Approximate Carleman theorems and a Denjoy–Carleman maximum principle

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Abstract

We give an extension of the Denjoy–Carleman theorem, which leads to a generalization of Carleman's theorem on the unique determination of probability measures by their moments. We also give complex versions of Carleman's theorem extending Theorem 4.1 of [2].

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1 Introduction

Given a subinterval I (bounded or unbounded) of $\mathbb{R}$, and a sequence $(M_n)_{n\geq 0}$ of positive numbers, write $\mathcal{C}_I(M_n)$ for the family of all C^∞-functions $f : I \to \mathbb{C}$ satisfying

$$|f^{(n)}(x)| \leq c_f \rho_f^n M_n \qquad (x \in I,\ n \geq 0), \tag{1}$$

where c_f and ρ_f are constants depending on f. Recall the Denjoy–Carleman theorem ([3], p. 97).

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Theorem A. *Let $(M_n)_{n\geq 0}$ be a positive sequence satisfying*

$$M_0 = 1, \quad M_n^2 \leq M_{n-1}M_{n+1} \ (n \geq 1) \quad \text{and} \quad \sum_{n=1}^{\infty} M_n^{-1/n} = \infty.$$

Let $f \in \mathcal{C}_I(M_n)$, where I is an interval containing 0, and suppose that

$$f^{(n)}(0) = 0 \text{ for all } n \geq 0.$$

Then f is identically equal to 0 on I.

In this paper we extend the Denjoy–Carleman theorem (when $I = \mathbb{R}$) by proving that f is constant when the condition $f^{(n)}(0) = 0$ for all n is replaced by the weaker condition $\lim_{n\to\infty} |f^{(n)}(0)|^{1/n} = 0$. More generally, we prove that if $\limsup_{n\to\infty} |f^{(n)}(0)|^{1/n} \leq C$, then (1) automatically implies a stronger form of itself, with $\rho_f = C$ and $M_n \equiv 1$.

We subsequently use these ideas to obtain a generalization of Carleman's theorem on the unique determination of probability measures by their moments. In the last section we also discuss complex versions of Carleman's theorem, generalizing Theorem 4.1 of [2].

2 A Denjoy–Carleman maximum principle

The following theorem is our main result.

Theorem 2.1 *Let $(M_n)_{n\geq 0}$ be positive sequence satisfying*

$$M_0 = 1, \quad M_n^2 \leq M_{n-1}M_{n+1} \ (n \geq 1) \quad \text{and} \quad \sum_{n=1}^{\infty} M_n^{-1/n} = \infty. \tag{2}$$

Let $f \in \mathcal{C}_{\mathbb{R}}(M_n)$, and suppose that

$$\limsup_{n\to\infty} |f^{(n)}(0)|^{1/n} \leq C. \tag{3}$$

Then, for all integers $m, n \geq 0$,

$$\sup_{x\in\mathbb{R}} |f^{(n+m)}(x)| \leq C^n \sup_{x\in\mathbb{R}} |f^{(m)}(x)|.$$

As a special case of this result, we obtain a generalization of the Denjoy–Carleman theorem (for $I = \mathbb{R}$).

Corollary 2.2 *Let (M_n) be as in the theorem, let $f \in \mathcal{C}_{\mathbb{R}}(M_n)$, and suppose that* $\lim_{n\to\infty} |f^{(n)}(0)|^{1/n} = 0$*. Then f is constant.*

Proof Applying the theorem with $C = 0$, we find that $f' \equiv 0$.

$\square$

In the course of the proof of Theorem 2.1, we shall need a result about entire functions. Recall that an entire function h is said to be of *exponential type* τ if

$$\limsup_{|z|\to\infty} \frac{\log |f(z)|}{|z|} = \tau.$$

The following result is well known; the second part is often called Bernstein's theorem.

Theorem B. ([1], Theorems 2.4.1 and 11.1.2) *Let h be an entire function of exponential type τ. Then h' is also of exponential type τ. If, further, h is bounded on $\mathbb{R}$, then so is h', and*

$$\sup_{x\in\mathbb{R}} |h'(x)| \le \tau \sup_{x\in\mathbb{R}} |h(x)|.$$

Proof of Theorem 2.1 Define $h : \mathbb{C} \to \mathbb{C}$ by

$$h(z) = \sum_{k\ge 0} \frac{f^{(k)}(0)}{k!} z^k.$$

From (3), given $\epsilon > 0$, there exists a constant A_ϵ such that

$$|f^{(k)}(0)| \le A_\epsilon (C+\epsilon)^k \quad (k \ge 0).$$

Therefore

$$\sum_{k\ge 0} \Big| \frac{f^{(k)}(0)}{k!} z^k \Big| \le \sum_{k\ge 0} \frac{A_\epsilon (C+\epsilon)^k}{k!} |z|^k = A_\epsilon e^{(C+\epsilon)|z|} \quad (z \in \mathbb{C}).$$

It follows that h is an entire function of exponential type at most C. We shall show that $f = h|_{\mathbb{R}}$. Assuming this, and noting also that f is bounded on $\mathbb{R}$ (by definition of $\mathcal{C}_{\mathbb{R}}(M_n)$), the result follows upon repeated application of Theorem B.

It remains to prove that $f = h|_{\mathbb{R}}$. Observe that, for $n \geq 0$ and $z \in \mathbb{C}$,

$$|h^{(n)}(z)| = \Big|\sum_{k\geq 0} \frac{f^{(n+k)}(0)}{k!} z^k\Big| \leq \sum_{k\geq 0} \frac{A_\epsilon (C+\epsilon)^{n+k}}{k!} |z|^k = A_\epsilon (C+\epsilon)^n e^{(C+\epsilon)|z|}.$$

In particular, given $R > 0$,

$$\sup_{x\in[-R,R]} |h^{(n)}(x)| \leq A_\epsilon (C+\epsilon)^n e^{(C+\epsilon)R} \quad (n \geq 0).$$

Now, using the fact that $M_0 = 1$ and $M_n^2 \leq M_{n-1}M_{n+1}$ $(n \geq 1)$, we have $M_n \geq M_1^n$ for all $n \geq 1$. Hence

$$\sup_{x\in[-R,R]} |h^{(n)}(x)| \leq A_\epsilon e^{(C+\epsilon)R} \Big(\frac{C+\epsilon}{M_1}\Big)^n M_n \quad (n \geq 0).$$

This shows that $h|_{[-R,R]} \in \mathcal{C}_{[-R,R]}(M_n)$. Define $g = f|_{[-R,R]} - h|_{[-R,R]}$. Then $g \in \mathcal{C}_{[-R,R]}(M_n)$, and further, by the construction of h, we have $g^{(n)}(0) = 0$ for all $n \geq 0$. Applying Theorem A, we get $g \equiv 0$ on $[-R, R]$. As this holds for each $R > 0$, we deduce that $f = h|_{\mathbb{R}}$, as desired.

□

3 An extension of Carleman's theorem

Let us first state Carleman's theorem ([3], p.126).

Theorem C. *Let μ, ν be Borel probability measures on $\mathbb{R}$, all of whose moments are finite. Suppose that, for each $n \geq 0$,*

$$S_n := \int_{-\infty}^{+\infty} t^n d\mu(t) = \int_{-\infty}^{+\infty} t^n d\nu(t),$$

and further that

$$\sum_{n=1}^{\infty} S_{2n}^{-1/2n} = \infty.$$

Then $\mu = \nu$.

As an application of the ideas of the previous section, we obtain the following approximate version of Carleman's theorem.

Theorem 3.1 *Let μ, ν be positive Borel measures on $\mathbb{R}$, all of whose moments are finite. Suppose that, for each $n \geq 0$, one has*

$$\int_{-\infty}^{+\infty} t^n d\mu(t) = \int_{-\infty}^{+\infty} t^n d\nu(t) + c_n, \tag{4}$$

where $\limsup_{n\to\infty} |c_n|^{1/n} \leq C$. *Suppose further that* $S_n := \int_{-\infty}^{\infty} t^n \, d\mu(t)$ *satisfies*

$$\sum_{n=1}^{\infty} S_{2n}^{-1/2n} = \infty. \tag{5}$$

Then $\mu = \nu + \sigma$, where σ is a signed measure supported on $[-C, C]$.

As a corollary, we obtain a generalization of Carleman's theorem.

Corollary 3.2 *Let μ, ν be positive Borel measures on $\mathbb{R}$, all of whose moments are finite. Suppose that, for each $n \geq 0$, one has*

$$\int_{-\infty}^{+\infty} t^n d\mu(t) = \int_{-\infty}^{+\infty} t^n d\nu(t) + c_n,$$

where $\lim_{n\to\infty} |c_n|^{1/n} = 0$. *Suppose further that* $S_n := \int_{-\infty}^{\infty} t^n \, d\mu(t)$ *satisfies (5). Then there exists $c \in \mathbb{R}$ such that $\mu = \nu + c\delta_0$. If, in addition, μ and ν are probability measures, then $\mu = \nu$.*

Proof: By the theorem, $\mu = \nu + \sigma$, where σ is supported on $\{0\}$. Thus $\sigma = c\delta_0$ for some $c \in \mathbb{R}$. If both μ and ν are probability measures, then necessarily $c = 0$, and so $\mu = \nu$.

$\square$

For the proof of Theorem 3.1, we shall need the following version of the Paley–Wiener theorem.

Theorem D. ([5], Theorem 7.23) *Let h be an entire function such that*

$$|h(z)| \leq Ae^{C|\operatorname{Im} z|} \qquad (z \in \mathbb{C}), \tag{6}$$

where A and C are constants. Then h is the Fourier–Laplace transform of a distribution supported on $[-C, C]$.

Proof of Theorem 3.1: For $n \geq 0$, define

$$M_n = \frac{1}{m_0} \int_{-\infty}^{+\infty} |t|^n d(\mu + \nu)(t),$$

where $m_0 = \mu(\mathbb{R}) + \nu(\mathbb{R})$. We claim that the sequence $(M_n)_{n \geq 0}$ satisfies the condition (2). Indeed, that $M_0 = 1$ is clear, and $M_n^2 \leq M_{n-1} M_{n+1}$ for $n \geq 1$ follows from Hölder's inequality. The verification of the remaining condition $\sum_{n \geq 1} M_n^{-1/n} = \infty$ is a bit more technical, and is postponed at the end of the proof.

Assuming this for the moment, define $f : \mathbb{R} \to \mathbb{C}$ by

$$f(x) = \frac{1}{m_0} \int_{-\infty}^{+\infty} e^{-itx} d(\mu - \nu)(t) \qquad (x \in \mathbb{R}). \tag{7}$$

Then $f \in C^\infty(\mathbb{R})$ and, for each $n \geq 0$,

$$f^{(n)}(x) = \frac{1}{m_0} \int_{-\infty}^{+\infty} (-it)^n e^{-itx} d(\mu - \nu)(t) \qquad (x \in \mathbb{R}).$$

In particular,

$$|f^{(n)}(x)| \leq M_n \qquad (x \in \mathbb{R},\, n \geq 0).$$

So, $f \in \mathcal{C}_{\mathbb{R}}(M_n)$. Also,

$$f^{(n)}(0) = \frac{(-i)^n}{m_0} \int_{-\infty}^{+\infty} t^n d(\mu - \nu)(t),$$

so from (4) we have $\limsup_{n \to \infty} |f^{(n)}(0)|^{1/n} \leq C$. As in the proof of Theorem 2.1, $f = h|_{\mathbb{R}}$, where h is an entire function of exponential type at most C. A simple application of the Phragmén–Lindelöf principle shows that h satisfies the estimate (6) (see e.g. [3, p.28]). Hence, using Theorem D, we see that h is the Fourier–Laplace transform of a distribution u supported on $[-C, C]$. Thus f is just the Fourier transform of u. But f was defined as the Fourier transform of $(\mu - \nu)/m_0$. So, by the uniqueness theorem for Fourier transforms of tempered distributions, $u = (\mu - \nu)/m_0$. In particular, $\mu - \nu$ is supported on $[-C, C]$, as required.

It remains to justify the claim that $\sum_{n \geq 1} M_n^{-1/n} = \infty$. Set $\alpha_k = S_{k+1}/S_k$ for $k \geq 0$. By Hölder's inequality, we have $S_n^2 \leq S_{n-1} S_{n+1}$ for $n \geq 1$. Therefore $(\alpha_k)_{k \geq 0}$ is an increasing sequence and $S_n \geq S_k \alpha_k^{n-k}$ for $n \geq k$. Let

$\alpha_\infty = \lim_{n\to\infty} \alpha_n$. Also note that there exists a positive constant λ such that $M_{2n} \leq \frac{2}{m_0} S_{2n} + \lambda$, since $M_{2n} = \frac{2}{m_0} S_{2n} + O(1)$.

If $\alpha_\infty = \infty$, fix k such that $\alpha_k \geq 1$. Now, since $S_{2n} \geq S_k \alpha_k^{2n-k}$ for $2n \geq k$, we get:

$$\begin{aligned} M_{2n} &\leq \frac{2}{m_0} S_{2n} + \lambda \\ &= \frac{2}{m_0} S_{2n} + \lambda \frac{\alpha_k^k}{S_k} \alpha_k{}^{-2n} \frac{S_k}{\alpha_k^k} \alpha_k^{2n} \\ &\leq S_{2n} \left(\frac{2}{m_0} + \lambda \frac{\alpha_k^k}{S_k} \right). \end{aligned}$$

It follows that

$$\begin{aligned} M_{2n}^{-1/2n} &\geq S_{2n}^{-1/2n} \left(\frac{2}{m_0} + \lambda \frac{\alpha_k^k}{S_k} \right)^{-1/2n} \\ &\geq S_{2n}^{-1/2n} \min \left(1, \left(\frac{2}{m_0} + \lambda \frac{\alpha_k^k}{S_k} \right)^{-1/2} \right) \end{aligned}$$

for all $n \geq 1$, which clearly implies that $\sum_{n\geq 1} M_{2n}^{-1/2n} = \infty$ whenever $\sum_{n\geq 1} S_{2n}^{-1/2n} = \infty$.

If $\alpha_\infty < \infty$, we have $\alpha_k \leq \alpha_\infty$ for every $k \geq 0$. It follows that

$$S_n = S_0 \alpha_0 \cdots \alpha_{n-1} \leq S_0 \alpha_\infty^n$$

for all $n \geq 0$. We get in this way

$$M_{2n} \leq \frac{2}{m_0} S_0 \alpha_\infty^{2n} + \lambda \leq \left(\frac{2}{m_0} S_0 + \lambda \right) (\alpha_\infty + 1)^{2n}.$$

In particular, we have $\lim_{n\to\infty} M_{2n}^{-1/2n} \neq 0$ since $M_{2n}^{-1/2n} \geq \frac{(\frac{2}{m_0} S_0 + \lambda)^{-1/2n}}{\alpha_\infty + 1}$. Therefore, we also obtain $\sum_{n\geq 1} M_{2n}^{-1/2n} = \infty$.

□

4 Complex versions of Carleman's theorem

Corollary 3.2 provides conditions for the uniqueness of probability measures whose moments do not differ too much. In this section we will present complex versions of Carleman's theorem, as initiated in [2], Theorem 4.1.

To that end, we first recall the following complex analysis result ([2], Theorem 1.1 and [4], Theorem 1.7).

Theorem E. *Let $(a_n)_{n\geq 0}$ be a sequence of complex numbers.*
1. If, for some $r \geq 0$,

$$\sum_{\substack{k=0\\ k \text{ even}}}^{n} \binom{n}{k} a_k = O(n^r) \quad \text{and} \quad \sum_{\substack{k=0\\ k \text{ odd}}}^{n} \binom{n}{k} a_k = O(n^r) \qquad \text{as } n \to \infty,$$

then $a_n = 0$ for all $n > r$.
2. If, for some $\beta > 1$,

$$\sum_{\substack{k=0\\ k \text{ even}}}^{n} \binom{n}{k} a_k = O(\beta^n) \quad \text{and} \quad \sum_{\substack{k=0\\ k \text{ odd}}}^{n} \binom{n}{k} a_k = O(\beta^n) \qquad \text{as } n \to \infty,$$

then $a_n = O(\alpha^n)$, where $\alpha = \sqrt{\beta^2 - 1}$.

Let us now state our first result.

Theorem 4.1 *Let μ, ν be positive Borel measures on $\mathbb{R}$, all of whose moments are finite. Suppose that*

$$\int_{-\infty}^{+\infty} (1 + it)^n d\mu(t) = \int_{-\infty}^{+\infty} (1 + it)^n d\nu(t) + d_n$$

where $\limsup_{n\to\infty} |d_n|^{1/n} \leq 1$. *Suppose further that $Z_n := \int_{-\infty}^{+\infty} (1 + it)^n d\mu(t)$ satisfies*

$$\sum_{n=1}^{\infty} |Z_{2n}|^{-1/2n} = \infty. \tag{8}$$

Then there exists $c \in \mathbb{R}$ such that $\mu = \nu + c\delta_0$. If, in addition, μ and ν are probability measures, then $\mu = \nu$.

Proof: First observe that, for each $n \geq 0$,

$$\int_{-\infty}^{+\infty} (1 + it)^n d(\mu - \nu)(t) = \sum_{k=0}^{n} \binom{n}{k} i^k \int_{-\infty}^{+\infty} t^k d(\mu - \nu)(t).$$

By hypothesis, the left-hand side is $O(\beta^n)$ as $n \to \infty$ for each $\beta > 1$. Hence, taking real and imaginary parts of the right-hand side and applying the second assertion of Theorem E, it follows that $\int_{-\infty}^{+\infty} t^n d(\mu - \nu)(t) = O(\alpha^n)$ as $n \to \infty$, for each $\alpha > 0$. In other words,

$$\int_{-\infty}^{\infty} t^n \, d\mu(t) = \int_{-\infty}^{\infty} t^n \, d\nu(t) + c_n \quad (n \geq 0)$$

where $\lim_{n\to\infty} |c_n|^{1/n} = 0$. Moreover, it has been proved in [2] that (8) implies (5). Applying Corollary 3.2, we get the desired result.

$\square$

The above theorem includes examples such as $d_n = e^{\sqrt{n}}$ with a faster increase than the polynomial growth required in Theorem 4.1 of [2].

Our second result is a variant that does not require a growth condition of type (8).

Theorem 4.2 *Let μ, ν be positive Borel measures on $\mathbb{R}$, all of whose moments are finite. Suppose that there exists a constant $r \geq 0$ such that*

$$\int_{-\infty}^{+\infty} (1 + it)^n d|\mu - \nu|(t) = O(n^r) \quad \text{as } n \to \infty. \tag{9}$$

Then there exists $c \in \mathbb{R}$ such that $\mu = \nu + c\delta_0$. If, in addition, μ and ν are probability measures, then $\mu = \nu$.

Proof: As before, for each $n \geq 0$,

$$\int_{-\infty}^{+\infty} (1 + it)^n d|\mu - \nu|(t) = \sum_{k=0}^{n} \binom{n}{k} i^k \int_{-\infty}^{+\infty} t^k d|\mu - \nu|(t).$$

By hypothesis, the left-hand side is $O(n^r)$ as $n \to \infty$. Hence, taking real and imaginary parts of the right-hand side and applying the first assertion of Theorem E, it follows that $\int_{-\infty}^{+\infty} t^n d|\mu - \nu|(t) = 0$ for all $n > r$. In particular, if $n_0 > r$ is even, we get $\int_{-\infty}^{+\infty} t^{n_0} d|\mu - \nu|(t) = 0$, and thus $\mu - \nu$ is supported on $\{0\}$. The result follows.

$\square$

We finish by remarking that there is no hope of replacing $|\mu - \nu|$ by $\mu - \nu$ in (9). Indeed, by Theorem E, we see that

$$\int_{-\infty}^{+\infty} (1 + it)^n d(\mu - \nu)(t) = O(n^r) \Longleftrightarrow \int_{-\infty}^{+\infty} t^n d(\mu - \nu)(t) = 0, \, n > r,$$

and it is a well-known fact (cf. [3], pp. 128–129) that two probability measures on $\mathbb{R}$ whose moments are finite and equal are not necessarily the same.

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