Monograph:

Banks, S.P (1980) Nonlinear Pertubations of Dynamical Systems With Bounded Inputs. Research Report. ACSE Report 129 . Department of Control Engineering, University of Sheffield, Mappin Street, Sheffield

NONLINEAR PERTUBATIONS OF DYNAMICAL

SYSTEMS WITH BOUNDED INPUTS

by

S. P. BANKS

Research Report No. 129
October 1980

Department of Control Engineering
University of Sheffield
Mappin Street,
SHEFFIELD

Abstract

The nonlinear variations of constants formula is used to derive state estimates when a nonlinear system is subject to bounded inputs. Both input-output and Lyapunov type methods are examined.

1. Introduction

In recent papers, Cook (1980a,b) has studied the effect on the states and outputs of a system when the input is subject to boundedness restrictions. The type of systems considered are of the form

$$\dot{x} = Ax + Bu + \psi(x,u,t) \qquad x \in \mathbb{R}^n$$

where A, B are matrices of appropriate sizes and ψ satisfies an inequality of the form

$$(2.1) \qquad ||\psi\,(x,u,t)|| \leq \alpha + \beta||x||$$

for $||x|| \leq \gamma$ and $u \in \Omega$ (the constraint set for the inputs).
In this paper we shall generalize these results to the case of nonlinear systems subject to nonlinear pertubations of the form

$$\dot{x} = f(x,t) + Bu + \psi\,(x,u,t)$$

by imposing certain restrictions on the unperturbed free system

$$\dot{x} = f(x,t).$$

In order to obtain bounds on the states we shall use the nonlinear variations of constants formula due to Alekseev (1961), which we shall discuss in section 2. The application of this formula to obtain state bounds assuming a bounded control is presented in section 3 and a Lyapunov type approach to the same problem is then discussed in section 4.
In section 5 a simple example is presented to illustrate the theory.

2. System Description

We shall assume that the system which we are considering may be written in the form

$$(2.1) \qquad \dot{x}(t) = f(x,t) + Bu + \psi\,(x,u,t)$$

where it is supposed, for simplicity, that the unperturbed free system

$$(2.2) \qquad \dot{y}(t) = f(y,t)$$

has sufficient conditions imposed on f to ensure the existance and uniqueness of solutions. Let $y(t;y_o,t_o)$ denote the solution of equations (2.2) with initial condition $y(t_o) = y_o$. Then, using the Alekseev nonlinear variation of constants formula (cf.Brauer, 1966) we have the following result:

Lemma 2.1 The solutions of equations (2.1) and (2.2) are related by the formula (where we consider each equation to have the same initial condition x_o):

$$(2.3) \qquad x(t;x_o,t_o) = y(t;x_o,t_o) + \int_{t_o}^{t} \Phi(t,s,x(s;x_o,t_o))\{Bu(s) + \psi(x(s;x_o,t_o), u(s),s)\}ds . \quad \square$$

In equation (2.3), Φ is the matrix function given by

$$\Phi(t,t_o,x_o) = \frac{\partial}{\partial x_o} \left[x(t;x_o,t_o)\right]$$

and it is the fundamental solution of the variational system

$$\dot{Z} = f_x\left[t,x(t,x_o,t_o)\right] Z .$$

In order to obtain a suitable bound on Φ we introduce the "logarithmic norm" of Lozinskii (1958);

$$\mu(A) = \lim_{h\to 0+} \frac{||I + hA|| - 1}{h}$$

where $||\cdot||$ denotes a particular matrix norm. If we use the standard vector norm on $\mathbb{R}^n$ and the corresponding induced matrix norm, then $\mu(A)$ is just the largest eigenvalue of $\frac{1}{2}(A+A^*)$, which we denote by $\lambda(A)$. We then have (cf. Coppel, 1965):

Lemma 2.2 The fundamental matrix $\Phi(t,t_o)$ of the linear system

$$\dot{Z} = A(t)Z$$

such that $\Phi(t_o,t_o) = I$ satisfies

$$||\Phi(t,t_o)|| \leq \exp\left[\int_{t_o}^{t} \mu(A(u))du\right] , \quad (t \geq t_o). \quad \square$$

Hence we see that if

$$\mu(f_x[t,x]) < \alpha_1(t) \quad \forall\ x ,t$$

then,

$$(2.4) \qquad ||\Phi(t,t_o,x_o)|| \leq \exp\left[\int_{t_o}^{t} \alpha_1(u)du\right].$$

We also have from Brauer (1966), the result:

<u>Lemma 2.3</u> Let $x_o, y_o \in \mathbb{R}^n$ and denote by ζ the straight line between x_o and y_o i.e.

$$\zeta(\lambda)=x_o+\lambda(y_o-x_o) \quad \text{for} \quad 0\leq\lambda\leq 1.$$

Then the system (2.2) has solutions through x_o, y_o, which satisfy

$$||y(t;y_o,t_o)-y(t;x_o,t_o)|| \leq \max_{0\leq\lambda\leq 1} ||\Phi(t,t_o,\zeta(\lambda))||.||y_o-x_o|| . \quad \square$$

We therefore have the following corollary:

<u>Corollary 2.4</u> If $0\in\mathbb{R}^n$ is a critical point of equation (2.2)

(i.e. $f(0,t) = 0 \ \forall \ t\geq t_o$), then we have

$$||y(t;y_o,t_o)|| \leq \max_{y'\in\zeta(y_o)} ||\Phi(t,t_o,y')||.||y_o||,$$

where

$$\zeta(y_o) = \{ y : y=\lambda y_o \text{ for some } \lambda\in[0,1]\}. \quad \square$$

In what follows, we shall assume that the origin is a critical point of equation (2.2), thus enabling us to obtain bounds on Φ.

3. <u>Obtaining State Bounds</u>

Suppose first that the control input is constrained simply by

$$||u|| \leq k$$

and that ψ satisfies the inequality (2.1) for $||x||\leq\gamma$. It follows, therefore, from (2.3) that

$$||x(t;x_o,t_o)|| \leq \exp\left[\int_{t_o}^{t} \alpha_1(u)du\right]||x_o|| + \int_{t_o}^{t} \exp\left[\int_{s}^{t}\alpha_1(u)du\right].$$

$$\{||B||k + \alpha + \beta\,\gamma\} \, ds,$$

provided $||x|| \leq \gamma$. Hence, if $\xi(t,s)=\exp\left[\int_s^t \alpha_1(u)du\right]$, we have

$$\gamma \leq \xi(t,t_o)||x_o|| + (||B||k + \alpha + \beta\gamma) \int_{t_o}^t \xi(t,s)ds.$$

We therefore have

<u>Lemma 3.1</u> Suppose that, for $t_o \leq t \leq t_o + \delta$,

$$1 - \beta \int_{t_o}^t \xi(t,s)ds > 0$$

and that

$$(3.1) \qquad \{\sup_{t\in[t_o,t_o+\delta]} (1 - \beta \int_{t_o}^t \xi(t,s)ds)^{-1}\}\{\sup_{t\in[t_o,t_o+\delta]} (\xi(t,t_o)||x_o|| + (||B||k + \alpha) \int_{t_o}^t \xi(t,s)ds)\} \leq \gamma,$$

then if $||x_o|| \leq \gamma$, we have

$$||x(t;x_o,t_o)|| \leq \gamma \quad \text{for} \quad t\in[t_o,t_o+\delta]. \quad \square$$

<u>Corollary 3.2</u> With the same notation as above, if $\alpha_1(u)$ is constant and negative, say $\alpha_1(u) = -\alpha_2$, $\alpha_2>0$, then if

$$\alpha_2 > \beta+1$$

and

$$||B||k + \alpha \leq \gamma\,(\alpha_2-\beta-1),$$

we have that $||x_o|| \leq \gamma$ implies

$$||x(t;x_o,t_o)|| \leq \gamma \quad \text{for } t \in [t_o,\infty).$$

<u>Proof</u> This follows easily from the lemma since condition (3.1) reduces to the condition

$$\left(\frac{\alpha_2}{\alpha_2-\beta}\right)\left(\frac{||x_o||}{\alpha_2} + (||B||k+\alpha)\frac{1}{\alpha_2}\right) \leq \gamma$$

which is satisfied if

$$||B||k+\alpha \leq \gamma(\alpha_2-\beta-1)$$

provided $||x_o|| \leq \gamma$. □

Following Cook (1980b) we can also obtain state bounds by assuming that the state vector x is partitioned into m subvectors $x_1,\ldots,x_m$, where $x_i \in \mathbb{R}^{\ell_i}$, $\sum_{i=1}^{m} \ell_i = n$, and that the nonlinearity ψ is similarly partitioned, with

(3.2)

$$||\psi_i|| \leq \alpha_i + \sum_j \beta_{ij}||x_j|| \quad , (\forall\ ||x_k|| \leq \gamma_k \quad , u \in \Omega).$$

We can also write Φ in the form

$$\Phi = (\Phi_{ij})_{1\leq i\leq m, 1\leq j\leq m}$$

where each Φ_{ij} is a matrix of appropriate size.

Thus, using lemma 2.1 and corollary 2.4, we can write

$$||x_i(t;x_o,t_o)|| \leq \sum_{j=1}^{m} \max_{x' \in \zeta(x_o)} ||\Phi_{ij}(t,t_o,x')|| \, ||x_{oj}||$$

$$+ \int_{t_o}^{t} \{ \sum_{j=1}^{m} ||\Phi_{ij}(t,s,x(s;x_o,t_o))||(||B||k+\alpha_j) ds$$

$$+ \sum_{j=1}^{m} \lambda_{ij}(t) \sup_{t_o\leq s\leq t} ||x_j(s)||$$

where

$$\lambda_{ij}(t) = \int_{t_o}^{t} \sum_{k=1}^{m} ||\Phi_{ij}(t,s,x(s;x_o,t_o))||\beta_{kj} ds,$$

provided

$$\sup_{o\leq\tau\leq t} ||x_i(\tau;x_o,t_o)|| \leq \gamma_i \quad , \quad 1\leq i\leq m.$$

Hence, if

$$||\Phi_{ij}(t,s,y)|| \leq \phi_{ij}(t,s)$$

for all y such that

$$||y|| \leq \{ \sum_{i=1}^{m} (\gamma_i)^2 \}^{\frac{1}{2}}$$

then we have

$$||x_i(t;x_o,t_o)|| \leq \sum_{j=1}^{m} \phi_{ij}(t,t_o)||x_{oj}|| + \int_{t_o}^{t} \sum_{j=1}^{m} \phi_{ij}(t,s)(||B||k+\alpha_j)ds$$

$$+ \sum_{j=1}^{m} \lambda_{ij}(t) \sup_{t_o \leq s \leq t} ||x_j(s;t_o,x_o)||$$

provided $\sup_{o \leq \tau \leq t} ||x_i(\tau;x_o,t_o)|| \leq \gamma_i$.

Therefore, defining the vector ζ by

$$\zeta_i(t) = \sum_{j=1}^{m} \phi_{ij}(t,t_o)||x_{oj}|| + \int_{t_o}^{t} \sum_{j=1}^{m} \phi_{ij}(t,s)(||B||k+\alpha_j)ds$$

and denoting the matrix (λ_{ij}) by Λ we have

Theorem 3.3 If

$$\Theta(t) = \{I - \sup_{o \leq \tau \leq t} \Lambda(\tau)\}^{-1} \sup_{o \leq \tau \leq t} \zeta(\tau)$$

exists and

$$\Theta_i(t) \leq \gamma_i \quad , \quad 1 \leq i \leq m,$$

then

$$\sup_{o \leq \tau \leq t} ||x_i(\tau)|| \leq \Theta_i(t) \quad , \quad 1 \leq i \leq m.$$

Proof The proof follows in the same way as Cook, 1980b. □

In particular, if $\Phi=(\Phi_{ij})$ is a diagonal matrix, each element of which satisfies a condition of the form (2.4), i.e.

$$||\Phi_{ij}(t,t_o,x_o)|| \leq \exp\left[\int_{t_o}^{t} \alpha_i(u) \, du \right]$$

then, from theorem 3.3 and lemma 3.1 we have

Corollary 3.4 If, for $t_o \le t \le t_o + \delta$

$$1-\beta \int_{t_o}^{t} \xi_i(t,s)ds > 0 \qquad (\xi_i = \exp[\int_s^t \alpha_i(u)dx])$$

and if

$$\Theta_i(t) \le \gamma_i \quad , \quad 1 \le i \le m,$$

where

$$\Theta(t) = \mathrm{diag}(\{1-\beta \int_{t_o}^{t} \xi_i(t,s)ds\}^{-1}) \sup_{o \le \tau \le t} \zeta(\tau)$$

then,

$$\sup_{o \le \tau \le t} ||x_i(\tau)|| \le \Theta_i(t), \quad 1 \le 1 \le m. \ \square$$

With this result it is possible to separate out the stable and unstable parts of Φ , and consider finite time bounds on the unstable parts.

4. Lyapunov Method

Consider again the system

(4.1) $$\dot{x}(t) = f(x) + Bu + \psi\,(x,u,t).$$

where ψ satisfies (2.1).

In order to apply a vector Lyapunov type argument, we shall define the matrix function $F(x)$ by

$$F_{ij}(x) = \frac{f_{ij}(x_j)}{x_j} \quad , \quad x_j \neq 0 \quad (i,j=1,\ldots\ldots,n)$$

where we assume that $f_i(x)$ is of the form

$$f_i(x) = f_{i1}(x_1) + f_{i2}(x_2)+\ldots.+f_{in}(x_n) \quad (i-1,\ldots.,n)$$

and that

$$\lim_{\alpha \to 0} \frac{f_{ij}(\alpha)}{\alpha} = \ell_{ij} \quad \text{(say)}$$

exists. We then define

$$F_{ij}(0) = \ell_{ij}.$$

Equation (4.1) may now be written in the form

(4.2) $\dot{x}(t) = F(x)x + Bu + \psi(x,u,t)$

and so if $= ||x||$ and ψ satisfies (2.1) then using the fact that

$$\frac{d}{dt}||x(t)|| = \frac{d}{dt}\left(\sum_{i=1}^{n} x_i^2(t)\right)^{\frac{1}{2}}$$

$$= \left(\sum_{i=1}^{n} |x_i(t)|\frac{d}{dt}|x_i(t)|\right) \Big/ \left(\sum_{j=1}^{n} x_j^2(t)\right)^{\frac{1}{2}}, \text{ a.e.}$$

$$\leq \left(\sum_{i=1}^{n} |x_i(t)||\dot{x}_i(t)|\right) / \left(\sum_{j=1}^{n} x_j^2(t)\right)^{\frac{1}{2}}, \text{ a.e.}$$

$$\leq ||\dot{x}_i(t)||, \text{a.e. (by Cauchy-Schwartz inequality)}$$

we have,

$$\dot{\xi} \leq ||F(x)||\xi + ||B||k + \alpha + \beta\xi$$

provided $||x(t)|| \leq \gamma$. Suppose now that F satisfies

$$||F(x)|| \leq G(||x||)$$

for some differentiable function $G: \mathbb{R}^+ \to \mathbb{R}^+$.

Then,

$$\dot{\xi} \leq G(\xi)\xi + \beta\xi + ||B||k+\alpha \;, \; \xi_o = \xi(t_o) = ||x(t_o)||$$

and so, if

(4.3) $$C_1 \leq G'(\xi)\xi + G(\xi) + \beta \leq C_2 \quad , \quad \forall \xi$$

where C_1, C_2 are constants, we have

(4.4) $$||\Phi(t,t_o,\xi_o)|| \leq \exp[C_2(t-t_o)]$$

where Φ is the variational solution of the system

$$\dot{\eta} = G(\eta)\eta \quad , \quad \eta(t_o) = \xi_o .$$

Defining

(4.5) $$\dot{\nu}(t) = \eta(t) + \int_{t_o}^{t} \Phi(t,s,\nu(s))(||B||k+\alpha)ds$$

we have

$$\dot{\nu}(t) = G(\nu)\nu + \beta\nu + ||B||k + \alpha \;, \quad \nu(t_o) = \zeta_o ,$$

and hence

$$\dot{\xi}(t) - \dot{\nu}(t) \leq G(\xi)\xi - G(\nu)\nu + \beta(\xi-\nu).$$

However, if G is analytic and can therefore be expanded in a Taylor series, we have

$$\begin{aligned} \dot{\xi}-\dot{\nu} \leq \beta(\xi-\nu)+ G(\xi)\xi- G(\nu)\nu &= \{G'(\xi)\xi+G(\xi) - G'(\nu)\nu- G(\nu)\}(\xi-\nu) \\ &\quad +\beta(\xi-\nu) \\ &\leq (C_2-C_1)(\xi-\nu) + \beta(\xi-\nu) \\ &= (C_2-C_1+\beta)(\xi-\nu) , \qquad (4.6) \end{aligned}$$

using (4.3). An application of Gronwall's lemma now proves the following result:

Theorem 4.1 Under the above assumptions, we have

$$||x(t)|| \leq \exp\left[C_2(t-t_0)\right]||x_0|| + \int_{t_0}^{t} \exp\left[C_2(t-s)\right](||B||k+\alpha)\, ds$$

$$= e(t) \text{ , say}$$

provided

$$\sup_{0\leq\tau\leq t} e(\tau) \leq \gamma .$$

Proof We merely note that, from (4.4) and (4.5),

$$|\nu(t)| \leq e(t), \quad t\geq t_0$$

and the result follows from (4.6) . □

5. Example

We shall consider, as a simple example of the theory, the system defined by the equations

(5.1) $$\begin{aligned} \dot{x}_1 &= -3x_1 + x_2 \\ \dot{x}_2 &= -x_1 - 3g(x_2) + x_1^3 + u \end{aligned}$$

or

(5.2) $$\dot{x} = f(x) + \psi(x) + Bu$$

where

$$x = (x_1,x_2)^T \ , \ f(x) = (-3x_1+x_2,-x_1-3g(x_2))^T$$

$$\psi(x) = (0,x_1^3)^T \ , \ B = (0\ 1)^T \ ,$$

We associate with (5.2) the unperturbed system

$$(5.3) \qquad \dot{y} = f(y)$$

and note that, for a given solution y(t) of (5.3), the variational system of (5.3) is

$$\dot{Z} = \begin{pmatrix} -3 & 1 \\ -1 & -3g'(y_2) \end{pmatrix} Z = A(t)Z, \text{ say.}$$

Hence, if

$$g'(y_2) \geq 1 \quad \text{for all } |y_2| \leq 1$$

we have

$$\mu(A(t)) \leq -3$$

provided $|y_2| \leq 1$. Hence, we see from corollary 3.2 that if the control is bounded by 1 , i.e. $|u|<1$ and we estimate ψ for $||x|| \leq 1$ by

$$||\psi|| = ||(0, x_1^3)|| \leq ||x||$$

(and so we may take $\alpha=0$, $\beta=1$) then if $||x_o|| \leq 1$ we have

$$||x(t)|| \leq 1 \quad \text{for} \quad t \geq 0.$$

We shall illustrate the use of the Lyapunov like method using the same example; however, g will now be assumed to be sector bounded and we shall derive a kind of finite-time stability for this system. Thus let g satisfy

$$-3x_2 \leq g(x_2) \leq 3x_2 \ , \quad x_2 \in \mathbb{R}.$$

Now,

$$F(x) = \begin{pmatrix} -3 & 1 \\ -1 & -3g(x_2)/x_2 \end{pmatrix}$$

and so

$$||F(x)|| \leq \sqrt{10} .$$

Hence, if $C_2 = \sqrt{10}+1$, theorem 4.1 implies that if $x_o=0$ and $t_1=(\log(C_2+1)/C_2$, then

Notes & References

1. The developments are discussed in e.g. Myers, G.J., Software Reliability Wiley, 1976; Welsh, J., Mckeag, M., Structured System Programming, Prentice Hall, 1980; Zelkowitz, M.V., Shaw, A.C., Gannon, J.D., Principles of Software Engineering and Design, Prentice Hall, 1979; Wegner, P., Programming with ADA, Prentice Hall, 1980.

2. Pyle, I.C.; 'Methods for the design of control software', Software for Computer Control, Proceedings Second IFAC/IFIP Symposium on Software for Computer Control, Prague 1979, Oxford, Pergamon 1979, pp51-57.

3. Wirth, N.; 'Towards a discipline of real-time programming', Software Practice and Experience, v.7., no. 1 p.

4. Some indication of the approaches adopted can be found by examining the papers presented by IFAC/IFIP conference in Prague (ref. 2). An example of approach(c) is the language CUTLASS being developed by the CEGB see papers in Proceedings of International Conference on Distributed Computer Control Systems, Birmingham 1977, Trends in On-Line Computer Control Systems, Sheffield 1979.