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ON THE ORDERS OF OPTIMAL SYSTEM INFINITE ZEROS

by

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Abstract

It is shown that the orders of the infinite zeros of optimal linear regulators are simply twice the integer structural invariants of the C^* transformation group. This indicates that a conjecture due to Kouvaritakis is 'almost always' valid.

It is well-known [1] that the stabilizable and detectable system $S(A,B,C)$ in R^n defined by

$$\dot{x}(t) = Ax(t) + Bu(t) \quad , \quad x(t) \in R^n$$

$$y(t) = C\,x(t) \quad , \quad y(t) \in R^m \quad , \quad u(t) \in R^{\ell} \tag{1}$$

with state feedback controller minimizing the performance criterion

$$J = \int_0^\infty \{y^T(t)\,Qy(t) + p^{-1}\,u^T(t)\,R\,u(t)\}\,dt \tag{2}$$

(where both Q and R are symmetric and positive definite and $p > 0$) has closed-loop poles equal to the left-half plane solutions of the equation

$$|\,I_\ell + p\,G^T(-s)\,G(s)| = 0 \tag{3}$$

where

$$G(s) = Q^{\frac{1}{2}}\,C(sI_n - A)^{-1}\,B\,R^{-\frac{1}{2}} \tag{4}$$

Without loss of generality we will take $Q = I_m$ and $R = I_\ell$ when G is simply the transfer function matrix of $S(A,B,C)$.

The basic analysis of [1] has been extended to provide a number of computation techniques [2] - [4] which have yielded several important results covering the structure of the root-locus as $p \to +\infty$. More precisely [4]:

<u>Theorem 1</u>: If $S(A,B,C)$ is left-invertible, there exists integers $q \geq 1$, $\{d_j\}_{1 \leq j \leq q}$ and $k_1 < k_2 < \ldots < k_q$ such that its optimal root-locus has unbounded branches as $p \to \infty$ of the form

$$s_{j\ell r}(p) = p^{1/2k_j}\,\eta_{j\ell r} + \alpha_{jr} + \varepsilon_{j\ell r}(p)$$

$$\lim_{p \to +\infty} \varepsilon_{j\ell r}(p) = 0 \ , \ 1 \leq \ell \leq k_j \ , \ 1 \leq r \leq d_j \ , \ 1 \leq j \leq q \qquad (5)$$

where each α_{jr} is pure imaginary and the $\eta_{j\ell r}$, $1 \leq \ell \leq k_j$ take the form $\eta_{j\ell r} = \lambda_{jr} \mu_{j\ell}$ where λ_{jr} is real and strictly positive and the $\mu_{j\ell}$, $1 \leq \ell \leq k_j$, are the distinct left-half-plane $2k_j^{th}$ roots of $(-1)^{k_j+1}$.

As noted in [4], it is seen that the optimal root-locus has $k_i d_i$ infinite zeros of (even) order $2k_i$, $1 \leq i \leq q$, and, moreover, the spatial arrangement of the asymptotes of order 2r are the left-half-plane solutions of $x^{2r} = +1$ if r is odd or $x^{2r} = -1$ if r is even. These observations clearly indicate that the conjectures 4.1 and 4.3 in [2] are correct! The purpose of this paper is to show that conjecture 4.2 in [2], which states that, in the case of $m = \ell$, the unbounded solutions of

$$|I_m + p\ G(s)| = 0 \qquad (6)$$

as $p \to \infty$ have orders $k_1, k_2, \ldots, k_q$ with 'multiplicities' $d_1, d_2, \ldots, d_q$ respectively, is also true in a certain generic sense. In effect, we will show that the orders of the infinite zeros of the optimal root-loci are, almost always, equal to twice the orders of the infinite zeros of the system G and that, when this situation holds, there are exactly as many $2q^{th}$ order optimal system infinite zeros as there are q^{th} order infinite zeros of G.

The basic idea behind the analysis lies in the use of the results in [5] in a form similar to that used in [6]. We consider only the case of $m = \ell$ (i.e. the same number of inputs and outputs) as it is only in this particular case that problem (6) makes sense and assume that S(A,B,C) is invertible. The basic structural representation of G required is given in [6] (equation (24)), namely, for any choice of F and K,

$$G(s) = \{I_m + H_2(s)\}\ C(sI_n - A - BF - KC)^{-1}B\ \{I_m + H_1(s)\} \qquad (7)$$

where H_1 and H_2 are strictly proper. It is noted [6] (equation (17)) that we can find F and K and nonsingular mxm matrices M and N such that

$$N\, C(sI_n - A - BF - KC)^{-1} B\, M = \mathrm{diag}\{\frac{1}{s^{n_1}},\ldots,\frac{1}{s^{n_m}}\} \triangleq D_1(s)\ldots \qquad (8)$$

where $n_1, n_2, \ldots, n_m$ are the integer structural invariants of the C^* transformation group introduced in [5]. The following important result is proved in [6]:

<u>Theorem 2</u>: Under mild (generic) assumptions on $\Gamma = NM$, and if the ordered list $n_1 \leq n_2 \leq \ldots \leq n_m$ has q' distinct entries $m_1 < m_2 < \ldots < m_{q'}$ of multiplicity d_i', $1 \leq i \leq q'$, then the system $S(A,B,C)$ has $m_i d_i'$ infinite zeros of order m_i, $1 \leq i \leq q'$.

Remark: The assumptions on Γ can be found in [6]. They are not always satisfied (indeed [7], the orders of the infinite zeros need not be equal to the m_i, $1 \leq i \leq q'$) but the introduction of a forward path controller will always remedy this situation, and this controller can be chosen on a random basis with probability one of success.

We now state the main result of this paper:

<u>Theorem 3</u>: $q = q'$, $d_i = d_i'$ $(1 \leq i \leq q)$ and $k_i = m_i$ $(1 \leq i \leq q)$

Remark: Combining this result with theorem 2 and its following remark, it is clear that the conjecture due to Kouvaritakis is <u>generically</u> valid but not always valid. More generally, interpreting $Q^{\frac{1}{2}}$ and $R^{-\frac{1}{2}}$ as constant pre-and past-multipliers in (4) we see that his conjecture is valid for 'almost all' choices of weighting matrices Q and R.

<u>Proof of Theorem 3</u>: Using (7) and (8) in (3) and introducing the notation $\tilde{E}(s) \triangleq E^T(-s)$ yields the equivalent relation

$$0 = \left| I_m + p(I_m+\tilde{H}_1)(M^T)^{-1}\tilde{D}_1(N^T)^{-1}(I_m+\tilde{H}_2)(I_m+H_2)N^{-1}D_1M^{-1}(I_m+H_1)\right|$$

$$= \left|(I_m+\tilde{H}_1)(MM^T)^{-1}(I_m+H_1)\right| . \left| M^T(I_m+\tilde{H}_1)^{-1}(I_m+H_1)^{-1}M\right.$$

$$\left. + \, p\,\tilde{D}_1(N^T)^{-1}(I_m+\tilde{H}_2)(I_m+H_2)N^{-1}D_1\right| \qquad (9)$$

or, noting that H_1, H_2 (and hence $\tilde{H}_1, \tilde{H}_2$) are strictly proper and defining the symmetric, positive-definite matrices $M_1 = M^TM$, $N_1 = NN^T$ and $\hat{N}_1 = N_1^{-1}$, we can replace (9) by

$$0 = \left| M_1 + O(s^{-1}) + p\,\tilde{D}_1(\hat{N}_1 + O(s^{-1}))\,D_1\right| \qquad (10)$$

on the unbounded branches of the root-locus. Here the notation $O(s^{-(k+1)})$ is used to describe any mapping such that $\lim_{|s|\to\infty} s^k\, O(s^{-(k+1)}) = 0$ on some domain.

Write, without loss of generality,

$$D_1(s) = \text{block diag}\,\{s^{-m_i} I_{d_i'}\} \quad 1\leq i\leq q'$$

$$\triangleq \begin{pmatrix} s^{-m_1} I_{d_1'} & 0 \\ 0 & D_2(s) \end{pmatrix} \qquad (11)$$

and also

$$\hat{N}_1 = \begin{pmatrix} \hat{N}_{11} & \hat{N}_{12} \\ \hat{N}_{21} & \hat{N}_{22} \end{pmatrix} \text{ and } M_1 = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_2 \end{pmatrix} \qquad (12)$$

where $\hat{N}_{11}$ and M_{11} are of dimension $d_1' x d_1'$. Note that $\hat{N}_{11}$, M_{11}, $\hat{N}_{22}$ and M_2 are all symmetric and positive-definite and hence nonsingular. Also note that ps^{-2m_1} cannot have a zero cluster point on any unbounded branch of the optimal root-locus otherwise (10) would then reduce to $0 = |M_1|$ which is impossible. We conclude therefore that ps^{-2m_1} can only have finite or infinite cluster points. Let λ be a <u>finite</u> cluster point of ps^{-2m_1} and use Schurs formula on (10) and the observation that $p\,D_2(s)$ (and hence $p\,\tilde{D}_2(s)$) are $O(s^{-1})$ in such circumstances to obtain a relation of the form

$$0 = |M_2 + O(s^{-1})|\,.\,|M_{11} + O(s^{-1}) + (-1)^{m_1} \frac{p}{s^{2m_1}} (\hat{N}_{11} + O(s^{-1}))$$

$$- (M_{12} + O(s^{-1}))(M_2 + O(s^{-1}))^{-1}(M_{21} + O(s^{-1}))| \qquad (13)$$

Letting both $|s|$ and p tend to infinite in the defined sense, we obtain a relation of the form

$$0 = |M_2|\,.\,|M_{11} + (-1)^{m_1} \lambda \hat{N}_{11} - M_{12} M_2^{-1} M_{21}| \qquad (14)$$

and hence (by the nonsingularity of M_2 and $\hat{N}_{11}$) that there are precisely d_1' possible non-zero values of λ obtained by solving (14) or, equivalently,

$$0 = \left|M_1 + (-1)^{m_1} \lambda \begin{pmatrix} \hat{N}_{11} & 0 \\ 0 & 0 \end{pmatrix}\right| \qquad (15)$$

Clearly the optimal system root-locus has d_1' m_1 $2m_1^{th}$ under infinite zeros exactly and we must hence have $d_1 = d_1'$ and $k_1 = m_1$.

Consider now those unbounded branches of the optimal root-locus where ps^{-2m_1} is <u>unbounded</u>. Using (11) and (12) in (10) and using Schurs' formula yields

$$\begin{vmatrix} M_{11}+O(s^{-1}) + ps^{-2m_1}(-1)^{m_1}(\hat{N}_{11}+O(s^{-1})), & M_{12}+O(s^{-1})+(-1)^{m_1} p\, s^{-m_1}(\hat{N}_{12}+O(s^{-1}))D_2(s) \\ M_{21}+O(s^{-1}) + ps^{-m_1}\tilde{D}_2(s)(\hat{N}_{21}+O(s^{-1})), & M_2+O(s^{-1}) + p\,\tilde{D}_2(s)\,(\hat{N}_{22}+O(s^{-1}))D_2(s) \end{vmatrix}$$

$$= |\,M_{11} + O(s^{-1}) + ps^{-2m_1}(-1)^{m_1}(\hat{N}_{11} + O(s^{-1}))|\,.\,|M_2 +$$

$$O(s^{-1}) + p\,\tilde{D}_2(s)(\hat{N}_{22} + O(s^{-1}))D_2(s) - \psi(s)\,| \qquad (16)$$

where $\psi(s)$ has the form

$$\psi(s) = (M_{21} + O(s^{-1}) + ps^{-m_1}\tilde{D}_2(s)\ (\hat{N}_{21} + O(s^{-1}))(M_{11} + O(s^{-1}) + ps^{-2m_1}(-1)^{m_1}(\hat{N}_{11} + O(s^{-1}))^{-1}(M_{12} + O(s^{-1}) + (-1)^{m_1}ps^{-m_1}(\hat{N}_{12} + O(s^{-1}))D_2(s))$$

which has an obvious decomposition of the form

$$\psi(s) = \psi_1(s,p) + p\ \tilde{D}_2(s)\ \psi_2(s,p)D_2(s)$$

where $\psi_1(s,p) = O(s^{-1})$ and $\psi_2(s,p) = (-1)^{m_1}\hat{N}_{21}\hat{N}_{11}^{-1}N_{12} + O(s^{-1})$ on all branches of the root-locus where ps^{-2m_1} is unbounded. The first determinant in (16) is clearly non-zero on these branches so, writing $\hat{N}_2 = \hat{N}_{22} - (-1)^{m_1}\hat{N}_{21}\hat{N}_{11}^{-1}\hat{N}_{12}$ (which is symmetric and positive definite due to the symmetry and positivity of $\hat{N}_1$), we can write (16) in the form

$$|M_2 + O(s^{-1}) + \tilde{D}_2(s)\ (\hat{N}_2 + O(s^{-1}))D_2(s)| = 0 \qquad (19)$$

whenever ps^{-2m_1} is unbounded. This relation has a clear similarity to (10) and hence we can use an induction argument to verify that the optimal root-locus has precisely $d_i' m_i$ $2m_i^{th}$ order infinite zeros, $1 \leq i \leq q'$. Theorem 3 follows by comparing this result with theorem 2.

Remark: The proof of the theorem can obviously be converted into a computational technique but it is not clear how N and M can easily be computed in practice. It is probably better to approach the computational problem more directly (e.g. as in [4]).

Finally we state the following theorem based on the observation that, during the proof of theorem 3, the orders multiplicities and asymptotic directions of the optimal root-locus depended only on N and M (and matrices derived from them) and hence are independent of F and K.

<u>Theorem 4</u>: The orders, multiplicities and asymptotic directions of the infinite zeros of the optimal root-locus of S(A,B,C) generated by the performance index (2) are identical to those of $S(A-BF_o-K_o C,B,C)$ with the same performance index and F_o and K_o arbitrary state feedback and output injection maps.

Remark: This result has clear connections with the corollary to theorem 1 in [6] and suggests that careful choice of F_o and K_o could be used to simplify theoretical or numerical computations. For example, if ν is an $\{A,B\}$ - invariant subspace in the kernel of C, the choice of F_o such that ν is $(A - BF_o)$- invariant enables us to replace S by a system $\bar{S}$ in the quotient space R^n/ν.

<u>References</u>

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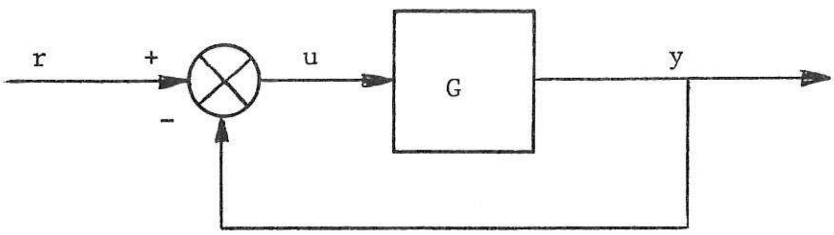
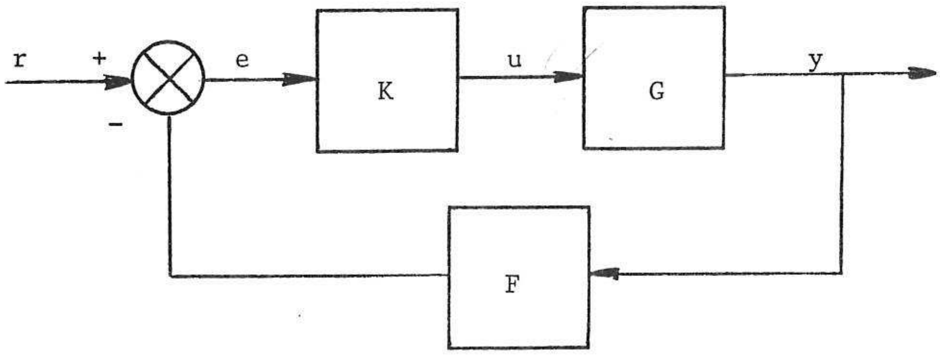


(a)



(b)

Fig. 1

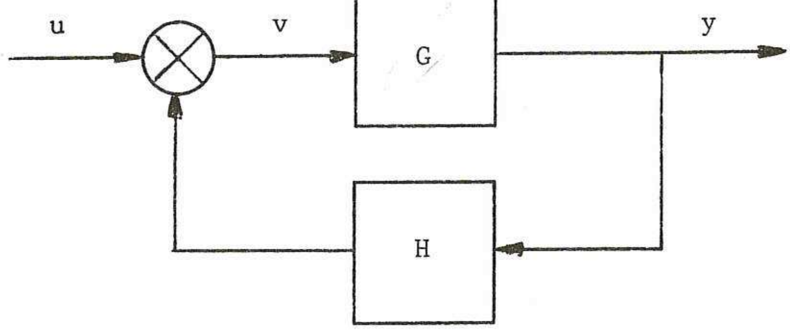


Fig. 2. Plant G with feedback perturbation H