 & y_3 for n=4 oscillators ($\varepsilon = 0.07$, $\alpha_L = 0.3$)

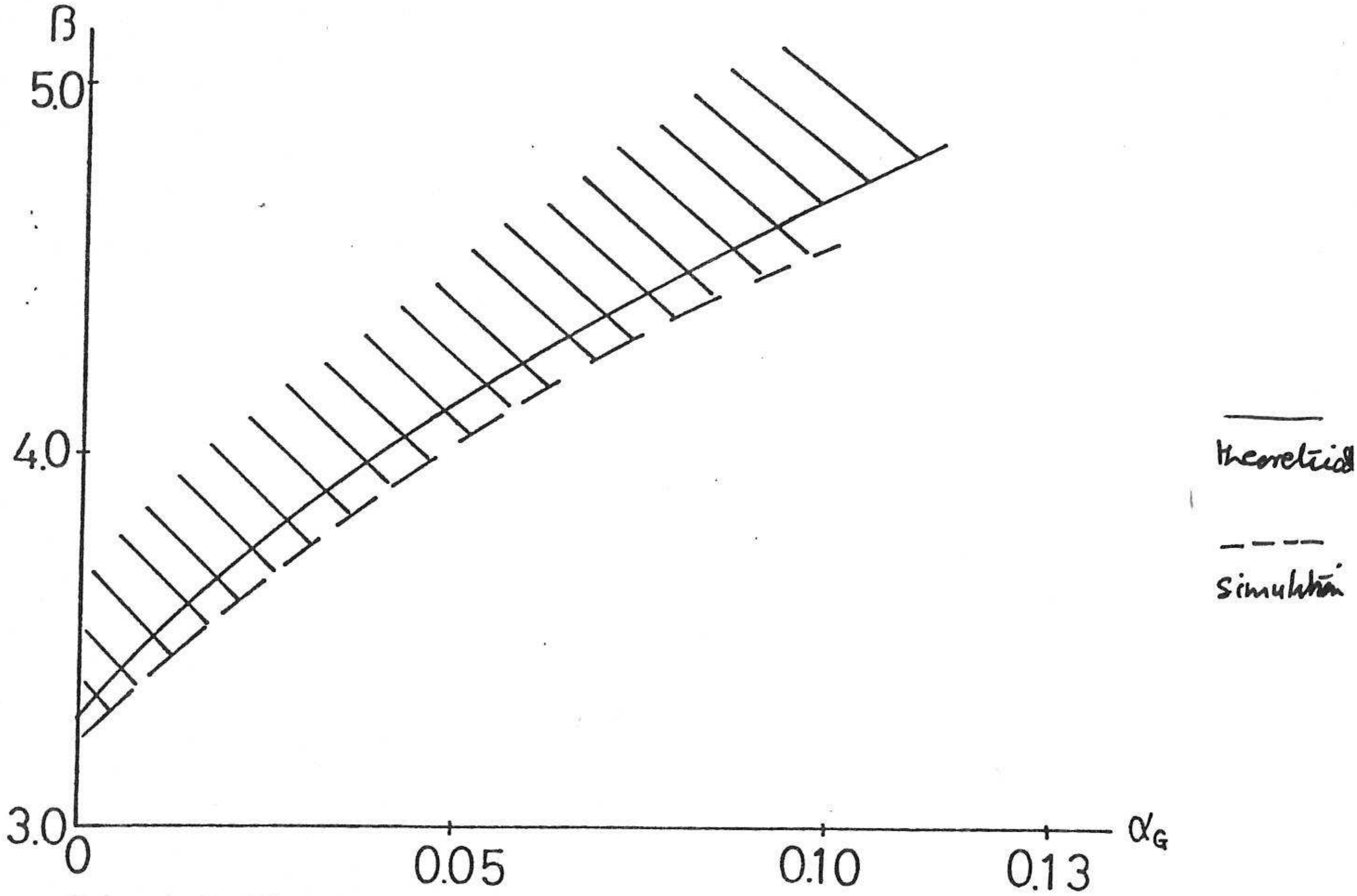


Fig. 8 Stable region of β for nonresonant double mode y_2 & y_4 for $n=4$ oscillators ($\varepsilon=0{\cdot}08$, $\alpha_L=0{\cdot}3$)

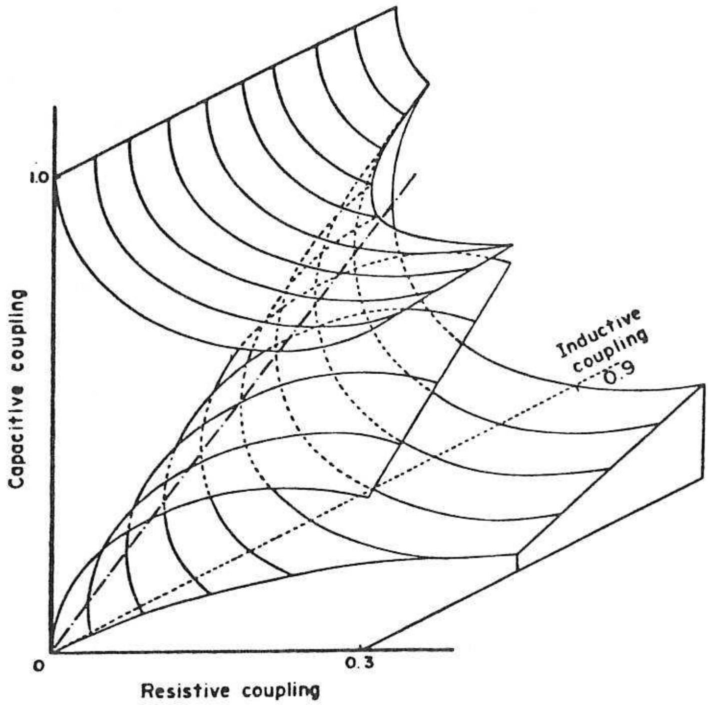
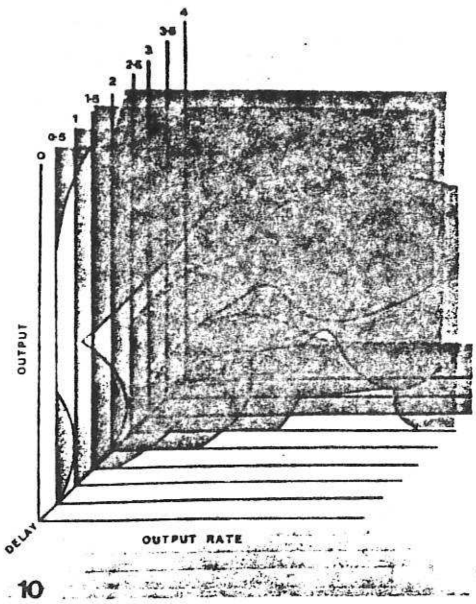


Fig 9. *The boundary contours in the RCL space separating stable and unstable regions for the anti-phase condition. For non-zero values of L there are two stable regions in the RC plane*



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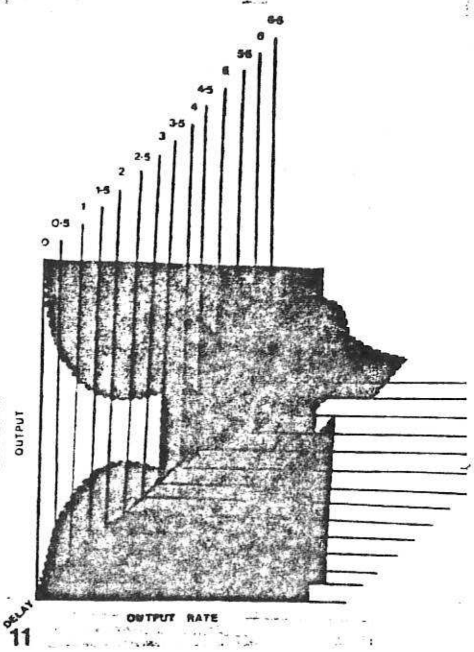

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Figure 10. In-phase mode stability boundaries for a parameter space of output, output-rate and time-delay. NB. Solid line denotes advancing boundary front. Dotted line denotes receding boundary front.

Figure 11. Anti-phase mode stability boundaries for a parameter space of output, output-rate and time-delay.

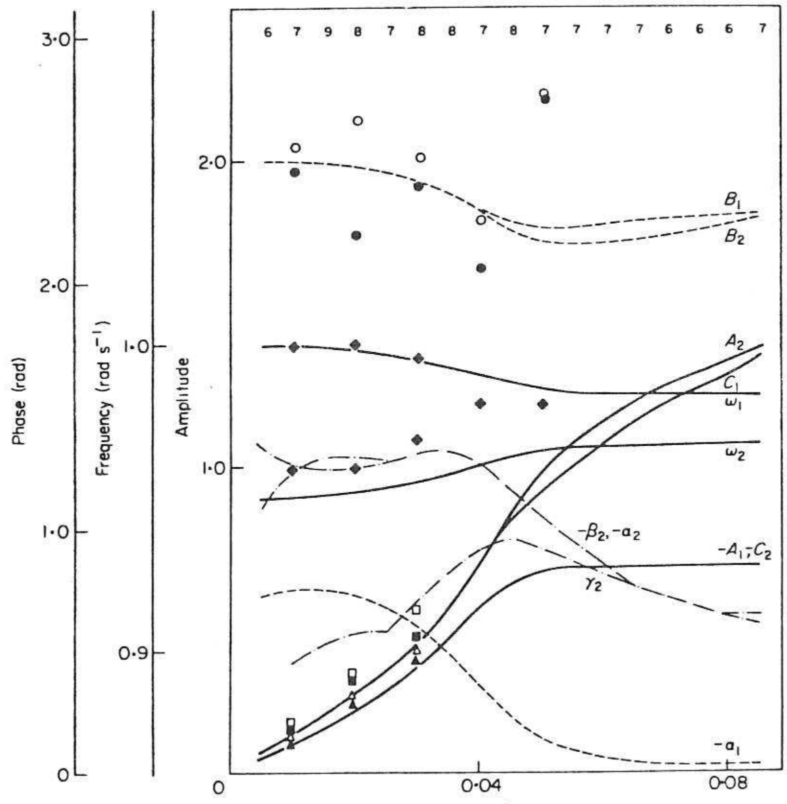


Fig 13. Modulation harmonic balance and spectrum analyser results for $\varepsilon = 0{\cdot}1$, and varying resistive coupling.

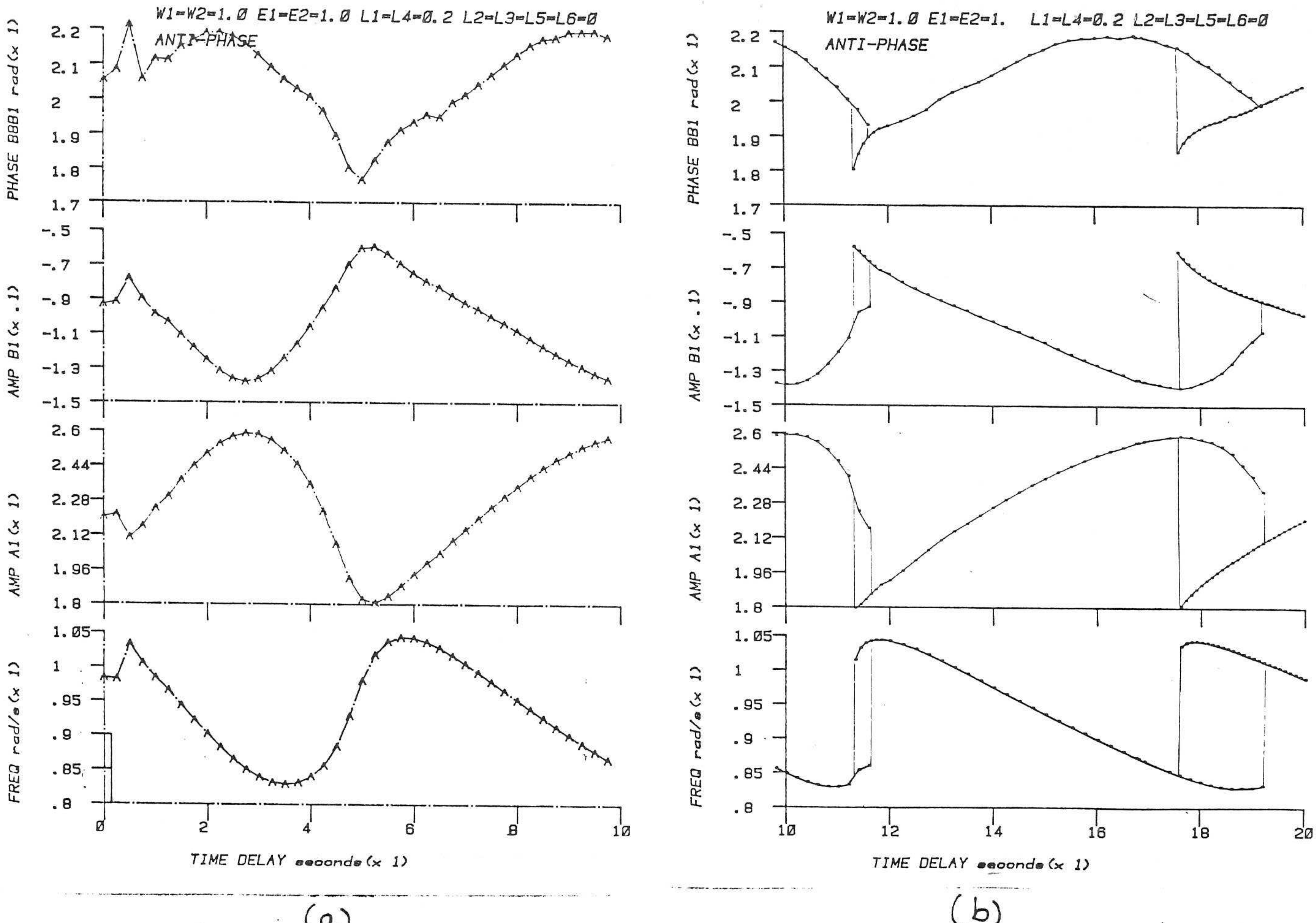


Fig.12. Variation of entrainment variables showing jump phenomena for lage delays (anti-phase mode)

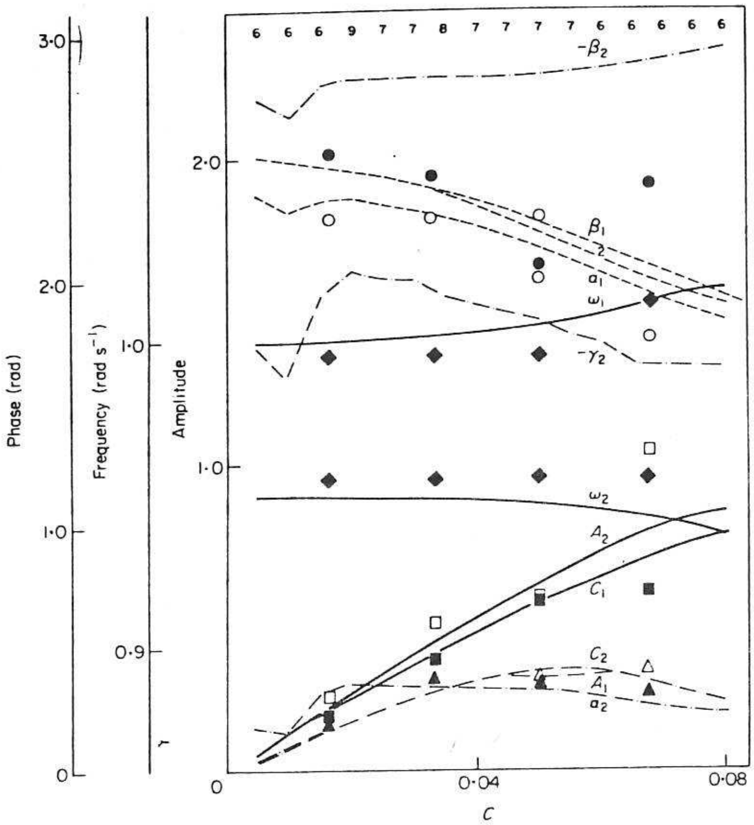


Fig 14. Modulation harmonic balance and spectrum analyser results for $\varepsilon = 0{\cdot}1$, and varying capacitive coupling.

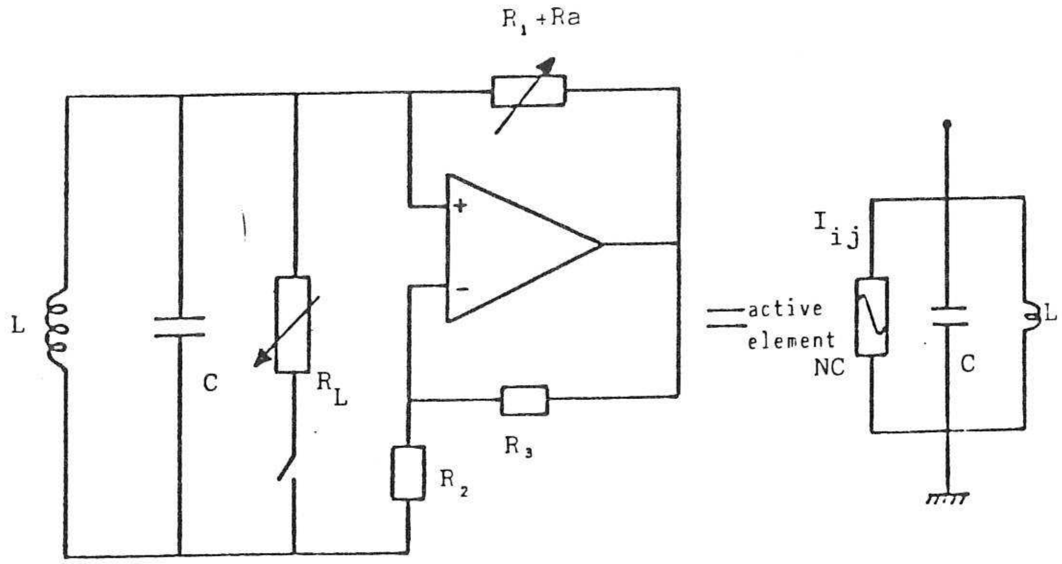


Fig. 16. Oscillator circuit with cubic nonlinear characteristic

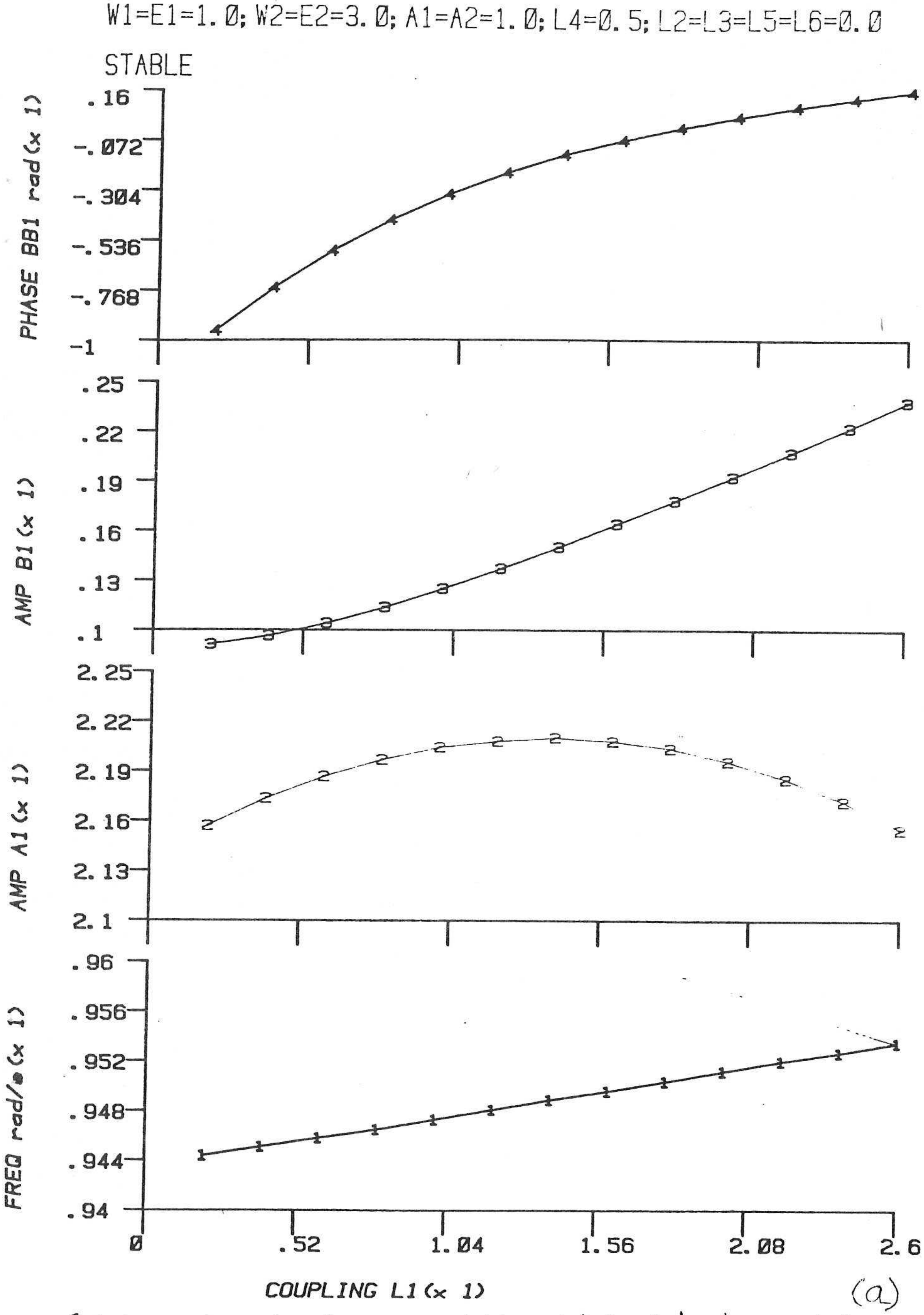
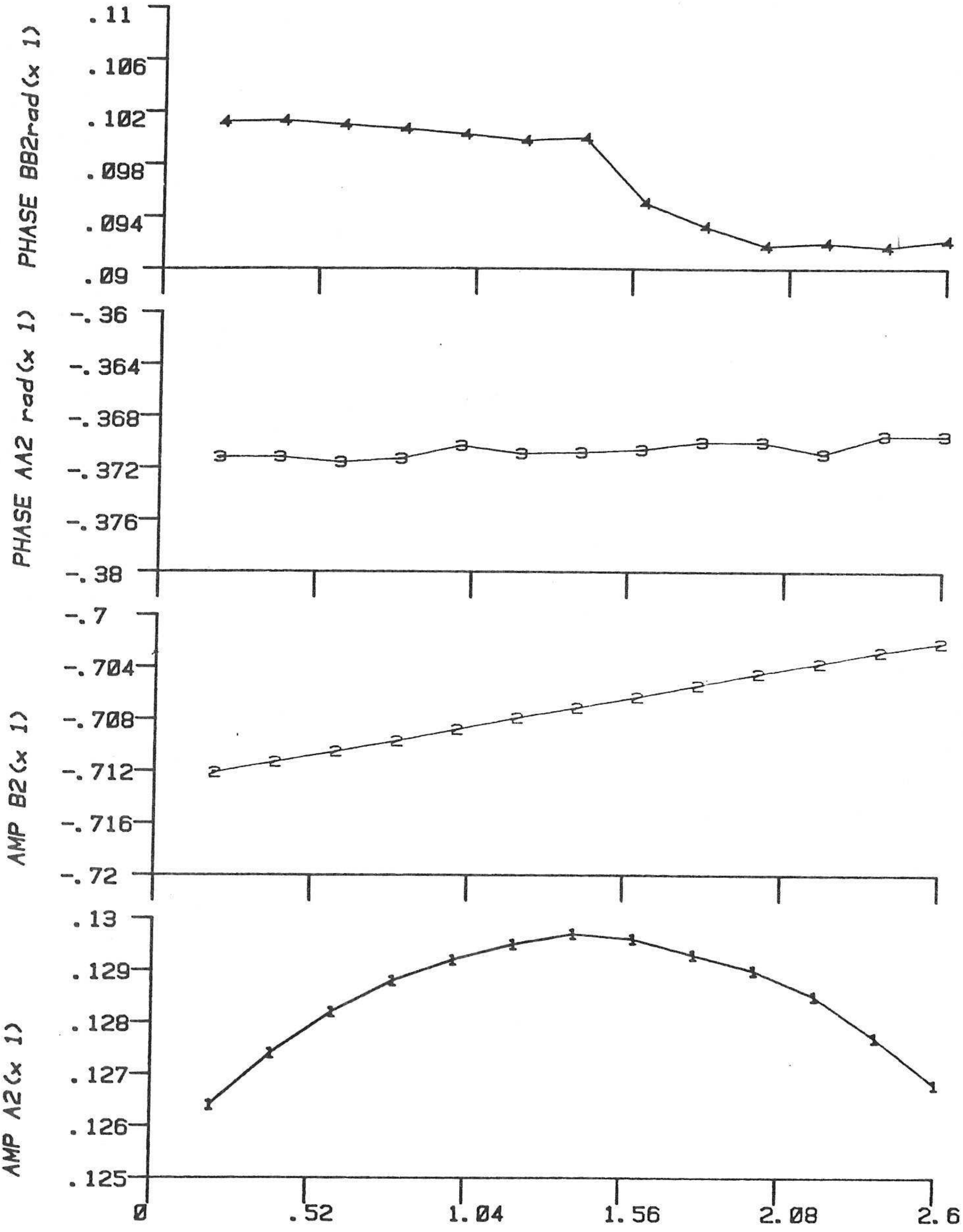


Fig 15. Sub-harmonic entrainment variables obtained by harmonic balancing for varying coupling delays (stable mode).

W1=E1=1.0; W2=E2=3.0; A1=A2=1.0; L4=0.5; L2=L3=L5=L6=0.0.

STABLE



(b)

Fig 15 (cont.)

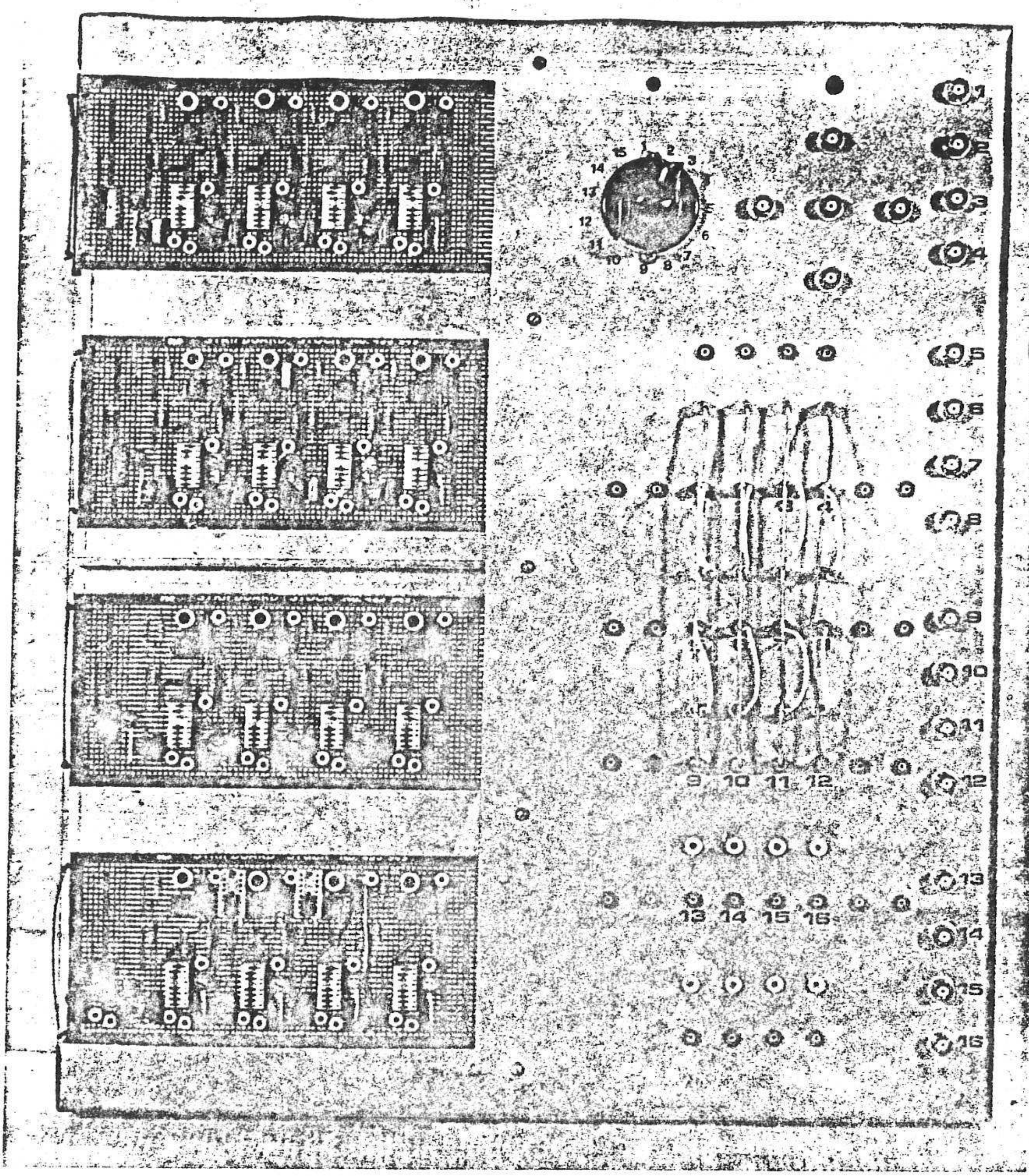


Fig. 17. 16 oscillator coupled model using electronic implementation of van der Pol third and fifth power characteristics.

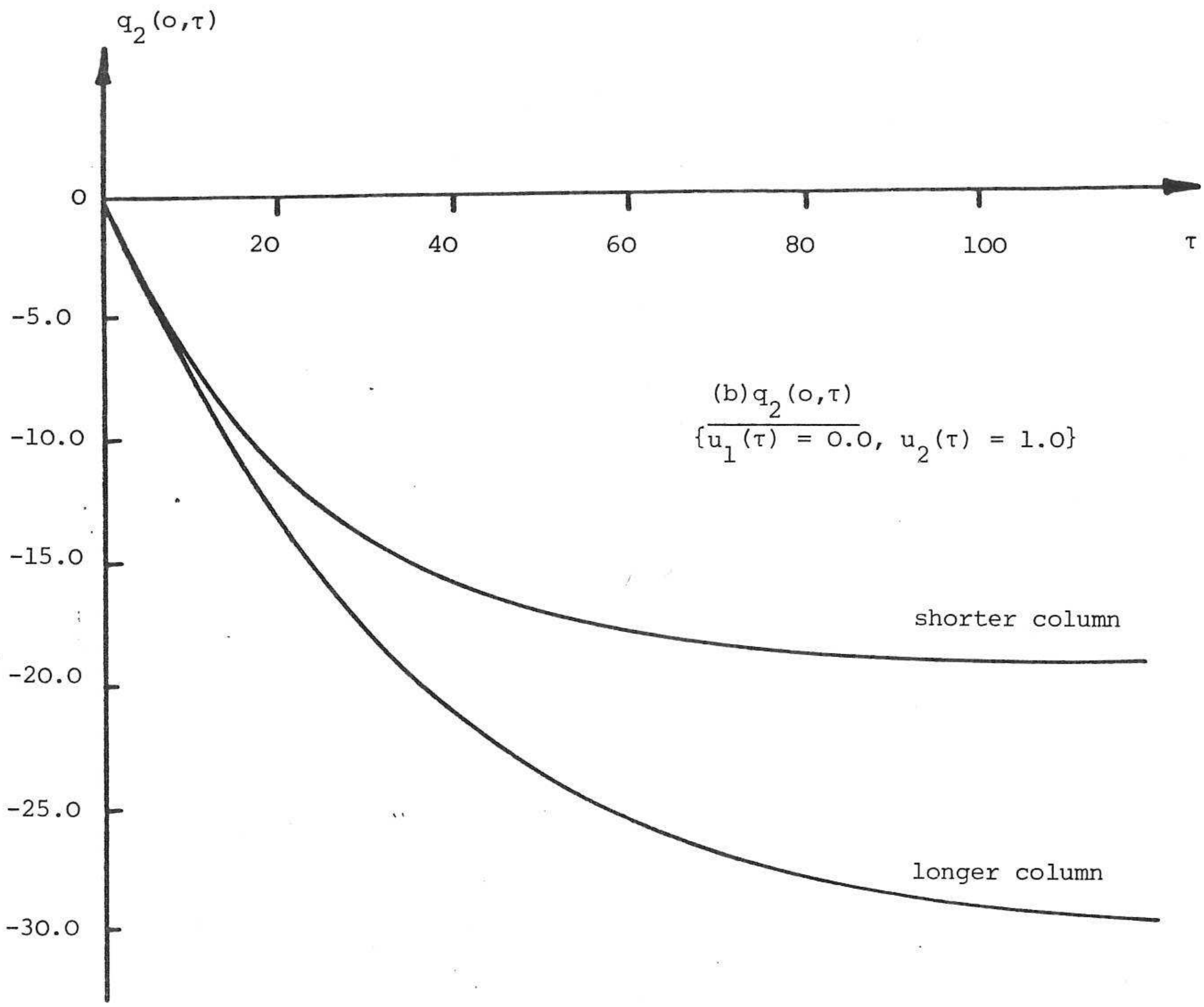


Fig. 6(b) Unit-step responses

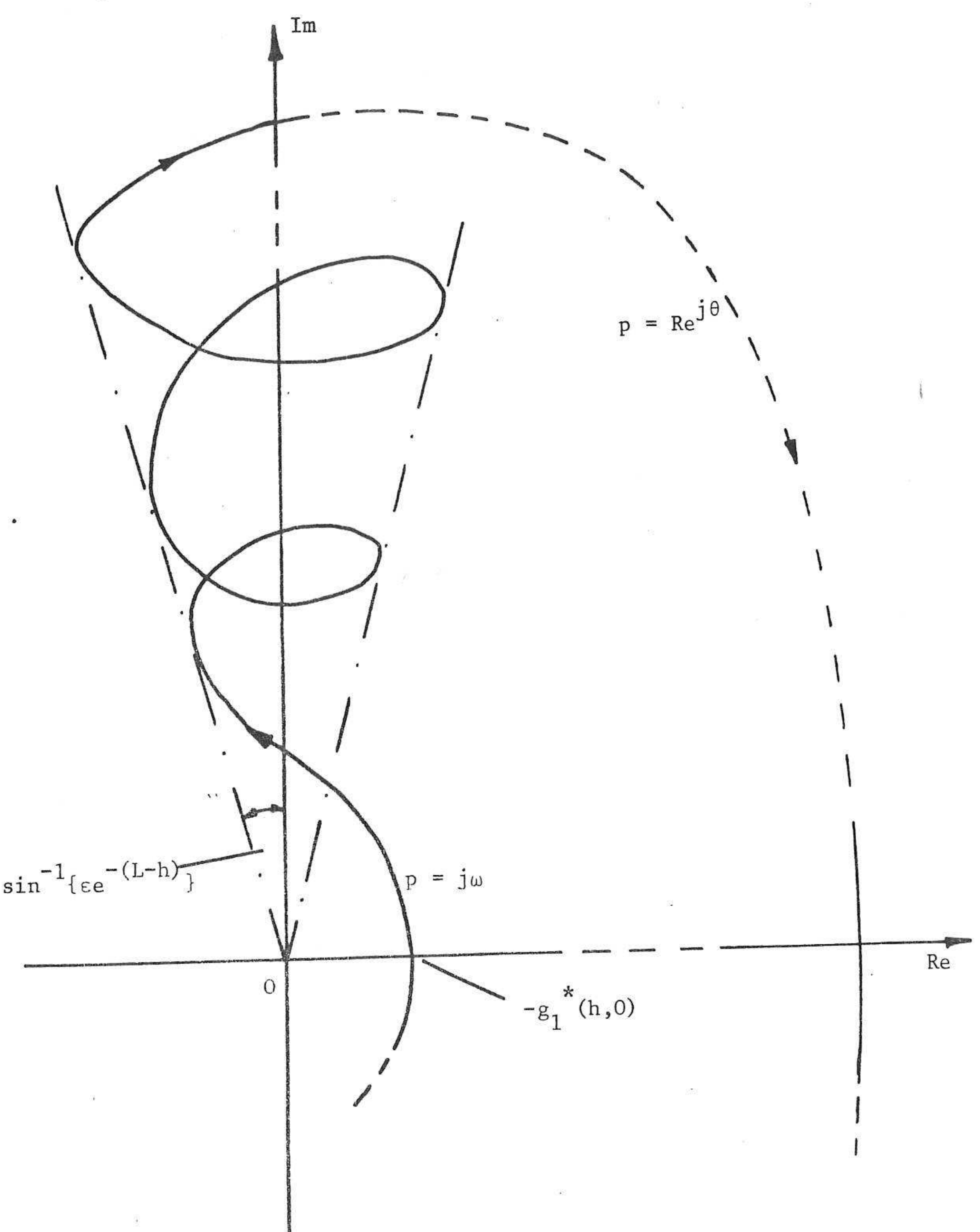


Fig. 7. Wave effects on inverse Nyquist locus of $-g_1^*(h,p)$: $\{g_1^*(h,0)<0\}$

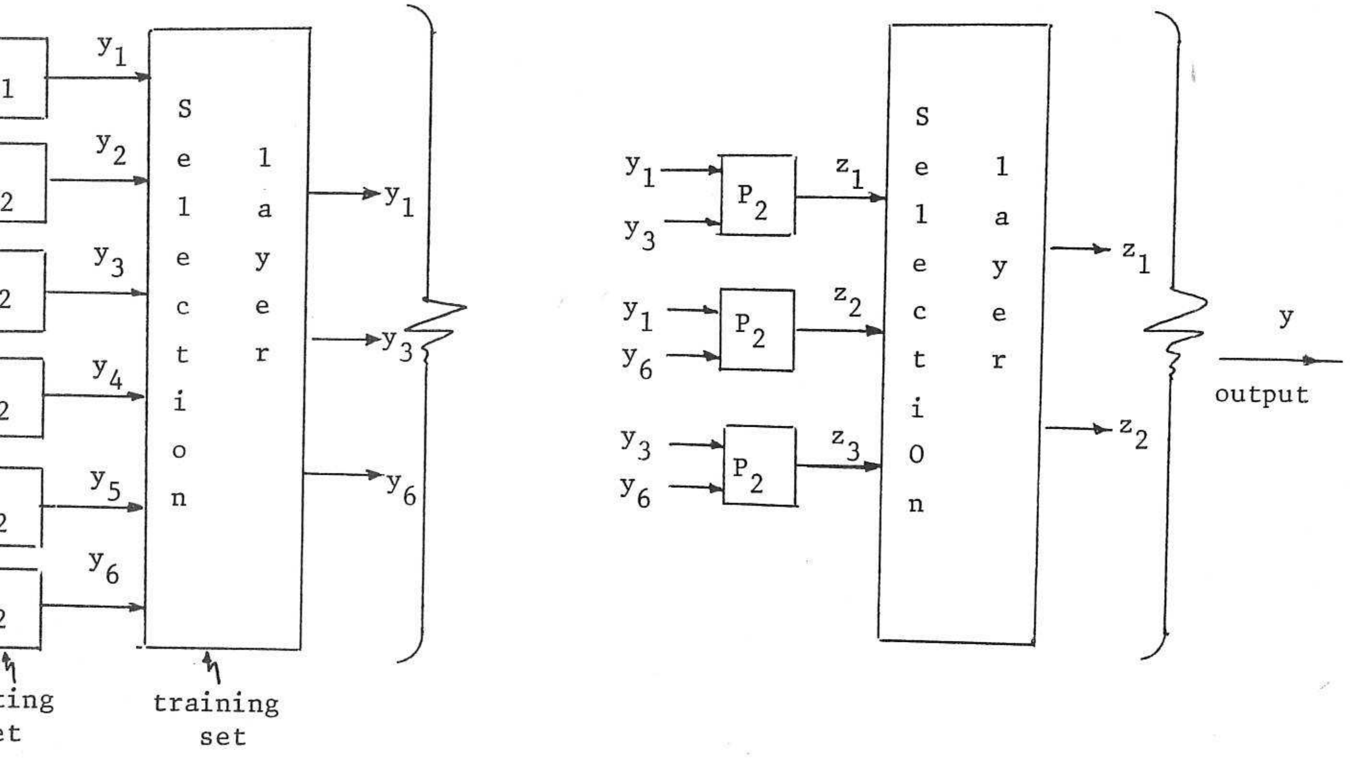


Fig.9. The GMDH