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STABILITY ASSESSMENT USING CONTRACTION CONDITIONS FOR UNKNOWN MULTIVARIABLE FEEDBACK SYSTEMS

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Abstract

The purpose of this paper is to illustrate, using specific examples, the stability region, the contraction condition region and the monotonic condition region in parameter plane. It is our intention to show that contraction condition region is a substantial part of the complete stability region for high gain (continuous) or for fast sampling (discrete) systems and hence gives considerable insight into system robustness.

We consider an unknown continuous multivariable system G(s), satisfying certain conditions, namely,

(a) $G(s) = C(sI - A)^{-1}B$ is minimum phase

(b) $|CB| \neq 0$.

Remark 1: These conditions are equivalent to the requirement that $|G(s)| \neq 0$ and

$$G^{-1}(s) = s A_o + A_1 + A_o H(s)$$

$$|A_o| \neq 0 \text{ , } H(o) = 0 \tag{1}$$

where H(s) is proper and stable.

Under these conditions (|1|,|2|,|3|), a multivariable first-order type model given by

$$G_A^{-1}, = s\tilde{A}_o + \tilde{A}_1 \tag{2}$$

is an adequate model for the purposes of designing a high-performance feedback controller and the system G(s) will be stable for P + I controller (|4|) given by

$$K(s) = \tilde{A}_o \operatorname{diag}\{K_j+C_j+K_jC_j/s\}_{1<j<m} - \tilde{A}_1 \tag{3}$$

if the contraction constant λ defined by

$$\lambda = \left\|\operatorname{diag.}\left\{\frac{s}{(s+K_j)(s+C_j)}\right\}_{1<j<m} (s\tilde{A}_o^{-1}(\tilde{A}_o-A_o)+\tilde{A}_o^{-1}(\tilde{A}_1-A_1) - \tilde{A}_o^{-1}A_oH(s)))\right\| \tag{4}$$

is strictly less than 1. The ideal values of $\tilde{A}_o$ and $\tilde{A}_1$ are usually

$$A_o = (CB)^{-1}, \quad A_1 = -(CA^{-1}B)^{-1}, \tag{5}$$

the tuning parameters must satisfy

$$K_j > 0, \quad C_j > 0 \quad (1 < j < m)$$

and H(s) must satisfy (1).

Remark 2: Defining $\tilde{H}(s)$ as

$$\tilde{H}(s) = \tilde{A}_o^{-1}(G^{-1}(s) - G_A^{-1}(s)) \tag{6}$$

it can be shown from (1) and (2) that H(s) is related to $\tilde{H}(s)$ by

$$\tilde{H}(s) = \tilde{A}_o^{-1}(s(A_o-\tilde{A}_o) + (A_1-\tilde{A}_1) + A_oH(s)) \tag{7}$$

and $\quad \tilde{H}(s) = H(s)$ when $A_o = \tilde{A}_o$, $A_1 = \tilde{A}_1$.

The contraction conditions in terms of $\tilde{H}(s)$ in

$$\lambda \triangleq \left\|\operatorname{diag}\left\{\frac{s}{(s+K_j)(s+C_j)}\right\}_{1<j<m} (-\tilde{H}(s))\right\| < 1 \tag{8}$$

The contraction condition (8) is a sufficient but not necessary condition. It is our aim, however, to show with examples that the region defined by the contraction condition in the parameter space of matrices $(\tilde{A}_o, \tilde{A}_1)$ can be a substantial part of the real stability region. We also hope to find the region where a monotonic condition (see|5|) is satisfied. In such cases A_o is assumed to be known exactly.

Remark 3: By suitable choice of $\tilde{A}_1$ (|5|) we can sometimes make the feedback operator $\tilde{H}$ monotonic and sign definite and in these case the contraction condition becomes (|5|)

$$\lambda' \triangleq ||A_o^{-1}(G^{-1}(o) - \tilde{A}_1||_m \max_{1<j<m} \gamma(K_j, C_j) < 1 \tag{9}$$

where $\gamma(K,C)$ is an upper bound for the norm of a minimal realization of $\frac{s}{(s+k)(s+C)}$ in the Banach space $C_\infty^m(o,\infty)$.

Now consider an unknown controllable and observable m-input m-output linear time-invariant discrete system with state-space form

$$x_{k+1} = \Phi x_K + \Delta u_K \;,\; x_K \in R^n$$
$$y_K = C x_K \;,\; y_K \in R^m \;,\; u_K \in R^m \;,\; K > 0 \tag{10}$$

with mxm z-transfer function matrix

$$G(z) = C(zI_m - \Phi)^{-1}\Delta \tag{11}$$

whose inverse can be represented in the form

$$G^{-1}(z) = B_o(z-1) + B_1 + B_o H(z)$$
$$H(z) \text{ is proper and stable, } H(1) = 0 \;,\; |B_o| \neq 0 \tag{12}$$

and a multivariable first-order model takes the form

$$G_A^{-1}(z) = (z-1)\tilde{B}_o + \tilde{B}_1 \tag{13}$$

In this case the suggested controller (|5|) has the multivariable proportional plus summation form

$$K(z) = \tilde{B}_o \operatorname{diag}\{1 - K_j C_j + \frac{(1-K_j)(1-C_j)z}{(z-1)}\}_{1<j<m} - \tilde{B}_1 \tag{14}$$

The contraction condition in this case is (|8|)

$$\lambda = ||\operatorname{diag}\{\frac{(z-1)}{(z-K_j)(z-C_j)}\}_{1<j<m} ((z-1)\tilde{B}_o^{-1}(\tilde{B}_o - B_o) + \tilde{B}_o^{-1}(\tilde{B}_1 - B_1) - \tilde{B}_o^{-1}B_o H(z))|| < 1 \tag{15}$$

where typical values of B_o and B_1 are

$$B_o = (C\Delta)^{-1} \; , \; B_1 = (C(I_m - \Phi)^{-1}\Delta)^{-1}$$

and we assume that $|K_j| < 1$, $-1 < C_j < 1$ $(1 < j < m)$

Remark 4: The contraction condition (95) can be written as

$$\lambda = \left|\left|\mathrm{diag}\left\{\frac{(z-1)}{(z-K_j)(z-C_j)}\right\}_{1<j<m} (-\tilde{H}(z))\right|\right| < 1 \tag{16}$$

where $\tilde{H}(z)$ is given by

$$\tilde{H}(z) = \tilde{B}_o^{-1}(G^{-1}(z) - G_A^{-1}(z))$$

$$= - \tilde{B}_o^{-1}((z-1)(\tilde{B}_o-B_o)+(\tilde{B}_1-B_1)-B_oH(z)) \tag{17}$$

when $\tilde{B}_o = B_o$ and $\tilde{B}_1 = B_1$ (nominal values) it is clear that

$$\tilde{H}(z) = H(z)$$

Remark 5: In those cases when $\tilde{B}_o = B_o$, we can sometimes choose $\tilde{B}_1$ to make the feedback operator $\tilde{H}(z)$ monotonic and sign definite (see |5|). In such cases the contraction condition becomes

$$\lambda' \triangleq ||B_o^{-1}(G^{-1}(1)-\tilde{B}_1)||_m \max_{1<j<m} \gamma(K_j,C_j) < 1 \tag{18}$$

where the function γ: $R^2 \to R$ is any upper bound for the norm in ℓ_∞^m of a minimal realization of $(z-1)_{/(z-K)(Z-C)}$

We will now obtain the actual stability domain the contraction condition region and the monotonic condition region in parameters plane for two examples.

Actual stability region

Example 1: Continuous system

For simplicity consider the single input/single output system with state-space form

$$\dot{x}(t) = \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u(t)$$

$$y(t) = |1 \quad 1|\, x(t) \tag{19}$$

and transfer function

$$G(s) = \frac{2(s+1)}{(s-1)(s+3)}$$

The closed-loop characteristic polynomial is

$$\rho_c(s) = |sI_n - A + BKC| \tag{20}$$

where, for simplicity, we will take proportional control action only of the form

$$K(s) = k\,\tilde{A}_o - \tilde{A}_1 \tag{21}$$

The closed-loop system is asymptotically stable if, and only if, the roots of ρ_c be in the open left-half complex plane. Therefore, after a little manipulation, the actual stability region is described by the relation

$$k\tilde{A}_o - \tilde{A}_1 > 1.5 \tag{22}$$

Example 2: Discrete System

Take Example 1 subjected to synchronized piecewise constant input and output sampling of frequency $h^{-1} = 10$. The induced discrete model takes the form

$$x_{k+1} = \begin{pmatrix} 1.105 & 0 \\ 0 & 0.741 \end{pmatrix} x_k + \begin{pmatrix} 0.105 \\ 0.086 \end{pmatrix} u_k$$

$$y_k = |1 \quad 1|\, x_K \quad , \quad K > 0 \tag{23}$$

The open-loop system has poles at the points 1.105 , 0.741 and one zero at the point 0.905 inside the unit circle. (i.e. the system is minimum phase).

We will take proportional control action only of the form (taking $C_j = 1$ and $K_j = 0$ in (14))

$$K(z) = \tilde{B}_o - \tilde{B}_1 \tag{24}$$

and note that the closed-loop system is asymptotically stable iff, the roots of $|zI_m - \Phi + \Delta KC|$ lie in the open unit circle in the complex plane. By usning transformation $z = \frac{1+W}{1-W}$ and routine algebra, for actual stability region in parameters $(\tilde{B}_o, \tilde{B}_1)$, we require that

$$1.45 < (\tilde{B}_o - \tilde{B}_1) < 10.07 \tag{25}$$

<u>Contraction condition region</u>

Using (8), the contraction condition region for example 1 in parameters $(\tilde{A}_o, \tilde{A}_1)$ space becomes

$$\text{Max} \left| \frac{\tilde{A}_o^{-1} | (\tilde{A}_o - 0.5)s^2 + (\tilde{A}_o + \tilde{A}_1 - 1)s + (\tilde{A}_1 + 1.5)|}{s^2 + (K+1)s + K} \right|_{s=j\omega} < 1$$

That is, we need

$$\text{Max} \left\{ \frac{(c - a\omega^2)^2 + b^2\omega^2}{(K-\omega^2)^2 + (K+1)^2\omega^2} \right\} < 1 \tag{26}$$

where $a = \tilde{A}_o^{-1}(\tilde{A}_o - 0.5)$

$b = \tilde{A}_o^{-1} (\tilde{A}_o + \tilde{A}_1 - 1)$

and $c = \tilde{A}_o^{-1}(\tilde{A}_1 + 1.5)$

Remark 6: By varying ω, we can find parameter $(\tilde{A}_o, \tilde{A}_1)$ numerically, such that (26) is not satisfied.

If we take, for illustrative purposes, $\tilde{A}_o = A_o = 0.5$ and $\tilde{A}_1 = A_1 = -1.5$ (nominal values) then

$$\mathrm{Max}\ \left\{ \frac{(C-a\omega^2)^2 + b^2\omega^2}{(K-\omega^2)^2+(K+1)^2\omega^2} \right\}$$

occurs at $\omega^2 = K$ and to satisfy the contraction condition we require $K > 3$. This should be compared with the exact result $K > 0$ obtained from equation (22).

For finite K and at high frequency (i.e. letting $\omega \to \infty$ in (26)) we also see that we need

$$\tilde{A}_o > 0.25 \qquad (27)$$

to satisfy the contraction condition i.e. A_o must be estimated to a certain accuracy (c.f. |4|)

For example 2 (discrete system), the contraction condition from (15) becomes

$$\mathrm{Max}\ \max_{\substack{|z| = 1 \\ |z| = R}} \left| \frac{az^2 + bz + c}{z^2 + zd} \right| < 1 \qquad (28)$$

where $a = \tilde{B}_o^{-1}(\tilde{B}_o - B_o)$

$c = \tilde{B}_o^{-1}(0.905(\tilde{B}_o - B_o) - 0.905(\tilde{B}_1 - B_1) - 0.346)$

$b = \tilde{B}_o^{-1}(\tilde{B}_1 - B_1 + 0.345(B_o) - 1.905(\tilde{B}_o - B_o))$

$d = -0.905$

and the parameter R can be set to an arbitrary large value.

Remark 7: Taking $R \gg 1$, it is easy to verify that the maximum occurs on $|z| = 1$.

By writing $z = e^{i\theta}$ ($|z| = 1$), the contraction condition can be written as

$$\left\{ \frac{(a\mathrm{Cos}2\theta + b\mathrm{Cos}\theta + C)^2 + (a\mathrm{Sin}2\theta + b\mathrm{Sin}\theta)^2}{(\mathrm{Cos}\theta + d)^2 + \mathrm{Sin}^2\theta} \right\} < 1 \qquad (29)$$

Remark 8: By varying $\theta(-\pi < \theta < \pi)$ we can find $(\tilde{B}_o, \tilde{B}_1)$ such that the contraction condition (29) is not satisfied.

Monotonic condition region:

For example 1, it is assumed that $\tilde{A}_o$ is known exactly, that is $\tilde{A}_o = A_o =$ 0.5 and the feedback operator $\tilde{H}$ is

$$\tilde{H}(s) = A_o^{-1}(G^{-1}(s) - G_A^{-1}(s))$$

$$= 2\,(\frac{(s+3)(s-1)}{2(s+1)} - 0.5s - \tilde{A}_1)$$

$$= 1 - \tilde{2}A_1 - \frac{4}{s+1} \qquad (30)$$

which is clearly monotonic and sign-definite if $1 - 2\tilde{A}_1 < 0$, that is, we require

$$\tilde{A}_1 > 0.5$$

In this case, the monotonic condition from (9) is simply

$$\lambda' \triangleq ||2(-1.5 - \tilde{A}_1|| \max \gamma(k,C) < 1 \qquad (31)$$

We can take (see |5|) $\gamma = 1/K$, for an upper bound of a minimal realization of $\frac{s}{s+K}$ and (31) becomes

$$\lambda' = |2(-1.5 - \tilde{A}_1|\ \frac{1}{K} < 1 \qquad (32)$$

For example, when K = 10, the values of $0.5 < \tilde{A}_1 < 3.5$ will satisfy the monotonic condition.

Remark 9: Taking $\tilde{A}_1 = 0.5$ in (32), we require K > 4 to satisfy monotonic condition. This should be compared with the exact value of K > 0 and the standard contraction value K > 3.

For example 2, (discrete system), the feedback operator

$$\tilde{H}(z) = 0.059 - 0.191\,\tilde{B}_1 - \frac{0.033}{z-0.905}$$

when $\tilde{B}_o = B_o = -5.23$

$\tilde{H}$ is monotonic and sign-definite if $\tilde{B}_1 > 0.309$. The monotonic condition from (18) is simply

$$\lambda' \triangleq \left|0.191(\tilde{B}_1 + 1.5)\right| \quad \gamma(K,C) < 1 \tag{33}$$

We can take $\gamma(K,C) = \dfrac{1}{1 - |K|}$ (see 5).

and taking $\underset{=}{K} = 0$, $\gamma = 1$.

The expression (33) reduces to

$$\lambda' \triangleq |0.191(\tilde{B}_1 + 1.5)| < 1 \tag{34}$$

and the values of $0.309 < \tilde{B}_1 < 3.7$ will satisfy the monotonic condition. This should be compared with exact result $-4.84 < \tilde{B}_1 < 3.7$ obtained from equation (25).

Results and Conclusions:

The stability region, the contraction condition region and the monotonic condition region may be plotted in the $(\tilde{A}_o\ \tilde{A}_1)$ space for example 1. This has been done in Fig. 1 for K = 10. It is seen from the Figure that the contraction condition region does not contain any points for which $\tilde{A}_o < 0.25$ (see also expression (27)).

It is known (|1|,|3|,|4|) that by increasing gain K, the contraction condition region will increase and hence the monotonic condition region. This is illustrated in Fig. 2 where K = 50.

The regions of stability, the contraction and monotonic are plotted in the $(\tilde{B}_o\tilde{B}_1)$ plane for example 2, in Fig. 3. for $h^{-1} = 10$.

It is clear from the examples that the contraction condition region is a substantial part of the actual stability region and increases as gain increases. (for continuous system). We have also shown in the Figures 1 2 and 3, the monotonic condition region. Monotonicity is an important property, which can be useful in designing a simple controller for unknown systems (see |6|,|7|).

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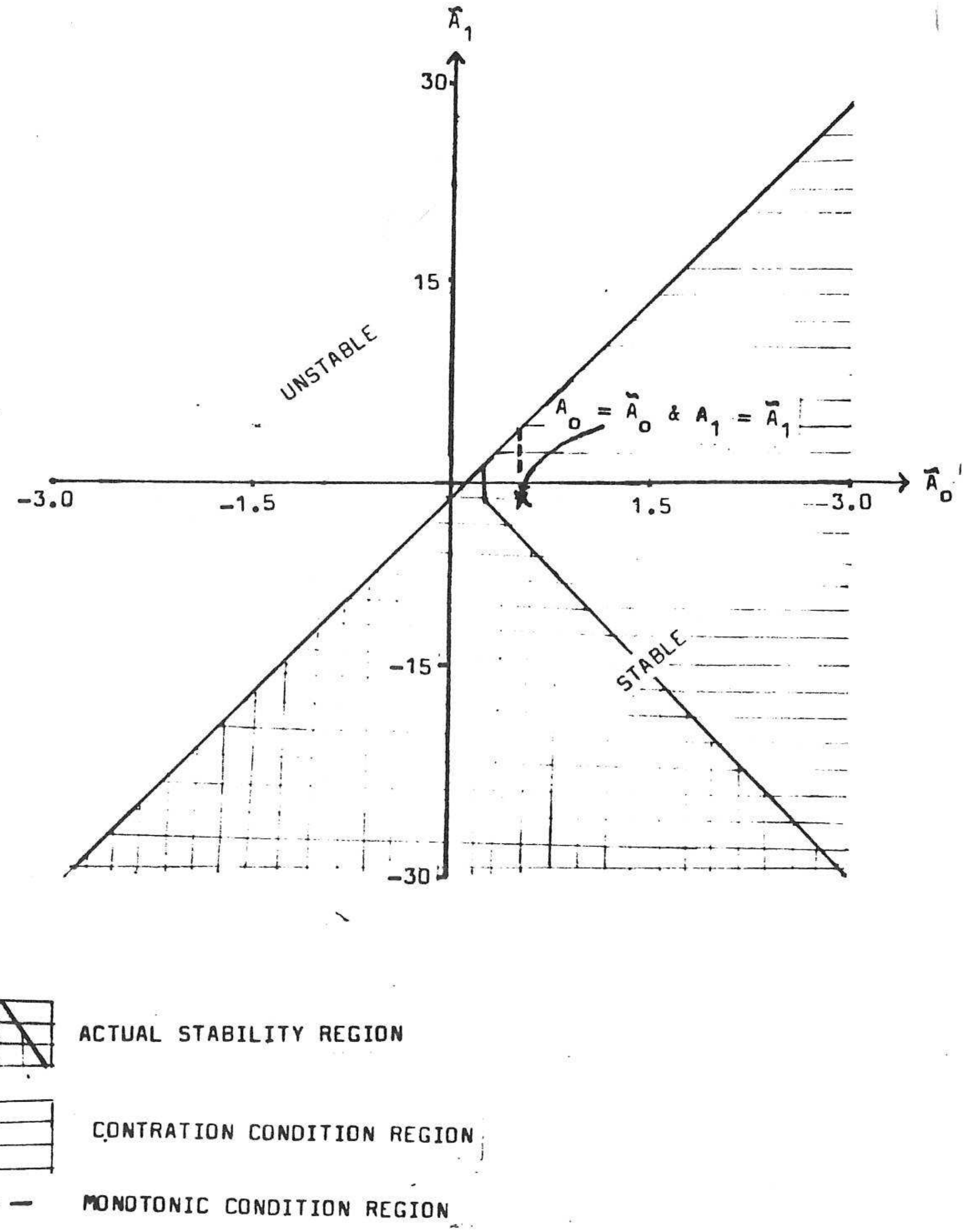
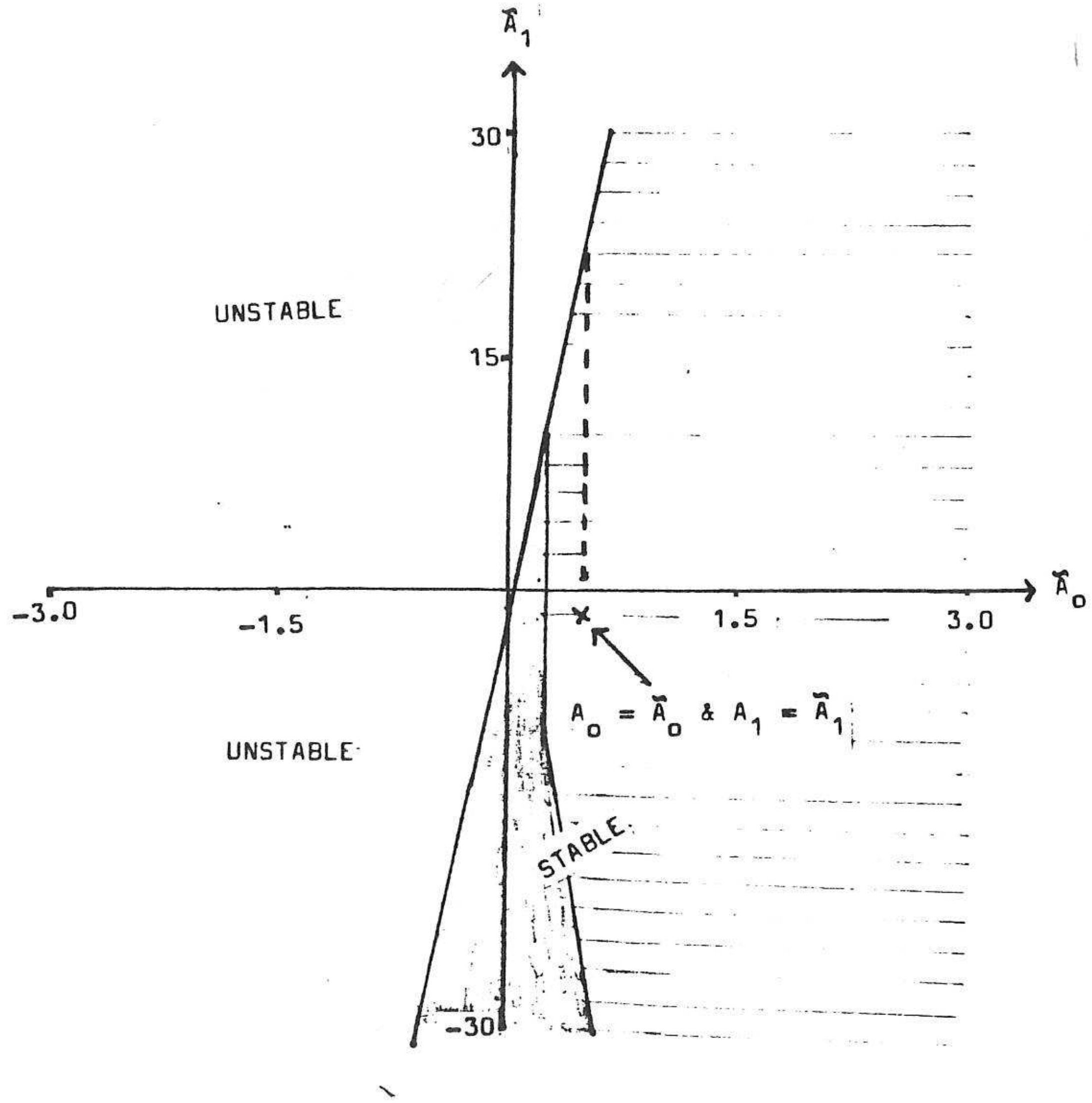
k=10



$\tilde{A}_1$
30
15
-15
-30
-3.0
-1.5
1.5
3.0
$\tilde{A}_0$
UNSTABLE
STABLE
$A_0 = \tilde{A}_0$ & $A_1 = \tilde{A}_1$
ACTUAL STABILITY REGION
CONTRATION CONDITION REGION
MONOTONIC CONDITION REGION

Fig. 1

k=50



ACTUAL STABILITY REGION

CONTRACTION CONDITION REGION

MONOTONIC CONDITION REGION

Fig. 2

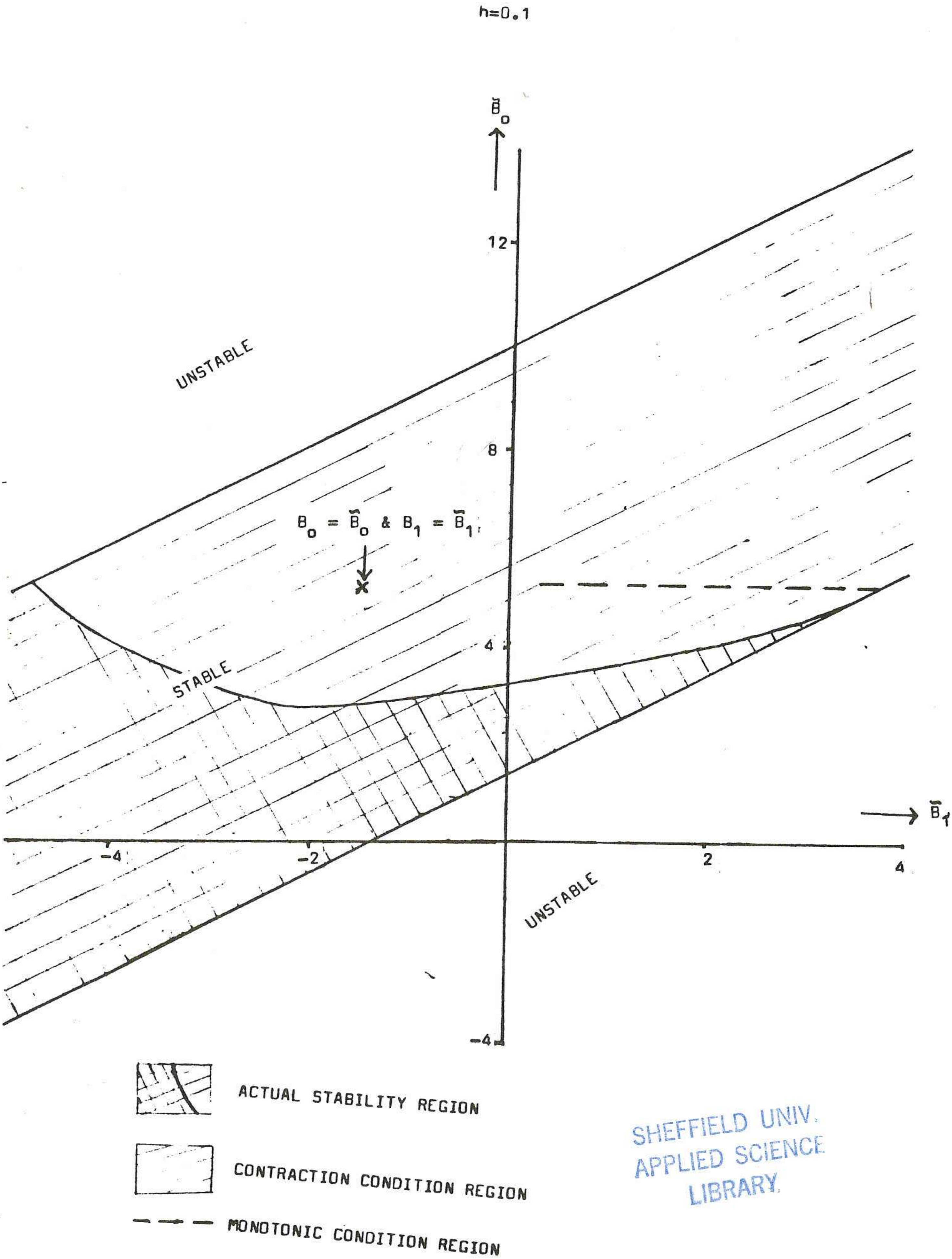

h=0.1
$\tilde{B}_o$
12
8
4
-4
UNSTABLE
$B_o = \tilde{B}_o$ & $B_1 = \tilde{B}_1$
STABLE
$\tilde{B}_1$
-4
-2
2
4
UNSTABLE
ACTUAL STABILITY REGION
CONTRACTION CONDITION REGION
MONOTONIC CONDITION REGION

Fig. 3

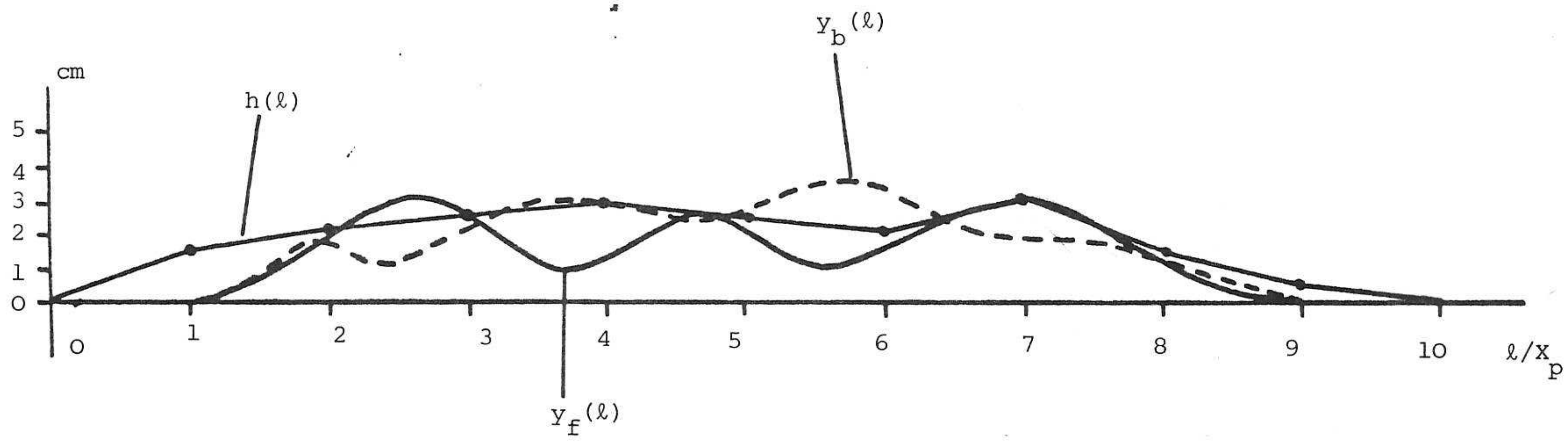


Fig. 14 Single-pass response of three-dimensional model