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A MULTIVARIABLE FEEDBACK CONTROL PROBLEM
WITH APPLICATION TO STRIP SHAPE CONTROL
FOR SENDZIMIR MILLS

by


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Research Report No. 153

July 1981
ABSTRACT

Consideration is given to the design of feedback controllers for a plant with transfer function matrix \( G(s) = \text{diag}(g_i(s)) G_m \) where \( G_m \) is a square and constant matrix and \( g_i(s) \) is a scalar transfer function, \( 1 \leq i \leq m \). Particular emphasis is placed on the case when \( g_i(s) = g(s), 1 \leq i \leq m \), and \( G_m \) is singular or 'almost' singular and the robustness of the design with respect to errors in \( G_m \) is represented in terms of a system of strict inequalities. An application to strip shape control for Sendzimir mills is indicated.
1. **Introduction**

We consider an $m$-input/$m$-output system with transfer function matrix (TFM)

$$ G(s) = \text{diag}(g_1(s), \ldots, g_m(s))G_m $$

...(1)

where $G_m$ is a real, constant $m \times m$ matrix and $g(s)$ is a strictly proper, stable transfer function (TF). The stability assumption is not necessary but is motivated by the objective of applying the analysis to strip shape control for Sendizimir mills which can be approximated (1) by a TFM of the form of (1). The objective of the analysis is to design a unity negative feedback system (2) with forward path controller $K(s)$ as illustrated in Fig.1 to ensure the stability and satisfactory transient performance characteristics required. In the following analysis we will distinguish between the cases when $G_m$ is nonsingular and when $G_m$ is singular or almost singular. In both cases it is shown that the robustness of the design can be represented by a system of strict inequalities.

2. **The Case of $|G_m| \neq 0$**

It is natural to set

$$ K(s) = G_m^{-1} \text{diag}(k_1(s), \ldots, k_m(s)) $$

...(2)

where $k_i(s)$ is a proper scalar TF, $1 \leq i \leq m$. It is immediately verified that

$$ |I_m + G(s)K(s)| = \prod_{i=1}^{m} \left(1 + g_i(s)k_i(s)\right) $$

...(3)

and that the closed-loop TFM
\[ H_c(s) = (I_m + G(s)K(s))^{-1}G(s)K(s) = \text{diag}[h_i(s)]_{1 \leq i \leq m} \]

\[ h_i(s) = g_i(s)k_i(s)/(1+g_i(s)k_i(s)), \quad 1 \leq i \leq m \quad \ldots (4) \]

indicating that the closed-loop multivariable system is non-interacting and also stable if, and only if, the scalar feedback systems \( h_i(s) \) shown in Fig.2 are stable.

The simplicity of the above analysis is deceptive as it relies crucially upon the invertibility of \( G_m \). If \( G_m \) is singular it is certainly necessary to modify the approach. It is also necessary to change the approach if \( G_m \) is 'almost singular' as small errors in estimation of elements of \( G_m \) could then lead to large errors in elements of \( G_m^{-1} \). The resulting control system is hence very sensitive to such modelling errors and possibly unstable!

3. The Case of \( G_m \) 'Almost Singular'

In the remainder of the paper we assume that \( g_i(s) = g(s), 1 \leq i \leq m, \)
when

\[ G(s) = g(s)G_m \quad \ldots (5) \]

Suppose that \( G_m \) is diagonalizable by the \( m \times m \) nonsingular transformation \( T \) to give

\[ T^{-1}G_m T = \text{diag}[\lambda_1, \lambda_2, \ldots, \lambda_m] \quad \ldots (6) \]

and that the eigenvalues satisfy the separation condition

\[ \mu_2 \overset{\Delta}{=} \min_{1 \leq i \leq \ell} |\lambda_i| > \max_{\ell+1 \leq i \leq m} |\lambda_i| \overset{\Delta}{=} \mu_1 \quad \ldots (7) \]

for some \( \ell \). It is trivially verified that \( \{\lambda_i\}_{1 \leq i \leq \ell} \) is invariant under
complex conjugation. Intuitively the eigenvalues $\lambda_{k+1, \ldots, \lambda_m}$ can be identified with the zero eigenvalues and the 'small' eigenvalues that are sensitive to modelling errors. Write also

$$T = \begin{bmatrix} T_1 & T_2 \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

where $T_1$ is the $m \times \ell$ matrix of eigenvectors of $T$ corresponding to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ and $V_1$ is the $\ell \times m$ matrix of 'dual' eigenvectors corresponding to these eigenvalues. Note that $T^{-1}T = I_m$ indicates that

$$V_1 T_1 = I_\ell, \quad V_2 T_2 = I_{m - \ell}, \quad V_1 T_2 = 0, \quad V_2 T_1 = 0 \quad \ldots (9)$$

The physical interpretation of this decomposition is seen by writing

$$G(s) = T \text{ diag } \{\lambda_i\}_{1 \leq i \leq m} T^{-1} g(s)$$

$$= T_1 \begin{bmatrix} g(s) \text{ diag } \{\lambda_i\}_{1 \leq i \leq \ell} \\ \end{bmatrix} V_1$$

$$+ T_2 \begin{bmatrix} g(s) \text{ diag } \{\lambda_i\}_{\ell + 1 \leq i \leq m} \\ \end{bmatrix} V_2$$

\ldots (10)

or, with the obvious identification of $G_1$ and $G_2$,

$$G(s) = T_1 G_1(s) V_1 + T_2 G_2(s) V_2$$

\ldots (11)

which is illustrated in block form in Fig.3. Clearly the 'G_1 loop' corresponding to the large 'insensitive' eigenvalues dominates the response characteristics of the system as, roughly speaking,

$$y = Gu = y_1 + y_2 \quad \ldots (12)$$

where $y_1 = T_1 G_1 V_1 u = 0(u_2) \gg 0(u_1) = T_2 G_2 V_2 u = y_2$. As $G_2$ is stable it is tempting therefore to ignore this loop in design. This notion can
be formalized by choosing
\[ K(s) = T_2 K_2(s)V_1 \quad , \quad K_2(s) = \text{diag}\{k_i(s)\}_{1 \leq i \leq 2} \quad \ldots (13) \]
when it is easily verified that
\[ T^{-1}K(s)T = \begin{pmatrix} K_1(s) & 0 \\ 0 & 0 \end{pmatrix} \quad \ldots (14) \]
and hence that the return-difference takes the form
\[ |I_m + G(s)K(s)| = |I_2 + G_2(s)K_2(s)| = \prod_{i=1}^{2} (1 + g(s)\lambda_i \tilde{k}_i(s)) \quad \ldots (15) \]
and that the closed-loop TFM
\[ H_c(s) = (I_m + GK)^{-1} \tilde{G}K \]
\[ = T_1 (I_2 + G_2(s)K_2(s))^{-1} G_1(s)K_1(s)V_1 \quad \ldots (16) \]
Remembering that \( G_2 \) is stable it is clear that \( K \) stabilizes \( G \) if, and only if, \( K_1 \) stabilizes \( G_1 \) in the configuration of Fig.4. This configuration has closed-loop TFM
\[ \tilde{H}_c = (I_2 + G_1 K_1)^{-1} G_1 K_1 = \text{diag}\left\{ \frac{g_k_{1,i}}{1 + g_1 \lambda_i \tilde{k}_i} \right\}_{1 \leq i \leq 2} \quad \ldots (17) \]
and (16) can be written as
\[ H_c(s) = T_1 \tilde{H}_c(s)V_1 \quad \ldots (18) \]
A particularly simple form is obtained by choosing
\[ \lambda_i \tilde{k}_i(s) = k(s) \quad , \quad 1 \leq i \leq 2 \quad \ldots (19) \]
when
\[ \tilde{H}_c(s) = \frac{g(s)k(s)}{1 + g(s)k(s)} I_2 \quad , \quad H_c(s) = \frac{g(s)k(s)}{1 + g(s)k(s)} T_1 V_1 \quad \ldots (20) \]
Finally, although the sensitivity of the design to the small eigenvalues (and hence the 'near' singularity of $G_m$) has been reduced, we have lost some performance as illustrated by noting that the choice of reference demand $r(s)$ of the form
\[ r(s) = T_2\hat{r}(s) \quad \text{...(21)} \]
yields the output from zero initial conditions
\[ y = H_c r = T_1 \hat{H}_c \hat{V}_1 T_2 \hat{r} \equiv 0 \quad \text{(by (9))} \quad \text{...(22)} \]
that the closed-loop system does not respond to demands in the subspace spanned by the eigenvectors corresponding to the 'small, sensitive' eigenvalues. In contrast the response to the demand $r = T_1 \hat{r}$ in the subspace spanned by the eigenvectors of the 'large, insensitive' eigenvalues is simply (using (9))
\[ y = H_c r = T_1 \hat{H}_c \hat{V}_1 T_1 \hat{r} = T_1 \hat{H}_c \hat{r} = T_1 \hat{y} \neq 0 \quad \text{...(23)} \]
where $\hat{y}$ is the response of the $l \times l$ feedback system of Fig.4 to the demand $\hat{r}$. In fact, if $\hat{y}$ is the 'ideal' response $\hat{y} = \hat{r}$, it is seen that the response $y$ to $r$ is the ideal response $y = r$ in the subspace spanned by the eigenvectors corresponding to the 'large' eigenvalues can be made to be arbitrarily good.

This analysis can clearly be continued to examine in more detail such concepts as steady-state response, interaction etc. As this note is essentially a theoretical preparation for application to the strip shape control problem \(^{(1)}\), such considerations are not pursued here.

4. Robustness of the Design with respect to Errors in $G_m$

Given the design technique of section 3 a stable feedback regulator can be simply designed. If however (as in ref (1)) $G_m$ is known to be
inaccurate it is important to produce a means of predicting the maximum errors in elements of $G_m$ that can be tolerated without spoiling feedback stability i.e. just how robust is the final design? Suppose therefore that $K$ has been designed for $G = G_m \hat{g}$ but that $G_m$ is subjected to a 'matrix error' $\Delta$. In such a situation the stability of the implemented feedback scheme is described by the return-difference

$$
|I_m + (G_m + \Delta)\hat{g}(s)K|
\equiv |I_m + Kg(G_m + \Delta)|
\equiv |I_m + KG + Kg\Delta|
\equiv |I_m + KG| \cdot |I_m + (I + KG)^{-1}Kg\Delta|
\equiv |I_m + CK| \cdot |I_m + (I + KG)^{-1}Kg\Delta|
\ldots (24)
$$

A necessary and sufficient condition for $\Delta$ to retain stability is hence that

$$
|I_m + (I + KG)^{-1}Kg\Delta| \neq 0 \quad \forall \quad \text{Re } s \geq 0 \quad \ldots (25)
$$

A simple calculation using (9) and (11) yields

$$(I + KG)^{-1}Kg = T_1(I_{kG} + K_{1G})^{-1}K_1gV_1 \quad \ldots (26)$$

and hence, using the identity $|I_m + AB| = |I_m + BA|$ valid\(^{(2)}\) for any $m \times k$ matrix $A$ and $k \times m$ matrix $B$, equation (25) can be replaced to the condition

$$
|I_{kG} + (I_{kG} + K_{1G})^{-1}K_1gV_1\Delta T_1| \neq 0
\quad \forall \quad \text{Re } s \geq 0
\quad \ldots (27)
$$

This expression is rather complicated but it can be replaced by a sufficient condition based upon the observation that a diagonally (row) dominant matrix is nonsingular. More precisely, a sufficient condition for the error $\Delta$ to be such that stability is retained is that
\[ 1 > \sum_{j=1}^{\ell} |F_{rj}(s)| \text{ for } 1 < r < \ell \quad \forall \ s \in D^\Delta \{s : \text{Re } s > 0\} \quad \ldots (28) \]

where the \( \ell \times \ell \) TFM \( F(s) \) is defined by
\[
F(s) \triangleq (I_L + K(s)C(s))^{-1}K_1(s)g(s)\mathbf{V}_L\mathbf{T}_1 \quad \ldots (29)\]

The frequency dependent condition (28) can be replaced by the frequency independent condition
\[ 1 > \sum_{j=1}^{\ell} \sup_{\text{Re } s > 0} |F_{rj}(s)| \quad , \quad 1 < r < \ell \quad \ldots (30) \]

Noting that \( F \) is strictly proper and analytic and bounded in the interior of \( D \) the suprema are achieved on the imaginary axis i.e. equation (28) is valid if
\[ 1 > \sum_{j=1}^{\ell} \sup_{\omega > 0} |F_{rj}(i\omega)| \quad , \quad 1 < r < \ell \quad \ldots (31) \]

This is the basic robustness relation used in the following development where it is shown to generate a set of strict linear inequalities describing the magnitude of errors \( \Delta \) that retain stability.

Although linear in the perturbation \( \Delta \), \( F \) is a fairly complex function in general. It can however be written in the element form
\[
F_{rj}(s) = \sum_{p=1}^{m} \sum_{q=1}^{m} f_{rjpq}(s) \Delta_{pq} \quad \ldots (32) \]

for suitable choice of \( f_{rjpq}(s) \) and clearly
\[ \sup_{\omega > 0} |F_{rj}(i\omega)| \leq \sum_{p=1}^{m} \sum_{q=1}^{m} \sup_{\omega > 0} |f_{rjpq}(i\omega)| |\Delta_{pq}| \quad \ldots (33) \]
It follows that (31) is satisfied if

$$1 > \sum_{j=1}^{\xi} \sum_{p=1}^{m} \sum_{q=1}^{m} \sup_{\omega > 0} |f_{rjpq}(i\omega)| |\Delta_{pq}|, \quad 1 \leq r \leq \ell \quad \ldots (34)$$

or, equivalently, if

$$1 > \sum_{p=1}^{m} \sum_{q=1}^{m} c_{rpq} |\Delta_{pq}|, \quad 1 \leq r \leq \ell \quad \ldots (35)$$

where the scalars

$$c_{rpq} = \sum_{j=1}^{\xi} \sup_{\omega > 0} |f_{rjpq}(i\omega)| \geq 0 \quad \ldots (36)$$

$$1 \leq r \leq \ell, \quad 1 \leq p \leq m, \quad 1 \leq q \leq m \quad \ldots$$

Equations (35) and (36) describe a computable class of perturbations or errors $\Delta$ that guarantee the retention of closed-loop stability. They are expressed in terms of $\xi$ linear inequalities in $m^2$ variables which could conceivably cause problems if $m$ is large. We can however derive a more conservative estimate by noting that (35) is valid if

$$\max_{1 \leq r \leq \ell} |\Delta_{pq}| < 1 / \max_{1 \leq r \leq \ell} \left( \sum_{p=1}^{m} \sum_{q=1}^{m} c_{rpq} \right) \quad \ldots (37)$$

the RHS being easily computed.

Finally we note one particular case when calculating can be simplified. Consider the choice of $K(s)$ via (13) and (19) when it is trivially verified that

$$(I + K_1 G_1)^{-1} K_1 G = \frac{g(s)k(s)}{1 + g(s)k(s)} \text{diag}\{\lambda_1^{-1}, \ldots, \lambda_{\xi}^{-1}\} \quad \ldots (38)$$
and hence that

\[ F_{rj}(s) = \frac{g(s)k(s)}{1+g(s)k(s)} \lambda_r^{-1} \sum_{p=1}^{m} \sum_{q=1}^{m} (V_{1})_{rp} \Delta_{pq} (T_{1})_{qj} \]

\[ 1 \leq r \leq \ell \quad , \quad 1 \leq j \leq \ell \]

\[ \ldots (39) \]

and

\[ f_{rj)pq}(s) = \frac{g(s)k(s)}{1+g(s)k(s)} \lambda_r^{-1} (V_{1})_{rp} (T_{1})_{qj} \]

\[ \ldots (40) \]

yielding

\[ c_{rpq} = \sup_{\omega > 0} \left| \frac{g(i\omega)k(i\omega)}{1 + g(i\omega)k(i\omega)} \right| d_{rpq} \]

\[ \ldots (41) \]

where

\[ d_{rpq} = \sum_{j=1}^{\ell} |\lambda_r^{-1}| |(V_{1})_{rp} (T_{1})_{qj}| \]

\[ \ldots (42) \]

Note that \( c_{rpq} \) is deduced easily from \( V_{1}, T_{1}, \{\lambda_r\}_{1 \leq r \leq \ell} \) and the single TF \( gk/(1+gk) \). Note also the following observations:

(i) the coefficient \( c_{rpq} \) is proportional to \( \lambda_r^{-1} \) indicating that large (resp. small) eigenvalues lead to small (resp. large) values of \( c_{rpq} \). Small eigenvalues tend hence to increase the sensitivity of the control system by reducing the permissible perturbations \( \Delta \).

(ii) the properties of the single TF \( gk/(1+gk) \) clearly affect stability. If, for example, it possesses a strong resonance, all \( c_{rpq} \) will be large hence increasing sensitivity to the perturbation \( \Delta \).
5. **Illustrative Example**

Take

\[ G(s) = \frac{1}{s+1} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \cdots (43) \]

from which

\[ G_m = \frac{1}{s+1} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad , \quad g(s) = \frac{1}{s+1} \quad \cdots (44) \]

and \( \lambda_1 = 1, \lambda_2 = 0. \) Take \( \ell = 1, \) whence

\[ T_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad , \quad v_1 = \frac{1}{2} \begin{bmatrix} 1 & 1 \end{bmatrix} \quad \cdots (45) \]

Considering, for simplicity, the case of proportional control, we have

\[ K(s) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \frac{1}{s+1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad k \quad \cdots (46) \]

where \( k_1(s) = k \) is constant. Equation (19) is trivially satisfied with \( k = k_1(s) = k(s) \) and hence the closed-loop system is stable if

\[ k+1 > 0 \quad \cdots (47) \]

with closed-loop TFM

\[ H_c(s) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \frac{k}{s+1+k} \quad \cdots (48) \]

We obtain from (41) and (42) that

\[ c_{111} = c_{112} = c_{121} = c_{122} = \frac{1}{2} \frac{k}{1+k} \quad \cdots (49) \]

and hence that stability is retained under all perturbations \( \Delta \) to \( G_m \) satisfying

\[ 1 > \frac{1}{2} \frac{k}{1+k} \left( |\Delta_{11}| + |\Delta_{12}| + |\Delta_{21}| + |\Delta_{22}| \right) \quad \cdots (50) \]
Alternatively

$$\max_{1<p<2, 1<q<2} |\Delta_{pq}| < \frac{1}{2} (1+k^{-1})$$  \hspace{1cm} \text{...(51)}

Note that the use of low gains $k$ tends to reduce sensitivity.

6. Conclusions

The special case considered has been shown to be amenable to analysis both in the standard, nonstandard and robustness sense. The approach taken is nonunique and could possibly be improved. This may become apparent when it is applied to the strip shape control problem.

References

