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by

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NUMERICAL OPTIMISATION OF MULTIPASS PROCESSES

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INTRODUCTION

Multipass processes (1) are a new class of system which were introduced by Edwards (2) to represent processes which can be characterised by repetitive action. Such systems are illustrated by consideration of machining operations where the material, or workpiece, involved is processed by a sequence of passes of the processing tool and exhibit the property that the state of the system generated on the (n-1)-th pass acts as a forcing term on the n-th pass and, hence, contributes to the dynamics of this pass. The rolling of metal strip, the ploughing of agricultural land and the longwall cutting of coal are all examples of multipass processes (2).

Until recently, there has been no available control theory for multipass processes and work to date (2-3) has emphasised frequency domain considerations of stability and control. In this paper, the linear quadratic problem for differential multipass processes is considered.

Linear differential multipass processes are modelled by state equations of the form

$$\dot{x}_k(t) = Ax_k(t) + Bu_k(t) + Cx_{k-1}(t)$$
$$t \in [0,T],\ k=1,\ 2,\ \dots,\ N, \qquad (1)$$

with boundary conditions

$$x_o(t) = f(t),\ t \in [0,T],\ x_k(o) = x_{ko},$$
$$k=1,\ 2,\ \dots,\ N, \qquad (2)$$

where A, B and C are real nxn, nxℓ and nxn matrices, respectively ($\ell < n$). The quadratic optimisation problem for such systems takes the form

$$\min J = \tfrac{1}{2} \sum_{k=1}^{N} \int_o^T \{x_k^T(t) Q x_k(t) + u_k^T(t)\ Ru_k(t)\}dt \qquad (3)$$

subject to the constraints (1) and (2), where Q and R are real, positive definite nxn and ℓxℓ matrices, respectively. Although casual, feedback solutions to this problem have recently been found (4), they present severe numerical problems.

This paper presents an iterative technique for computing the optimal state and control trajectories $x_k(\cdot)$, $u_k(\cdot)$, k=1, 2, ..., N, solving problem (1) - (3). The computed control is semi-closed loop in nature and the algorithm is based on recently developed ideas (5) for the solution of linearly constrained minimum norm problems in Hilbert space. The approach is related to current work on quadratic optimisation of large scale systems and the underlying concepts have previously found successful application in the linear quadratic problem for time delay systems (5).

LINEARLY CONSTRAINED QUADRATIC OPTIMISATION

Linearly constrained quadratic optimisation problems take the general form

$$\min\ \{\|x\|^2 : L_i x = b_i,\ 1 \le i \le m\} \qquad (4)$$

where x is regarded as a point in a real Hilbert space H (with inner product <., .> and induced norm $\|\cdot\| = \langle .,.\rangle^{\frac{1}{2}}$) and L_i, $1 \le i \le m$, are bounded linear operators mapping H into real Hilbert spaces H_i, $1 \le i \le m$, respectively. A recent paper (5) extends an algorithm (6) for the calculation of feasible solutions of constrained differential-algebraic systems to give a systematic computational procedure for the solution of problem (4).

Defining the closed and convex linear varieties

$$D_i = \{x \in H : L_i x = b_i\}\ 1 \le i \le m.$$

then the following theorem (5) provides an iterative solution of problem (4) as the weak limit of a sequence of optimisation problems.

Theorem 1 Suppose that $D_1 \cap D_2 \cap \dots \cap D_m$ is nonempty and define the sequence of linear varieties

$$k_j = D_{(j-1)\bmod(m-1)+2},\ j \ge 1.$$

Then the sequences $T_i \triangleq \{r_1^{(i)}, r_2^{(i)}, \dots\}$ i = 1, 2, defined by the relations

$$r_j^{(1)} \in D_1,\ r_j^{(2)} \in K_j,\ j \ge 1$$

$$\|r_1^{(1)}\|^2 = \min\{\|r\|^2 : r \in D_1\}$$

$$\|r_\ell^{(2)} - r_\ell^{(1)}\|^2 = \min\{\|r - r_\ell^{(1)}\|^2 : r \in K_\ell\}\quad \ell \ge 1,$$

$$\|s_\ell - r_\ell^{(2)}\|^2 = \min\{\|s - r^{(2)}\|^2,\ s \in D_1\},\ \ell \ge 1,$$

$$\|s_\ell - r_\ell^{(1)}\|^2 \lambda_\ell^* = \|r_\ell^{(2)} - r_\ell^{(1)}\|^2,\ \lambda_\ell^* \ge 1,\ \ell \ge 1,$$

$$1 \le \lambda_\ell \le \lambda_\ell^*,\ \ell \ge 1,$$

$$r_{\ell+1}^{(1)} = r_\ell^{(1)} + \lambda_\ell\{s_\ell - r_\ell^{(1)}\},\ \ell \ge 1,$$

are well-defined and converge weakly to the unique point $r_\infty \in D_1 \cap D_2 \cap \ldots \cap D_m$ solving problem (4). Moreover, if L_i is compact, $\{L_i r_j^{(1)}\}_{j \geq 1}$ converges strongly to b_i and, if H is finite-dimensional, both T_1 and T_2 converge strongly to r_∞. Finally, for all $j \geq 1$, $r_j^{(1)} \in D_1$ solves the minimum norm problem

$$\| x_\infty \|^2 = \min\{ \| x \|^2 : x \in D_1, L_i x = L_i r_j^{(1)},$$
$$2 \leq i \leq m\}$$

In practical terms (5) the result generates a sequence $T_1 = \{r_j^{(1)}\} \subseteq D_1$ converging weakly to the unique solution of (4). The flexibility in the choice of λ_i, $i \geq 1$ inherent in the algorithm is of great practical significance in that λ_i can be regarded as an accelerating extrapolation factor (5,6) if $\lambda_i \simeq \lambda^*_i >> 1$. Alternatively, the choice of $1 \leq \lambda_i << \lambda^*_i$ will present the growth of numerical errors if they are a problem.

Consider, now, the linear quadratic problem (5)

$$\min \{ \| z \|^2_{\hat{H}_o} : z = z_o + \hat{L}_o z + \sum_{i=1}^{M} \hat{L}_i (y_o^{(i)} + \Delta_i z)\} \tag{5}$$

where $\hat{H}_o, \hat{H}_1, \ldots, \hat{H}_m$ are real Hilbert spaces with inner products $\langle \cdot , \cdot \rangle_{\hat{H}_i}$, $o \leq i \leq m$, inducing the norms $\| x_i \|_{\hat{H}_i} = \langle x_i, x_i \rangle^{1/2}_{\hat{H}}$,

$o \leq i \leq m$, and $\hat{L}_o : \hat{H}_o \to \hat{H}_o$,

$\hat{L}_i : \hat{H}_i \to \hat{H}_o$, $\Delta_i : \hat{H}_o \to \hat{H}_i$, $1 \leq i \leq m$, are linear and bounded. The vectors $y_o^{(i)} \in \hat{H}_i$, $1 \leq i \leq m$, are taken to be fixed and it is assumed that

$$\langle y_o^{(i)}, \Delta_i z \rangle_{\hat{H}} = 0 \text{ for all } z \in \hat{H}_o, \quad 1 \leq i \leq m \tag{6}$$

The following theorem (5) indicates that this problem has an equivalent formulation in terms of problem (4).

<u>Theorem 2</u> Let $\gamma_i > o$, $1 \leq i \leq m$, be chosen such that $\sum_{i=1}^{m} \gamma_i \| \Delta^*_i \Delta_i \|_{\hat{H}_i} < 1$ and define the product space $Y = \hat{H}_o \times \hat{H}_1 \times \ldots \times \hat{H}_m$.

Then

$$\langle (z, y_1, \ldots, y_m), (\tilde{z}, \tilde{y}_1, \ldots, \tilde{y}_m) \rangle_Y =$$
$$\langle z, \left[I_o - \sum_{i=1}^{m} \gamma_i \Delta^*_i \Delta_i \right] \tilde{z} \rangle_{\hat{H}_o} + \sum_{i=1}^{m} \gamma_i \langle y_i, \tilde{y}_i \rangle_{\hat{H}_i}$$

is a non-degenerate inner product on Y inducing the norm

$$\| (z, y_1, \ldots, y_m) \|_Y = \langle (z, y_1 \ldots, y_m), (z, y_1, \ldots, y_m) \rangle^{1/2}_Y$$

Moreover, Y is a Hilbert space and problem (5) is equivalent to the well defined minimum norm problem on Y.

$$\min \| (z, y_1, \ldots, y_m) \|^2_Y \tag{7}$$

subject to

$$z = z_o + \hat{L}_o z + \sum_{i=1}^{m} \hat{L}_i y_i \tag{8}$$

and

$$y_i = y_o^{(i)} + \Delta_i z, \quad 1 \leq i \leq m \tag{9}$$

Theorem 2 provides a decomposition of problem (5) in terms of (7) - (9), which belongs to the general class of linear quadratic problems (4), and can therefore be solved via the iterative scheme defined in Theorem 1. These concepts have previously been used to develop numerical algorithms for quadratic optimisation of linear differential-delay systems (5). In this paper, their application is extended to the numerical optimisation of differential multipass processes.

<u>QUADRATIC OPTIMISATION OF MULTIPASS PROCESSES</u>

Consider the Hilbert spaces

$\hat{H}_o = L_2^n [0,T] \times L_2^\ell [0,T] \times \ldots \times L_2^n [0,T] \times L_2^\ell [0,T]$, of 2 N-tuples $(x_1, u_1, \ldots, x_n, u_N)$, and $\hat{H}_1 = L_2^n [0,T] \times \ldots \times L_2^n [0,T]$, of N-tuples $(x_1, \ldots x_N)$ with inner products

$$\langle (x_1, u_1, \ldots, x_N, u_N), (\tilde{x}_1, \tilde{u}_1, \ldots, \tilde{x}_N, \tilde{u}_N) \rangle_{\hat{H}_o} =$$
$$\frac{1}{2} \sum_{k=1}^{N} \int_0^T \{ x_k^T(t) Q \tilde{x}_k(t) + u_k^T(t) R \tilde{u}_k(t) \} dt \quad \text{and}$$

$$\langle (v_1, \ldots, v_N), (\tilde{v}_1, \ldots, \tilde{v}_N) \rangle_{\hat{H}_1} =$$
$$= \frac{1}{2} \sum_{k=1}^{N} \int_0^T v_k^T(t) Q \tilde{v}_k(t) dt$$

and associated norms $\| (x_1, u_1, \ldots, x_N, u_N) \|^2_{\hat{H}_o} =$

$\langle (x_1, u_1, \ldots, x_N, u_N), (x_1, u_1, \ldots, x_N, u_N) \rangle^{1/2}_{\hat{H}_1}$ and

$\| v_1, \ldots, v_N) \|_{\hat{H}_1} = \langle v_1, \ldots, v_N), (v_1, \ldots, v_N) \rangle^{1/2}_{\hat{H}_1}$,

respectively. Defining $z(t) = (x_1(t), u_1(t), \ldots, x_N(t), u_N(t))$, $z_o(t) = (x_{1O}, 0, \ldots, x_{NO}, 0)$, $y_o(t) = (f(t), 0, \ldots, 0)$, $t \in [0,T]$, $\hat{L}_o : \hat{H}_o \to \hat{H}_o$ by

$\hat{L}_o z(t) = \hat{L}_o(x_1(t), u_1(t), \ldots, x_N(t), u_N(t)) =$

$$\left(\int_0^t \{Ax_1(s)+Bu_1(s)\}ds, u_1(t), \ldots, \int_0^t \{Ax_N(s)+B u_N(s)\}ds, u_N(t)\right),$$

$\Delta : \hat{H}_o \to \hat{H}_1$ by $\Delta z(t) = \Delta(x_1(t), u_1(t), \ldots, x_N(t), u_N(t)) = (0, x_1(t), \ldots, x_{N-1}(t))$ and

$\hat{L}_1 : \hat{H}_1 \to \hat{H}_o$ by

$\hat{L}_1(v_1(t), \ldots, v_N(t)) = \left(\int_0^t Cv_1(s)ds, 0, \ldots, \int_0^t Cv_N(s)ds, 0\right)$, then, with this abstract setting, the linear quadratic optimisation problem (1) - (3) is

$$\min_{z \in \hat{H}_o} \| z \|^2_{\hat{H}_o} \tag{10}$$

subject to

$$z = z_o + \hat{L}_o z + \hat{L}_1 [y_o + \Delta z] \tag{11}$$

Note that

$$\langle y_o, \Delta z\rangle_{\hat{H}_1} = \langle (f, o, \ldots, o), (o, x_1, \ldots, x_{n-1})\rangle_{\hat{H}_1}$$

$$= 0, \text{ for all } z \in \hat{H}_o,$$

and hence the results of the previous section apply to this problem.

Applying Theorem 2, then (10) - (11) has an equivalent representation on the product Hilbert space $Y = \hat{H}_o \times \hat{H}_1$ of pairs (z,y) with inner product

$$\langle (z,y), (\tilde{z}, \tilde{y})\rangle_Y = \langle z, [I_o - \gamma\Delta^*\Delta]\tilde{z}\rangle_{\hat{H}_o} + \gamma\langle y, \tilde{y}\rangle_{\hat{H}_1}$$

where $\gamma > 0$ is chosen such that $\gamma \|\Delta^*\Delta\| < 1$.

Defining $\|(z,y)\|_Y = \langle (z,y), (z,y)\rangle_Y^{\frac{1}{2}}$, then problem (10) - (11) is equivalent to

$$\min \| (z,y) \|^2_Y \tag{12}$$

subject to

$$z = z_o + \hat{L}_o z + \hat{L}_1 y \tag{13}$$

and

$$y = y_o + \Delta z \tag{14}$$

For the problem considered here, it is clear that $\|\Delta^*\Delta\| = 1$ and a choice of γ, $0 < \gamma < 1$ will suffice.

The norm on Y is just

$$\| (z,y) \|^2_Y = \| z \|^2_{\hat{H}_o} - \gamma \| \Delta z \|^2_{\hat{H}_1} + \gamma \| y \|^2_{\hat{H}_1}$$

$$= \tfrac{1}{2} \sum_{k=1}^N \int_0^T \{x_k^T(t) Qx_k(t) + u_k^T(t) Ru_k(t)\}dt$$

$$- \gamma \tfrac{1}{2} \sum_{k=1}^{N-1} \int_0^T x_k^T(t)Qx_k(t)dt + \gamma \tfrac{1}{2} \sum_{k=1}^N \int_0^T y_k^T(t)Qy_k(t)dt$$

$$= \tfrac{1}{2}(1-\gamma) \sum_{k=1}^{N-1} \int_0^T x_k^T(t)Qx_k(t)dt + \tfrac{1}{2} \int_0^T x_N^T(t)Qx_N(t)dt$$

$$+ \tfrac{1}{2} \sum_{k=1}^N \int_0^T u_k^T(t)Ru_k(t)dt + \tfrac{1}{2}\gamma \sum_{k=1}^N \int_0^T y_k^T(t)Qy_k(t)dt$$

and the constraints (13) and (14) are

$$\dot{x}_k(t) = Ax_k(t) + Bu_k(t) + Cy_k(t), \; x_k(0) = x_{ko}, \quad k = 1, 2, \ldots, N, \tag{15}$$

and

$$y_1(t) = f(t), \; y_k(t) = x_{k-1}(t), \; k = 2, 3, \ldots, N, \tag{16}$$

respectively. The linear quadratic multipass problem (1) - (3) is therefore equivalent to the linearly constrained optimisation problem

$$\min \hat{J} = \tfrac{1}{2} \sum_{k=1}^N \int_0^T x_k^T(t)(1-\gamma)Qx_k(t)dt + \tfrac{1}{2} \int_0^T x_N^T(t)Qx_N(t)dt + \tfrac{1}{2} \sum_{k=1}^N \int_0^T \{u_k^T(t)Ru_k(t) + y_k^T(t)\gamma Qy_k(t)\}dt \tag{17}$$

subject to (15) and (16). Note that the modified state equations (15) and, indeed, the algebraic constraints (16) are now decoupled.

Defining the closed and convex linear varieties

$$D_1 = \{(x_1, u_1, y_1, \ldots, x_N, u_N, y_N) \in Y : x_k(t) = x_{ko} + \int_0^t (Ax_k(s) + Bu_k(s) + Cy_k(s))ds, \; k = 1, 2, \ldots, N, \; t \in [0,T]\}$$

$$D_2 = \{(x_1, u_1, y_1, \ldots, x_N, u_N, y_N) \in Y : y_1(t) = f(t), y_k(t) = x_{k-1}(t), \; k = 2, 3, \ldots, N, \; t \in [0,T]\}$$

then problem (15) - (17) is just

$$\min \{\| z,y) \|^2_Y : (z,y) \in D_1 \cap D_2$$

which can be solved by the algorithm of Theorem 1 where, in this case $K_j = D_2$, $j \geq 1$.

Defining $r_\ell^{(i)} = (x_{1(\ell)}^{(i)}, u_{1(\ell)}^{(i)}, y_{1(\ell)}^{(i)}, \ldots\ldots,$

$x_{N(\ell)}^{(i)}, u_{N(\ell)}^{(i)}, y_{N(\ell)}^{(i)}), \; i=1, 2, \; \ell \geq 1$, then, in computational terms $r_1^{(i)}$ solves

$$\min \tfrac{1}{2} \sum_{k=1}^{N-1} \int_o^T x_k^T(t)(1-\gamma)Qx_k(t)\,dt$$

$$+ \tfrac{1}{2} \int_o^T x_N^T(t)Qx_N(t)\,dt$$

$$+ \tfrac{1}{2} \sum_{k=1}^{N} \int_o^T \{u_k^T(t)Ru_k(t) + y_k^T(t)\gamma Qy_k(t)\}dt$$

subject to the state equations (15), which separates into N-1 independent, identical subproblems for (x_1, u_1, y_1), ..., $(x_{N-1}, u_{N-1}, y_{N-1})$ with a further subproblem for (x_N, u_N, y_N).

K-th subproblem (K=1, 2, 3, ..., N-1):

$$\min \tfrac{1}{2} \int_o^T \{x_k^T(t)(1-\gamma)Qx_k(t) + u_k^T(t)Ru_k(t)$$

$$+ y_k^T(t)\gamma Qy_k(t)\}dt$$

subject to

$$\dot{x}_k(t) = Ax_k(t) + Bu_k(t) + Cy_k(t), \; x_k(o) = x_{ko}.$$

This has solution

$$u_{k(1)}^{(1)}(t) = -R^{-1}B^TK(t)x_{k(1)}^{(1)}(t)$$

$$y_{k(1)}^{(1)}(t) = \frac{1}{\gamma} Q^{-1}C^TK(t)x_{k(1)}^{(1)}(t) \qquad (18)$$

where K(t) solves the matrix Riccati equation

$$\dot{K}(t) = -K(t)A - A^TK(t) + K(t)BR^{-1}B^TK(t)$$

$$+ \frac{1}{\gamma} K(t)CQ^{-1}C^TK(t) - (1-\gamma)Q, \; K(T) = O \qquad (19)$$

N-th subproblem:

$$\min \tfrac{1}{2} \int_o^T \{x_N^T(t)Qx_N(t) + u_N^T(t)Ru_N(t) + y_N^T(t)\gamma Qy_N(t)\}dt$$

subject to

$$\dot{x}_N(t) = Ax_N(t) + Bu_N(t) + Cy_N(t), \; x_N(o) = x_{NO}$$

This has solution

$$\left.\begin{aligned} u_{N(1)}^{(1)}(t) &= -R^{-1}B^T\hat{K}(t)x_{N(1)}^{(1)}(t) \\ y_{N(1)}^{(1)}(t) &= -\frac{1}{\gamma} C^TQ^{-1}\hat{K}(t)x_{N(1)}^{(1)}(t) \end{aligned}\right\} \qquad (20)$$

where $\hat{K}(t)$ solves the matrix Riccati equation

$$\dot{\hat{K}}(t) = -\hat{K}(t)A - A^T\hat{K}(t) + \hat{K}(t)BR^{-1}B^T\hat{K}(t)$$

$$= \frac{1}{\gamma} \hat{K}(t)CQ^{-1}C^T\hat{K}(t) - Q, \; \hat{K}(T) = O \qquad (21)$$

The iterates $r_\ell^{(2)}$, $\ell \geq 1$, solve

$$\min \tfrac{1}{2} \sum_{k=1}^{N-1} \int_o^T [x_k(t) - x_{k(\ell)}^{(1)}(t)]^T (1-\gamma)Q[x_k(t) - x_{k(\ell)}^{(1)}(t)]dt$$

$$+ \tfrac{1}{2} \int_o^T [x_N(t) - x_{N(\ell)}^{(1)}(t)]^T Q[x_N(t) - x_{N(\ell)}^{(1)}(t)]dt$$

$$+ \tfrac{1}{2} \sum_{k=1}^{N} \int_o^T [u_k(t) - u_{k(\ell)}^{(1)}(t)]^T R[u_k(t) - u_{k(\ell)}^{(1)}(t)]dt$$

$$+ \tfrac{1}{2} \sum_{k=1}^{N} \int_o^T [y_k(t) - y_{k(\ell)}^{(1)}(t)]^T \gamma Q[y_k(t) - y_{k(\ell)}^{(1)}(t)]dt$$

subject to the algebraic constraints

$$y_1(t) = f(t), y_k(t) = x_{k-1}(t), \; k = 2, 3, \ldots, N,$$

$$t \in [0,T].$$

This has solution

$$u_{k(\ell)}^{(2)}(t) = u_{k(\ell)}^{(1)}(t), \; k = 1, 2, \ldots, N,$$

$$x_{k(\ell)}^{(2)}(t) = (1-\gamma)x_{k(\ell)}^{(1)}(t) + \gamma y_{k+1(\ell)}^{(1)}(t), \; k=1, 2, \ldots, N-1$$

$$x_{N(\ell)}^{(2)}(t) = x_{N(\ell)}^{(1)}(t) \qquad (22)$$

$$y_{1(\ell)}^{(2)}(t) = f(t), \; y_{k(\ell)}^{(2)} = x_{k-1(\ell)}^{(2)}(t), \; k = 2, 3, \ldots, N$$

Finally, the iterates $s = (x_{1(\ell)}^S, u_{1(\ell)}^S, y_{1(\ell)}^S, \ldots$

$\ldots x_{N(\ell)}^S, u_{N(\ell)}^S, y_{N(\ell)}^S)$, $\ell \geq 1$, solve

$$\min \tfrac{1}{2} \sum_{k=1}^{N-1} \int_o^T [x_k(t) - x_{k(\ell)}^{(2)}(t)]^T (1-\gamma)Q[x_k(t) - x_{k(\ell)}^{(2)}(t)]dt$$

$$+ \tfrac{1}{2} \int_o^T [x_N(t) - x_{N(\ell)}^{(2)}(t)]^T Q[x_N(t) - x_{N(\ell)}^{(2)}(t)]dt$$

$$+ \tfrac{1}{2} \sum_{k=1}^{N} \int_o^T [u_k(t) - u_{k(\ell)}^{(2)}(t)]^T R[u_k(t) - u_{k(\ell)}^{(2)}(t)]dt$$

$$+ \tfrac{1}{2} \sum_{k=1}^{N} \int_o^T [y_k(t) - y_{k(\ell)}^{(2)}(t)]^T \gamma Q[y_k(t) - y_{k(\ell)}^{(2)}(t)]dt$$

subject to the state equations (15). Again this separates into N-1 similar subproblems for (x_1, u_1, y_1), ..., $(x_{N-1}, u_{N-1}, y_{N-1})$ with a further subproblem for (x_N, u_N, y_N).

k-th subproblem (k=1, 2, ..., N-1):

$$\min \tfrac{1}{2}\int_0^T \{[x_k(t) - x^{(2)}_{k(\ell)}(t)]^T (1-\gamma)Q[x_k(t) - x^{(2)}_{k(\ell)}(t)]$$

$$+ [u_k(t) - u^{(2)}_{k(\ell)}(t)]^T R[u_k(t) - u^{(2)}_{k(\ell)}(t)]$$

$$+ [y_k(t) - y^{(2)}_{k(\ell)}(t)]^T \gamma Q [y_k(t) - y^{(2)}_{k(\ell)}(t)]\}\, dt$$

subject to

$$\dot{x}_k(t) = Ax_k(t) + Bu_k(t) + Cy_k(t),\ x_k(o) = x_{ko}.$$

This has solution

$$\left.\begin{aligned} u^s_{k(\ell)}(t) &= -R^{-1}B^T[K(t)x^s_{k(\ell)}(t) - g_{k(\ell)}(t)] + u^{(2)}_{k(\ell)}(t) \\ y^s_{k(\ell)}(t) &= -\tfrac{1}{\gamma}Q^{-1}C^T[K(t)x^s_{k(\ell)}(t) - g_{k(\ell)}(t)] + y^{(2)}_{k(\ell)}(t) \end{aligned}\right\} \quad (23)$$

where $K(t)$ solves equation (19) and $g_{k(\ell)}(t)$ solves the tracking equation

$$\dot{g}_{k(\ell)}(t) = -A^T g_{k(\ell)}(t) + K(t)BR^{-1}B^T g_{k(\ell)}(t) + \tfrac{1}{\gamma}K(t)CQ^{-1}C^T g_{k(\ell)}(t)$$

$$+ K(t)Bu^{(2)}_{k(\ell)}(t) + K(t)Cy^{(2)}_{k(\ell)}(t) + (1-\gamma)Qx^{(2)}_{k(\ell)}(t)$$

$$g_{k(\ell)}(T) = 0 \quad (24)$$

N-th subproblem:

$$\min \tfrac{1}{2}\int_0^T \{[x_N(t)-x^{(2)}_{N(\ell)}(t)]^T Q[x_N(t)-x^{(2)}_{N(\ell)}(t)]$$

$$+ [u_N(t) - u^{(2)}_{N(\ell)}(t)]^T R[u_N(t) - u^{(2)}_{N(\ell)}(t)]$$

$$+ [y_N(t) - y^{(2)}_{N(\ell)}(t)]^T \gamma Q[y_N(t) - y^{(2)}_{N(\ell)}(t)]\}\, dt$$

subject to

$$\dot{x}_N(t) = Ax_N(t) + Bu_N(t) + Cy_N(t),\ x_N(o) = x_{NO}.$$

This has solution

$$u^s_{N(\ell)}(t) = -R^{-1}B^T[\hat{K}(t)x^s_{N(\ell)}(t) - \hat{g}_{N(\ell)}(t)] + u^{(2)}_{N(\ell)}(t)$$

$$y^s_{N(\ell)}(t) = -\tfrac{1}{\gamma}Q^{-1}C^T[\hat{K}(t)x^s_{N(\ell)}(t) - \hat{g}_{N(\ell)}(t)] + y^{(2)}_{N(\ell)}(t) \quad (25)$$

where $\hat{K}(t)$ solves equation (21) and $\hat{g}_{N(\ell)}(t)$ solves the tracking equation

$$\dot{\hat{g}}_{N(\ell)}(t) = -A^T\hat{g}_{N(\ell)}(t) + \hat{K}(t)BR^{-1}B^T\hat{g}_{N(\ell)}(t) + \tfrac{1}{\gamma}\hat{K}(t)CQ^{-1}C^T\hat{g}_{N(\ell)}(t)$$

$$+ \hat{K}(t)Bu^{(2)}_{N(\ell)}(t) + \hat{K}(t)Cy^{(2)}_{N(\ell)}(t) - Qx^{(2)}_{N(\ell)}(t) \quad (26)$$

$$\hat{g}_{N(\ell)}(T) = 0$$

An iterative method for solving the linear quadratic multipass problem (1) - (3) has been developed which simply involves a single integration of the 2 Riccati equations (19) and (21) plus sequential application of the algebraic relations (22), together with integration of the N state equations (15) and N tracking vector equations (24) and (26).

APPLICATION TO A SELF-STEERED TRACTOR

The iterative algorithm developed in the previous section is now illustrated with an application to a linear differential multipass process model of a self-steered tractor (2). The state equation is

$$\dot{x}_k(t) = \begin{bmatrix} 0 & 0 \\ -1600 & -40 \end{bmatrix} x_k(t) + \begin{bmatrix} 0 \\ 1600 \end{bmatrix} u_k(t)$$

$$+ \begin{bmatrix} 0 & 0 \\ 1600 & 0 \end{bmatrix} x_{k-1}(t),\ t \in [0,1] \quad (27)$$

corresponding to a system in which the damping ratio and undamped natural frequency of the single pass loop (2) are 0.5 and 40, respectively, and the pass length, i.e. length of the furrow, is unity. The optimisation problem is concerned with minimising the quadratic cost functional

$$J = \tfrac{1}{2}\sum_{k=1}^{10}\int_0^1 \{x_k^T(t)x_k(t) + u_k^2(t)\}dt$$

where $x_k(t)$, k=1, 2, ..., 10, $t \in [0,1]$, satisfies the state equation (27) with boundary conditions

$$x_o(t) = f(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ t \in [0,1],\ \text{and}$$

$$x_k(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ k=1, 2, \ldots, 10.$$

The solution obtained is that of the approximating discrete time system and a discretisation step of h=0.01 is employed.

With a choice of $\gamma = 0.5$ and

$$\lambda_i = \begin{cases} 1.0, & i = 5, 10, 15, \ldots \\ \lambda^*_i, & \text{otherwise} \end{cases}, \quad \text{i.e. full}$$

extrapolation except at every 5th iteration where the extrapolation factor is set to zero to prevent the growth of numerical errors, the algorithm converged with respect to J in 9 iterations, resulting in an optimal value of $J^* = 0.416$. The optimal state and control trajectories are presented in Figures 1-3 and Table 1 displays the rate of convergence and variation in λ_i with iteration.

i	J	λ_i
1	0.4291	2.32
2	0.3179	1.17
3	0.4098	2.47
4	0.3970	1.23
5	0.4037	1.0
6	0.4180	4.74
7	0.4154	3.15
8	0.4159	1.22
9	0.4158	2.94
10	0.4159	1.0

TABLE 1

It is worth noting that when no extrapolation was employed (i.e. $\lambda_i = 1.0$ throughout) the algorithm required 26 iterations for convergence. Finally, the choice of $\gamma = 0.5$ was found to be optimal in the sense that for $\gamma > 0.5$ the algorithm converged more slowly, whereas choices of $\gamma < 0.5$ resulted in increased values of λ^*_i at each iteration and had the effect of introducing numerical errors into the computation. This necessitated the resetting of λ_i to unity more frequently than at every 5 iterations with a corresponding decrease in the convergence rate.

CONLUSIONS

This paper presents a systematic computational procedure for the solution of the quadratic optimisation problem for linear differential multipass processes. The algorithm is iterative in nature and is based on recently developed ideas for the solution of linearly constrained minimum norm problems in Hilbert space. The computational scheme has guaranteed convergence and the resulting solution is of the semi-closed loop variety Finally, the iterative technique is illustrated with an application to a linear differential multipass process model of a self-steered tractor and it is demonstrated that convergence to the optimal solution can be obtained in a small number of iterations.

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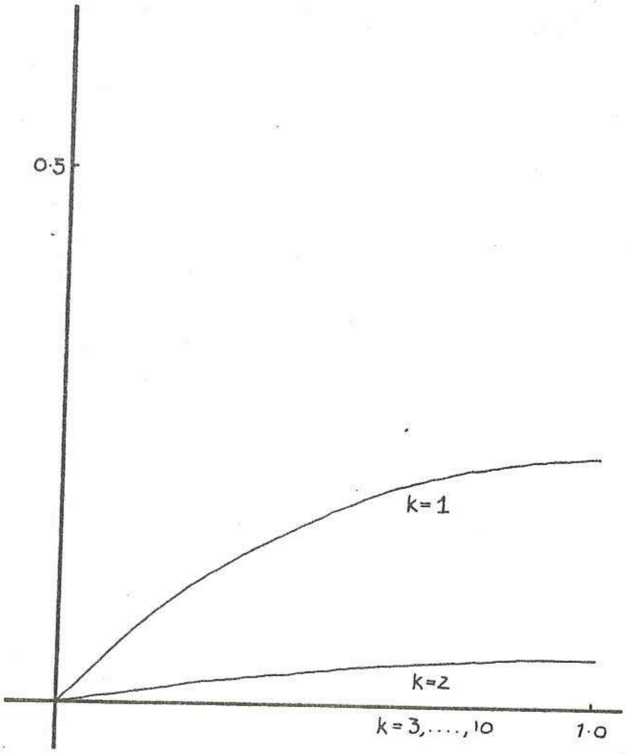

FIGURE 1 $x_{k_1}(t)$

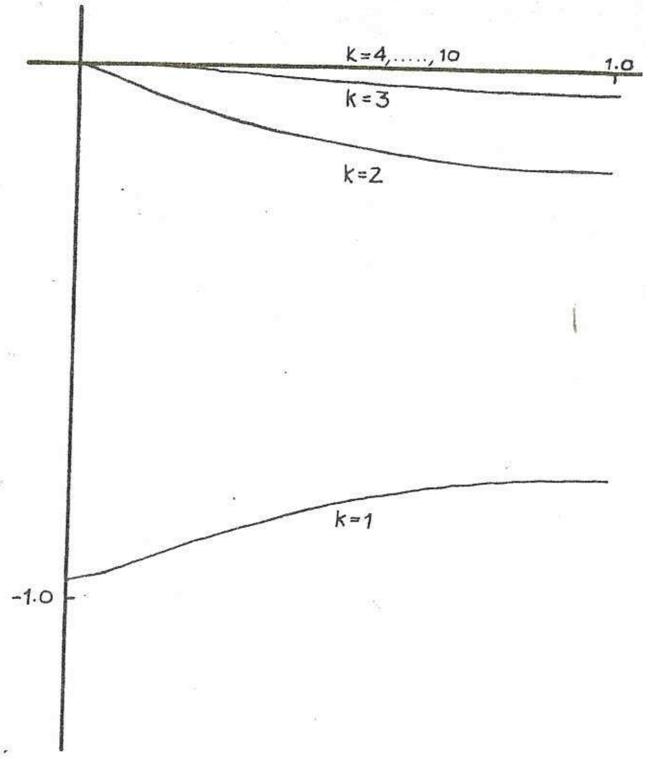

FIGURE 3 $u_k(t)$

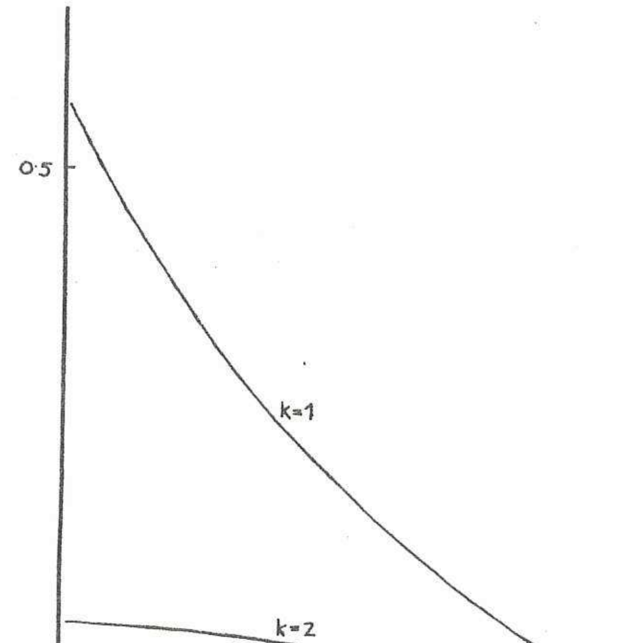

FIGURE 2 $x_{k_2}(t)$