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A STRUCTURE THEOREM FOR SUBGROUPS OF $GL_n$ OVER COMPLETE LOCAL NOETHERIAN RINGS WITH LARGE RESIDUAL IMAGE

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Abstract. Given a complete local Noetherian ring $(A, m_A)$ with finite residue field and a subfield $k$ of $A/m_A$, we show that every closed subgroup $G$ of $GL_n(A)$ such that $G \mod m_A \supseteq SL_n(k)$ contains a conjugate of $SL_n(W(k)_A)$ under some small restrictions on $k$. Here $W(k)_A$ is the closed subring of $A$ generated by the Teichmüller lifts of elements of the subfield $k$.

1. Introduction

Let $k$ be a finite field of characteristic $p$ and let $W(k)$ be its Witt ring. Then, by the structure theorem for complete local rings (see Theorem 29.2 of [4]), every complete local ring with residue field containing $k$ is naturally a $W$-algebra. More precisely, given a complete local ring $(A, m_A)$ and a field homomorphism $\varphi : k \to A/m_A$, there is a unique homomorphism $\varphi : W(k) \to A$ of local rings which induces $\varphi$ on residue fields. The homomorphism $\varphi$ is completely determined by its action on Teichmüller lifts: if $x \in k$ and $\hat{x} \in W(k)$ is its Teichmüller then $\varphi(\hat{x})$ is the Teichmüller lift of $\varphi(x)$.

In this article, we consider an 'analogous' property for subgroups of $GL_n$ over complete local Noetherian rings. From here on the index $n$ is fixed and assumed to be at least 2. First a small bit of notation before we state our result formally: Given a complete local ring $(A, m_A)$ and a finite subfield $k$ of the residue field $A/m_A$, denote by $W(k)_A$ the image of the natural local homomorphism $W(k) \to A$ from the structure theorem. Alternatively, $W(k)_A$ is the smallest closed subring of $A$ containing the Teichmüller lifts of elements of the subfield $k$.

Main Theorem. Let $(A, m_A)$ be a complete local Noetherian ring with maximal ideal $m_A$ and finite residue field $A/m_A$ of characteristic $p$. Suppose we are given a subfield $k$ of $A/m_A$ and a closed subgroup $G$ of $GL_n(A)$. Assume that:

- The cardinality of $k$ is at least 4. Furthermore, assume that $k \neq \mathbb{F}_4$ if $n = 2$ and that $k \neq \mathbb{F}_5$ if $n = 3$.
- $G \mod m_A \supseteq SL_n(k)$.

Then $G$ contains a conjugate of $SL_n(W(k)_A)$.

For an application, set $W_m := W(k)/p^n$ and $G := SL_n(W_m)$ with $k$ as in the above theorem. Then the above result implies that $W_m$, with the natural representation $\rho : G \to SL_n(W_m)$, is the universal deformation ring for deformations of $\varphi := \rho \mod p : G \to SL_n(k)$ in the category of complete local Noetherian rings with residue field $k$. (See Remark 4.5.)

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We now outline the structure of this article (and introduce some notation along the way). If $M$ is a module over a commutative ring $A$, then $M(M)$, resp. $M_0(M)$, denotes the $GL_n(A)$-module of $n$ by $n$ matrices over $M$, resp. $n$ by $n$ trace 0 matrices over $M$, with $GL_n(A)$ action given by conjugation. The bi-module structure on $M$ is of course given by $amb := abm$ for all $a, b \in A$, $m \in M$. A typical application of this consideration is when $B = A/J$ for some ideal $J$ with $J^2 = 0$. Then $GL_n(B)$ acts on $M(J)$ and $M_0(J)$, and this action is compatible with the action of $GL_n(A)$.

Given $A$, $B$ and $J$ as above, we can understand subgroups of $SL_n(A)$ if we know enough about extensions of $SL_n(B)$ by $M_0(J)$. We give a brief description of the process involved (in terms of group extensions) in section 2. Determining extensions in general can be a complicated problem but, for the proof of the main theorem, we only need to look at extensions of $SL_n(W(k)/p^m)$ by $M_0(k)$. To carry out the argument we need some control over $H^1(SL_n(W(k)/p^m), M_0(k))$ and $H^2(SL_n(W(k)/p^m), M_0(k))$. Some care is needed when $p$ divides $n$; the necessary calculations are carried out in section 3.

We remark that the condition on the residual image of $G$ is necessary for the calculations used here to work. There are results due to Pink (see [9]) characterising closed subgroups of $SL_2(A)$ when the complete local ring $A$ has odd residue characteristic. (The proof depends on matrix/Lie algebra identities that only work when $n = 2$.) For explicit descriptions of some classes of subgroups of $SL_2(A)$, see Böckle [1].

A different aspect of the size of closed subgroups of $GL_n(A)$ with large residual image is studied by Boston in [7]. In a sense our result complements that of Boston: we give a lower bound for the size of closed subgroups assuming the image modulo $m_A$ is big enough, while Boston’s result there, loc. cit, says such subgroups will contain $SL_n(A)$ if the image modulo $m_A^2$ is big enough.

## 2. Twisted semi-direct products

Let $G$ be a finite group. Given an $\mathbb{F}_p[G]$-module $V$ and a normalised 2-cocyle $x : G \times G \to \mathbb{V}$, we can then form the twisted semi-direct product $V \rtimes_x G$. Here, normalised means that $x(g, e) = x(e, g) = 0$ for all $g \in G$ where we have denoted the identity of $G$ by $e$. Recall $V \rtimes_x G$ has elements $(v, g)$ with $v \in V$, $g \in G$ and composition

$$(v_1, g_1)(v_2, g_2) \in (x(g_1, g_2) + v_1 + g_1v_2, g_1g_2),$$

and that the cohomology class of $x$ in $H^2(G, V)$ represents the extension

$$0 \to V \xrightarrow{v \mapsto (v, e)} V \rtimes_x G \xrightarrow{(v, g) \mapsto g} G \to e.\tag{2.1}$$

The conjugation action of $V \rtimes_x G$ on $V$ is the one given by the $G$ action on $V$ i.e. $(u, g)v := (u, g)(v, e)(u, g)^{-1} = (gv, e)$ holds for all $u, v \in V$, $g \in G$.

We record the following result for use in the next section.

**Proposition 2.1.** With $G$, $V$ and $x : G \times G \to \mathbb{V}$ as above, let $\phi : V \rtimes_x e \to V$ be the map $(v, e) \to -v$. Then under the transgression map

$$\delta : \text{Hom}_G(V \rtimes_x e, V) \to H^1(V \rtimes_x e, V)^G \to H^2(G, V),$$

$\delta(\phi)$ is the class of $x$.

**Proof.** Let $\pi : V \rtimes_x G \to V$ be the map given by $\pi(v, g) := -v$. Thus $\pi|_{V \rtimes_x e} = \phi$ and $\pi(ab) = \pi(a) + a\pi(b)a^{-1}$ whenever $a$ or $b$ is in $V \rtimes_x e$. The map $\partial \pi : G \times G \to V$
given by $\partial \pi (g_1, g_2) := \pi (a_1) + a_1 \pi (a_2) a_1^{-1} - \pi (a_1 a_2)$ where $a_i \in V \rtimes_x G$ lifts $g_i$ is then well defined and $\delta (\phi)$ is the class of $\partial \pi$. (See Proposition 1.6.5 in [8].) Taking $a_i := (0, g_i)$ we see that $\partial \pi (g_1, g_2) = x(g_1, g_2)$. \hfill \Box

For the remainder of this section, we assume that we are given an $\mathbb{F}_p [G]$-module $M$ of finite cardinality and an $\mathbb{F}_p [G]$-submodule $N \subseteq M$ such that the map

\begin{equation}
H^2 (G, N) \rightarrow H^2 (G, M)
\end{equation}

is injective, and fix a normalised 2-cocycle $x : G \times G \rightarrow N$. As we shall see, assumption \ref{2.2} pretty much determines $N \rtimes_x G$ as a subgroup of $M \rtimes_x G$ up to conjugacy.

Suppose we are given a subgroup $H$ of $M \rtimes_x G$ extending $G$ by $N$ i.e. the sequence

\begin{equation}
0 \rightarrow N \rightarrow H \xrightarrow{(m,g) \rightarrow g} G \rightarrow e
\end{equation}

is exact. By assumption \ref{2.2}, the extension \ref{2.3} must correspond to $x$ in $H^2 (G, N)$. Hence there is an isomorphism $\theta : N \rtimes_x G \rightarrow H$ such that the diagram

\begin{equation}
\begin{array}{ccc}
0 & \rightarrow & N \\
\downarrow & & \downarrow \theta \\
0 & \rightarrow & H \\
\downarrow & & \downarrow \\
0 & \rightarrow & G \\
\end{array}
\end{equation}

commutes, and this allows us to define a map $\xi : G \rightarrow M$ so that the relation $\theta (0, g) = (\xi (g), g)$ holds for all $g \in G$.

**Proposition 2.2.** With notation and assumptions as above, we have:

(i) $\theta (n, g) = (n + \xi (g), g)$ for all $n \in N, g \in G$.

(ii) The map $\xi : G \rightarrow M$ is a 1-cocycle.

(iii) If $H^1 (G, M) = 0$ then $\theta$ is conjugation by $(m, e)$ for some $m \in M$.

**Proof.**

(i) This is a simple computation using the relation $(n, g) = (n, e)(0, g)$.

(ii) Let $g_1, g_2 \in G$. Using part (i), we get

\begin{align*}
\theta ((0, g_1)(0, g_2)) &= \theta ((x(g_1, g_2), g_1 g_2)) = (x(g_1, g_2) + \xi (g_1 g_2), g_1 g_2), \quad \text{and} \\
\theta ((0, g_1)(0, g_2)) &= (\xi (g_1), g_1)(\xi (g_2), g_2) = (x(g_1, g_2) + \xi (g_1) + g_1 \xi (g_2), g_1 g_2).
\end{align*}

Therefore we must have $\xi (g_1 g_2) = \xi (g_1) + g_1 \xi (g_2)$.

(iii) If $H^1 (G, M) = 0$ then there exists an $m \in M$ such that $\xi (g) = gm - m$ for all $g \in G$. One then uses part (i) to check that

$$(m, e)^{-1}(n, g)(m, e) = (n + gm - m, g) = \theta (n, g).$$

We now give—with a view to motivating the calculations in the next section—a sketch of how we use the above proposition to prove a particular case of the main theorem. Suppose that we have an Artinian local ring $(A, m_A)$ with residue field $k$, and suppose that we are given a subgroup $G \leq SL_n (A)$ with $G \mod m_A = SL_n (k)$. We’d like to know if a conjugate of $G$ contains $SL_n (W(k)_A)$.

Suppose that $J$ is an ideal of $A$ killed by $m_A$. To simplify the discussion further, let’s assume that the quotient $A/J$ is $W_m := W(k)/p^m$, that $W(k)_A = W(k)/p^{m+1}$, and that $G \mod J = SL_n (W_m)$. The assumption that $W(k)_A = W(k)/p^{m+1}$ gives us a choice $k \subseteq J$, and we can set up an identification of $SL_n (A)$ with a twisted semi-direct product $M_0 (J) \rtimes_x SL_n (W_m)$ so that the subgroup $SL_n (W(k)_A)$ gets identified with $M_0 (k) \rtimes_x SL_n (W_m)$. In order to apply Proposition \ref{2.2} and conclude

that $G$ is, up to conjugation, $M \rtimes_x SL_n(W_m)$ for some $\mathbb{F}_p[SL_n(W_m)]$-submodule $M$ of $\mathbb{M}_0(J)$, we need to verify that:

- Assumption 2.2 holds for $\mathbb{F}_p[SL_n(W_m)]$-submodules of $\mathbb{M}_0(J)$ (Theorem 3.1);
- $H^1(SL_n(W_m),\mathbb{M}_0(J)) = (0)$. This is a consequence of known results when $m = 1$ (Theorem 3.2) and Proposition 3.10 in ‘good’ cases. Extra arguments (cf, for instance, Proposition 3.8) are needed when $p$ divides $n$.

We can then conclude that a conjugate of $G$ contains $SL_n(W(k)_A)$ provided $\mathbb{M}_0(k)$ contains $M$. This is derived from the injectivity of $H^2$s (in particular Corollary 3.13); see claim 4.3 in section 4.

3. Cohomology of $SL_n(W/p^m)$

We fix, as usual, a finite field $k$ of characteristic $p$ and set $W_m := W/p^m$ where $W := W(k)$ is the Witt ring of $k$. From here on we assume $n \geq 2$. Our aim is to verify that assumption 2.2 holds. More precisely, we have the following:

**Theorem 3.1.** Let $k$ be a finite field of characteristic $p$ and cardinality at least 4. Suppose $N \subseteq M$ are $\mathbb{F}_p[SL_n(W_m)]$-submodules of $\mathbb{M}_0(k)^r$ for some integer $r \geq 1$. Then the induced map on second cohomology $H^2(SL_n(W_m), N) \to H^2(SL_n(W_m), M)$ is injective.

The proof of Theorem 3.1 relies on knowledge of the first cohomology of $SL_n(W_m)$ with coefficients in $\mathbb{M}_0(k)$. There are a couple more $SL_n(W_m)$ modules to consider when $p$ divides $n$, and we introduce these: Write $S$ for the subspace of scalar matrices in $\mathbb{M}_0(k)$. Thus $S = (0)$ unless $p$ divides $n$ in which case $S = \{\lambda I : \lambda \in k\}$. If $p/n$ we define $V := \mathbb{M}_0(k)/S$.

The first cohomology of $SL_n(W_m)$ with coefficients in $\mathbb{M}_0(k)$ or $V$ is well understood when $m = 1$, and we refer to Cline, Parshall and Scott [3, Table 4.5] for the following the following result. (For results on $H^2(SL_n(k),\mathbb{M}_0(k))$ see [2, 11].)

**Theorem 3.2.** Assume that the cardinality of $k$ is at least 4.

- Suppose $(n,p) = 1$. Then $H^1(SL_n(k),\mathbb{M}_0(k))$ is always 0 except for $H^1(SL_2(\mathbb{F}_5),\mathbb{M}_0(k))$ which is a 1-dimensional $k$-vector space.
- Suppose $p|n$. Then $H^1(SL_n(k),V)$ is a 1-dimensional $k$-vector space.

Throughout this section, we will denote by $\Gamma$ the kernel of the mod $p^m$-reduction map $SL_n(W_{m+1}) \to SL_n(W_m)$. We have suppressed the dependence on $m$ in our notation; this shouldn’t create any great inconvenience. If $M \in \mathbb{M}_0(W)$ is a trace 0, $n \times n$-matrix with coefficients in $W$ then $I + p^m M \mod p^{m+1}$ is in $\Gamma$, and this sets up a natural identification of $\mathbb{M}_0(k)$ and $\Gamma$ compatible with $SL_n(W_m)$-action. The extension of Theorem 3.2 to the group $SL_n(W_m)$ for arbitrary $m$, carried out in subsections 3.2 and 3.3, then relies on the injectivity of transgression maps from $H^1(\Gamma, -)^{SL_n(W_m)}$ to $H^2(SL_n(W_m), -)$.

We end—before we go into the main computations of this section—by reviewing the structure of $\mathbb{M}_0(k)$, and therefore of $\Gamma$, as an $\mathbb{F}_p[SL_n(k)]$-module. For $1 \leq i,j \leq n$, $e_{ij}$ denotes the matrix unit which is 0 at all places except at the $(i,j)$-th place where it is 1.

**Lemma 3.3.** Assume that $k \neq \mathbb{F}_2$ if $n = 2$. 

ones on the diagonal. As an $F$ direct sum of copies of $F$ (3.1) 0 that follows that $X$ is a subspace of $S$, or $X = M_0(k)$. Thus $M_0(k)/S$ is a simple $F_p[SL_n(k)]$-module, and the sequence

\[ 0 \to S \to M_0(k) \to \mathbb{V} \to 0 \]

is non-split when $p|n$.

(ii) If $\phi : M_0(k) \to M_0(k)$ is a homomorphism of $F_p[SL_n(k)]$-modules then there exists a $\lambda \in k$ such that $\phi(A) = \lambda A$ for all $A \in M_0(k)$.

(iii) Suppose $p|n$ and $\phi : M_0(k) \to \mathbb{V}$ is a homomorphism of $F_p[SL_n(k)]$-modules. Then $\phi(S) = (0)$ and the induced map $\phi : \mathbb{V} \to \mathbb{V}$ is multiplication by a scalar in $k$.

**Proof.** Let $U$ be the subgroup $SL_n(k)$ consisting of upper triangular matrices with ones on the diagonal. As an $F_p[U]$-module the semi-simplification of $M_0(k)$ is a direct sum of copies of $F_p$ and $M_0(k)^U = S + ke_{1n}$. Therefore if the $F_p[SL_n(k)]$-submodule $X$ is not a subspace of $S$ then $X$ contains a matrix $aI + be_{1n}$ with $b \neq 0$.

Suppose first that $a = 0$. By considering the action of diagonal matrices, we see that $X$ must in fact contain the full $k$-span of $e_{1n}$. Conjugation by $SL_n(k)$ then implies that $X \supseteq ke_{ij}$ whenever $i \neq j$. Now, under the action of $SL_n(k)$, we can conjugate $e_{ij} + e_{ji}$ with $i \neq j$ to $e_{ii} - e_{jj}$ when $p$ is odd and to $e_{ii} - e_{jj} + e_{ij}$ when $p = 2$. In any case, we can conclude that $X \supseteq k(e_{ii} - e_{jj})$ whenever $i \neq j$. It follows that $X$ must be the whole space $M_0(k)$.

Suppose now $a \neq 0$. Thus $S \neq 0$ and $p$ divides $n$. When $n \geq 3$ the relation

\[ (I + e_{21})(aI + be_{1n})(I - e_{21}) = aI + be_{1n} + be_{2n} \]

implies $be_{2n}$ and, consequently, $be_{1n}$ are in $X$, and so $X = M_0(k)$. When $n = 2$—so $p = 2$ and $k$ has at least 4 elements—we can find a $0 \neq \lambda \in k$ with $\lambda^2 \neq 1$. Conjugating by \[ \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix} \], we see that $aI + b\lambda^2 e_{1n} \in X$. This gives $0 \neq b(\lambda^2 - 1)e_{1n} \in X$ and so $X = M_0(k)$.

Now for part (ii). Since $\phi$ commutes with the action of $SL_n(k)$, the subspaces $M_0(k)^U$ and $M_0(k)^U$ are invariant under $\phi$. When $p$ divides $n$ the first of these gives $\phi(S) \subseteq S$; if $p$ does not divide $n$, then $M_0(k)^U = ke_{1n}$ and so we must have $\phi(e_{1n}) = \lambda e_{1n}$ for some $\lambda \in k$. In any case, we can find a $\lambda \in k$ such that the $F_p[SL_n(k)]$-module homomorphism $\phi - [\lambda] : M_0(k) \to M_0(k)$ given by $A \to \phi(A) - \lambda A$ has non-trivial kernel. We can then conclude, by part (i) and a simple dimension count, that the kernel has to be the whole space $M_0(k)$, and therefore $\phi$ must be multiplication by $\lambda$.

For part (iii), that $S \subseteq \ker \phi$ follows from part (i). The second part is proved along the same lines as the proof of part (ii) by considering $\phi(e_{1n})$. \hfill \Box

### 3.1. Determination of $H^1(SL_n(W_m), k)$

Let $k$ have cardinality $p^d$. Our aim is to show that $H^1(SL_n(W_m), k)$ vanishes, subject to some mild restrictions on $k$. We do this inductively using inflation–restriction after dealing with the base case $m = 1$ by adapting Quillen’s result in the general linear group case (see section 11 of [10]).

To start off we impose no restrictions other than $n \geq 2$. Denote by $T$ the subgroup of diagonal matrices in $SL_n(k)$ and write $(t_1, t_\ldots, t_n)$ for the diagonal matrix with $(i, i)$-th entry $t_i$. The image of the homomorphism $T \to (k^*)^{n-1}$ given
by

\[(t_1, \ldots, t_n) \rightarrow (t_2/t_1, \ldots, t_n/t_{n-1})\]

has index \( h := \text{hcf}(n, p^d - 1) \) in \((k^*)^{n-1}\). Taking this into account and following the remark at end of section 11 of [10], the proof covering the general linear group case only needs a small modification at one place\(^1\) to give the following:

**Theorem 3.4.** Let \( k \) be a finite field of characteristic \( p \) and cardinality \( p^d \). Then \( H^i(SL_n(k), \mathbb{F}_p) = 0 \) for \( 0 < i < d(p - 1)/h \) where \( h := \text{hcf}(n, p^d - 1) \).

For a fixed \( n \), Theorem 3.4 implies the vanishing of \( H^1(SL_n(k), k) \) and \( H^2(SL_n(k), k) \) for fields with sufficiently large cardinality. To get a stronger result for \( H^1 \) and \( H^2 \) covering fields with small cardinality, we will need to carry out a slightly more detailed analysis.

In order to show \( H^*(SL_n(k), \mathbb{F}_p) = 0 \) it is enough to check that \( H^*(\mathbb{U}, \mathbb{F}_p)^T = 0 \) where \( \mathbb{U} \) is the subgroup of upper triangular matrices with ones on the diagonal. Fix an algebraic closure \( \overline{\mathbb{F}}_p \) of \( \mathbb{F}_p \) containing \( k \). Since \( \mathbb{T} \) is an abelian group of order prime to \( p \), the \( \overline{\mathbb{F}}_p[\mathbb{T}] \)-module \( H^*(\mathbb{U}, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p \) is isomorphic to a direct sum of characters; we will then have to check that none of these can be the trivial character.

Let \( \Delta^+ \) be the set of characters \( a_{ij} : \mathbb{T} \rightarrow k^\times \) given by \( a_{ij}(t_1, \ldots, t_n) := t_i/t_j \) where \( 1 \leq i < j \leq n \). The analysis in [10] section 11 shows that the Poincaré series of \( H^*(\mathbb{U}) \) as a representation of \( \mathbb{T} \), denoted by \( P.S.(H^*(\mathbb{U})) \), satisfies the bound

\[
(3.2) \quad P.S.(H^*(\mathbb{U})) := \sum_{i \geq 0} \text{cl}(H^i(\mathbb{U}, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p)z^i \ll \prod_{a \in \Delta^+} \prod_{b=0}^{d-1} \frac{1 + a^{-p^b}z}{1 - a^{-p^b}z^2}
\]

in \( R_{\mathbb{F}_p}(\mathbb{T})[[z]] \). Here \( R_{\mathbb{F}_p}(\mathbb{T}) \) is the Grothendieck group for representations of \( \mathbb{T} \) over \( \mathbb{F}_p \), and \( \text{cl}(V) \) is the class of a \( \mathbb{F}_p[\mathbb{T}] \)-module \( V \) in \( R_{\mathbb{F}_p}(\mathbb{T}) \): given \( \mathbb{F}_p[\mathbb{T}] \)-modules \( V_0, V_1, V_2, \ldots \) and \( W_0, W_1, W_2, \ldots \), the bound

\[
\sum_{i \geq 0} \text{cl}(W_i)z^i \ll \sum_{i \geq 0} \text{cl}(V_i)z^i
\]

in \( R_{\mathbb{F}_p}(\mathbb{T})[[z]] \) expresses the property that \( W_i \) is isomorphic to an \( \mathbb{F}_p[\mathbb{T}] \)-submodule of \( V_i \) for every integer \( i \geq 0 \). Thus the right hand side of (3.2) tells us which characters might occur in the decomposition of the \( \mathbb{F}_p[\mathbb{T}] \)-module \( H^*(\mathbb{U}, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p \).

Note that our choice of a positive root system \( \Delta^+ \) is different from the one in [10]; the choice made there leads to a sign discrepancy in the upper bound (3.2) (but doesn’t affect any of the results derived from it). If we use the ordering on \( \Delta^+ \) given by \((i', j') \leq (i, j)\) if either \( i' < i \), or \( i' = i \) and \( j \leq j' \), then with notation as in [10] we have a central extension

\[
0 \rightarrow k_a \rightarrow \mathbb{U}/\mathbb{U}_a \rightarrow \mathbb{U}/\mathbb{U}_{a'} \rightarrow 1
\]

with \( T \)-action and the argument in [10] carries through verbatim.

It is then straightforward to work out the coefficients of \( z \) and \( z^2 \) on the right hand side of (3.2) and we can conclude the following: If \( \chi : \mathbb{T} \rightarrow \mathbb{F}_p^\times \) is a character occurring in \( \text{cl}(H^i(\mathbb{U}, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p) \), \( i = 1, 2 \), then \( \chi^{-1} \) is either

- a Galois conjugate of a positive root i.e. \( \chi^{-1} = a^pb \) for some positive root \( a \in \Delta \) and integer \( 0 \leq b < d \), or

\(^1\)The congruence just before Lemma 16 changes to a congruence modulo \((p^d - 1)/h\).
a product $\alpha \alpha'$ where $\alpha, \alpha'$ are Galois conjugates of positive roots and $\alpha \neq \alpha'$. (This case happens only when $i = 2$.)

Thus, taking Galois conjugates as needed, we need to determine when $a_{ij}$ or $a_{ij}a_{kl}^p$ is the trivial character, where $a_{ij}, a_{kl} \in \Delta^+$ and $0 < b < d$ in the case $(i, j) = (k, l)$. The first case is immediate: $a_{ij}$ is never the trivial character except when $k = \mathbb{F}_2$, or $n = 2$ and $k = \mathbb{F}_3$.

Now for the second case. We now have integers $1 \leq i < j \leq n, 1 \leq k < l \leq n, 0 \leq b < d$ with $b \neq 0$ if $(i, j) = (k, l)$ such that the following relation holds:

$$(3.3) \quad \frac{t_i}{t_j} \left(\frac{t_k}{t_l}\right)^p = 1 \quad \text{for all} \quad (t_1, \ldots, t_n) \in \mathbb{T}.$$

We will determine for which fields the above relation holds by specialising suitably. We exclude $k = \mathbb{F}_2$ in what follows.

Firstly, let’s consider the case when $i, j, k$ and $l$ are distinct. Thus $n \geq 4$. We can specialise $3.3$ to $t_k = t_l = 1$ and $t_i = t_j^{-1} = t$ for $t \in k^\times$. We then get $t^2 = 1$ for all $t \in k^\times$—which implies $k$ can only be $\mathbb{F}_3$. Furthermore, if $n \geq 5$ we have an even better specialisation: we can choose $t_j = t_k = t_l = 1$ and $t_i$ freely, and conclude $3.3$ never holds.

Next, suppose the cardinality of $(i, j, k, l)$ is 3. If we suppose $(i, j, k, l) = (i, k, l)$ (the case $(i, j, k, l) = (j, k, l)$ is similar), then specialisation to $t_j = t_k = t_l = t^{-1}$ and $t_i = t^2$ implies that $t^2 = 1$ for all $t \in k^\times$ i.e. $k$ is a subfield of $\mathbb{F}_4$. If in addition $n \geq 4$ we can take $t_k = t_l = 1$ and then there is a free choice for either $t_i$, so $3.3$ cannot hold.

Finally consider the case when the cardinality of $(i, j, k, l)$ is 2. We must then have $i = k, j = l$ and $1 \leq b < d$. Taking $t_i = t = t_j^{-1}$, we get $t^{2(1+p^b)} = 1$ for all $t \in k^\times$, and so $2(1 + p^b) = p^d - 1$. This only works when $k = \mathbb{F}_9$. Moreover, when $n \geq 3$, we can set $t_j = 1$ and then the relation $3.3$ implies $t^{p^b + 1} = 1$ for all $t \in k^\times$.

So $p^b + 1 = p^d - 1$ and $k$ is necessarily $\mathbb{F}_4$. Therefore in the case $(i, j) = (k, l)$ the relation $3.3$ holds only when $n = 2$ and $k = \mathbb{F}_9$.

We have thus proved the first part of the following:

**Theorem 3.5.** Let $k \neq \mathbb{F}_2$ be a finite field of characteristic $p$ and let $n \geq 2$ be an integer. Further, assume that

- if $n = 4$ then $k$ is not $\mathbb{F}_3$;
- if $n = 3$ then $k \neq \mathbb{F}_4$;
- if $n = 2$ then $k$ is not $\mathbb{F}_3$ or $\mathbb{F}_9$.

Then $H^1(SL_n(k), \mathbb{F}_p)$ and $H^2(SL_n(k), \mathbb{F}_p)$ are both trivial. Furthermore, under the same assumptions on $k$, we have $H^1(SL_n(W_m), k) = (0)$ for all integers $m \geq 1$.

The second part is proved by induction using inflation-restriction and the vanishing of $H^1(SL_n(k), k)$ from the first part. With $\Gamma = \ker(SL_n(W_{m+1}) \to SL_n(W_m))$ we have

$$0 \to H^1(SL_n(W_m), k) \to H^1(SL_n(W_{m+1}), k) \to H^1(\Gamma, k)^{SL_n(W_m)}.$$

Now the natural identification of $M_0(k)$ with $\Gamma$ compatible with $SL_n(W_m)$-actions sets up an isomorphism between $H^1(\Gamma, k)^{SL_n(W_m)}$ and $\text{Hom}_{\Gamma}([SL_n(k)]^m, M_0(k), k)$. The latter vector space is easily seen to be $(0)$ by a dimension count using Lemma $3.3$, and the theorem follows.
3.2. Determination of $H^1(SL_n(W_m), M_0(k))$. The result here is that all cohomology classes come from $H^1(SL_n(k), M_0(k))$. More precisely:

**Proposition 3.6.** Suppose that $k$ has cardinality at least 4 and that $k \neq \mathbb{F}_4$ when $n = 3$. The inflation map $H^1(SL_n(W_m), M_0(k)) \to H^1(SL_n(W_{m+1}), M_0(k))$ is then an isomorphism for all integers $m \geq 1$.

By the inflation–restriction exact sequence, the above proposition follows if we can show that the transgression map

$$\delta : H^1(\Gamma, M_0(k))^{SL_n(W_m)} \to H^2(SL_n(W_m), M_0(k))$$

is injective. Since $H^1(\Gamma, M_0(k))^{SL_n(W_m)}$ has dimension 1 as a $k$-vector space by Lemma 3.3, we just need to check that $\delta$ is not the zero map.

Recall that we have a natural identification of $\Gamma$ with $M_0(k)$ given by $\phi(I + p^mA) := A \mod p$. Hence by Proposition 3.1 we see that $\delta(-\phi)$ must be the class of the extension

$$I \to \Gamma \to SL_n(W_{m+1}) \to SL_n(W_m) \to I.$$ 

Therefore the required conclusion follows if the above extension is non-split, and we address this below.

**Proposition 3.7.** Assume that $k$ has cardinality at least 4 and that if $n = 3$ then $k \neq \mathbb{F}_4$. Then the extension

$$I \to \Gamma \to SL_n(W_{m+1}) \to SL_n(W_m) \to I$$

does not split for any integer $m \geq 1$.

**Proof.** This should be well known, but it is hard to find a reference in the form we need. We therefore sketch a proof for completeness. The case when $n = 2$ and $p \geq 5$ is discussed in [13]. For the non-splitting of the above sequence when $k = \mathbb{F}_p$ see [11]; for non-splitting in the $GL_n$ case see [12].

If $R$ is a commutative ring and $r \in R$ then we write $N(r)$ for the elementary nilpotent $n \times n$ matrix in $M(R)$ with zeroes in all places except at the $(1,2)$-th entry where it is $r$. Note that $N(r)^2 = 0$ and that

$$(I + N(r))^k = I + kN(r) = I + krN(1)$$

for every integer $k$.

Suppose there is a homomorphism $\theta : SL_n(W_m) \to SL_n(W_{m+1})$ which splits the above exact sequence 3.3. We fix a section $s : W_m \to W_{m+1}$ that sends Teichmüller lifts to Teichmüller lifts. For instance, if we think in terms of Witt vectors of finite length then we can take $s$ to be the map $(a_1, \ldots, a_m) \to (a_1, \ldots, a_m, 0)$. Finally, take the map $A : W_m \to M_0(k)$ so that the relation

$$\theta(I + N(x)) = (I + p^mA(x))(I + N(s(x)))$$

holds for all $x \in W_m$ (and we have abused notation and identified $p^mW_{m+1}$ with $p^m(k)$).

Now $\theta(I + N(x))$ has order dividing $p^m$ in $SL_n(W/p^{m+1})$ for any $x \in W_m$. Writing $N$ and $A$ in lieu of $N(s(x))$ and $A(x)$, we have

$$(I + N)^k(I + p^mA)(I + N)^{-k} = I + p^m(A + kNA - kAN - k^2NAN)$$

for any integer $k$, and a small calculation yields

$$\theta(I + N(x))^p = (I + \alpha p^m(NA - AN) - \beta p^mNAN)(I + p^mN).$$
where $\alpha = p^m(p^m - 1)/2$ and $\beta = p^n(p^m - 1)(2p^m - 1)/6$. Hence if either $p \geq 5$, or $p$ divides 6 and $m \geq 2$, then $\theta(I + N(1))$ cannot have order $p^m$—a contradiction.

From here on $p$ divides 6 and $m = 1$; so $\theta : SL_n(k) \to SL_n(W/p^2)$ and $s(x) = \hat{x}$.

Applying $\theta$ to $(I + N(x))(I + N(y)) = I + N(x + y)$ and multiplying by $N(1)$ on the left and right then gives $N(1)A(x)N(1) + N(1)A(y)N(1) = N(1)A(x + y)N(1)$, and therefore

$$a_{21}(x + y) = a_{21}(x) + a_{21}(y)$$

for all $x, y \in k$.

Suppose now $p = 3$. The expression (3.5) for $\theta(I + N(x))^p$ then becomes

$$I + pxN(1) + px^2N(1)A(x)N(1) = I.$$ Comparing the $(1, 2)$-th entries on both sides we get $x^2a_{21}(x) + x = 0$ for all $x \in k$.

Thus for $x \neq 0$ we have $a_{21}(x) = -x^{-1}$. This contradicts the linearity of $a_{21}$ if $k \neq \mathbb{F}_3$.

Before we consider the case $p = 2$ specifically, we make some relevant simplifications by considering the action of $\Gamma$, the subgroup of diagonal matrices in $SL_n(k)$. For $t = (t_1, \ldots, t_n) \in SL_n(k)$ we define $\hat{t} := (t_1, \ldots, t_n) \in SL_n(W/p^2)$. We must then have $\theta(t) = B(t)\hat{t}$ where $B : T \to \Gamma$ is a 1-cocycle. Since $H^1(T, \Gamma) = 0$ we can assume, after conjugation by a matrix in $\Gamma$ if necessary, that $\theta(t) = \hat{t}$. The homomorphism condition applied to $\theta(t(I + N(x))t^{-1})$ then gives

$$(I + pA(t_1x/t_2))(I + (t_1x/t_2)N(1)) = (I + ptA(x)t^{-1})(I + \hat{t}xN(1)\hat{t}^{-1})$$

where $t = (t_1, \ldots, t_n)$. Hence $A(t_1x/t_2) = tA(x)t^{-1}$ for all $t \in T$ and $x \in k$. By considering specialisations $t_1 = t_2 = 1$ for $n \geq 4$ and $t = (\lambda, \lambda, \lambda^{-2})$ when $n = 3$, we conclude that $a_{ij}(x) = 0$ if $i \neq j$ and $i \geq 3$ or $j \geq 3$ provided $k$ has cardinality at least 4 and $k \neq \mathbb{F}_4$ when $n = 3$.

We now go back to assuming $p = 2$ and $m = 1$. Relation (3.5) then becomes

$$I + px(N(1)A(x) + A(x)N(1)) + px^2N(1)A(x)N(1) + pxN(1) = I,$$

and we get $a_{21}(x) = 0$ and $a_{11}(x) + a_{22}(x) = 1$ whenever $x \neq 0$. Hence if $k$ has cardinality at least 4 and $k \neq \mathbb{F}_4$ when $n = 3$, then $\theta(I + N(x))$ is an upper-triangular matrix and so $a_{ii}(x + y) = a_{ii}(x) + a_{ii}(y)$ for $i = 1, \ldots, n$ and $x, y \in k$.

Since $k$ has at least 4 elements we can choose $x, y \in k$ with $xy(x + y) \neq 0$, and this gives

$$1 = a_{11}(x + y) + a_{22}(x + y) = (a_{11}(x) + a_{11}(y)) + (a_{22}(x) + a_{22}(y)) = 1 + 1$$

—a contradiction. □

3.3. $H^1$ when $n$ and $p$ are not coprime. Suppose now that $p$ divides $n$. Thus $M_0(k)$ is reducible and we have the exact sequence

$$0 \to \mathbb{S} \xrightarrow{i} M_0(k) \xrightarrow{\pi} V \to 0.$$ (3.6)

We then have the following analogue of Proposition (3.8).

**Proposition 3.8.** Assume that $p$ divides $n$ and that the cardinality of $k$ is at least 4. The inflation map $H^1(SL_n(W_m), V) \to H^1(SL_n(W_{m+1}), V)$ is then an isomorphism for all integers $m \geq 1$. 

Denote by $Z$ the subgroup of $\Gamma$ consisting of the scalar matrices $(1 + p^m \lambda)I$. We then have an exact sequence
\[
(3.7) \quad I \to \Gamma/Z \to SL_n(W_{m+1})/Z \xrightarrow{\text{mod } p^m} SL_n(W_m) \to I.
\]
Under the natural identification $\phi : \Gamma \to M_0(\mathbf{k})$ given by $\phi(I + p^m A) := A \mod p$ of $\Gamma$ with $M_0(\mathbf{k})$, the groups $Z$, resp. $\Gamma/Z$, correspond to $S$, resp. $\mathcal{V}$. If we set $\psi : \Gamma/Z \to \mathcal{V}$ to be the map induced by $\phi \mod S$, then Proposition 2.4 shows that $\delta(-\psi)$ is the cohomology class of the extension 3.7 under the transgression map
\[
\delta : H^1(\Gamma/Z, \mathcal{V})^{SL_n(W_m)} \to H^2(SL_n(W_m), \mathcal{V}).
\]
Now, by Lemma 3.3 the map
\[
H^1(\Gamma/Z, \mathcal{V})^{SL_n(W_m)} \to H^1(\Gamma, \mathcal{V})^{SL_n(W_m)}
\]
is an isomorphism of 1-dimensional $\mathbf{k}$-vector spaces. Thus the conclusion of Proposition 3.8 holds exactly when the extension 3.7 is non-split.

In many cases the required non-splitting follows from a simple modification of the proof of Proposition 3.8. More precisely, we have the following:

**Lemma 3.9.** Suppose $p|n$, and assume that either $p \geq 5$ or $m \geq 2$. Then the extension
\[
I \to \Gamma/Z \to SL_n(W_{m+1})/Z \to SL_n(W_m) \to I
\]
does not split.

**Proof.** We give a sketch: Suppose $\theta : SL_n(W_m) \to SL_n(W_{m+1})/Z$ is a section. Then, with $N(1)$ the elementary nilpotent matrix described in the proof of Proposition 3.7 we have $\theta(I + N(1)) = (I + p^n A)(I + N(1))$ modulo $Z$ for some $A \in M_0(\mathbf{k})$. Because elements in $Z$ are central, relation 3.9 holds modulo $Z$ and the lemma easily follows. \qed

We now deal with the case $m = 1$ and complete the proof of Proposition 3.8. Consider the commutative diagram
\[
\begin{array}{ccc}
H^1(\Gamma, M_0(\mathbf{k}))^{SL_n(W_m)} & \xrightarrow{\delta} & H^2(SL_n(W_m), M_0(\mathbf{k})) \\
\downarrow \pi^* & & \downarrow \pi^* \\
H^1(\Gamma, \mathcal{V})^{SL_n(W_m)} & \xrightarrow{\delta} & H^2(SL_n(W_m), \mathcal{V})
\end{array}
\]
where $\pi^*$ is the map induced by the projection $\pi : M_0(\mathbf{k}) \to \mathcal{V}$. Now, the map $\pi^*$ on the left hand side of the square is an isomorphism by Lemma 3.3. Since the cardinality of $\mathbf{k}$ is at least 4 (and remembering that we are also assuming $p|n$), the top row of the square 3.8 is an injection by Proposition 3.6. Furthermore, Theorem 3.5 implies $H^2(SL_n(W_m), \mathbf{k}) = (0)$ and therefore the map $\pi^*$ on the right hand side of the square is an injection. Hence the bottom row of the square 3.8 is also an injection and we can conclude the proposition.

**Remark 3.10.** As we saw in course of the proof, Proposition 3.8 implies the following extension of Lemma 3.9.

**Corollary 3.11.** Assume that $p$ divides $n$ and $\mathbf{k}$ has cardinality at least 4. Then the sequence
\[
I \to \Gamma/Z \to SL_n(W_{m+1})/Z \to SL_n(W_m) \to I
\]
does not split for any integer $m \geq 1$. 
We end this subsection with a description of the relations between the cohomology groups with coefficients $\mathbb{M}_0(k)$, $S$ and $V$:

**Proposition 3.12.** Suppose that $p$ divides $n$ and that $k$ has at least 4 elements. Then, with $i$ and $\pi$ as in the exact sequence 3.6, the map $H^1(SL_n(W_m), \mathbb{M}_0(k)) \rightarrow H^1(SL_n(W_m), V)$ is an isomorphism and $0 \rightarrow H^2(SL_n(W_m), S) \xrightarrow{i^*} H^2(SL_n(W_m), \mathbb{M}_0(k)) \xrightarrow{\pi^*} H^2(SL_n(W_m), V)$ is exact.

**Proof.** The long exact sequence obtained from 3.6 shows that we just need to check $H^2(SL_n(W_m), \mathbb{M}_0(k)) \xrightarrow{\pi^*} H^2(SL_n(W_m), V)$ is an isomorphism. This holds when $m = 1$ because both $H^1(SL_n(k), S)$ and $H^2(SL_n(k), S)$ are 0 by Theorem 3.3. For general $m$ we can use induction because in the commutative diagram

$$
\begin{array}{ccc}
H^1(SL_n(W_m), \mathbb{M}_0(k)) & \xrightarrow{\pi^*} & H^1(SL_n(W_m), V) \\
\downarrow & & \downarrow \\
H^1(SL_n(W_{m+1}), \mathbb{M}_0(k)) & \xrightarrow{\pi^*} & H^1(SL_n(W_{m+1}), V)
\end{array}
$$

the vertical inflation maps are isomorphisms by Proposition 3.6 and Proposition 3.8. □

### 3.4. Proof of Theorem 3.1
Recall that we want to show the injectivity of $H^2(SL_n(W_m), N) \rightarrow H^2(SL_n(W_m), M)$ whenever $N \subseteq M$ are $F_p[SL_n(W_m)]$-submodules of $\mathbb{M}_0(k)^r$ for some integer $r \geq 1$.

We will write $H^r(X)$ to mean $H^r(SL_n(W_m), X)$. Note that it is enough to show that $H^2(M) \rightarrow H^2(\mathbb{M}_0(k)^r)$ is injective for all $F_p[SL_n(W_m)]$-submodules $M$ of $\mathbb{M}_0(k)^r$. If $(n, p) = 1$ then $\mathbb{M}_0(k)^r$ is semi-simple and the desired injectivity is immediate. So we will suppose $p$ divides $n$ from here on.

Consider the commutative diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & M \cap S^r \\
\downarrow i & & \downarrow i \\
M & \xrightarrow{\pi} & M/(M \cap S^r) & \longrightarrow & 0
\end{array}
$$

(3.9)

$$
\begin{array}{ccc}
0 & \longrightarrow & S^r \\
\downarrow i & & \downarrow j \\
\mathbb{M}_0(k)^r & \xrightarrow{\pi} & V^r & \longrightarrow & 0
\end{array}
$$

where the $i$’s are inclusions. Thus $j$ is necessarily an injection. Taking cohomology and using Proposition 3.12 we get a commutative diagram

$$
\begin{array}{ccc}
H^2(M \cap S^r) & \longrightarrow & H^2(M) & \longrightarrow & H^2(M/(M \cap S^r)) \\
\downarrow i^* & & \downarrow j^* & & \downarrow j^* \\
H^2(S^r) & \longrightarrow & H^2(\mathbb{M}_0(k)^r) & \longrightarrow & H^2(V^r)
\end{array}
$$

(3.10)

in which the horizontal rows are exact. Now the maps $H^2(M \cap S^r) \xrightarrow{i^*} H^2(S^r)$ and $H^2(M/(M \cap S^r)) \xrightarrow{j^*} H^2(V^r)$ are injective since $S^r$ and $V^r$ are semi-simple and $i$, $j$ are injections. A straightforward diagram chase then shows that $i^* : H^2(M) \rightarrow H^2(\mathbb{M}_0(k)^r)$ is an injection, and this completes the proof of Theorem 3.1. □

As a consequence, we have the following:
Corollary 3.13. Let \( k \) be a finite field of characteristic \( p \) and cardinality at least 4, and let \( M, N \) be two \( \mathbb{F}_p[SL_n(W)] \)-submodules of \( M_0(k)^r \) for some integer \( r \geq 1 \).
Suppose we are given \( x \in H^2(SL_n(W), M) \) and \( y \in H^2(SL_n(W), N) \) such that \( x \) and \( y \) represent the same cohomology class in \( H^2(SL_n(W), M_0(k)^r) \). Then there exists \( z \in H^2(SL_n(W), M \cap N) \) such that \( x = z \), resp. \( y = z \), holds in \( H^2(SL_n(W), M) \), resp. \( H^2(SL_n(W), N) \).

Proof. Consider the exact sequence
\[
0 \to M \cap N \xrightarrow{m \mapsto m \circ m} M \oplus N \xrightarrow{m \mapsto n - n} M + N \to 0.
\]
By Theorem 3.1, we get a short exact sequence
\[
0 \to H^2(M \cap N) \to H^2(M) \oplus H^2(N) \to H^2(M + N).
\]
Since \( H^2(M + N) \to H^2(M_0(k)^r) \) is injective, it follows that \( x \oplus y \) is zero in \( H^2(M + N) \) and therefore must be in the image of \( H^2(M \cap N) \).

4. Proof of the main theorem

From here on, we assume that we are given finite fields \( k \subseteq k' \) of characteristic \( p \).
Let \( C \) be the category of complete local Noetherian rings \( (A, m_A) \) with residue field \( A/m_A = k' \) and with morphisms required to be identity on \( k' \). We will abbreviate \( W(k) \) and \( W(k)_A \) for \( A \) an object in \( C \) to \( W \) and \( W_A \) respectively. Recall that \( W_A \) is the closed subring of \( A \) generated by the Teichmüller lifts of elements of \( k \); it is not an object in \( C \) unless \( k = k' \). Throughout this section we assume that the finite field \( k \) satisfies the hypothesis of the main theorem:

Assumption 4.1. The cardinality of \( k \) is at least 4. Furthermore, \( k \neq \mathbb{F}_5 \) if \( n = 2 \) and that \( k \neq \mathbb{F}_4 \) if \( n = 3 \).

Suppose we are given a local ring \( (A, m_A) \) in \( C \) and a closed subgroup \( G \) of \( GL_n(A) \) such that \( G \mod m_A \supseteq SL_n(k) \). We want to show that \( G \) contains a conjugate of \( SL_n(W_A) \). Now, without any loss of generality, we may assume that \( G \mod m_A = SL_n(k) \). The quotient \( G/(G \cap SL_n(A)) \) is then pro-\( p \). This implies that \( G \cap SL_n(A) \mod m_A \) is a normal subgroup of \( SL_n(k) \) with index a power of \( p \). Now \( PSL_n(k) \) is simple since the cardinality of \( k \) is at least 4. Consequently we must have \( G \cap SL_n(A) \mod m_A = SL_n(k) \). Along with the fact that \( A \) is the inductive limit of Artinian quotients \( A/m_A^3 \), we see that the main theorem follows from the following proposition:

Proposition 4.2. Let \( \pi : (A, m_A) \to (B, m_B) \) be a surjection of Artinian local rings in \( C \) with \( m_A \ker \pi = 0 \), and let \( H \) be a subgroup of \( SL_n(A) \) such that \( \pi H = SL_n(W_B) \). Assume that \( k \) satisfies assumption 4.1. Then we can find a \( u \in GL_n(A) \) such that \( \pi u I = I \) and \( uH^{-1} \supseteq SL_n(W_A) \).

For the proof of the above proposition, let’s set \( G := \pi^{-1} SL_n(W_B) \cap SL_n(A) \) where \( \pi^{-1} SL_n(W_B) \) is the pre-image of \( SL_n(W_B) \) under the map \( \pi : GL_n(A) \to GL_n(B) \). We then have an exact sequence
\[
0 \to M(\ker \pi) \xrightarrow{j} \pi^{-1} SL_n(W_B) \xrightarrow{\pi} SL_n(W_B) \to I
\]
with \( j(v) = I + v \) for \( v \in M(\ker \pi) \), and this restricts to
\[
0 \to M_0(\ker \pi) \xrightarrow{j} G \xrightarrow{\pi} SL_n(W_B) \to I.
\]
Note that $\mathcal{M}(\ker \pi) \cong \mathcal{M}(\mathbb{k}) \otimes_{\mathbb{k}} \ker \pi$ and $\mathcal{M}_0(\ker \pi) \cong \mathcal{M}_0(\mathbb{k}) \otimes_{\mathbb{k}} \ker \pi$ as $\mathbb{k}[\text{SL}_n(W_B)]$-modules.

In what follows we will abbreviate $H^*(\text{SL}_n(W_B), X)$ to simply $H^*(X)$. For $X \subseteq \text{SL}_n(A)$, we set $\mathcal{M}_0(X)$ to be the set of matrices $v \in \mathcal{M}_0(\ker \pi)$ such that $j(v) \in X$. We then have the following:

**Claim 4.3.** $\mathcal{M}_0(\text{SL}_n(W_A)) \subseteq \mathcal{M}_0(H)$.

Let’s assume the above claim and carry on with the proof of Proposition 4.2. Fix a section $s : \text{SL}_n(W_B) \to \text{SL}_n(W_A)$ that sends identity to identity and set $x : \text{SL}_n(W_B) \times \text{SL}_n(W_B) \to \mathcal{M}_0(\text{SL}_n(W_A))$ to be the resulting 2-cocycle representing the extension

\[
0 \to \mathcal{M}_0(\text{SL}_n(W_A)) \xrightarrow{j} \text{SL}_n(W_A) \to \text{SL}_n(W_B) \to I.
\]

The section $s$ and cocycle $x$ thus set up an identification

\[
\varphi : \pi^{-1}\text{SL}_n(W_B) \to \mathcal{M}_0 \rtimes_s \text{SL}_n(W_B),
\]

and we have the following commutative diagram (cf. diagram 2.3)

\[
\begin{array}{cccccc}
0 & \longrightarrow & \mathcal{M}_0(H) & \longrightarrow & \mathcal{M}_0(H) \rtimes_s \text{SL}_n(W_B) & \longrightarrow & \text{SL}_n(W_B) & \longrightarrow & I \\
\downarrow & & \downarrow & & \downarrow \varphi & & \text{id} & & \\
0 & \longrightarrow & \mathcal{M}_0(H) & \longrightarrow & \varphi H & \longrightarrow & \text{SL}_n(W_B) & \longrightarrow & I.
\end{array}
\]

Suppose first that $(p, n) = 1$. Our assumptions on $\mathbb{k}$ imply that we can combine Theorem 3.2 and Proposition 3.6 to conclude that $H^1(\mathcal{M}_0(\mathbb{k})) = 0$. Consequently, we get $H^1(\mathcal{M}_0(\ker \pi)) = 0$. Furthermore, $H^2(\mathcal{M}_0(H)) \to H^2(\mathcal{M}_0(\ker \pi))$ is an injection by Theorem 3.1. Hence we can apply Proposition 2.2 and conclude that $\mathcal{M}_0(H) \rtimes_s \text{SL}_n(W_B) = \varphi u H u^{-1}$ for some $u \in G$ (cf. sequence 2.2) with $\pi(u) = I$.

Suppose now $p$ divides $n$. Since $H^1(\mathbb{k}) = 0$ by Theorem 3.5, we get the following exact sequence

\[
0 \to \mathbb{k} \to H^1(\mathcal{M}_0(\mathbb{k})) \to H^1(\mathcal{M}(\mathbb{k})) \to 0 \to H^2(\mathcal{M}_0(\mathbb{k})) \to H^2(\mathcal{M}(\mathbb{k}))
\]

from $0 \to \mathcal{M}_0(\mathbb{k}) \to \mathcal{M}(\mathbb{k}) \to \mathbb{k} \to 0$. Now since $\dim_{\mathbb{k}} H^1(\mathbb{V}) = 1$ by Theorem 3.2 and Proposition 3.8 we must also have $\dim_{\mathbb{k}} H^1(\mathcal{M}_0(\mathbb{k})) = 1$ by Proposition 3.12. Hence $H^1(\mathcal{M}(\mathbb{k})) = 0$ and, consequently, $H^1(\mathcal{M}(\ker \pi)) = 0$. Along with Theorem 3.1 the above exact sequence also shows that $H^2(\mathcal{M}_0(H)) \to H^2(\mathcal{M}(\ker \pi))$ is an injection. Hence $\mathcal{M}_0(H) \rtimes_s \text{SL}_n(W_B) = \varphi u H u^{-1}$ for some $u \in \pi^{-1}\text{SL}_n(W_B)$ (cf. sequence 2.2) with $\pi(u) = I$ by Proposition 2.2.

In any case, we have found a $u \in \text{GL}_n(A)$ with $\pi(u) = I$ and $\varphi u H u^{-1} = \mathcal{M}_0(H) \rtimes_s \text{SL}_n(W_B)$. Finally,

\[
\varphi \text{SL}_n(W_A) = \mathcal{M}_0(\text{SL}_n(W_A)) \rtimes_s \text{SL}_n(W_B) \subseteq \mathcal{M}_0(H) \rtimes_s \text{SL}_n(W_B)
\]

as $\mathcal{M}_0(\text{SL}_n(W_A)) \subseteq \mathcal{M}_0(H)$ by our claim 4.3 and the proposition follows.

We now establish the claim to complete the argument.

**Proof of Claim 4.3.** There is nothing to prove if $W_A \xrightarrow{\mathcal{M}} W_B$ is an injection (as $\mathcal{M}_0(\text{SL}_n(W_A))$ is then 0). Therefore we may suppose that we have a natural identification of $W_{m+1} \xrightarrow{\mathcal{M}} W_m$ with $W_{m+1} \to W_m$ for some integer $m \geq 1$, and consequently an identification of $\mathcal{M}_0(\text{SL}_n(W_A))$ with $\mathcal{M}_0(\mathbb{k})$. We will freely use these identifications in what follows.
As in the proof of the proposition, let \( x \in H^2(\mathcal{M}_0(k)) \) represent the extension and let \( y \in H^2(\mathcal{M}_0(H)) \) represent the extension

\[
0 \to \mathcal{M}_0(H) \to H \to SL_n(W_R) \to I.
\]

Then \( x \) and \( y \) represent the same cohomology class in \( H^2(\mathcal{M}_0(\ker \pi)) \). By Corollary \([13, \text{Chapter IV}(3)]\), there is an \( z \in H^2(\mathcal{M}_0(k) \cap \mathcal{M}_0(H)) \) such that \( x \) and \( z \) (resp. \( y \) and \( z \)) represent the same cohomology class in \( H^2(\mathcal{M}_0(k)) \) (resp. \( H^2(\mathcal{M}_0(H)) \)).

Suppose the claim \( \mathcal{M}_0(k) \subseteq \mathcal{M}_0(H) \) is false. Then we must have \( \mathcal{M}_0(k) \cap \mathcal{M}_0(H) \subseteq \mathcal{S} \) by Lemma \([3, 3]\). Now, if \( \mathcal{M}_0(k) \cap \mathcal{M}_0(H) = 0 \) then \( x \) will be zero, contradicting non-splitting of the extension \([3, 3]\).

Thus \( \mathcal{M}_0(k) \cap \mathcal{M}_0(H) \) must be a non-zero submodule of \( \mathcal{S} \), and we must therefore have \( p \) dividing \( n \). Now the image of \( x \) in \( H^2(\mathcal{M}_0(k)/\mathcal{S}) \) represents the extension

\[
0 \to \mathcal{M}_0(k)/\mathcal{S} \to SL_n(W_{m+1})/Z \to SL_n(W_m) \to I.
\]

Since this is non-split by Corollary \([3, 11]\) the image of \( x \) in \( H^2(\mathcal{V}) \) is not 0. This contradicts the fact that \( x \) is itself in the image of \( H^2(\mathcal{S}) \to H^2(\mathcal{M}_0(k)) \).

\[\square\]

**Remark 4.4.** It is well known that the mod-\( p \) reduction map \( SL_2(\mathbb{Z}/p\mathbb{Z}) \to SL_2(\mathbb{F}_p) \) has a homomorphic section when \( p \) is 2 or 3. (See the exercises at the end of \([13, \text{Chapter IV}(3)]\).) Thus the conclusion of the main theorem fails when \( n = 2 \) and \( k = \mathbb{F}_5 \).

The main theorem also fails when \( n = 2 \) and \( k = \mathbb{F}_5 \). To see this, choose \( 0 \neq \xi \in H^1(SL_2(\mathbb{F}_5), \mathcal{M}_0(\mathbb{F}_5)) \) and consider the subgroup

\[
G := \{(I + \xi(A))A \mid A \in SL_2(\mathbb{F}_5)\}
\]

of \( SL_2(\mathbb{F}_5[\epsilon]) \) where \( \mathbb{F}_5[\epsilon] \) is the ring of dual numbers (so \( \epsilon^2 = 0 \)). Clearly, \( G \mod \epsilon = SL_2(\mathbb{F}_5) \). If \( G \) can be conjugated to \( SL_2(\mathbb{F}_5) \) in \( GL_2(\mathbb{F}_5[\epsilon]) \) then the cocycle \( \xi \) must vanish in \( H^1(SL_2(\mathbb{F}_5), \mathcal{M}(\mathbb{F}_5)) \). This cannot happen as the sequence

\[
0 \to \mathcal{M}_0(\mathbb{F}_5) \to \mathcal{M}(\mathbb{F}_5) \to \mathbb{F}_5 \to 0
\]

splits.

**Remark 4.5.** Fix a finite field \( k \) satisfying assumption \([1, 1]\) and an integer \( m \geq 1 \). The main theorem then determines the universal deformation ring for \( G := SL_n(W_m) \) with standard representation completely. (See \([2, 6]\) for background on deformation of representations.)

To describe this fully, let \( \rho : G \to SL_n(W_m) \) be the natural representation and set \( \overline{\rho} := \rho \mod p \). We work inside the category of complete local Noetherian rings with residue field \( k \) from here on. Let \( R \) be the universal deformation ring for deformations of \((G, \overline{\rho})\) in this category and let \( \rho_R : G \to GL_n(R) \) be the universal representation.

By universality, there is a morphism \( \pi : R \to W_m \) such that \( \pi \circ \rho_R \) is strictly equivalent to \( \rho \). By our main theorem \( X \rho_R(G)X^{-1} \supseteq SL_n(W_R) \) for some \( X \) in \( GL_n(R) \); here, we can insist that \( X \) reduces to the identity modulo \( m_R \). Now \( \pi|_{W_R} : W_R \to W_m \) along with

\[
|SL_n(W_m)| = |G| \geq |\rho_R(G)| \geq |SL_n(W_R)| \geq |SL_n(W_m)|
\]

implies that \( \pi|_{W_R} : W_R \to W_m \) is an isomorphism and that \( X \rho_R(G)X^{-1} = SL_n(W_R) \). Replacing \( \rho_R \) with the strictly equivalent representation \( X \rho_R X^{-1} \) if necessary, we can then assume that \( \rho_R : G \to GL_n(R) \) takes values in \( SL_n(W_R) \).

Writing \( i : W_m \to W_R \) for the inverse to \( \pi|_{W_R} \), we conclude that \( i \circ \rho \) is strictly equivalent to \( \rho_R \).
We will now verify that \( \rho : G \to SL_n(W_m) \) is the universal deformation. So given a lifting \( \rho_A : G \to GL_n(A) \) of \( \overline{\rho} : G \to SL_n(k) \), we need to show that there is a unique morphism \( i_A : W_m \to A \) such that \( i \circ \rho \) is strictly equivalent to \( \rho_A \). Uniqueness comes for free (it has to send 1 to 1). For existence, note that by universality there is a morphism \( \pi_A : R \to A \) such that \( \pi_A \circ \rho_R \) is strictly equivalent to \( \rho_A \). It is then an easy check to see that \( i_A := \pi_A \circ i \) works.

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References


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