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Abstract

In this paper we give definitions of basic concepts such as symmetries, first integrals, Hamiltonian and recursion operators suitable for ordinary differential equations on associative algebras, and in particular for matrix differential equations. We choose existence of hierarchies of first integrals and/or symmetries as a criterion for integrability and justify it by examples. Using our componentless approach we have solved a number of classification problems for integrable equations on free associative algebras. Also, in the simplest case, we have listed all possible Hamiltonian operators of low order.

Integrable ODEs on Associative Algebras

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1 Introduction

In the classical theory (Lie, Liouville, etc.) of ordinary differential equations (ODEs) there are remarkable results which relate the property of integrability of ODEs in quadratures with the existence of continuous symmetries and first integrals. Symmetries and first integrals may serve as a solid mathematical foundation for an algebraic theory of integrable equations. In the case of integrable partial differential equations (PDEs) such a theory does already exist and proved to be extremely efficient [1, 2, 3, 4]. The key property of integrable PDEs is the existence of infinite hierarchies of local infinitesimal symmetries generated by a recursion operator. Characteristic features of integrable Hamiltonian PDEs are multi-Hamiltonian structures and hierarchies of local conservation laws. These properties are well described in the fundamental monograph by P.Olver [5] where references on original publications are well presented.

A straightforward generalisation of these ideas to the case of ODEs is impossible, since the number of independent commuting symmetries and first integrals for a finite dimensional dynamical system is finite and bounded by its dimension. To overcome this obstacle we propose to study an intermediate object, namely, equations on free associative algebras. Here we are going to show that such equations are quite similar to PDEs. In particular they may have infinite hierarchies of first integrals or symmetries and there are plenty of reasons to choose these properties as an algebraic definition of integrability.

If we are given a matrix differential equation with no restrictions on the matrix dimension then we can treat it as an equation on an abstract associative algebra. For example, equations for the classical Euler top in n-dimensional space can be written in a matrix form

$$\dot{M} = [M, \Omega]\,. \tag{1}$$

Here M, Ω are real skew-symmetric $n \times n$ matrices of angular momentum and angular velocity, brackets $[\,,\,]$ denote usual matrix commutator and

$$M = J\Omega + \Omega J\,, \tag{2}$$

where J is a constant diagonal matrix of moments of inertia. The problem of integration of the general n dimensional case was solved by Manakov [6]. He made an important observation that the Euler equation (1), (2) is a stationary point of the N-wave equations, known to be integrable [7].

Equation (1) is a Hamiltonian system[1] with Hamiltonian

$$H_1 = \frac{1}{2}\mathrm{trace}(M\Omega)\,. \tag{3}$$

It is easy to check that

$$H_2 = -\mathrm{trace}(M^2 J^2)\,, \qquad H_3 = \mathrm{trace}(2J^4M^2 + J^2MJ^2M) \tag{4}$$

are also first integrals of (1). First integral H_2 generates a symmetry of the Euler equation

$$M_\tau = [M, \mathrm{grad}_M H_2] = J^2M^2 - M^2J^2\,. \tag{5}$$

In terms of M, J, Ω first integrals and symmetries have reasonably simple polynomial form. An attempt to find first integrals or symmetries in component notations, say by the method of indetermined coefficients, would fail for dimensions $n > 4$ because of enormous scale of computations involved. In order to avoid component-wise computations and make results suitable for any dimensional problem we could regard objects M, Ω, J as generators of an associative algebra (i.e. assume that all our objects are polynomials of non–commutative variables M, Ω, J with complex or real coefficients). This is not a free algebra because of the constraint (2). A more serious problem is that the Euler equation (1) does not define a t-derivation of the algebra. Indeed, we cannot determine $\frac{d\Omega}{dt}$ as an element of the algebra using the Euler equation (1) and the constraint (2). In other words the Euler equation is not an *evolutionary* equation. On the contrary, symmetry (5) with the condition $J_\tau = 0$ defines a derivation of a free algebra generated by elements M and J (we shall denote associative multiplication in this algebra by $\circ$). It is much easier to study (5)

[1] Indeed, it can be rewritten in the form $\dot{M} = \{M, H_1\} = [M, \mathrm{grad}_M H_1]$ with Poisson brackets defined as $\{F, G\} = \mathrm{trace}(\mathrm{grad}_M F, [M, \mathrm{grad}_M G])$.

than the original Euler equation. Since the Euler equation and (5) belong to the same hierarchy, they have common symmetries and first integrals.

In this paper we will use equation

$$M_t = M^2 \circ C - C \circ M^2 \tag{6}$$

which coincides with (5) if $C = J^2$, as a basic and instructive example.

Being defined on a free algebra, equation (6) has infinite hierarchies of polynomial symmetries and first integrals (see Section 1) and therefore is integrable according to our definition. Taking concrete finite and infinite dimensional associative algebras (such as algebras of differential or integral operators, matrix algebras or algebras with particular commutation relations) instead of the universal free associative algebra one can obtain a wide variety of equations which we expect to be integrable in a more conventional sense. For instance, in matrix realisations, symmetries and first integrals of equations on abstract associative algebras are inherited by the corresponding matrix ODEs and can be used for their integration in quadratures.

Our approach to differential equations on associative algebras requires proper definitions for such concepts as symmetry, first integral, Hamiltonian and recursion operators, etc. In the next section we give a definition of symmetries and first integrals suitable for equations on free associative algebras. Also we show that equation (6) is not a unique representative of one component integrable equations and formulate a few classification results (we call them Results, rather than Proposition or Theorem, because they were obtained by straightforward computations).

Fréchet derivative, gradient, Hamiltonian and recursion operators for equations on free associative algebras are defined in Section 3. In the same section we formulate a simple classification result for Hamiltonian operators. In particular, we have shown that equation (6) is a tri–Hamiltonian system. Ratios of the corresponding Hamiltonian operators yield two recursion operators. Existence of two independent recursion operators enables us to build up a complete two-index hierarchy of symmetries for equation (6).

Many important *multi–component* integrable equations on associative algebras can be obtained as reductions of *one–component* equations. Here we give just one example. Let M and C in (6) be represented by matrices of the form

$$M = \begin{pmatrix} 0 & u_1 & 0 & 0 & \cdot & 0 \\ 0 & 0 & u_2 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & u_{N-1} \\ u_N & 0 & 0 & 0 & \cdot & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 0 & \cdot & 0 & J_N \\ J_1 & 0 & 0 & \cdot & 0 & 0 \\ 0 & J_2 & 0 & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & J_{N-1} & 0 \end{pmatrix}, \tag{7}$$

where u_k and J_k are block matrices (of any dimension) or even generators of a bigger free algebra. It follows from equation (6) that u_k satisfy the non-abelian Volterra equation

$$\frac{d}{dt} u_k = u_k \circ u_{k+1} \circ J_{k+1} - J_{k-1} \circ u_{k-1} \circ u_k, \quad k \in \mathbb{Z}_N\,. \tag{8}$$

In Section 4 we consider other examples (modified Volterra equation, discrete Burgers type equation, etc) and reductions to two–component systems of equations. Also we solve a simplest classification problem for two–component equations with one extra symmetry and study some of their properties.

We conclude our paper by a discussion of possible developments and future research.

1.1 First integrals and symmetries for equations on free associative algebras

Equation (6) is naturally defined on a free algebra $\mathcal{M}$ generated by elements M and C over the field $\mathbb{C}$ with an associative (but noncommutative) multiplication denoted as $\circ$. Elements of the field will be denoted by Greek letters $\alpha, \beta, \gamma \ldots \in \mathbb{C}$, elements of $\mathcal{M}$ will be denoted by Latin letters. Also we assume that $\mathcal{M}$ contains a unity element which we denote *Id*.

All basic objects such as symmetries, first integrals, recursion and Hamiltonian operators can be naturally defined for equations on associative algebras. We formulate and illustrate definitions for equations on algebra $\mathcal{M}$ generated by two elements M (variable) and C (constant), but all these definitions can be extended to any associative algebra with a finite number of generators straightforwardly.

Definition 1 *For a differential equation $M_t = F$ on algebra $\mathcal{M}$ we say that $G \in \mathcal{M}$ is a generator of an infinitesimal symmetry if the flows $M_\tau = G$ and $M_t = F$ commute (i.e if $\frac{dG}{dt}$ evaluating according the equation and $\frac{dF}{d\tau}$ evaluating according $M_\tau = G$ give the same element of $\mathcal{M}$).*

Remark. In terms of the (non-commutative) differential algebra, equation $M_t = F$ is nothing but a derivation D_F of algebra $\mathcal{M}$, which maps M to F and symmetry $M_\tau = G$ is another derivation D_G, commuting with D_F.

To introduce the concept of first integrals, we need an analog of trace, which is not yet defined in algebra $\mathcal{M}$. As the matter of fact, in our calculations we use only two property of the trace, namely linearity and a possibility to perform cyclic permutations in monomials. Therefore, instead of a trace, we define an equivalence relation for elements of $\mathcal{M}$ in a standard way.

Definition 2 *Two elements f_1 and f_2 of $\mathcal{M}$ will be called equivalent, denoted $f_1 \sim f_2$, iff f_1 can be obtained from f_2 by cyclic permutations in its monomials.*

For example, $\alpha M \circ C \circ M \sim \alpha C \circ M \circ M$, and of course, any commutator is equivalent to zero.

Definition 3 *Element h of $\mathcal{M}$ will be called a first integral of our differential equation, if $\frac{dh}{dt} \sim 0$. First integrals h_1 and h_2 are said to be equivalent if $h_1 - h_2 \sim 0$.*

Here there is an obvious similarity with a definition for conserved densities in the theory of evolutionary PDEs [5]. In both cases, first integrals and conserved densities are defined as elements of equivalence classes, the difference is in the choice of the equivalence relation.

According to our definition, $R_n = M^n\,,\; n = 1, 2, \ldots$ are nontrivial first integrals of equation (6). Indeed, $R_n \not\sim 0$ and $\frac{d}{dt} M^n = \sum_{k=1}^{n} M^{k-1} \circ (M^2 \circ C - C \circ M^2) \circ M^{n-k}$. It is easy to see that each element of the sum is equivalent to zero.

The r.h.s. of equation (6) is a (double) homogeneous polynomial with respect to scaling $\sigma : C \to \mu C\,,\; M \to \nu M$. For such type homogeneous polynomials the exponents of μ and ν we shall call weights. As a linear space, algebra $\mathcal{M}$ can be decomposed in a direct sum $\mathcal{M} = \oplus \mathcal{M}_{nm}$ where $a \in \mathcal{M}_{nm}$ if $\sigma : a \to \mu^n \lambda^m a$. It is easy to see that

Lemma 1 *Let I be a first integral and G a generator of a symmetry for a (double) homogeneous equation. Then any (double) homogeneous component of I or G (i.e. a projection on $\mathcal{M}_{nm}$) is also a first integral or a generator of a symmetry respectively.*

A direct computation of first integrals for (6) with small weights is a simple problem, since we have only a few coefficients to determine. For example a general homogeneous polynomial of weights $(2, 2)$ is equivalent to

$$P_{2,2} = \alpha C^2 \circ M^2 + \beta C \circ M \circ C \circ M\,.$$

It is easy to check that $\frac{dP_{2,2}}{dt} \sim 0$ iff $\alpha = 2\beta$ and therefore

$$I_{2,2} = 2C^2 \circ M^2 + C \circ M \circ C \circ M \tag{9}$$

is a first integral of equation (6).

In a similar way one can easily find that apart from the obvious orbit integrals $I_{0,n} = M^n$, the following polynomials $I_{n,1} = C^n \circ M\,, I_{1,n} = C \circ M^n$ and

$$I_{3,2} = C^3 \circ M^2 + C^2 \circ M \circ C \circ M \qquad I_{2,3} = C^2 \circ M^3 + C \circ M \circ C \circ M^2, \tag{10}$$

are nontrivial first integrals of equation (6). It is interesting to note that the space of first integrals in each homogeneous component $\mathcal{M}_{kn}$ is one dimensional. We have found the above integrals (9) and (10) by a simple straightforward computation. We did not use the known Lax representation for equation (6) (not known for all equations), but if we do so, we would find that $I_{n,m}$ is equivalent to the projection of $(M + C)^{(n+m)}$ on $\mathcal{M}_{nm}$.

A similar computation gives us generators of homogeneous symmetries $S_{n,m} \in \mathcal{M}_{nm}$ for (6). For instance, to obtain $S_{2,2}$ we start with a general ansatz

$$Q_{2,2} = \alpha_1 C^2 \circ M^2 + \alpha_2 C \circ M \circ C \circ M + \alpha_3 C \circ M^2 \circ C + \alpha_4 M \circ C^2 \circ M + \alpha_5 M \circ C \circ M \circ C + \alpha_6 M^2 \circ C^2.$$

It is easy to verify that $Q_{2,2}$ gives a symmetry for (6) iff $\alpha_3 = \alpha_4 = 0$ and $\alpha_1 = \alpha_2 = -\alpha_5 = -\alpha_6$.

Having first integrals and symmetries of equation (6) we can use them for the integration of Euler's equation. Let $C = J^2$, with J being a diagonal matrix with positive eigenvalues, and let M be a skew-symmetric real matrix. Then one can show that $H_{n,m} = \mathrm{trace}(I_{n,m})$ is a first integral of the Euler equation. Of course, only half of the integrals survive such a reduction, since M is a skew symmetric matrix and therefore $H_{n,2k-1} = 0$. Also, the first integrals may become functionally dependent. For instance, in the case of the three dimensional Euler top only two first integrals are independent. We can choose $H_{0,2}$ and $H_{1,2}$ as a basic set of functionally independent first integrals. In the four dimensional case as a basic set we can choose $H_{0,2}, H_{1,2}, H_{2,2}$ and $H_{0,4}$. In both cases the Hamiltonian (3) of the Euler equation does not belong to the basic set (it is a function of the basic integrals).

Summarising the above exercise we would like to say the following: The Euler top on an abstract associative algebra is not an evolutionary equation and therefore it is difficult, if possible at all, to study its symmetries and first integrals directly. Meanwhile, it has an evolutionary symmetry (6) which is naturally defined on a free algebra. It is very easy to find first integrals and symmetries for (6) (only linear algebraic equations to be solved). In a standard matrix case, symmetries and first integrals of (6) are also symmetries and first integrals of the Euler equation, they are sufficient for integration of the Euler equation in quadratures.

1.2 Other examples of integrable evolutionary equations on $\mathcal{M}$.

Equation (6) possesses sequences of first integrals and symmetries. These properties enable us to integrate its matrix (finite dimensional) reductions. A natural question is: whether there exist other equations which possess such properties? In this section we show that the answer is affirmative. The definitions given above proved to be efficient and suitable for handling such problems.

Let us inquire when a general double homogeneous evolutionary equation on $\mathcal{M}$:

$$M_t = \alpha M^2 \circ C + \beta M \circ C \circ M + \gamma C \circ M^2 \tag{11}$$

possesses symmetries? According to Lemma 1, we can study symmetries in each homogeneous component independently. General equation (11) has an obvious sequence of symmetries $S_{1,n} = M \circ C^n - C^n \circ M$, $n = 1, 2, \ldots$. In the matrix case these symmetries would correspond to the invariance of our equation with respect to similarity transformations commuting with C. Symmetries $S_{1,n}$ we shall call trivial and omit them from the further consideration.

We checked the existence of homogeneous symmetries $S_{n,m} \in \mathcal{M}_{n,m}$ with $n + m \le 10$ for equation (11) by straightforward computations. To do so, we wrote a general homogeneous polynomial $P_{n,m} \in \mathcal{M}_{n,m}$ with indetermined coefficients, and commuted equation (11) with flow $M_\tau = P_{n,m}$. Then we equated the result of the commutation to zero and solved the resulting system of algebraic equations for the coefficients of $P_{n,m}$ and $\{\alpha, \beta, \gamma\}$. This system is nonlinear (all equations are quadratic), but quite simple. All steps in this study are algorithmic. In order to perform actual computations we wrote a symbolic code for Mathematica.

Result 1 *Up to scaling of t, every equation of the form (11) possessing a nontrivial symmetry $S_{n,m} \in \mathcal{M}$, $n + m \le 10$ coincides with one of the following equations:*

$$M_t = [M^2, C], \tag{12}$$
$$M_t = [M, M \circ C], \tag{13}$$
$$M_t = [C \circ M, M], \tag{14}$$
$$M_t = M \circ C \circ M. \tag{15}$$

The first equation of the list coincides with (6). Equations (13) and (14) are equivalent in the following sense. Let us define a formal involution $\star$ on $\mathcal{M}$ by

$$M^\star = M, \;\; C^\star = C, \;\; (A \circ B)^\star = B^\star \circ A^\star, \;\; (\alpha A + \beta B)^\star = \alpha A^\star + \beta B^\star, \;\; A, B \in \mathcal{M}, \;\; \alpha, \beta \in \mathbb{C}. \tag{16}$$

If an equation $M_t = P$ has a symmetry $M_\tau = S$ then equation $M_t = P^\star$ has symmetry $M_\tau = S^\star$. The r.h.s. of equation (14) is a result of the involution $\star$ applied to (13). Similarly, all results related to (13) can be easily rearranged for (14).

The form of symmetries for equations (13), (15) that we found with the help of our computer programme has given us a hint for the following statement, which can be easily proven.

Proposition 1 *Equation (13) has commuting symmetries*

$$M_{\tau_{n,m}} = [M,\, M^{n-1} \circ C^m]\,, \quad n, m \in \mathbb{N}\,. \tag{17}$$

Equation (15) has commuting symmetries

$$M_{\tau_{2,m}} = M \circ C^m \circ M\,, \quad m \in \mathbb{N}\,. \tag{18}$$

Another interesting problem is to describe equations of the form (11) which possess first integrals. Straightforward computations lead to the following

Result 2 *Equation (11) has first integrals of the form $I_{n,0} = M^n$ iff $\alpha + \beta + \gamma = 0$. If equation (11) has other first integrals $I_{n,m}$, $n + m \leq 10$, then up to scaling of t it coincides with (12).*

Equation (15) does not have first integrals, and equation (13) has only obvious orbit integrals, but both equations have a hierarchy of symmetries. It indicates the "C-integrability" of (15) and (13) (a similar situation for PDEs is described in [1]).

Indeed, if M, C are $n \times n$ matrices, then a solution of the Cauchy problem ($M(0) = M_0$) for equation (15) is [8]:

$$M(t) = (\mathbf{e} - M_0 C t)^{-1} M_0\,,$$

where $\mathbf{e}$ is the unit matrix.

Equation (13) in the matrix case can be integrated in the following way. Let M_0 be any constant matrix and T be a solution of the following linear differential equation with constant coefficients

$$T_t = M_0 T C\,, \qquad T|_{t=0} = \mathbf{e} \tag{19}$$

then $M(t) = T^{-1} M_0 T$ satisfies equation (13) and $M(0) = M_0$. Moreover, if $T(t, \tau_{n,m})$ is a simultaneous fundamental solution for equation (19) and $T_{\tau_{n,m}} = M_0^n T C^m$ (these equations are obviously compatible), then $M(t, \tau_{n,m}) = T^{-1} M_0 T$ satisfies equation $M_{\tau_{n,m}} = [M, M^{n-1} C^m]$ as well.

Equations (12) and (13) have cubic (in M) symmetries. The next natural question is: whether there exist other cubic double homogeneous equations

$$M_t = \alpha M^3 \circ C + \beta M^2 \circ C \circ M + \gamma M \circ C \circ M^2 + \delta C \circ M^3 \tag{20}$$

which possess symmetries or first integrals. The answer can be formulated as follows.

Result 3 *Up to scaling of t and formal involution $\star$, every equation of the form (20) possessing a nontrivial symmetry $S_{n,m} \in \mathcal{M}\,, n + m \leq 10$ coincides with one of the following equations:*

$$
\begin{aligned}
M_t &= [M^3,\, C]\,, & (21)\\
M_t &= [M,\, C \circ M^2]\,, & (22)\\
M_t &= [M \circ C \circ M,\, M]\,, & (23)\\
M_t &= M^3 \circ C - M^2 \circ C \circ M + M \circ C \circ M^2 - C \circ M^3,. & (24)
\end{aligned}
$$

The first two equations (21) and (22) we expected, since they are symmetries of equations (12) and (13) respectively.

Equation (24) is new, and has symmetries with weights $(2k - 1, 1)$, but does not possess symmetries of weights $(2k, 1)$. It is interesting to note that element $N = M^2$ satisfies the equation $N_t = [N^2, C]$ (12). In the scalar case we would consider these equations to be point-equivalent. In the matrix case if we know solutions of (24) we know solutions of equation (12) as well, but not vice-versa.

Moreover, we can find an equation for the "n–th root" of N. Indeed, it is easy to show that if M satisfies equation

$$M_t = M^{n+1} \circ C - M^n \circ C \circ M + M \circ C \circ M^n - C \circ M^{n+1},. \tag{25}$$

then $N = M^n$ satisfies equation (12). Equation (25) has symmetries with weights $(kn + 1, 1)\,, k = 0, 1, \ldots$, but does not have symmetries of weights $(p, 1)$ where $p \not\equiv 1 (\mathrm{mod}(n))$.

In the above list, equation (23) is maybe the most interesting one. It has symmetries of weights $(n, 1)$ for any n. This equation also can be related on a formal level with an extension of the hierarchy of equation (12). If algebra $\mathcal{M}$ contains (or is extended by) the inverse element

$N = M^{-1}$, then it follows from (23) that $N_t = [N^{-1}, C]$ and it is a symmetry of equation $N_\tau = [N^2, C]$ (12). Generally speaking, the inverse element may not exist (for example it does not exist in the 3×3 matrix case when M is skew–symmetric).

In a similar way one can verify that equation

$$M_t = [M,\, M^n \circ C \circ M^n] \tag{26}$$

is related with (23) by the substitution $N = M^n$. Similar to (25), this equation has lacunas of length n in the sequence of symmetries, namely it has symmetries of weights $(kn+1, 1)\,, k = 2, 3, \ldots$.

2 Hamiltonian and recursion operators on $\mathcal{M}$.

2.1 Basic definitions

In this section we give definitions for some basic objects such as the Fréchet and variational derivatives, Hamiltonian operators, etc. suitable for equations on free algebra $\mathcal{M}$. We shall use equation (6) to illustrate our definitions.

For any $a \in \mathcal{M}$ we define operators of left (L_a) and right (R_a) multiplications which map $\mathcal{M}$ into itself according the following rule: let b be any element of $\mathcal{M}$, then

$$L_a(b) = a \circ b, \quad R_a(b) = b \circ a\,.$$

It follows from the above definition and associativity of algebra $\mathcal{M}$ that $R_a L_b = L_b R_a$ and

$$L_{a\circ b} = L_a L_b, \quad R_{a \circ b} = R_b R_a, \quad L_{\alpha a + \beta b} = \alpha L_a + \beta L_b, \quad R_{\alpha a + \beta b} = \alpha R_a + \beta R_b. \tag{27}$$

Let us define an algebra $\mathcal{O}$ with generators L_M, L_C, R_M, R_C satisfying the following relations:

$$R_M L_M = L_M R_M, \quad R_M L_C = L_C R_M, \quad R_C L_M = L_M R_C, \quad R_C L_C = L_C R_C\,.$$

We shall call $\mathcal{O}$ as the algebra of local operators. Due to (27) any operator of multiplication on elements of $\mathcal{M}$ can be represented by a corresponding element of $\mathcal{O}$. Let us denote by id the unity element of $\mathcal{O}$, i.e. the operator of multiplication by Id.

The gradation of $\mathcal{M}$ induces a gradation structure $\mathcal{O} = \oplus \mathcal{O}_{n,m}$ on $\mathcal{O}$. If we scale $C \to \mu C$ and $M \to \nu M$, then elements of $\mathcal{O}_{n,m}$ gain the multiplier $\mu^n \nu^m$. And, of course, if $A \in \mathcal{O}_{n,m}$, then $A : \mathcal{M}_{p,q} \to \mathcal{M}_{p+n,q+m}$. we shall call elements of $\mathcal{O}_{n,m}$ homogeneous operators of the weight (n, m).

The Fréchet derivative a_* for any element $a \in \mathcal{M}$ belongs to $\mathcal{O}$.

Definition 4 *Let $a = a(M, C)$ be any element of $\mathcal{M}$. Then the Fréchet derivative $a_* \in \mathcal{O}$ of a is uniquely defined by:*

$$\frac{d}{d\epsilon} a(M + \epsilon\, \delta M, C)|_{\epsilon=0} = a_*(\delta M)\,.$$

For example, following this definition we can calculate the Fréchet derivative F_* of element $F(M, C) = M^2 \circ C - C \circ M^2$ which is the r.h.s. of equation (6):

$$\begin{gathered}\frac{d}{d\epsilon}(F(M + \epsilon\, \delta M), C))|_{\epsilon=0} = \frac{d}{d\epsilon}((M + \epsilon\, \delta M)^2 \circ C - C \circ (M + \epsilon\, \delta M)^2)|_{\epsilon=0} = \\ \delta M \circ M \circ C + M \circ \delta M \circ C - C \circ \delta M \circ M - C \circ M \circ \delta M = F_*(\delta M)\end{gathered}$$

Therefore

$$F_* = R_C R_M + L_M R_C - L_C R_M - L_C L_M \tag{28}$$

Let D_F and D_G be two derivations of algebra $\mathcal{M}$ corresponding to elements $F, G \in \mathcal{M}$ respectively (cf. Remark after the Definition 1). Then their commutator is also a derivation of the algebra $\mathcal{M}$

$$D_K = [D_F,\, D_G]\,,$$

with

$$K = G_*(F) - F_*(G)\,.$$

In particular, G is a generator of a symmetry for equation $M_t = F$ iff

$$G_*(F) = F_*(G)\,. \tag{29}$$

Usually equation (29) is called the defining equation for symmetries (c.f. [5]).

Any derivation D_F of $\mathcal{M}$ induces a derivation of $\mathcal{O}$:

$$D_F L_a = L_{a_\star(F)}, \qquad D_F R_a = R_{a_\star(F)} \tag{30}$$

for any $a \in \mathcal{M}$.

The formal involution $\star$ (16) induces an involution of $\mathcal{O}$:

$$L_a^\star = R_a \tag{31}$$

It follows from (31) that $R_a^\star = L_a$ and in particular $R_M^\star = L_M,\ \ L_C^\star = R_C$. Operator $Q \in \mathcal{O}$ is called symmetric or skew–symmetric if $Q^\star = Q$ or $Q^\star = -Q$ respectively. For example operator $ad_M = L_M - R_M$ is skew-symmetric. It is useful to remember that for any $a, b \in \mathcal{M}$ and $Q \in \mathcal{M}$

$$a \circ Q(b) \sim Q^\star(a) \circ b.$$

2.2 Hamiltonian structures on $\mathcal{M}$.

The general Hamiltonian equation on $\mathcal{M}$ has the form

$$M_t = \Theta(\mathrm{grad}_M(H(M,C))) \tag{32}$$

where $H(M,C)$ is a Hamiltonian of the equation, Θ is a Hamiltonian operator. Here we shall study local Hamiltonian operators, i.e. assume that $\Theta \in \mathcal{O}$.

For any $a \in \mathcal{M}$ we define the gradient $\mathrm{grad}_M(a) \in \mathcal{M}$ as follows:

Definition 5 *Let $a(M,C), \delta M \in \mathcal{M}$. Then $grad_M(a(M,C))$ is uniquely defined by:*

$$\frac{d}{d\epsilon} a(M + \epsilon\,\delta M, C)|_{\epsilon=0} \sim \delta M \circ grad_M(a(M,C))\,.$$

It is easy to check that if $H_1 \sim H_2\,, H_i \in \mathcal{M}$ then $\mathrm{grad}_M(H_1) = \mathrm{grad}_M(H_2)$ and in particular if $H_1 \sim 0$ then $\mathrm{grad}_M(H_1) = 0$. Moreover, the following analog[2] of the well known Theorem (cf. [9]) holds.

Proposition 2 *Let $a \in \mathcal{M}$, then $a \sim const\,, const \in \mathcal{M}$ iff $grad_M(a) = 0$.*

For example, let us verify that equation (6) can be written in a Hamiltonian form (32) with Hamiltonian $H(M,C) = C \circ M^2$ and Hamiltonian operator $\Theta = ad_M$ ($\Theta(a) = M \circ a - a \circ M$ for any $a \in \mathcal{M}$). It is enough to show that $\mathrm{grad}_M(H(M,C)) = M \circ C + C \circ M$. Indeed,

$$H(M + \epsilon\,\delta M, C) = H(M,C) + \epsilon C \circ \delta M \circ M + \epsilon C \circ M \circ \delta M + O(\epsilon^2)$$

and

$$\frac{d}{d\epsilon} H(M+\epsilon\,\delta M, C)|_{\epsilon=0} = C \circ \delta M \circ M + C \circ M \circ \delta M \sim \delta M \circ (M \circ C + C \circ M) = \delta M \circ \mathrm{grad}_M(H(M,C))\,.$$

Definition 6 *We shall call $\Theta \in \mathcal{O}$ a Hamiltonian operator, if the Poisson bracket*

$$\{a,b\} = grad_M a \circ \Theta(grad_M b), \qquad a, b \in \mathcal{M}$$

satisfies conditions

$$\{a,b\} + \{b,a\} \sim 0 \tag{33}$$

$$\{a,\{b,c\}\} + \{b,\{c,a\}\} + \{c,\{a,b\}\} \sim 0 \tag{34}$$

for any elements $a, b, c \in \mathcal{M}$.

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[2]The above defined grad_M is an analog of the variational derivative in the theory of PDEs.

With the help of the *substitution principle* [5] one can show that (33) implies

$$\Theta^{\star} = -\Theta \tag{35}$$

(i.e. Θ is a skew–symmetric operator with respect to the involution (31)). Moreover, the Jacoby identity (34) is equivalent to the following condition (same as (7.11) on page 428 in [5]):

$$Q_2 \circ X_3(Q_1) + Q_3 \circ X_1(Q_2) + Q_1 \circ X_2(Q_3) \sim 0, \tag{36}$$

where operators $X_i \in \mathcal{O}$ are defined as follows

$$X_i = D_{\Theta(Q_i)}(\Theta)\,.$$

It is easy to check that $\hat{\Theta} = ad_{C^k} = L_{C^k} - R_{C^k}$ is a Hamiltonian operator for any k. Indeed, it is a skew–symmetric operator and satisfies (36) since $X_i = 0$. Operator $\Theta_1 = ad_M = L_M - R_M$ also satisfies conditions (35), (36), and therefore, is a Hamiltonian operator on $\mathcal{M}$. Hamiltonian operators $\hat{\Theta}$ and Θ_1 are homogeneous and have weights $(k, 0)$ and $(0, 1)$ respectively. For any $\lambda \in \mathbb{R}$, operator

$$\Theta(\lambda) = \Theta_1 + \lambda\hat{\Theta} \tag{37}$$

is a Hamiltonian operator as well. In other words, operators $\hat{\Theta}$ and Θ_1 are compatible and form a Hamiltonian pencil $\Theta(\lambda)$. As usual, the Hamiltonian pencil can be obtained by a simple shift: if we replace M by $M + \lambda C^k$ in Θ_1, we find

$$ad_{M+\lambda C^k} = L_{M+\lambda C^k} - R_{M+\lambda C^k} = \Theta_1 + \lambda\hat{\Theta}\,.$$

We can employ the method of indetermined coefficients for searching Hamiltonian operators of relatively low weights. Conditions (35) and (36) yield systems of linear and quadratic equations for the coefficients of Θ.

Result 4 *Up to scaling of M and C, every homogeneous Hamiltonian operator of weights $(0, m)$, $m < 8$ and $(1, m)$, $m < 7$ coincides with one of the following:*

$$\begin{aligned}
\Theta_0 &= R_C - L_C\,, && (38)\\
\Theta_1 &= R_M - L_M\,, && (39)\\
\Theta_2 &= L_C R_M - L_M R_C\,, && (40)\\
\Theta_3 &= L_M^2 R_M - L_M R_M^2\,, && (41)\\
\Theta_4 &= L_M L_C L_M R_M - L_M R_M R_C R_M\,. && (42)
\end{aligned}$$

The shifts $C \to C + \lambda\, Id$ and $M \to M + \mu\, Id$ in Θ_2 gives a Hamiltonian pencil $\Theta_2 + \lambda\Theta_1 + \mu\Theta_0$ for the Euler top and its hierarchy (i.e. equation (6), for example). Thus equation (6) is a *three Hamiltonian* system.

A similar shift $C \to C + \lambda\, Id$ of Θ_4 gives a pencil $\Theta_4 + \lambda\Theta_3$. If we assume the element M to be invertible, then the following formal change of variables $\tilde{M} = M^{-1}$ relates operators Θ_1 and Θ_2 with Θ_3 and Θ_4 respectively.

2.3 Recursion operators.

Recursion operators, if they exist, give a convenient way for generating a hierarchy of symmetries (see, for instance, [5]). For PDEs, most of known recursion operators are pseudo-differential. Existence of local, i.e. differential, recursion operators usually indicates on C–integrability of the equation [4]. Here we begin with a definition of local recursion operators for ODEs on $\mathcal{M}$ and later extend it to a "non local" case.

Definition 7 *We say that $\Lambda \in \mathcal{O}$ is a recursion operator for an evolutionary equation*

$$M_t = F, \qquad M \in \mathcal{M}, \tag{43}$$

if it satisfies the following equation:

$$\Lambda_t = F_* \Lambda - \Lambda F_*\,, \tag{44}$$

where F_* is the Fréchet derivative of F

Operators satisfying equation (44) act on the space of symmetries of equation (43).

Proposition 3 *If Λ is a recursion operator and G is the generator of a symmetry $M_\tau = G$ for equation (43) then $G_1 = \Lambda(G)$ is also a generator of a symmetry $M_{\tau_1} = G_1$ for equation (43).*

Proof. Indeed, let us check that $(M_t)_{\tau_1}$ and $(M_{\tau_1})_t$ represent the same element of $\mathcal{M}$. We have: $(M_t)_{\tau_1} = F_*\Lambda(G)$ and

$$(M_{\tau_1})_t = \Lambda_t(G) + \Lambda(G_t) = (F_*\Lambda - \Lambda F_*)(G) + \Lambda G_* F = \Lambda(G_*F - F_*G) + F_*\Lambda\,.$$

Now, due to the condition (29), we have $(M_t)_{\tau_1} = (M_{\tau_1})_t$. $\Box$

If we have two operators Λ_1 and Λ_2 satisfying equation (44), then any linear combination $\alpha\Lambda_1 + \beta\Lambda_2$ with constant coefficients $\alpha, \beta \in \mathbb{C}$, a composition $\Lambda_1\Lambda_2$ and, in particular, any power Λ_1^n also satisfy (44).

Applying the sequence $\Lambda^k,\ k = 1, 2, \ldots$ to a single symmetry (or to equation (43) itself) one could generate a hierarchy of symmetries. In the case of integrable one–component evolution PDEs the whole hierarchy of symmetries is generated by a single recursion operator. In our case of equations on $\mathcal{M}$ the situation is rather different. As we have seen above (cf. (17)), integrable equations may have two-index hierarchy of symmetries. It gives us a hint that we should have *two* independent recursion operators, and each of the operators raises the corresponding index.

For example, $\Lambda = L_M$ is a local recursion operator for equation (13). Indeed, according (13), $\frac{d}{dt}\Lambda = L_{M^2C} - L_{MCM}$. The Fréchet derivative of the r.h.s. in this case is

$$F_* = R_{MC} + L_M R_C - R_{CM} - L_{MC}$$

and therefore

$$F_*\Lambda - \Lambda F_* = (R_{MC} + L_M R_C - R_{CM} - L_{MC})L_M - L_M(R_{MC} + L_M R_C - R_{CM} - L_{MC}) = L_{M^2C} - L_{MCM}\,.$$

The following sequence $M_{\tau_{1,n+2}} = L_{M^n}(M^2C - MCM) = M^{n+2}C - M^{n+1}CM$, $n \in \mathbb{N}$ is an infinite hierarchy of symmetries for equation (13). It is not the whole set of symmetries (Proposition 1, (22)) and we shall see that a "non local" recursion operator is responsible for the remaining part of the hierarchy.

In the theory of integrable PDEs a non local recursion operator, by definition, is a ratio of two local (i.e differential) operators. In our case we define the corresponding object as a ratio of two operators from $\mathcal{O}$.

One of possible ways to introduce non local recursion operators for multi–Hamiltonian equations belongs to Magri [10]:

Proposition 4 *If Θ and Θ_1 are two Hamiltonian operators for the same equation, then their ratio*

$$\Lambda = \Theta_1\Theta^{-1} \tag{45}$$

is a recursion operator.

Proof. Indeed, any Hamiltonian operator satisfies equation (c.f. (7.38), page 449, [5])

$$\Theta_t = F_*\Theta + \Theta F_*^\star\,. \tag{46}$$

If Θ and Θ_1 are two solutions of equation (46), then it is easy to check that the ratio $\Lambda = \Theta_1\Theta^{-1}$ satisfies equation (44). $\Box$

In order to illustrate this Proposition let us consider equation (6). It has three Hamiltonian operators $\Theta_0 = L_C - R_C = ad_C$ (38), $\Theta_1 = L_M - R_M = ad_M$ (39) and $\Theta_2 = L_M R_C - L_C R_M$ (40). Consequently, we have two independent recursion operators for (6):

$$\Lambda_1 = (L_M R_C - L_C R_M)ad_M^{-1}\,, \qquad \Lambda_2 = (L_M R_C - L_C R_M)ad_C^{-1}\,. \tag{47}$$

Here we propose a simple generalisation of Proposition 4, and make it suitable even for non–Hamiltonian equations.

Proposition 5 *Let $Q_1, Q_2 \in \mathcal{O}$ be two solutions of operator equation*

$$Q_t = F_*Q + QP\,, \tag{48}$$

where $P \in \mathcal{O}$ is a given operator. Then

$$\Lambda = Q_2 Q_1^{-1} \tag{49}$$

is a recursion operator for equation (43).

Proof. Indeed,

$$\Lambda_t = \frac{dQ_2}{dt}Q_1^{-1} - Q_2Q_1^{-1}\frac{dQ_1}{dt}Q_1^{-1} = (F_*Q_2+Q_2P)Q_1^{-1} - Q_2Q_1^{-1}(F_*Q_1+Q_1P)Q_1^{-1} = F_*\Lambda - \Lambda F_* \,.$$

□

Equation (48) is a generalisation of two well known operator equations. Namely, if $P = -F_*$, then (48) coincides with equation (44) for a recursion operator. If $P = F_*^\star$, then skew-symmetric solutions of (48) give us Hamiltonian operators (46). Here we would like to emphasise that in Lemma 5 we do not specify the nature of the operator P, the only requirement is an existence of two (or more) solutions Q_1 and Q_2. Such a construction may occur to be useful for theory of integrable PDEs as well.

It is clear that for a homogeneous equation (43) with $F \in \mathcal{M}_{n,m}$, operator P must belong to $\mathcal{O}_{n,m-1}$ and we could try to find simultaneously operators P, Q_1 and Q_2 by the method of indetermined coefficients. Actually, following this simple idea we have found recursion operators for equations (13),(15) and (23).

Equation (15) is not a Hamiltonian system, but it has a hierarchy of symmetries. It is easy to find that

$$Q_n = L_M L_C^n \,, \qquad n = 0, 1, \ldots$$

are solutions of equation (48) with $P = R_M R_C$. Therefore

$$\Lambda = L_M L_C (L_M)^{-1} \tag{50}$$

is a non local recursion operator for (15).

In a similar way one can show that besides of a local recursion operator $\Lambda_1 = L_M$, equation (13) has a non local recursion operator $\Lambda_2 = ad_M R_C ad_M^{-1}$ which corresponds to $Q_1 = ad_M, Q_2 = ad_M R_C$ and $P = L_{MC} - L_M R_C$

It is easy to check that Θ_3 (41), Θ_4 (42) are Hamiltonian operators for equation $M_t = [M \circ C \circ M, M]$ (23). Therefore

$$\Lambda_1 = \Theta_4\Theta_3^{-1} = (L_M L_C L_M R_M - L_M R_M R_C R_M)(L_M^2 R_M - L_M R_M^2)^{-1} = (L_M L_C - R_M R_C) ad_M^{-1} \tag{51}$$

is a recursion operator for (23). In the last part of the equality (51), i.e. after a cancellation, we have a ratio of $Q_1 = L_M L_C - R_M R_C$ and $Q_2 = ad_M$. Neither of these two factors is a Hamiltonian operator for (23), but they both satisfy the same equation (48), where $P = L_M L_C R_M - L_M R_M R_C$ is a local operator. Moreover, $Q_3 = L_M R_M R_C - L_M L_C R_M$ is a third independent solution of (48). Therefore $\Lambda_2 = Q_3 ad_M^{-1}$ is another recursion operator for equation (23).

In spite of a non local recursion operator is formally defined by expression (49) in Proposition 5, its action on symmetries is not well defined yet. Here there are two problems:

- Action of non local operator (49) can be correctly defined only on elements of the image space $\mathcal{M}_{Q_1} \subset \mathcal{M}$ of operator Q_1.
- If Q_1 has a non–trivial kernel, then action of Λ is not uniquely defined.

Similar picture we had in the theory of integrable PDEs [5], where often there was a problem to define the action of D_x^{-1} on differential polynomials.

As a first example we consider an action of recursion operator (50) on symmetries of equation (15). In this case the kernel space of operator $Q_1 = L_M$ is trivial. The action of Q_1^{-1} (and therefore of Λ) is correctly defined on subset $\mathcal{M}_{Q_1} = \{M \circ a; a \in \mathcal{M}\}$ by $Q_1^{-1}(M \circ a) = a$. Powers $\Lambda^k = L_M C^k L_M^{-1}$ of the recursion operator Λ are also correctly defined on $\mathcal{M}_{Q_1}$ and the r.h.s. of equation (15) belongs to $\mathcal{M}_{Q_1}$. Therefore we can generate the following infinite hierarchy of symmetries

$$S_{n,2} = \Lambda^{n-1}(M \circ C \circ M) = M \circ C^n \circ M \,, \quad n = 0, 1, 2, \ldots$$

for equation (15). Equation (15) has other symmetries, namely the trivial symmetry $M_{\tau_0} = M \circ C - C \circ M$ and the scaling symmetry $M_{s_1} = M + tM \circ C \circ M$. Action of Λ (50) on the generator of the trivial symmetry is not correctly defined (i.e. the result does not belong to $\mathcal{M}$). On the contrary, the action of Λ^k on the generator of the scaling symmetry is correctly defined and give the following hierarchy of time dependent symmetries

$$M_{s_k} = \Lambda^{k-1}(M + tM \circ C \circ M) = M \circ C^{k-1} + tM \circ C^k \circ M \,, \quad k = 0, 1, 2, \ldots \,.$$

As usual (c.f. [3]), time independent part of the above hierarchy defines a hierarchy of master symmetries. Namely,

$$S_{m+n,2} = S_{m,2*}(K_n) - K_{n*}(S_{m,2})\,,$$

where $K_n = M \circ C^n$ are generators of master symmetries.

In the case when operator Q_1 (see Proposition 5) has a nontrivial kernel we have to impose some extra conditions in order to make the action of Q_1^{-1} uniquely defined. As an example, let us consider equation (6) with recursion operator Λ_1 (47). We can choose $M^k\,, k = 1, 2, \dots$ as a basis in the kernel space of ad_M. Applying the recursion operator Λ_1 to the trivial symmetry generated by $S_{1,1} = M \circ C - C \circ M$ we obtain

$$\begin{aligned} G &= \Lambda_1(S_{1,1}) = \Theta_2(C) + \sum \alpha_k \Theta_2(M^k) \\ &= M \circ C^2 - C^2 \circ M^2 + \sum \alpha_k (M^{k+1} \circ C - C \circ M^{k+1})\,. \end{aligned}$$

We see that accounting elements of the kernel space we add symmetries $S_{1,k} = M^k \circ C - C \circ M^k \in \mathcal{M}_{1,k}$ to G. The function $S_{1,1}$, operators $\Theta_1 = ad_M, \Theta_2 = L_M R_C - L_C R_M$ are homogeneous and if we request that the function $G = S_{2,1} = \Lambda_1(S_{1,1})$ to be homogeneous too, we have to choose coefficients $\alpha_k = 0$. It is easy to see that $S_{2,1}$ is an element from the image space of ad_M, and therefore $S_{3,1} = \Lambda_1(S_{2,1}) = \Lambda_1^2(S_{1,1})$ is well defined and is the next member in the hierarchy of symmetries of equation (6), etc. Moreover, it is easy to check, that elements $S_{k,1}$ belong to the image of another Hamiltonian operator $\Theta_0 = ad_C$ and therefore the action of operator Λ_2 (47) on these generators is also correctly defined. The whole two–index hierarchy of symmetries for equation (6) can be obtained by the action of recursion operators $\Lambda_{n,m} = \Lambda_2^n \Lambda_1^m$ on the trivial symmetry

$$S_{n,m} = \Lambda_{m-1,n-1}(S_{1,1})\,. \tag{52}$$

The same hierarchy of symmetries can be obtained using the first integrals $I_{n,m}$

$$S_{n,m} = ad_M \mathrm{grad}_M(I_{n,m}) = \Theta_2 \mathrm{grad}_M(I_{n-1,m})\,.$$

In a similar way one can check that pairs of recursion operators for equations (13) and (23) give corresponding two–index hierarchies of symmetries.

3 Multicomponent equations

3.1 Breeding and reducing equations

In the previous sections we were dealing with evolutionary equations on a free algebra $\mathcal{M}$ with one constant (C) and one variable (M) generators. Our result indicates (see Result 1) that there are only three basic hierarchies of equations and one could think that it may not be worth to develop a sophisticated theory to serve these three exceptional cases. In this section we are going to demonstrate that many important integrable equations are nothing but particular cases of these three equations.

A way to breed equations is to regard M and C as $N \times N$ or even infinite dimensional matrices whose entries are generators of a free algebra $\mathcal{A}$ and to reduce the system obtained by imposing linear constrains on the entries compatible with the dynamics.

One example which gives the (non-abelian) Volterra equation (8) has been considered in Introduction. Deriving equation (8) from (6) we have made two steps: we have replaced M, C by $N \times N$ matrices whose entries belong to a free algebra and then we have imposed constrains by setting some of the entries equal to zero. These constrains are compatible with the dynamics. Indeed, let C be given and be of the form (7). If the initial conditions are such that matrix M has the form (7) then it will remain of the same form for any t. The constrains imposed are not compatible with all the symmetries $S_{n,m}$ of equation (6). Only symmetries $S_{n,n+1}$ survive under the reduction. First integrals of equation (8) can be obtained from the first integrals $I_{n,m}$ of (6) by taking a formal trace of the matrix corresponding to $I_{n,m}$ (i.e. summing up diagonal elements). For example

$$\rho_1 = \mathrm{Tr}(I_{1,1}) = \mathrm{Tr}(M \circ C) = \sum_{n \in \mathbb{Z}_N} J_n \circ u_n\,,$$

$$\begin{aligned}\rho_2 &= \mathrm{Tr}(I_{2,2}) = \mathrm{Tr}(2M^2 \circ C^2 + M \circ C \circ M \circ C) = \\ &= \sum_{n\in\mathbb{Z}_N} J_n \circ u_n \circ J_n \circ u_n + 2J_n \circ u_n \circ u_{n+1} \circ J_{n+1}\,.\end{aligned}$$

It is easy to see that $\mathrm{Tr}(I_{n,m}) = 0$ if $n \not\equiv m \,\mathrm{mod}(N)$.

The modified Volterra equation

$$\frac{d}{dt}u_k = u_k \circ (J_{k-1} \circ u_{k-1} - u_{k+1} \circ J_k) \circ u_k, \quad k \in \mathbb{Z}_N\,. \tag{53}$$

is a reduction of equation (23), where we have to assume M to be of the form (7) and matrix C is of the form $[C]_{p,q} = \delta^N_{p+2,q} J_q$ (here $\delta^N_{i,j} = 1$ if $i \equiv j \,\mathrm{mod}(N)$ and 0 otherwise). Again, symmetries and first integrals of equation (53) can be easily obtained from the corresponding symmetries and first integrals of equation (23).

Cyclic reduction (7) is also compatible with equation (13). The corresponding C–integrable system of equations for variables u_k is:

$$\frac{d}{dt}u_k = u_k \circ u_{k+1} \circ J_{k+1} - u_k \circ J_k \circ u_k, \quad k \in \mathbb{Z}_N\,. \tag{54}$$

Let us consider the simplest nontrivial case $N = 2$ and assume that $J_1 = J_2 = Id$. In this case equations (54) and (8) are reduced to

$$u_t = u \circ u - u \circ v\,, \quad v_t = v \circ v - v \circ u\,, \tag{55}$$

and

$$u_t = u \circ v - v \circ u\,, \quad v_t = v \circ u - u \circ v\,, \tag{56}$$

respectively. One more two component equation can be obtained from (8) if we assume $N = 3$, $J_1 = J_2 = J_3 = Id$ and $u_3 = -u_1 - u_2$:

$$u_t = u \circ u + u \circ v + v \circ u\,, \quad v_t = -v \circ v - u \circ v - v \circ u\,. \tag{57}$$

All these evolutionary equations are defined on a free algebra $\mathcal{A}$ with generators u, v over the field $\mathbb{C}$. Elements u and v satisfy a system of equations of the form

$$u_t = P(u, v)\,, \quad v_t = Q(u, v)\,, \tag{58}$$

where $P(u, v)$ and $Q(u, v)$ are elements of $\mathcal{A}$. The involution $\star$ (16) on $\mathcal{A}$ is defined by

$$u^\star = u\,, \quad v^\star = v\,, \quad (a \circ b)^\star = b^\star \circ a^\star\,, \quad a, b \in \mathcal{A}. \tag{59}$$

Equations, which are related to each other by linear transformations of the form

$$\hat{u} = \alpha u + \beta v\,, \quad \hat{v} = \gamma u + \delta v\,, \quad \alpha\delta - \beta\gamma \neq 0 \tag{60}$$

and involution (59), we shall call *equivalent*.

Equations which are equivalent to

$$u_t = P(u, v)\,, \quad v_t = Q(v) \tag{61}$$

we shall call *triangular*.

For example, (56) is a triangular equation. Indeed, after the following change of variables $\hat{u} = u, \hat{v} = u + v$ we obtain $\hat{u}_t = [\hat{u}, \hat{v}]$, $\hat{v}_t = 0$. It is easy to check that equation (57) is equivalent to

$$u_t = v^2\,, \quad v_t = u^2 \tag{62}$$

and not triangular. Equations (57) and (55) are not equivalent.

3.2 The simplest classification problem for equations on algebra $\mathcal{A}$

3.2.1 Quadratic equations with a cubic symmetry

Equations (55), (57) have infinite hierarchies of symmetries and first integrals. It is interesting to answer the question whether do exist other quadratic equations

$$\begin{aligned} u_t &= \alpha_1 u \circ u + \alpha_2 u \circ v + \alpha_3 v \circ u + \alpha_4 v \circ v, \\ v_t &= \beta_1 u \circ u + \beta_2 u \circ v + \beta_3 v \circ u + \beta_4 v \circ v \end{aligned} \tag{63}$$

which possess symmetries or first integrals? And, if so, how many classes of inequivalent and non–triangular equations do exist? A partial answer to these questions is given by

Theorem 1 *Any non-triangular equation (63) possessing a symmetry of the form*

$$\begin{aligned} u_\tau &= \gamma_1 u \circ u \circ u + \gamma_2 u \circ u \circ v + \gamma_3 u \circ v \circ u + \gamma_4 v \circ u \circ u + \\ &\quad \gamma_5 u \circ v \circ v + \gamma_6 v \circ u \circ v + \gamma_7 v \circ v \circ u + \gamma_8 v \circ v \circ v, \\ v_\tau &= \delta_1 u \circ u \circ u + \delta_2 u \circ u \circ v + \delta_3 u \circ v \circ u + \delta_4 v \circ u \circ u + \\ &\quad \delta_5 u \circ v \circ v + \delta_6 v \circ u \circ v + \delta_7 v \circ v \circ u + \delta_8 v \circ v \circ v, \end{aligned}$$

is equivalent to one of the following:

$$\begin{aligned} u_t &= u \circ u - u \circ v, \\ v_t &= v \circ v - u \circ v + v \circ u, \end{aligned} \tag{64}$$

$$\begin{aligned} u_t &= u \circ v, \\ v_t &= v \circ u, \end{aligned} \tag{65}$$

$$\begin{aligned} u_t &= u \circ u - u \circ v, \\ v_t &= v \circ v - u \circ v, \end{aligned} \tag{66}$$

$$\begin{aligned} u_t &= -u \circ v, \\ v_t &= v \circ v + u \circ v - v \circ u, \end{aligned} \tag{67}$$

$$\begin{aligned} u_t &= u \circ v - v \circ u, \\ v_t &= u \circ u + u \circ v - v \circ u, \end{aligned} \tag{68}$$

$$\begin{aligned} u_t &= v \circ v, \\ v_t &= u \circ u, \end{aligned} \tag{69}$$

It is a remarkable fact, that a requirement of existence of just one cubic symmetry selects a finite list of equations with no free parameters (or more precisely, all possible parameters can be removed by linear transformations (60)).

The next natural question is whether equations (64)-(69) have other symmetries, do they possess first integrals, whether the corresponding matrix equations are integrable? We have calculated all time independent polynomial homogeneous symmetries and first integrals of low orders for equations of the list. The dimensions of linear spaces of symmetries and first integrals are compiled in the following table:

	Number of Symmetries								Number of First Integrals											
Order	1	2	3	4	5	6	7	8	1	2	3	4	5	6	7	8	9	10	11	12
Equation (64)	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Equation (65)	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Equation (66)	0	1	1	1	1	1	1	1	0	1	0	1	0	1	0	1	0	1	0	1
Equation (67)	0	1	1	1	0	1	0	1	0	1	0	1	0	2	0	2	1	3	2	6
Equation (68)	0	1	1	1	0	1	0	1	1	1	1	2	2	4	4	8	11	20	27	52
Equation (69)	0	1	1	0	2	1	1	2	0	0	1	1	0	2	1	1	2	2	1	3

None of the equations from the list has a linear symmetry. The only symmetry of order two is the equation itself.

Equation (64) has a symmetry of order 3

$$\begin{aligned} u_\tau &= u \circ u \circ v\,, \\ v_\tau &= v \circ u \circ v - u \circ v \circ v\,. \end{aligned} \tag{70}$$

(and that is the reason why it belongs to the list) but it does not have any higher symmetries nor any first integrals (at least of orders less than nine and thirteen respectively).

Equation (65) has a hierarchy of symmetries

$$\left(\begin{array}{c} u_{\tau_n} \\ v_{\tau_n} \end{array} \right) = \Lambda^n \left(\begin{array}{c} u \circ v \\ v \circ u \end{array} \right) = \left(\begin{array}{c} u \circ v \circ (u-v)^n \\ v \circ u \circ (u-v)^n \end{array} \right), \quad n = 0, 1, 2, \ldots \tag{71}$$

where the corresponding recursion operator Λ is of the form

$$\Lambda = \left(\begin{array}{cc} R_u - R_v, & 0 \\ 0, & R_u - R_v \end{array} \right).$$

It is easy to verify that

$$\rho_n = (u-v)^n$$

is a sequence of first integrals for equation (65). Equation (65) can be written in a Hamiltonian form

$$\left(\begin{array}{c} u_t \\ v_t \end{array} \right) = \Theta \left(\begin{array}{c} \mathrm{grad}_u \rho_1 \\ \mathrm{grad}_v \rho_1 \end{array} \right)$$

where the Hamiltonian operator Θ is of the form

$$\Theta = \left(\begin{array}{cc} R_u R_u - L_u L_u, & L_u L_v + L_u R_v - L_v R_u + R_u R_v \\ L_u R_v - L_v L_u - L_v R_u - R_v R_u, & L_v L_v - R_v R_v \end{array} \right).$$

It is interesting to note that equation (65) is a stationary point of 1+1 dimensional integrable system [11]

$$\begin{aligned} u_t &= u_x + u \circ v\,, \\ v_t &= -v_x + v \circ u\,. \end{aligned}$$

Equations (66) and (65) have different sequences of orders of first integrals - it is an independent indication that these two equations are not equivalent, i.e. cannot be related by invertible transformation (60) and involution (59). It is easy to check that

$$\rho_{2n} = (u \circ v)^n$$

is a sequence of first integrals for equation (66). Its symmetries also can be written in terms of a recursion operator

$$\left(\begin{array}{c} u_{\tau_{2n}} \\ v_{\tau_{2n}} \end{array} \right) = \Lambda^n \left(\begin{array}{c} -u \circ v \\ v \circ v + u \circ v - v \circ u \end{array} \right), \quad n = 0, 1, 2, \ldots \tag{72}$$

$$\left(\begin{array}{c} u_{\tau_{2n+1}} \\ v_{\tau_{2n+1}} \end{array} \right) = \Lambda^n \left(\begin{array}{c} u \circ u \circ v - u \circ v \circ u \\ -u \circ v \circ v + v \circ u \circ v \end{array} \right), \quad n = 0, 1, 2, \ldots \tag{73}$$

where

$$\begin{aligned} \Lambda &= \left(\begin{array}{cc} R_u R_v - L_u R_v, & L_u R_u - L_u L_u \\ R_v R_v - L_v R_v, & L_u R_v - L_v L_u \end{array} \right) = \\ &= \left(\begin{array}{cc} R_u - L_u, & 0 \\ 0, & R_v - L_v \end{array} \right) \left(\begin{array}{cc} R_v, & L_u \\ R_v, & L_u \end{array} \right). \end{aligned}$$

Equation (69) is, maybe, the most interesting in the list. It is a Hamiltonian equation with

$$\Theta = \left(\begin{array}{cc} 0, & 1 \\ -1, & 0 \end{array} \right)$$

and Hamiltonian $H = \frac{1}{3}(v^3 - u^3)$. We have seen in the previous section that (69) is equivalent to (57) which is a reduction of equation (6). Therefore the Lax representation for (69) is known and its symmetries and first integrals can be easily found. A straightforward attempt to find a local recursion operator (i.e. to find solution of equation (44)) gives

$$\Lambda = \begin{pmatrix} L_u L_v - L_u R_v - L_v R_u + R_u R_v, & -L_u L_u + 2L_u R_u - R_u R_u \\ L_v L_v - 2L_v R_v + R_v R_v, & L_u R_v - L_v L_u + L_v R_u - R_v R_u \end{pmatrix} =$$

$$= \begin{pmatrix} L_u - R_u, & 0 \\ 0, & L_v - R_v \end{pmatrix} \begin{pmatrix} L_v - R_v, & -L_u + R_u \\ L_v - R_v, & -L_u + R_u \end{pmatrix} . \tag{74}$$

Equation itself and all known to us time independent symmetries belong to the kernel of this "recursion" operator Λ (74). Equation (69) is homogeneous and therefore it has a scaling symmetry

$$u_{\tau_s} = u + tv^2 , \quad v_{\tau_s} = v + tu^2 .$$

Applying Λ to this scaling symmetry we obtain a cubic symmetry

$$\begin{pmatrix} u_\tau \\ v_\tau \end{pmatrix} = \Lambda \begin{pmatrix} u + tv^2 \\ v + tu^2 \end{pmatrix} = \begin{pmatrix} -2u \circ u \circ v + 4u \circ v \circ u - 2v \circ u \circ u \\ 2u \circ v \circ v - 4v \circ u \circ v + 2v \circ v \circ u \end{pmatrix} \tag{75}$$

of equation (69).

A systematic description of symmetries and first integrals for equations (67) and (68) is still an open problem.

3.2.2 Quadratic equations with a quartic symmetry

Integrable equation (55) does not possess a cubic symmetry and therefore does not belong to the list (64)-(69), but it has a quartic symmetry

$$\begin{aligned} u_\tau &= u \circ v \circ u^2 - u \circ v^2 \circ u , \\ v_\tau &= v \circ u \circ v^2 - v \circ u^2 \circ v . \end{aligned} \tag{76}$$

Symmetry (76) and higher order symmetries of equation (55) can be generated by the following recursion operator:

$$\Lambda = \begin{pmatrix} L_u L_v , 0 & \\ 0, & L_v L_u \end{pmatrix} .$$

Proposition 6 *Apart from (65)-(68) the list of quadratic equations on $\mathcal{A}$ which possess a quartic symmetry includes the following inequivalent equations:*

$$\begin{aligned} u_t &= u \circ u - v \circ u , \\ v_t &= v \circ v - u \circ v ; \end{aligned} \tag{77}$$

$$\begin{aligned} u_t &= -u \circ v , \\ v_t &= v \circ v + u \circ v ; \end{aligned} \tag{78}$$

$$\begin{aligned} u_t &= -v \circ u , \\ v_t &= v \circ v + u \circ v ; \end{aligned} \tag{79}$$

$$\begin{aligned} u_t &= u \circ u - u \circ v - 2v \circ u , \\ v_t &= v \circ v - 2u \circ v - v \circ u ; \end{aligned} \tag{80}$$

$$\begin{aligned} u_t &= u \circ u - 2v \circ u , \\ v_t &= v \circ v - 2v \circ u ; \end{aligned} \tag{81}$$

$$\begin{aligned} u_t &= u \circ u - 2u \circ v , \\ v_t &= v \circ v + 4v \circ u . \end{aligned} \tag{82}$$

Here we do not claim that the list presented above is complete. Some extra work is required to cast Proposition 6 into the form similar to Theorem 1.

It is interesting to look at the sequence of dimensions for symmetries and first integrals of equations (77)-(82).

	Number of Symmetries								Number of First Integrals											
Order	1	2	3	4	5	6	7	8	1	2	3	4	5	6	7	8	9	10	11	12
Equation (77)	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
Equation (78)	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
Equation (79)	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
Equation (80)	0	1	0	1	0	1	1	1	0	0	0	1	0	1	0	1	1	1	0	2
Equation (81)	0	1	0	1	1	1	1	1	0	0	1	1	0	2	1	1	2	2	1	3
Equation (82)	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Equations (77), (78) and (79) have identical sequences for dimensions of symmetries and first integrals. Nevertheless we have checked that they are not equivalent, i.e. cannot be related by linear transformations (60) and involution (59).

Equation (82) possesses a symmetry of order four but it seems it does not possess any other symmetries or first integrals.

Instead of a conclusion

In this paper, we have made an attempt to bring the rich hidden structure of integrable PDEs [4] to a new domain, namely, to differential equations on free associative algebras (see also [12, 13, 14]). We have formulated basic definitions and shown their correctness and efficiency. Like integrable PDEs, ODEs on associative algebras may have infinite hierarchies of symmetries and first integrals and that asserts an algebraic definition of integrability. In the case of finite dimensional (matrix) representations of the algebra, the corresponding (matrix) systems of ODEs inherit these symmetries and first integrals and can be integrated in quadratures.

This study raises a lot of questions and open entirely new area for research. For instance, we foresee that the list of integrable equations on associative algebras with constrains should be a lot bigger. Quantum problems, with some commutation relations, naturally fall in this class. Another important and promising problem for further research is a systematic study of reductions of equations on associative algebras and, in particular corresponding matrix systems of ODEs. We expect that infinite dimensional realisations (for instance by operators in a Hilbert space) and their reductions may be of interest as well. We would like, also, to look at the theory of classical integrable tops from the point of such a componentless approach. That would require to study equations on associative algebras with a few constant elements related by algebraic constrains.

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