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ON THE OPTIMAL CONTROL OF RESIDUAL STRESSES IN HIGH TEMPERATURE MATERIALS

by

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Summary

When a material body changes temperature nonuniformly, stresses are induced due to nonhomogeneous straining and if these stresses exceed the plastic yield limit, a proportion of them may be 'locked into' the material in the form of residual stresses. In this paper we study of the problem of controlling the temperature field, using point controllers on the boundary of the material, in order to minimize the residual stresses. The optimal selection of the control points is also considered.

Key words: Optimal control, residual stresses, plastic behaviour, distributed parameter systems.

1. Introduction

In any material body an increase (or decrease) in temperature T is associated with a corresponding expansion (or contraction) of the body, the linear change of length being proportional to the coefficient of thermal expansion α. (The body will be assumed to be isotropic in this paper and so α is independent of the direction in which it is measured.) If follows that a body which has been heated to a fairly high temperature (for example in metal forming , and in particular hot rolling) will tend to contract more in the outer layers than in the interior, since the centre will cool down more slowly than the periphery. Hence, stresses will build up in the material as it cools down and when the body is finally at room temperature throughout, a 'residual' stress field will be present which will affect the loading characteristics of the body, and may also cause buckling if the body is divided into smaller pieces.

It is therefore of importance to minimize the residual stresses as much as possible. (Of course, in certain circumstances, prestressing can be important, but in this paper we shall be concerned only with the case where residual stresses are undesirable.) There are many ways of relieving residual stresses in a body once it has cooled both destructive and nondestructive. In thermal stress relieving, the body is reheated in the neighbourhood of a stressed region until the yield stress of the material is small enough to overcome the imposed stress field. However, this method has certain disadvantages associated with reheat cracking and possible weakening of the material.

A less ad hoc approach was used by Kusakabe et al [5] in the case of hot-rolled H-sections, and consists of locally reheating the body on the surface at various points before it has cooled down in order to attempt to achieve a uniform temperature field throughout the body. However, in this paper the authors did not consider the problem as an optimal control problem, but merely tried various control inputs and numerically optimized over a certain small number of such inputs.

In this paper we shall consider an isotropic body of any shape and we shall carry out a theoretical study of the optimal control problem of minimizing the average temperature in the body by using heat inputs at various points on its boundary. It will be shown that a Riccati equation can be developed for the optimal control and in the case when a certain operator has a simple spectral respresentation, the solution of this equation can be written down in terms of the 'exponential' of an infinite matrix.

Having obtained the optimal control we shall consider the problem of optimally selecting the points at which to apply the controls. This will be done by writing down the Kuhn-Tucker conditions for this problem.

2. Thermoelastic Equations

When a body is heated to a given temperature distribution and then allowed to cool down, the nonuniform temperature fields produced during cooling generate stresses in the material. If the body were perfectly elastic at all temperatures, these stresses would vanish when the body returns to its original temperature. However, if the thermally induced stresses exceed the yield stress of the material then the material flows plastically and the stresses are 'locked' into the body. Hence a discussion of residual stresses in heated materials necessitates not only a consideration of linear elasticity but also the effects of yielding on the stress field must be taken into account.

Consider first the equations of thermoelasticity. It can be shown [5] that, in the absence of heat sources in a body, the temperature field T and the volume change e (e = sum of normal strains) are related by

$$\nabla^2 T - \frac{1}{\mu}\dot{T} - \eta\dot{e} = 0$$

where η depends on the initial temperature T_o. In this paper we shall assume that the strains are very small and so we may neglect the term $\dot{e}$ in this equation. The temperature field may therefore be uncoupled from the stress field. Hence,

$$\mu\nabla^2 T = \dot{T}$$

The equations relating the temperature distribution and the stress field can be obtained from the Duhamel-Neumann relations ([5]), namely

$$\varepsilon_{ij} = \alpha T\delta_{ij} + \frac{1}{2G}(\sigma_{ij} - \frac{\nu}{1+\nu}\Theta\delta_{ij})$$

where $\Theta = \sigma_{ii}$. In this equation, $\varepsilon_{ij}, \sigma_{ij}$ represent the ijth strain and stress component respectively and G, ν are elastic constants. T is the temperature change from a stress-free state of the body. It follows ([8]) that

$$(\lambda+G)e,_i + G\nabla^2 u_i - \frac{\alpha E}{1-2\nu}\cdot T,_i = 0,$$

for i = 1,2,3, where

$$e = \varepsilon_{ii}\ ,\ \varepsilon_{ij} = \tfrac{1}{2}(u_{i,j}+u_{j,i}).$$

These equations can be solved by introducing the thermoelastic potential ψ such that

$$u_i = \psi,_i$$

(Note that in the above $f,_i$ represents $\partial f/\partial x_i$). This will then lead to the equation

$$\nabla^2\psi = \frac{1+\nu}{1-\nu}\alpha T. \qquad (2.1)$$

If the boundary of the body is stres-free then

$$\sigma_{ij}n_j = 0 \qquad (2.2)$$

where $\underline{n}$ is the unit outward normal to the surface.

Equation (2.1) can be solved in particular by the function

$$\psi(x) = -\frac{(1+\nu)\alpha}{4\pi(1-\nu)} \int_\Omega T(y)\frac{1}{r'}\,dv \tag{2.3}$$

where $r' = \|x-y\|$, $x,y\in\mathbb{R}^3$. However, this solution will not satisfy the boundary condition (2.2) and so we must solve an ordinary stress problem (i.e. independent of temperature) which has the stress boundary conditions

$$\sigma'_{ij} = -\left[\mu(\psi,_{ij} + \psi,_{ji}) + (\lambda\nabla^2\psi - T\gamma)\delta_{ij}\right]_{\partial\Omega} \tag{2.4}$$

where $\gamma=(3\lambda + 2\mu)\alpha$. (This boundary condition follows from the relation

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + (\lambda e - T\gamma)\delta_{ij}$$

and the definition of ψ.) For example, in the plain strain case, we can solve the biharmonic equation

$$\nabla^4\phi = 0$$

together with the boundary conditions

$$\phi,_{yy} = \sigma'_{xx}$$

$$\phi,_{xx} = \sigma'_{yy}$$

$$\phi,_{xy} = -\sigma'_{xy}$$

where σ'_{ij} is given by the right-hand side of equation (2.4). The potentials ψ and ϕ then give the complete solution to the problem.

However, the above solution is only valid as long as the yielding limit is not reached by the stress. As the material cools down and non-uniform temperature fields arise, there may exist a time when the stress at some points reach this limit and hence plastic regions develop in the body. The elastic-plastic boundaries are dynamic (i.e. change with time) and so the problem of an analytic determination of the complete stress field would be extremely difficult (if indeed it were possible at all). However, an approximate idea of the plastic regions might be obtained by solving the thermoelastic equations above and regarding the plastic regions as those where the maximum shear stresses violoate the yielding criterion. If the optimal control solution which we derive in this paper is successful, these plastic regions might be expected to be fairly small and so this procedure should give at least a first order approximation. Of course, this solution may also be compared with that obtained without control to get some idea of the improvement which the control achieves.

We shall take the point of view as in the previous paragraph since we are interested mainly in developing the optimal temperature control, which we will see can be done independently of the knowledge of the plastic zones for small strains.

3. <u>The Optimal Control Problem</u>

Consider a bounded body Ω subject to temperature control on the boundary $\partial\Omega$, as in fig. 3.1. We shall suppose that the control takes

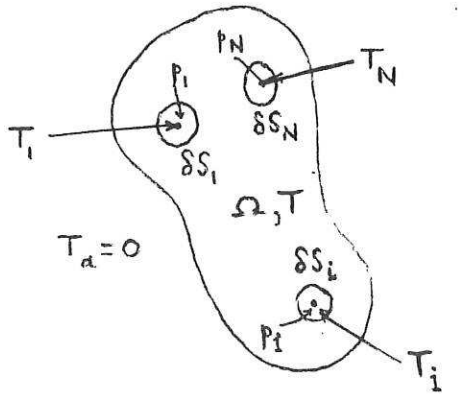

Fig. 3.1

the form of passive cooling distributed over $\partial\Omega$ apart from N small areas of surface δS_i surrounding points $p_i \in \partial\Omega (1 \leq i \leq N)$ where the temperature is controlled to T_i. It will be convenient to measure the temperature relative to the ambient air temperature T_a. Hence, in these units we may set $T_a = 0$. Let $\delta S = \bigcup_{i=1}^{N} \delta S_i$ and define

$$\phi_1(p) = \begin{cases} 1 & \text{if } p \in \partial\Omega \setminus \delta S \\ 0 & \text{if } p \in \delta S \end{cases}$$

$$\phi_2(p) = \begin{cases} 0 & \text{if } p \in \partial\Omega \setminus \delta S \\ 1 & \text{if } p \in \delta S \end{cases}$$

Then the boundary condition on $\partial\Omega$ is (by Newton's law of cooling)

$$HT\Big|_{\partial\Omega} = -\lambda \frac{\partial T}{\partial n}\Big|_{\partial\Omega} \phi_1 + \sum_{i=1}^{N} HT_i \phi_2$$

where H is the coefficient of heat transfer.

However, if $\psi \in C^{\infty}(\partial\Omega)$, then

$$\langle HT|_{\partial\Omega}, \psi \rangle = \langle (-\lambda \frac{\partial T}{\partial_n}|_{\partial\Omega}) \phi_1 , \psi \rangle + \sum_{i=1}^{N} \langle HT_i \phi_2 , \psi \rangle \tag{3.1}$$

where

$$\langle f, g \rangle = \int_{\partial\Omega} f(p) g(p) ds \quad (= \text{surface integral over } \partial\Omega)$$

and so if each region δS_i is small with area a_i , then

$$\langle HT|_{\partial\Omega}, \psi \rangle \simeq \langle -\lambda \frac{\partial T}{\partial_n}\Big|_{\partial\Omega} , \psi \rangle + H \sum_{i-1}^{N} T_i \, a_i \, . \tag{3.2}$$

Therefore the boundary condition is approximately

$$HT|_{\partial\Omega} = -\lambda \frac{\partial T}{\partial n}\Big|_{\partial\Omega} + \sum_{i=1}^{N} HT_i a_i \, \delta(p - p_i) \tag{3.3}$$

where

$$\langle \delta(p - p_i) , \psi \rangle = \psi(p_i) \quad \text{for } \psi \in C^{\infty}(\partial\Omega)$$

provided that the inner products and the expression (3.3) are now interpreted in the distribution sense. Since the temperatures T_i are the controls we shall denote them by u_i and then (3.3) becomes

$$HT|_{\partial\Omega} = -\lambda \frac{\partial T}{\partial_n}\Big|_{\partial\Omega} + \sum_{i=1}^{N} Hu_i a_i \, \delta(p - p_i) \tag{3.4}$$

The dynamic constraint for this problem is, of course, the heat conduction equation :

$$\frac{\partial T}{\partial t} = \mu\nabla^2 T \tag{3.5}$$

together with the boundary condition (3.4). However, the normal control problem takes the form

$$\dot{x} = Ax + Bu \qquad (*)$$

and so we would like to define an operator B so that the heat conduction equation (3.5) becomes

$$\frac{\partial T}{\partial t} = \mu\nabla^2 T + Bu \tag{3.6}$$

where $u = (u_1,\ldots,u_N)^T$.

In order to determine the interpretation (3.6), it is necessary first to define what is meant by a solution of (3.5) which satisfies the boundary condition (3.4). Since the boundary value of T is not a smooth function, we can no longer expect to obtain a classical solution to (3.5). The correct interpretation for a solution of (3.5) is now that if a <u>weak solution</u>; to define such a solution we need Green's theorem which states that

$$\int_\Omega (\phi\nabla^2\psi - \psi\nabla^2\phi)\,dv = -\int_{\partial\Omega}(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n})\,dS \tag{3.7}$$

for any sufficiently smooth functions ϕ, ψ. Consider now a smooth function T*, take the inner product (over Ω) with (3.5), and apply (3.7). Then,

$$\int_\Omega \frac{\partial T}{\partial t}.T^*\,dv = \mu\int_\Omega \nabla^2 T.T^*\,dv$$

$$= \mu\int_\Omega T\,\nabla^2 T^*\,dv - \mu\int_{\partial\Omega}(T^*\frac{\partial T}{\partial n} - \frac{T\partial T^*}{\partial n})\,dS$$

$$= \mu\int_\Omega T.\nabla^2 T^*\,dv + \mu\int_{\partial\Omega} T^* HT\,ds - \mu\sum_{i=1}^{N} T^*(p_i)Hu_i a_i$$

$$+ \mu\int_{\partial\Omega} T\frac{\partial T^*}{\partial n}\,ds \tag{3.8}$$

Now consider the problem defined by the equation

$$\frac{\partial T^*}{\partial t} = -\mu\nabla^2 T^* + f \tag{3.9}$$

with the boundary condition $\left.\frac{\partial T^*}{\partial n}\right|_{\Omega} = 0$,

for some sufficiently smooth forcing function f.

It is well known that this has a unique classical solution $T^*(x,t)$ for $x\varepsilon\Omega$, $t\geq 0$. Using this solution in (38) and integrating with respect to t over the interval $[0,t_1]$, we obtain

$$\int_o^{t_1}\int_\Omega f(x,t)T dv\, dt - \int_o^{t_1}\int_{\partial\Omega}\mu T^* HT\, dS dt + \int_o^{t_1}\mu\sum_{i=1}^{N} T^*(p_i,t)Hu_i a_i dt$$

$$+ \mu\int_\Omega T^*(x,0)T_o(x)\, dv = 0 , \tag{3.10}$$

where

$$T(x,0) = T_o(x)$$

$$T^*(x,t_1) = 0$$

are respectively the initial and final conditions of the equations (3.5) and (3.9).

Definition 3.1 A weak solution of (3.5) and (3.4) is a function $T(x,t)$ which satisfies (3.10).

It should be noted that the concept of weak solution is the appropriate setting when studying distributional solutions (see [4]). However, for control theoretical reasons, it is not particularly convenient to use weak solutions and so we consider the problem defined by the partial differential equation

$$\frac{\partial T}{\partial t} = \mu\nabla^2 T + \mu H\sum_{i=1}^{N} u_i a_i\ \delta(p-p_i) \tag{3.11}$$

subject to the boundary condition

$$HT\Big|_{\partial\Omega} = \frac{-\lambda\partial T}{\partial n}\Big|_{\partial\Omega} . \qquad (3.12)$$

Intuitively, of course, equations (3.11) and (3.12) represent the problem of heat flow in a body subject to passive cooling on the whole of the boundary, together with control injected at the points p_i. Note that if equation (3.11) is multiplied by T* and integrated over Ω and t (from 0 to t_1), then we again arrive at (3.10) and so the weak solutions of (3.5) and (3.4) are the same as those of (3.11) and (3.12). However, (3.11) is now in the appropriate form for control purposes; the main difficulty is that the operator B in equation (*) is now an unbounded operator.

In studying the control problem defined by equations (3.11), (3.12), it is convenient to introduce the semigroup S(t) generated by the operator A with domain

$$D(A) = \{T \in L^2(\Omega): \; T \in H^2[\Omega] ,$$

$$HT\Big|_{\partial\Omega} = \frac{-\lambda\partial T}{\partial n}\Big|_{\partial\Omega} \}$$

defined by

$$AT(x) = \mu \nabla^2 T(x)$$

(cf. [2], [10]) The <u>mild solution</u> of (3.11) can then be defined as the solution of the integral equation

$$T(t) = S(t)T_o + \int_o^t S(t-s)\mu H \cdot \sum_{i=1}^{N} (u_i a_i \delta(p-p_i))ds \qquad (3.13)$$

However, extreme care must be taken in the interpretation of (3.13), since $\delta(p-p_i) \notin L^2[\Omega]$. In fact $\delta(p-p_i) \in H^{-\frac{1}{2}-\varepsilon}[\Omega]$ and so the operator B is a continuous linear map from $\mathbb{R}^N$ to $H^{-\frac{1}{2}-\varepsilon}[\Omega]$; i.e.

$B \in \mathcal{L}(\mathbb{R}^N, H^{-\frac{1}{2}-\varepsilon}[\Omega])$. ($\varepsilon$ is arbitrarily small)

(For a discussion of the spaces H^s, see [1].) In order to justify the equation (3.13) we note that

$$S(t) \in \mathcal{L}(H^{-\frac{1}{2}-\varepsilon}[\Omega],\ L^2[\Omega]\ )\ ,\ t>0$$

and we have

$$||S(t)T||_{L^2[\Omega]} \leq \frac{M}{t^{\frac{1}{4}+\frac{1}{2}\varepsilon}}\ ||T||_{H^{-\frac{1}{2}-\varepsilon}} \tag{3.14}$$

(cf. [2]). The condition (3.14) means that the semigroup generated by A is 'smoothing' and shows that equation (3.13) is well-defined. The following result is proved in [2].

Lemma 3.2 A mild solution of (3.11)(3.12) is a weak solution of (3.5) and (3.4). □

Hence, we can work entirely with mild solutions of (3.11). The final ingredient necessary to define our optimal control problem is the cost functional. This can be derived by remembering that we require the temperature variation throughout the body to be as small as possible. Also, it is shown in [5] that it is preferable to achieve this with as low temperature inputs at the points p_i is possible. Now, the average temperature over the body at time t is

$$\frac{1}{V} \int_\Omega T(t,x)\ dv$$

where V is the volume of the body. Hence, we would like to minimise the difference

$$V_T = T - \frac{1}{V} \int_\Omega T(t,x)\ dv$$

in some sense, over Ω. The most obvious way to do this is to minimise $||V_T||_{L^2(\Omega)}$. Denote by K the operator defined on $L^2(\Omega)$ by

$$(KT)(x) = \frac{1}{V} \int_\Omega T(x')\ dv$$

(i.e. KT is a constant function on Ω). This operator is well-defined since $L^2(\Omega) \subseteq L^1(\Omega)$, provided Ω is bounded, and it is, in fact, bounded, since

$$\left(\int_\Omega \left(\frac{1}{V}\int_\Omega T(x')dv'\right)^2 dv\right)^{\frac{1}{2}} = \frac{1}{V^{\frac{1}{2}}}\left|\int_\Omega T(x')dv'\right|$$

$$\leq \|T\|_{L^2(\Omega)}$$

Now,

$$\|V_T\|_{L^2(\Omega)} = \left\langle (I-K)T,(I-K)T\right\rangle_{L^2(\Omega)}$$

and since I-K is bounded, we may define the transpose K^* of K. thus,

$$\|V_T\|_{L^2(\Omega)} = \left\langle T,(I-K)^*(I-K)T\right\rangle_{L^2(\Omega)}$$

putting $M = (I-K)^*(I-K)$, we see that it is desired to minimise the quantity

$$\left\langle T,MT\right\rangle_{L^2(\Omega)}$$

over some fixed time interval $[0,t_1]$, say. Note that M is clearly a positive operator.

Now, we also require to minimise the control effort, $u \in \mathbb{R}^N$, and so, if $R \in \mathcal{L}(\mathbb{R}^N, \mathbb{R}^N)$ is positive-definite and symmetric, then we can take

$$J(T,u) = \left\langle T(t),MT(t)\right\rangle_{L^2(\Omega)} + \int_0^{t_1}\{\left\langle T(s),MT(s)\right\rangle_{L^2(\Omega)} + \left\langle u,Ru\right\rangle_{\mathbb{R}^N}\}ds$$

to be the correct cost functional.

4. The Riccati Equation

The optimal control for the problem posed in section 3 can now be found using the following theorem.

Theorem 4.1 [2] The optimal control which minimizes J(T,u) is given by the feedback

$$u^*(t) = -R^{-1}B^*Q(t)T(t)$$

where Q(t) is the unique solution of

$$Q(t) = U^*(t_1,t)MU(t_1,t) + \int_t^{t_1} U^*(s,t)\left[M+Q(s)BR^{-1}B^*Q(s)\right]U(s,t)ds$$

and $U(t,s)$ is an 'evolution operator' which satisfies the equation

$$U(t,s) = S(t-s) - \int_s^t S(t-p)BR^{-1}Q(p)U(p,s)dp \; .$$

Moreover,

$$Q(t) \in \mathcal{L}(H^{-\frac{1}{2}-\varepsilon}(\Omega), L^2(\Omega)) \; , \; U(t,s) \in \mathcal{L}(H^{-\frac{1}{2}-\varepsilon}(\Omega), L^2(\Omega))$$

for almost all t,s. □

It can be readily shown that Q also satisfies the differential equation

$$\frac{d}{dt}\langle Q(t)h,k\rangle + \langle Q(t)h,Ak\rangle + \langle Ah,Q(t)k\rangle = \langle Q(t)BR^{-1}B^*Q(t)h,k\rangle - \langle Mh,k\rangle \tag{4.1}$$

with the final condition $Q(t_1) = M$, where $h,k \in D(A)$.

Now, it is easy to see that the operator K is, in fact, self adjoint and so

$$M = (I-K)^2 = I - 2K + K^2.$$

But $K^2 = K$ (i.e. K is idempotent) and so

$$M = I - K \; .$$

Consider finally the dual operator B^* of B. Now

$$Bu = \mu H \sum_{i=1}^{N} u_i a_i \delta(p-p_i) \quad \text{for} \quad u = (u_1,\ldots,u_N)^T \in \mathbb{R}^N \; ,$$

and so if $v(x) \in H^{\frac{1}{2}+\varepsilon}(\Omega)$, we have

$$\begin{aligned}
\langle v \, , \, Bu\rangle &= \langle v, \mu H \sum_{i=1}^{N} u_i a_i \delta(p-p_i) \rangle \\
&= \mu H \sum_{i=1}^{N} a_i \langle v \, , \, u_i \delta(p-p_i) \rangle \\
&= \mu H \sum_{i=1}^{N} a_i u_i \, v(p_i) \\
&= \sum_{i=1}^{N} \langle \mu H a_i \, v(p_i), \, u_i \rangle_{\mathbb{R}} \\
&= \langle Cv \, , \, u \rangle_{\mathbb{R}^N}
\end{aligned}$$

where the duality is with respect to the pairing between $H^{\frac{1}{2}+\varepsilon}(\Omega)$ and

$H^{-\frac{1}{2}-\varepsilon}(\Omega)$, except in the last two lines,

and

$$B^*v = Cv = (\mu H a_i v(p_i))^T \in \mathbb{R}^N.$$

Hence, apart from multiplying constants, B* is the map which evaluates the argument at the N points $p_1,\ldots,p_N$.

In order to write the Riccati equation in a more easily understandable form, we must now use Schwartz's kernel theorem [7,9], which states that a distribution on Ω may be written as am integral operator with an appropriate kernel function. Thus, we may write

$$(Q(t)T)(x) = \int_\Omega \kappa(x,y,t)T(y)dy.$$

Hence, the Riccat equation (4.1) becomes

$$\int_\Omega\int_\Omega \kappa_t(x,y,t)h(y)k(x)dydx + \int_\Omega\int_\Omega \mu\nabla^2 k(x)\kappa(x,y,t)h(y)dydx$$
$$+ \int_\Omega\int_\Omega \mu\nabla^2 h(x)\,\kappa(x,y,t)k(y)dydx$$
$$= \int_\Omega\int_\Omega k(x)\kappa(x,y,t)\,\mu^2H^2 \sum_{i=1}^N a_i\delta(y-p_i)\{\sum_{j=1}^N \bar{r}_{ij}a_j\int_\Omega \kappa(p_j,y_1,t)h(y_1)dy_1\}dydx$$

$$-\int_\Omega h(x)k(x)dx + \int_\Omega \frac{1}{V}h(x)dx\int_\Omega k(x)dx\ ,$$

where $R^{-1} = (\bar{r}_{ij})_{1\le i,j\le N}$. Hence, using Green's theorem (3.7), it follows that

$$\int_\Omega\int_\Omega \kappa_t(x,y,t)h(y)k(x)dydx + \int_\Omega\int_\Omega \mu k(x)\nabla_x^2\,\kappa(x,y,t)h(y)dydx$$

$$-\int_\Omega\int_{\partial\Omega}\mu\kappa(x,y,t)\frac{\partial k(x)}{\partial n_x}\,h(y)dS_x dy + \int_\Omega\int_{\partial\Omega}\mu\frac{\partial}{\partial n_x}\kappa(x,y,t)k(x)h(y)dS_x dy$$

$$+\int_\Omega\int_\Omega \mu k(x)\nabla_y^2\kappa(x,y,t)h(y)dxdy - \int_\Omega\int_{\partial\Omega}\mu\,\kappa(x,y,t)\frac{\partial h(y)}{\partial n_y}k(x)dS_y dx$$

$$+\int_\Omega\int_{\partial\Omega}\mu\frac{\partial}{\partial n_y}\kappa(x,y,t)h(y)k(x)dS_y dx \tag{4.2}$$

$$= \sum_{i=1}^N \left\{\sum_{j=1}^N \int_\Omega\int_\Omega \mu^2H^2\,k(x)\kappa(x,p_i,t)\kappa(p_j,y,t)a_ia_j\bar{r}_{ij}h(y)dydx\right\}$$

$$-\int_\Omega\int_\Omega h(x)k(y)\delta(x-y)dxdy + \frac{1}{V}\int_\Omega\int_\Omega h(x)k(y)dydx$$

Note that we have used the fact that Q is self-adjoint as the solution of a Riccati equation and so the kernel $\mathcal{K}(x,y,t)$ is symmetric in x and y .

However, $h,k \in D(A)$ and so

$$Hh\Big|_{\partial\Omega} = -\lambda \frac{\partial h}{\partial n}\Big|_{\partial\Omega}$$

$$Hk\Big|_{\partial\Omega} = -\lambda \frac{\partial k}{\partial n}\Big|_{\partial\Omega}$$

and so, if we choose $\mathcal{K}$ to satisfy

$$\begin{aligned} &H\mathcal{K}(x,y,t) + \lambda \frac{\partial}{\partial n_x} \mathcal{K}(x,y,t) = 0 \quad , \forall\ x \in \partial\Omega, y \in \Omega, t \in [o,t_1] \\ &H\mathcal{K}(x,y,t) + \lambda \frac{\partial}{\partial n_y} \mathcal{K}(x,y,t) = 0 \quad , \forall\ x \in \Omega, y \in \partial\Omega, t \in [o,t_1] \end{aligned} \tag{4.3}$$

it follows that (4.2) may be satisfied by the solution of the equation

$$\begin{aligned} &\mathcal{K}_t + \mu\nabla_x^2 \mathcal{K} + \mu\nabla_y^2 \mathcal{K} + \delta(x-y) - \frac{1}{V} \\ &= \sum_{i=1}^{N} \sum_{j=1}^{N} \mu^2 H^2 \mathcal{K}(x,p_i,t)\mathcal{K}(p_j,y,t) a_i a_j \bar{r}_{ij} \end{aligned} \tag{4.4}$$

subject to the boundary conditions (4.3) and the final condition

$$\mathcal{K}(x,y,t_1) = \delta(x-y) - \frac{1}{V} . \tag{4.5}$$

The optimal control u* is then given by

$$u^*(t) = -R^{-1} \begin{pmatrix} a_1 \int_\Omega (p_1,y,t)T(y,t)dv \\ \vdots \\ a_N \int_\Omega (p_N,y,t)T(y,t)dv \end{pmatrix} \cdot \mu H \tag{4.6}$$

5. <u>An example</u>

We shall now consider the example of a beam of uniform rectangular cross section with sides of length α, β, and we shall assume that the control can be applied uniformly along the length of the beam. The problem is then two-dimensional, and to solve the equation (4.4) for $\mathcal{K}(t,x_1,x_2,y_1,y_2)$, it is convenient to use the basis of D(A) defined by the eigenvalues of the operator $\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$, in the present case. It is easy to see that these eigenvectors (which, of course, should satisfy the boundary condition (3.12)) are given by

$$e^1_{mn}(x_1,x_2) = \left(-\frac{\lambda}{H}\frac{(m-\frac{1}{2})\Pi}{\alpha}\cdot\cos\left\{\frac{(m-\frac{1}{2})\Pi x_1}{\alpha}\right\} + \sin\left\{\frac{(m-\frac{1}{2}\Pi)x_1}{\alpha}\right\}\right)\cdot$$

$$\left(-\frac{\lambda}{H}\frac{(n-\frac{1}{2})\Pi}{\beta}\cos\left\{\frac{(n-\frac{1}{2})\Pi\, x_2}{\beta}\right\} + \sin\left\{\frac{(n-\frac{1}{2}\Pi)x_2}{\beta}\right\}\right) \qquad (5.1)$$

$$= \gamma^1_m \delta^1_n .$$

(for $1 \leq m,n < \infty$). However, these vectors are not orthogonal for $(m_1,n_1) \neq (m_2,n_2)$ and so we shall use the Gram-Schmidt procedure to obtain the new vectors

$$e_{mn} = \gamma_m \delta_n$$

where

$$\gamma_1 = \gamma^1_1, \quad \delta_1 = \delta^1_1$$

$$\gamma_m = \gamma^1_m - \sum_{i=1}^{m-1} \frac{\langle \gamma^1_m, \gamma_i \rangle \gamma_i}{\|\gamma_i\|^2}, \quad \delta_n = \delta^1_n - \sum_{i=1}^{n-1} \frac{\langle \delta^1_n, \delta_i \rangle}{\|\delta_i\|^2}\delta_i, \quad m,n \geq 2$$

and the inner products and norms are with respect to $L^2[0,\alpha]$ and $L^2[0,\beta]$. Of course, the new vectors e_{mn} still satisfy the boundary condition (3.12).

Now put

$$k(\underline{x},\underline{y},t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} a_{mnk\ell}(t) e_{mn}(x_1,x_2) e_{k\ell}(y_1,y_2).$$

Then, by (4.4), we obtain

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} \dot{a}_{mnk\ell}(t)\gamma_m\delta_n\gamma_k\delta_\ell$$
$$+ \mu \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} a_{mnk\ell}(t)\left\{\sum_{i=1}^{m}\xi_{mi}\gamma_i\delta_n\gamma_k\delta_\ell + \sum_{i=1}^{n}\gamma_m\eta_{ni}\delta_i\gamma_k\delta_\ell\right.$$

$$\left. + \sum_{i=1}^{k}\gamma_m\delta_n\xi_{ki}\gamma_i\delta_\ell + \sum_{i=1}^{\ell}\gamma_m\delta_n\gamma_k\eta_{\ell i}\delta_i\right\}$$

$$+ \delta(x-y) - \frac{1}{V}$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{N} \mu^2 H^2 \left\{ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} a_{mnk\ell}(t) \gamma_m \delta_n \gamma_k (p_i^{x}1) \delta_\ell (p_i^{x}2) \right\} X$$

$$\left\{ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \sum_{1=1}^{\infty} a_{mnk\ell}(t) \gamma_m (p_j^{x_1}) \; \delta_n (p_j^{x_2}) \; \gamma_k \delta_\ell \right\} a_i a_j . \bar{r}_{ij},$$

where $p_i^{x_1}$, $p_i^{x_2}$ are the x_1, x_2 components of p_i respectively, and

$$\frac{d^2 \gamma_m}{dx_1^2} = \gamma_m'' = \sum_{i=1}^{m} \xi_{mi} \gamma_i$$

$$\frac{d^2 \delta_n}{dx_2^2} = \delta_n'' = \sum_{i=1}^{n} \eta_{ni} \delta_i$$

for constants ξ_{mi}, η_{ni}.

Hence, taking the inner product with $\gamma_m \delta_n \gamma_k \delta_\ell$, we have

$$\dot{a}_{mnk\ell} + \mu \{ \sum_{i=m}^{\infty} a_{ink\ell} \xi_{im} + \sum_{i=n}^{\infty} a_{mik\ell} \eta_{in}$$

$$+ \sum_{i=k}^{\infty} a_{mni\ell} \xi_{ik} + \sum_{i=\ell}^{\infty} a_{mnki} \eta_{i\ell} \} \tag{5.2}$$

$$+ \delta_m^k \; \delta_n^\ell - \frac{1}{V} I_m J_n I_k J_\ell$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{N} \mu^2 H^2 \{ \sum_{k_1=1}^{\infty} \sum_{\ell_1=1}^{\infty} a_{mnk_1\ell_1}(t) \; \gamma_{k_1} (p_i^{x}1) \; \delta_{\ell_1} (p_i^{x}1) \; I_{k_1} J_{\ell_1} \} X$$

$$\{ \sum_{m_1=1}^{\infty} \sum_{n_1=1}^{\infty} a_{m_1 n_1 k\ell}(t) \gamma_{m_1} (p_j^{x}1) \; \delta_{n_1} (p_j^{x}2) \; I_{m_1} J_{n_1} \} a_i a_j \bar{r}_{ij} \; ,$$

where $$I_m = \int_0^\alpha \gamma_m dx_1$$

$$J_n = \int_0^\beta \delta_n dx_2$$

and $$\delta_m^k = \begin{cases} 1 & \text{if } m = k \\ 0 & \text{if } m \neq k \end{cases}$$

together with the final condition

$$a_{mnk\ell}(t_1) = \delta_m^k \; \delta_m^\ell - \frac{1}{V} I_m J_n I_k J_\ell .$$

Consider now a general index pair (i,j) and associate with i the numbers i_1, i_2 such that

$$(i_1-1)^2 < i \le i_1^2 \quad \text{and} \quad i_2 = \left[(k+1)/2\right]$$

where $1 \le k \le 2i_1 - 1$ and $[\alpha]$ is the integer part of α. Then put

$$m_i = i_1 \ , \quad n_i = i_2 \quad \text{if } k \text{ is even}$$

$$m_i = i_2 \ , \quad n_i = i_1 \quad \text{if } k \text{ is odd}$$

Similarly associate with j the numbers k_j, ℓ_j defined in an analogous way. Then we define the matrix $\mathcal{A}$ with $(i,j)^{th}$ element

$$\mathcal{A}_{i,j} = a_{m_i n_i k_j \ell_j} \ .$$

Similarly, we introduce the matrices $\mathcal{J}$, Ξ where

$$\mathcal{J}_{i,j} = I_{m_i} J_{n_i} I_{k_j} J_{\ell_j} / V$$

$$\Xi_{i,j} = (\varepsilon^{m_i}_{k_j} \delta^{\ell_j}_{n_i} \xi_{m_i, k_j} + \varepsilon^{n_i}_{\ell_j} \delta^{k_j}_{m_i} \eta_{n_i, \ell_j})\mu$$

where we have introduced the notation

$$\varepsilon^p_q = \begin{cases} 1 & \text{if } p \ge q \\ 0 & \text{otherwise} \end{cases}$$

and, finally, we introduce the matrix $\mathcal{Y}$ defined by

$$\mathcal{Y}_{ij} = \mu^2 H^2 \sum_{\alpha=1}^{N} \sum_{\beta=1}^{N} \{\gamma_{k_j}(p_\alpha^{x_1}) \delta_{\ell_j}(p_\alpha^{x_2}) I_{k_j} J_{\ell_j} \times$$

$$\gamma_{m_i}(p_\beta^{x_1}) \delta_{n_i}(p_\beta^{x_2}) I_{m_i} J_{n_i}\} a_\alpha a_\beta \bar{r}_{\alpha\beta} \ .$$

Then, it can be seen that equation (5.2) becomes

$$\dot{\mathcal{A}} + \Xi' \mathcal{A} + \mathcal{A} \Xi + I - \mathcal{J} = \mathcal{A} \mathcal{Y} \mathcal{A} \qquad (5.3)$$

where I is the infinite identity matrix, together with the final condition

$$\mathcal{A}(t_1) = I - \mathcal{J} .$$

Note that $Q(t) \in \mathcal{L}(H^{-\frac{1}{2}+\varepsilon}(\Omega), L^2(\Omega))$ and $D(A) \subseteq H^{-\frac{1}{2}+\varepsilon}(\Omega)$, so $Q(t) \in \mathcal{L}(D(A), L^2(\Omega))$. However, $\mathcal{A}$ is a representation of Q with respect to orthonormal bases of D(A) and $L^2(\Omega)$, so that

$$\mathcal{A} \in \mathcal{L}(\ell_1^2, \ell^2) \quad .$$

$\ell_1^2 \subseteq \ell^2$ is the subspace defined by the commutative diagram

$$\begin{array}{ccc} D(A) & \subsetneq & L_2 \\ \simeq \downarrow & & \downarrow \simeq \\ \ell_1^2 & \subseteq & \ell^2 \end{array}$$

It is well-known that the Riccati equation can be replaced by an equivalent linear equation together with an initial condition. This can be achieved by putting

$$\mathcal{A} = YX^{-1}$$

where

$$\frac{d}{dt}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} -\Xi' & \mathcal{Y} \\ I-\mathcal{J} & \Xi \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} \qquad (5.4)$$

or $\qquad \dot{Z} = F\,Z$, say, and

$$Y(o) = (I-\mathcal{J})X(o).$$

It is convenient to set $X(o) = I$, and then $Y(o) = I-\mathcal{J}$. Since the solution of (5.4) exists in $\mathcal{L}(\ell_1^2, \ell^2) \oplus \mathcal{L}(\ell_1^2, \ell^2)$, we can rearrange the terms as we desire. Hence, if

$$F = \begin{bmatrix} F_1 & F_2 \\ F_3 & F_4 \end{bmatrix} \in \mathcal{L}\,(\ell_1^2 \oplus \ell_1^2\,, \ell^2 \oplus \ell^2)$$

where $\quad F_r = (f_{ij}^r)_{1\le i,j<\infty}\;, \quad 1\le r\le 4.\qquad$ we shall define

$$\overline{F} = (\overline{f}_{ij})_{1\le i,j<\infty}$$

where

$$\overline{f}_{ij} = \begin{pmatrix} f^1_{ij} & f^2_{ij} \\ f^3_{ij} & f^4_{ij} \end{pmatrix}, \qquad 1\le i,j<\infty \;.$$

Making a similar obvious definition of $\overline{Z}$, we see that (5.4) is equivalent to the system

$$\frac{d\bar{Z}}{dt} = \bar{F}\bar{Z}, \quad \bar{Z}(o) = \begin{pmatrix} I \\ I-\mathcal{J} \end{pmatrix}.$$

Hence,

$$\bar{Z}(t) = \exp(\bar{F}t)\bar{Z}(o),$$

where $\exp(\bar{F}t)$ is the semigroup generated by $\bar{F} \in \mathcal{L}(\ell_1^2 \oplus \ell_1^2, \ell^2 \oplus \ell^2)$.
If now $\sim$ represents the obvious inverse isomorphism of $\ell^2 \simeq \ell^2 \oplus \ell^2$,

then

$$Z(t) = \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} = \widetilde{\exp(\bar{F}t)\bar{Z}(o)},$$

and $\mathcal{A}$ can be determined from X and Y; i.e.

$$\mathcal{A} = Y(t_1)X(t_1)^{-1} = E_2E_1^{-1}$$

where $\widetilde{\exp(\bar{F}t_1)\bar{Z}(o)} = \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}$.

6. Optimal Selection of the Control Points

The optimal control problem has been reduced in the preceding sections to the problem of solving a nonlinear partial differential equation (4.4) and in the case when the spectrum of the operator A can be found, the solution can be written explicity as an infinite matrix 'exponential'. This optimal solution, of course, will depend on the placing of the control actuators, i.e. on the points p_i, $1 \le i \le N$. It is therefore of considerable interest to be able to optimally select these points to give the best performance. This can be achieved by first recalling the result,

Lemma 6.1 ([2]). The optimal cost for the control problem in sections 3,4 is given by

$$J^*(p_1,\ldots,p_N) = \langle T_o, Q(t_o)T_o \rangle,$$

where Q is the solution of the Riccati equation (4.1) (which depends on p_i) and T_o is the initial temperature distribution. □

We shall suppose that the surface of the body under consideration consists of a number of subsets of certain algebraic varieties. Then, we can assume that it is defined by a set of j inequalities and m-j equalities,

$$\bar{g}_i(x) \leq 0 \quad , \quad 1 \leq i \leq j$$

$$\bar{g}_i(x) = 0 \quad , \quad j+1 \leq i \leq m \ ,$$

where $x \in \mathbb{R}^3$ (or $x \in \mathbb{R}^2$ for a two-dimensional body).

It follows that we must minimize the objective function

$$f(r) = f(p_1,\dots,p_N) = J^*(p_1,\dots,p_N)$$

where $r = (p_{11},p_{12},p_{13},p_{21},p_{22},p_{23},\dots,p_{N1},p_{N2},p_{N3}) \in \mathbb{R}^{3N}$ (or $\mathbb{R}^{2N}$ for a two-dimensional problem), subject to the constraints

$$g_{\ell+(\ -1)N}(r) = \bar{g}_i(p_\ell) \leq 0 \ , \quad 1 \leq \ell \leq N,\ 1 \leq i \leq j$$

$$g_{\ell+(i-1)N}(r) = 0 \qquad 1 \leq \ell \leq N, j < i \leq m$$

and, of course, $p_\ell = (r_{3(\ell-1)+1}, r_{3(\ell-1)+2}, r_{3(\ell-1)+3})$.

Introducing jN slack variables r_{si} , $1 \leq i \leq jN$, we have the problem

minimize $f(r)$

subject to $g_{\ell+(i-1)N}(r) + r_{s,\ell+(i-1)N} = 0, \quad 1 \leq \ell \leq N, 1 \leq i \leq j$

$g_{\ell+(i-1)N}(r) = 0 \qquad , \quad 1 \leq \ell \leq N, j \leq i \leq m$

$r_{s,k} \geq 0 \ , \qquad 1 \leq k \leq jN$

Forming the Lagrangian

$$F(r,r_s,\lambda) = \lambda_o f(r) - \sum_{i=1}^{jN} \lambda_i \left[r_{si} + g_i(r)\right]$$

$$- \sum_{i=jN+1}^{mN} \lambda_i \, g_i(r) \ ,$$

where $(\lambda_i)_{1 \leq i \leq mN}$ is a Lagrange multiplier, we have the necessary conditions

$$\lambda_o \frac{\partial f(r^*)}{\partial r_j} - \sum_{i=1}^{mN} \lambda_i \frac{\partial g_i(r^*)}{\partial r_j} = 0 \ , \quad j=1,\dots,3N$$

$$g_i(r^*) = 0 \quad , \quad i \in I \tag{6.1}$$

$$\lambda_i = 0 \quad , \quad i \in \hat{I}$$

for an optimal solution r^*, where I, $\hat{I}$ are the index sets corresponding to active and inactive constraints, respectively (cf Hadley [3]).

Now,

$$f(r) = J^*(p_1,\ldots,p_N) = \langle T_o, Q(t_o)T_o \rangle$$
$$= \int_\Omega \int_\Omega \kappa(x,y,t_o)T_o(x)T_o(y)dydx$$

and κ is a function or r. Hence, the necessary conditions (6.1) become

$$\lambda_o \int_\Omega \int_\Omega \frac{\partial}{\partial r_j} \kappa(x,y,t_o;r^*)\ T_o(x)T_o(y)dydx - \sum_{i=1}^{mN} \lambda_i \frac{\partial g_i(r^*)}{\partial r_j} = 0,\ j=1,\ \ldots,\ 3N$$

$$g_i(r^*) = 0, \quad i\varepsilon I \tag{6.2}$$
$$\lambda_i = 0, \quad i\varepsilon \hat{I}.$$

Returning to the example of section 5, we have a two dimensional region defined by the constraints

$$x_1x_2(x_1-\alpha)(x_2-\beta) = 0$$

together with the inequalities

$$\alpha \geq x_1 \geq 0$$
$$\beta \geq x_2 \geq 0.$$

We can therefore define the constraints as follows (for simplicity we assume that N = 1):

$$g_1(x_1,x_2) = -x_1 \leq 0$$
$$g_2(x_1,x_2) = x_1-\alpha \leq 0$$
$$g_3(x_1,x_2) = -x_2 \leq 0$$
$$g_4(x_1,x_2) = x_2-\beta \leq 0$$
$$g_5(x_1,x_2) = x_1\ x_2(x_1-\alpha)(x_2-\beta) = 0.$$

Now let

$$T_o(x_1,x_2) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} t_{ij}\gamma_i(x_1)\ \delta_j(x_2)$$

Then,

$$\int_\Omega \int_\Omega \frac{\partial}{\partial r_j} \kappa(x,y,t_o;r^*)T_o(x)T_o(y)dydx$$

$$= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} \frac{\partial}{\partial r_j} a_{mnk\ell}(t_o,r^*)\ t_{mn}t_{k\ell}$$

$$= \mathrm{tr}\left(\frac{\partial \mathbf{A}}{\partial r_j} \mathcal{T}'_o\right)$$

$$= \mathrm{tr}\left(\left[\frac{\partial E_2}{\partial r_j} E_1^{-1} - E_2 E_1^{-1} \frac{\partial E_1}{\partial r_j} E_1^{-1}\right] \mathcal{T}'_o\right)$$

$$= \mathrm{t_r}\left(\left[\frac{\partial E_2}{\partial r_j} E_1^{-1} - \mathcal{K}\frac{\partial E_1}{\partial r_j} E_1^{-1}\right] \mathcal{T}'_o\right) = \Theta_j(r^*), \text{ say,}$$

where $\mathcal{T}_o = (\mathcal{T}_{oij}) = (t_{m_i n_i} t_{k_j \ell_j})$,

and so the necessary conditions (6.2) become

$$\Theta_1(r^*) - (-\lambda_1+\lambda_2+\lambda_5 r_2^*(r_1^*-\alpha)\cdot(r_2^*-\beta) + \lambda_5 r_1^* r_2^* (r_2^* - \beta)) = 0$$

$$\Theta_2(r^*) - (-\lambda_3+\lambda_4+\lambda_5 r_1^* (r_1^*-\alpha)(r_2^*-\beta)+\lambda_5 r_1^* r_2^* (r_1^*-\alpha)) = 0.$$

These equations must be solved for r_1^*, r_2^* together with the equations

$$g_i(r^*) = 0$$

for the active constraints g_i. This can be done by numerical search techniques by first allowing only the g_5 constraint to be active (in which case $\lambda_1=\lambda_2=\lambda_3=\lambda_4= 0$) and checking to see if the r^* satisfies the remaining constraints. If not we allow antother constraint to become active and repeat the procedure, until all constraints are active. (This numerical procedure can be achieved by assuming that $\mathcal{K}(s,u,t)$ is represented in terms of a finite number of modes e_{mn})

7. Determining the Stress Field

As we stated earlier, we shall not attempt to obtain a complete solution for the dynamic plastic boundaries, but merely solve the thermoelastic equations in terms of the temperature T. After finding $\mathcal{K}$ and the optimal points p_i we can solve (3.11) for T. For simplicity we shall again return to the example of section 5 and assume as before that N=1. Hence (3.11) becomes

$$\frac{\partial T}{\partial t} = \mu\nabla^2 T + \mu H u \mathbf{a} \delta(p-\overline{p})$$

(dropping the suffix i and replacing p_1 by $\overline{p}$), and so by (4.6),

$$\frac{\partial T}{\partial t} = \mu\nabla^2 T - (\mu Ha)^2 \int_\Omega \varkappa(\overline{p},y,t)T(y,t)dv\ \delta(p-\overline{p}) ,$$

where we have normalized R (which is now just a scalar) to 1.

Using the basis e_{mn} of D(A) as before, we obtain

$$\dot{T}_{mn}(t) = \mu \sum_{i=m}^{\infty} T_{in}\xi_{im} + \mu \sum_{i=n}^{\infty} T_{mi}\eta_{in}$$

$$- (\mu Ha)^2 \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} \Big(\sum_{m_1=1}^{\infty} \sum_{n_1=1}^{\infty} a_{m_1 n_1 k\ell}\gamma_{m_1}(\overline{p}_1)\ \delta_{n_1}(\overline{p}_2)\Big)T_{k\ell}(t)\gamma_m(\overline{p}_1)\delta_n(\overline{p}_2)$$

where

$$T = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} T_{ij}\gamma_i\delta_j ,$$

and so

$$\dot{\Upsilon} = \Xi'\Upsilon + (\Gamma\Gamma'\mathcal{A}(t))\Upsilon$$

where Ξ and $\mathcal{A}$ are as in section 5, and Υ, Γ are column vectors defined by

$$\Upsilon_i = T_{m_i n_i} , \quad \Gamma_i = \gamma_{m_i}(\overline{p}_1)\delta_{n_i}(\overline{p}_2)$$

(i, m_i, n_i are as defined earlier). Hence

$$\Upsilon = \mathcal{U}(t,0)\Upsilon_0$$

where $\mathcal{U}(t,s)$ is the evolution operator generated by $\Xi'\Upsilon + (\Gamma\Gamma'\mathcal{A}(t))$

where again $\mathcal{U}(t,s)$ exists in $\mathcal{L}(\ell^2,\ell^2)$ as before. Having determined T , it follows from (2.3) that

$$\psi(x) = \frac{-(1+\nu)\alpha}{4\pi(1-\nu)} \int_\Omega T(y)\ \frac{1}{r'}\ dv .$$

Let

$$\sigma_{\partial\Omega}(x) = -\big[\mu(\psi_{,ij}(x)+\psi_{,ij}(x))+(\lambda\nabla^2\psi(x)-T(x)\gamma)\delta_{ij}\big]$$

for $x\in\partial\Omega$. Then $\sigma_{\partial\Omega}$ is a tensor-valued function defined on the boundary of Ω.

It remains therefore to solve the biharmonic equation

$$\nabla^4 \phi = 0$$

subject to the boundary conditions

$$\phi_{,yy} = (\sigma_{\partial\Omega})_{xx}$$
$$\phi_{,xx} = (\sigma_{\partial\Omega})_{yy}$$
$$\phi_{,xy} = -(\sigma_{\partial\Omega})_{xy} \quad .$$

However, this is now an elementary exercise in partial differential equations and so we shall not bore the reader by writing out the details. The general stress field is therefore given by

$$\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2G\,\varepsilon_{ij} + (\partial^2\phi/\partial_{x_i}\partial_{x_j})(-1)^{i+j} \quad , \quad i,j=1,2 \tag{7.1}$$

where $\varepsilon_{ij} = (u_{i,j}+u_{j,i})/2$ and $u_i = \partial\psi/\partial x_i$.

This stress field is only valid, as we states earlier, as long as no yielding takes place in the material. However, some indication of the plastic zones can be obtained by finding the points $x\varepsilon\Omega$ where the stress values which are larger than those given by the yield criterion. Therefore, using Von Mises yield criterion, for example, the yield set $\Omega_y(t)\subseteq\Omega$ is given approximately by those points $x\varepsilon\Omega$ where

$$\left\{ \left(\frac{\sigma_1^{\,p}-\sigma_2^{\,p}}{2}\right)^2 + \left(\frac{\sigma_2^{\,p}}{2}\right)^2 + \left(\frac{\sigma_1^{\,p}}{2}\right)^2 \right\}^{\frac{1}{2}} \geq \frac{\sigma_y(T)}{\sqrt{2}}$$

where σ_y is the yield stress of the material (which, of course, depends on the temperature) and $\sigma_i^{\,p}$ are the principal values of the stress field given by (7.1).

8. Conclusions

In this paper we have given a theoretical study of an optimal control problem in the area of residual stress relieving. As we have seen, it is possible to write down the Riccati equation and solve it in a particular case in terms of the 'exponential' of an infinite matrix. Of course, numerically it is sufficient to consider a finite number of modes in the expansions of the various functions involved and so these functions are calcuable and lead to an expression for the optimal cost in terms of the points at which the control is applied. We then applied Kuhn-Tucker theory to derive necessary conditions for the optimal selection of these points. Finally, the derivation of the termal stress field was considered and an approximate method of determining the plastic regions was discussed. Since our interest is mainly in the optimal control problem, we did not persue the complete determination of the plastic regions, which, of course, would be necessary to obtain an accurate picture of the residual stresses. (This, in itself would be an interesting and complex problem.) However, if the optimal control is effective, the residual stresses should be small and the plastic regions should not be too important.

It is hoped that this paper has shown that this area of engineering has some interesting applications for the abstract theory of optimal control of partial differential equations, where the control is naturally restricted to the boundary. The reasoning requires some fairly sophisticated ideas of mathematics, but, as we have seen, these are necessary for a proper interpretation of the boundary control, and hopefully this will not deter the engineer from reading the material presented here.

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