THE WEISS CONJECTURE ON ADMISSIBILITY OF OBSERVATION OPERATORS FOR CONTRACTION SEMIGROUPS

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We prove the conjecture of George Weiss for contraction semigroups on Hilbert spaces, giving a characterization of infinite-time admissible observation functionals for a contraction semigroup, namely that such a functional C is infinite-time admissible if and only if there is an $M > 0$ such that $\|C(sI - A)^{-1}\| \le M/\sqrt{\operatorname{Re} s}$ for all s in the open right half-plane. Here A denotes the infinitesimal generator of the semigroup. The result provides a simultaneous generalization of several celebrated results from the theory of Hardy spaces involving Carleson measures and Hankel operators.

1 Introduction and main result

In this paper we are concerned with linear systems of the following kind:

$$\begin{aligned} \dot{x}(t) &= Ax(t), \qquad t \ge 0, \\ y(t) &= Cx(t). \end{aligned} \tag{1}$$

Here $x(t) \in H$, where H is a Hilbert space, is the state of the system at time $t \ge 0$ and $y \in L^2_{loc}(0, \infty)$ is the output of the system. The space H is called the *state space*. In (1), both A and C are possibly unbounded operators. A is the infinitesimal generator of a C_0-semigroup $T(t)$ of contractions on H and C is assumed to be a linear bounded operator from $D(A)$, the domain of A to $\mathbb{C}$. However, in general C will not be a bounded operator from H to $\mathbb{C}$. By a solution of $\dot{x}(t) = Ax(t)$ with initial condition $x(0) = x_0 \in H$ we mean the continuous function

$$x(t) = T(t)x_0, \qquad t \ge 0.$$

These assumptions are not sufficient to guarantee that the output of the system is in $L^2_{loc}(0, \infty)$. In order to guarantee this an additional assumption is needed. Following Weiss [13] we introduce admissible observation operators for $T(t)$. Let X and Y be normed spaces; by $\mathcal{L}(X, Y)$ we denote the set of bounded linear operators from X to Y.

Definition 1.1 *Let $C \in \mathcal{L}(D(A), \mathbb{C})$. Then C is called an* admissible observation operator *for $T(t)$, if for some (and hence any) $t > 0$, there is some $K > 0$ such that*

$$\|CT(\cdot)x\|_{L^2(0,t)} \le K\|x\|, \qquad x \in D(A).$$

The admissibility of C guarantees that we can extend the mapping $x_0 \to CT(\cdot)x_0$ to a bounded linear operator from H to $L^2(0,\infty)$. In order to guarantee that the output of the system is in $L^2(0,\infty)$ a slightly stronger assumption, called infinite-time admissibility, is needed.

Definition 1.2 *Let $C \in \mathcal{L}(D(A),\mathbb{C})$. Then C is called an* infinite-time admissible observation operator for $T(t)$, *if there is some $K > 0$ such that*

$$\|CT(\cdot)x\|_{L^2(0,\infty)} \leq K\|x\|, \qquad x \in D(A).$$

If C is bounded, i.e., $C \in \mathcal{L}(H,\mathbb{C})$, then C is always admissible, but C may not be infinite-time admissible. However, if A is exponentially stable, then the notions of admissibility and infinite-time admissibility are equivalent. In this article, we show the following equivalent condition for infinite-time admissibility. By $\mathbb{C}_+$ we denote the set $\{s \in \mathbb{C} \mid \operatorname{Re} s > 0\}$.

Theorem 1.3 *Let $T(t)$ be a C_0-semigroup of contractions on a separable Hilbert space H with infinitesimal generator A and let $C \in \mathcal{L}(D(A),\mathbb{C})$. Then the following statements are equivalent.*

1. *There exists a constant $M > 0$ such that*

$$\|C(sI - A)^{-1}\| \leq \frac{M}{\sqrt{\operatorname{Re} s}}, \qquad s \in \mathbb{C}_+.$$

2. *C is infinite-time admissible.*

It is already known that (2) implies (1), see Weiss [14]. Moreover, it was conjectured in Weiss [14, 15] that (1) implies (2) as well. In Weiss [14, 15] it is proved that (1) and (2) are equivalent for normal semigroups and for exponentially stable right-invertible semigroups, and in Partington and Weiss [10] it is shown that the equivalence of (1) and (2) holds for the right-shift on $L^2(0,\infty)$; the methods introduced there can be adapted to more general situations, as we shall see. See Grabowski and Callier [5] for a related equivalent condition for infinite-time admissibility.

It should be noted, that it is easy to see that Theorem 1.3 holds for $C \in \mathcal{L}(D(A),\mathbb{C}^n)$ as well. Moreover, Theorem 1.3 holds for bounded C_0-semigroups which are similar to a contraction C_0-semigroup. In Grabowski and Callier [5] it is noted that if there exists an admissible observation operator $\tilde{C}$ for $T(t)$ such that $(T(t),\tilde{C})$ is exactly observable then $T(t)$ is similar to a contraction C_0-semigroup. However, not every bounded C_0-semigroup is similar to a contraction C_0-semigroup; examples are given for example in Packel [9] and Simard [11].

One of the interesting features of Theorem 1.3 is that it contains many celebrated results of Function Theory as easy corollaries. For example, it is known that Theorem 1.3 for the right-shift semigroup is equivalent (in an elementary way) to Fefferman's duality theorem and to Bonsall's theorem [2, 3] on the boundedness of Hankel operators; another special case, in which $T(t)$ is a contraction semigroup of normal operators, is equivalent to the Carleson measure theorem (see [14, 15]), and we discuss this further in Example 3.4.

As a corollary of Theorem 1.3 we obtain the following equivalent condition for admissibility.

Corollary 1.4 *Let $T(t)$ be a C_0-semigroup on a separable Hilbert space H with infinitesimal generator A satisfying*

$$\|T(t)\| \leq e^{\alpha t}, \qquad t \geq 0,$$

for some constant $\alpha \in \mathbb{R}$, and let $C \in \mathcal{L}(D(A), \mathbb{C})$. Then the following statements are equivalent.

1. *There exist constants $M, \sigma > 0$ such that*

$$\|C(sI - A)^{-1}\| \leq \frac{M}{\sqrt{\operatorname{Re} s}}, \qquad \operatorname{Re} s > \sigma.$$

2. *C is admissible.*

2 On a special contraction semigroup

In this section, we prove that the implication (1) to (2) holds for a special contraction semigroup. The semigroup studied here is a model for completely non-unitary contraction semigroups, i.e., every completely non-unitary contraction semigroup is unitarily equivalent to one of the semigroups studied in this section. The definition of a completely non-unitary contraction semigroup will be given in Section 3. Further results concerning the model of such a semigroup can be found in Sz. Nagy and Foiaş [12].

Throughout this section we assume the following: E and F are Hilbert spaces, $\theta : \mathbb{C}_+ \to \mathcal{L}(E, F)$ is a holomorphic function satisfying $\|\theta(s)\| \leq 1$ on $\mathbb{C}_+$, $\Delta : i\mathbb{R} \to \mathcal{L}(E)$ is defined by

$$\Delta(i\omega) := [I_E - \theta(i\omega)^*\theta(i\omega)]^{1/2}, \qquad \omega \in \mathbb{R},$$

and the C_0-semigroup $T(t)$ on U with infinitesimal generator A is given by

$$\begin{aligned}
T(t)u &:= P_U[e^{-i\omega t}u(i\omega)], \qquad u \in U,\\
X &:= H^2(\mathbb{C}_+, F) \oplus \overline{\Delta L^2(i\mathbb{R}, E)},\\
U &:= X \ominus \{\theta f \oplus \Delta f \mid f \in H^2(\mathbb{C}_+, E)\},\\
Au &:= P_U[-i\omega u(i\omega)], \qquad u \in D(A),\\
D(A) &:= \{u \in U \mid i\omega u(i\omega) \in X\}.
\end{aligned}$$

Here P_U denotes the orthogonal projection from X onto U, $U_1 \oplus U_2$ denotes the direct sum of U_1 and U_2, and $X \ominus U$ denotes the orthogonal complement of U in X. Moreover, we assume that $C \in \mathcal{L}(D(A), \mathbb{C})$. $H^2(\mathbb{C}_+, E)$ denotes the Hardy space of E-valued functions on the right half-plane, which is a closed subspace of $L^2(i\mathbb{R}, E)$. By $\mathbb{C}_-$ we denote the set $\{s \in \mathbb{C} \mid \operatorname{Re} s < 0\}$, and $H^2(\mathbb{C}_-, E)$ is the corresponding Hardy space. The following lemma will be useful to us.

Lemma 2.1 *Let $c : i\mathbb{R} \to F \oplus E$ such that $\frac{c(i\omega)}{1-i\omega} \in U$ and let $y \in X$ such that $i\omega y(i\omega) \in X$ and $i\omega(P_U y)(i\omega) \in X$. Then*

$$\int_{-\infty}^{\infty} \langle c(i\omega), (P_U(y))(i\omega)\rangle \, d\omega = \int_{-\infty}^{\infty} \langle c(i\omega), y(i\omega)\rangle \, d\omega$$

Proof: We write $y = u + u^\perp$ with $u \in U$ and $u^\perp \in U^\perp$. Now $i\omega y(i\omega) \in X$ and $i\omega u(i\omega) = i\omega(P_U y)(i\omega) \in X$ implies

$$i\omega u^\perp(i\omega) \in X. \tag{2}$$

In order to show the statement it remains to show that $i\omega u^\perp(i\omega) \in U^\perp$, because this implies $(1 + i\omega)u^\perp(i\omega) \in U^\perp$ and so

$$\int_{-\infty}^{\infty} \langle c(i\omega), u^\perp(i\omega)\rangle \, d\omega = \int_{-\infty}^{\infty} \left\langle \frac{c(i\omega)}{1-i\omega}, (1+i\omega)u^\perp(i\omega) \right\rangle d\omega = 0.$$

Using the calculation

$$\|\theta f \oplus \Delta f\|^2 = \langle \theta f, \theta f\rangle + \langle 1 - \theta^*\theta f, f\rangle = \|f\|^2, \qquad f \in H^2(\mathbb{C}_+, E),$$

we see that $\{\theta f \oplus \Delta f \mid f \in H^2(\mathbb{C}_+, E)\}$ is closed, and so $U^\perp = \{\theta f \oplus \Delta f \mid f \in H^2(\mathbb{C}_+, E)\}$.

Thus using the fact that $u^\perp \in U^\perp$ there exists a function $f \in H^2(\mathbb{C}_+, E)$ such that

$$u^\perp = \theta f \oplus \Delta f.$$

Using (2), we get $i\omega(\theta(i\omega)f(i\omega) \oplus \Delta(i\omega)f(i\omega)) \in X$, which implies $i\omega\theta(i\omega)f(i\omega) \in L^2(i\mathbb{R}, F)$ and $i\omega\Delta(i\omega)f(i\omega) \in L^2(i\mathbb{R}, E)$. Thus we get

$$\begin{aligned}
\|i\omega f(i\omega)\|^2 &= \langle i\omega f(i\omega), (I - \theta(i\omega)^*\theta(i\omega))i\omega f(i\omega)\rangle + \|i\omega\theta(i\omega)f(i\omega)\|^2 \\
&= \|i\omega\Delta(i\omega)f(i\omega)\|^2 + \|i\omega\theta(i\omega)f(i\omega)\|^2,
\end{aligned}$$

which shows $i\omega f(i\omega) \in L^2(i\mathbb{R}, E)$.

Let $g \in H^2(\mathbb{C}_-, E)$ be such that $g(z)z \in H^2(\mathbb{C}_-, E)$. Then we have

$$\int_{-\infty}^{\infty} \langle f(i\omega)i\omega, g(i\omega)\rangle \, d\omega = -\int_{-\infty}^{\infty} \langle f(i\omega), i\omega g(i\omega)\rangle \, d\omega = 0.$$

Using the fact that $\{g \in H^2(\mathbb{C}_-, E) \mid g(z)z \in H^2(\mathbb{C}_-, E)\}$ is dense in $H^2(\mathbb{C}_-, E)$, we get $f(z)z \in H^2(\mathbb{C}_+, E)$ and so $i\omega u^\perp(i\omega) \in U^\perp$. This proves the lemma. ∎

The following proposition provides us with an integral representation of C.

Proposition 2.2 *C has a representation*

$$Cx = \int_{-\infty}^{\infty} \langle x(i\omega), c(i\omega)\rangle \, d\omega, \quad x \in D(A),$$

for some function $c : i\mathbb{R} \to F \oplus E$ such that $z \to \frac{c(i\omega)}{1-i\omega} \in U$.

The proof is a modification of the proof of Proposition 2.1 in Partington and Weiss [10].

Proof: Using that $C \in \mathcal{L}(D(A), \mathbb{C})$, we get that $C(I-A)^{-1}$ is a bounded functional on U. Thus the Riesz representation theorem implies the existence of a function $\tilde{c} \in U$ such that

$$C(I-A)^{-1}u = \langle u, \tilde{c} \rangle = \int_{-\infty}^{\infty} \langle u(i\omega), \tilde{c}(i\omega) \rangle \, d\omega, \qquad u \in U.$$

With the notation $x = (I-A)^{-1}u$, we get $x \in D(A) \subset U$ and

$$\begin{aligned} Cx &= \int_{-\infty}^{\infty} \langle (I-A)x(i\omega), \tilde{c}(i\omega) \rangle \, d\omega = \int_{-\infty}^{\infty} \langle x(i\omega) + P_U(i\omega x(i\omega)), \tilde{c}(i\omega) \rangle \, d\omega \\ &= \int_{-\infty}^{\infty} \langle x(i\omega), \tilde{c}(i\omega) \rangle \, d\omega + \int_{-\infty}^{\infty} \langle i\omega x(i\omega), P_U \tilde{c}(i\omega) \rangle \, d\omega \\ &= \int_{-\infty}^{\infty} \langle (1+i\omega)x(i\omega), \tilde{c}(i\omega) \rangle \, d\omega, \qquad \text{(using } \tilde{c} \in U\text{)}. \end{aligned}$$

Denoting $c(i\omega) = \tilde{c}(i\omega)(1-i\omega)$ the proposition is proved. ■

In the following let c be given as in Proposition 2.2. By $\mathcal{L}_b : L^2(\mathbb{R}) \to L^2(i\mathbb{R})$ we denote the bilateral Laplace transformation, which is given by

$$(\mathcal{L}_b f)(i\omega) := \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \, dt, \qquad f \in L^2(\mathbb{R}) \cap L^1(\mathbb{R}),$$

and which extends to $L^2(\mathbb{R})$ by continuity. $\mathcal{L}_b$ is an isomorphism from $L^2(\mathbb{R})$ to $L^2(i\mathbb{R})$, and from $L^2(\mathbb{R}_+)$ to $H^2(\mathbb{C}_+)$. We have the following characterization of admissibility.

Proposition 2.3 *The following statements are equivalent*

1. *C is infinite-time admissible.*
2. *c defines a bounded operator $\Gamma_c : U \to H^2(\mathbb{C}_-)$, by*

$$\Gamma_c u := P_-(\langle u, c \rangle), \qquad u \in U.$$

Here P_- denotes the orthogonal projection from $L^2(i\mathbb{R})$ onto $H^2(\mathbb{C}_-)$. Note, that originally Γ_c is only defined on $\{u \in U \mid \langle u, c \rangle \in L^2(i\mathbb{R})\}$.

Thus C is infinite-time admissible if and only if Γ_c is a bounded operator. Note that in the case when θ is inner and $\Delta = 0$, the operator Γ_c is a Hankel operator. The presence of Δ gives the operator something of a Toeplitz character as well, as we shall see more precisely later. Again the proof of this proposition is a variation on the proof of Lemma 2.2 in Partington and Weiss [10].

Proof: Using Lemma 2.1 and Proposition 2.2, and noting that $T(t)D(A) \subset D(A)$, for every $u \in D(A)$ we get

$$\begin{aligned}
|CT(t)u| &= \left| \int_{-\infty}^{\infty} \langle P_U(e^{-ts}u(s))(i\omega), c(i\omega)\rangle \, d\omega \right| \\
&= \left| \int_{-\infty}^{\infty} \langle e^{-i\omega t}u(i\omega), c(i\omega)\rangle \, d\omega \right| \\
&= \left| \int_{-\infty}^{\infty} e^{i\omega t}\langle c(i\omega), u(i\omega)\rangle \, d\omega \right| \\
&= \sqrt{2\pi}|\mathcal{L}_b^{-1}(\langle c, u\rangle)(t)|,
\end{aligned}$$

where $\mathcal{L}_b : L^2(\mathbb{R}) \to L^2(i\mathbb{R})$ is the bilateral Laplace transform. Thus we get

$$\int_0^{\infty} |CT(t)u|^2 \, dt = \|P_+(\langle c, u\rangle)\|^2 = \|P_-(\langle u, c\rangle)\|^2.$$

Thus the proposition is proved. ■

We now write the function c as

$$c(i\omega) = c_1(i\omega) \oplus c_2(i\omega), \qquad \omega \in \mathbb{R},$$

where $c_1 : i\mathbb{R} \to F$ and $c_2 : i\mathbb{R} \to E$, and we define $\phi_z : \mathbb{C}_+ \to F \oplus E$, $z \in \mathbb{C}_+$, by

$$\phi_z(i\omega) := \frac{c_1(\bar{z}) - c_1(i\omega)}{\bar{z} - i\omega} \oplus \frac{c_2(i\omega)}{i\omega - \bar{z}}, \qquad \omega \in \mathbb{R}.$$

Lemma 2.4 *We have $\phi_z \in U$ for every $z \in \mathbb{C}_+$.*

Proof: Let $z \in \mathbb{C}_+$. First of all, we show that $\phi_1(i\omega) := \frac{c_1(\bar{z})-c_1(i\omega)}{\bar{z}-i\omega} \in H^2(\mathbb{C}_+, F)$. Of course, $\phi_1(i\omega) \in F$ for every $\omega \in \mathbb{C}$. Using the fact that $\frac{c_1(s)}{1-s}$ is holomorphic, it is easy to see that ϕ_1 is holomorphic on $\mathbb{C}_+ \backslash \{\bar{z}\}$. At the point $s = \bar{z}$, the function ϕ_1 has a limit, and thus it is holomorphic on $\mathbb{C}_+$. On every vertical line in $\mathbb{C}_+$ the function ϕ_1 decays for large s like $\frac{c_1(s)}{1-s}$. Using the fact that $\frac{c_1(s)}{1-s}$ lies in $H^2(\mathbb{C}_+, F)$ we see that ϕ_1 lies in $H^2(\mathbb{C}_+, F)$ as well.

Next we show that $\phi_2(i\omega) := \frac{c_2(i\omega)}{i\omega-\bar{z}} \in \overline{\Delta L^2(i\mathbb{R}, E)}$. Since $\frac{c_2(i\omega)}{1-i\omega} \in \overline{\Delta L^2(i\mathbb{R}, E)}$, there exists a sequence $f_n \in L^2(i\mathbb{R}, E)$ such that $\Delta f_n \to \frac{c_2(i\omega)}{1-i\omega}$ as n tends to infinity. However, then $g_n(i\omega) := f_n(i\omega)\frac{1-i\omega}{i\omega-\bar{z}} \in L^2(i\mathbb{R}, E)$ and $\Delta g_n \to \phi_2$, which shows that $\phi_2 \in \overline{\Delta L^2(i\mathbb{R}, E)}$.

Thus we have proved that $\phi_z \in X$. It now remains to show that

$$\langle \phi_z, \theta f \oplus \Delta f\rangle = 0, \qquad f \in H^2(\mathbb{C}_+, E),$$

which follows from the calculation

$$\begin{aligned}
&\langle \phi_z, \theta f \oplus \Delta f\rangle \\
&= \int_{-\infty}^{\infty} \left\langle \frac{c_1(\bar{z}) - c_1(i\omega)}{\bar{z} - i\omega}, \theta(i\omega)f(i\omega) \right\rangle - \left\langle \frac{c_2(i\omega)}{\bar{z} - i\omega}, \Delta(i\omega)f(i\omega) \right\rangle d\omega \\
&= \int_{-\infty}^{\infty} \left\langle \frac{c_1(\bar{z})}{1 - i\omega}, \theta(i\omega)f(i\omega)\frac{1 + i\omega}{z + i\omega} \right\rangle d\omega \\
&\quad - \int_{-\infty}^{\infty} \left\langle \frac{c_1(i\omega)}{1 - i\omega} \oplus \frac{c_2(i\omega)}{1 - i\omega}, (\theta(i\omega) \oplus \Delta(i\omega))f(i\omega)\frac{1 + i\omega}{z + i\omega} \right\rangle d\omega \\
&= 0,
\end{aligned}$$

since $\frac{c_1(\bar{z})}{1-i\omega} \in H^2(\mathbb{C}_-, F)$, $\frac{c(i\omega)}{1-i\omega} \in U$ and $f(i\omega)\frac{1+i\omega}{z+i\omega} \in H^2(\mathbb{C}_+, F)$. ■

Lemma 2.5 *We have* $\|C(zI-A)^{-1}\| = \|\phi_z\|_X$ *for every* $z \in \mathbb{C}_+$.

The proof of this lemma is a modification the proof of Lemma 2.3 and Proposition 2.4 in Partington and Weiss [10].

Proof: For $u_1 \oplus u_2 \in U$, we have that

$$\begin{aligned} C(zI-A)^{-1}(u_1 \oplus u_2) &= C\left[\frac{u_1(\cdot) \oplus u_2(\cdot)}{z+\cdot}\right] \\ &= \int_{-\infty}^{\infty} \left\langle u_1(i\omega) \oplus u_2(i\omega), \frac{c_1(i\omega) \oplus c_2(i\omega)}{\bar{z}-i\omega} \right\rangle d\omega. \end{aligned}$$

Using the fact that the function $s \to 1/(\bar{z}-s)$ lies in $H^2(\mathbb{C}_-)$, we get

$$\int_{-\infty}^{\infty} \left\langle u_1(i\omega), \frac{c_1(\bar{z})}{\bar{z}-i\omega} \right\rangle d\omega = 0.$$

Thus we have

$$C(zI-A)^{-1}(u_1 \oplus u_2) = -\int_{-\infty}^{\infty} \langle u_1(i\omega) \oplus u_2(i\omega), \phi_z(i\omega)\rangle = \langle u_1 \oplus u_2, \phi_z\rangle.$$

Using Lemma 2.4, this proves the lemma. ■

We now need to inroduce functions of *bounded mean oscillation (BMO)*; more details can be found for example in Meyer [8] and Blasco [1].

Definition 2.6 *Let Z be a Hilbert space. The space $BMO(i\mathbb{R}, Z)$ is the space of all locally integrable functions f on $i\mathbb{R}$ with values in Z such that*

$$\sup_I \frac{1}{|I|} \int_I \|f(i\omega) - f_I\|\, d\omega < \infty,$$

where the supremum is taken over all intervals I of finite measure $|I| > 0$, and

$$f_I := \frac{1}{|I|} \int_I f(i\omega)\, d\omega.$$

Definition 2.7 *The space $BMOA(\mathbb{C}_+, Z)$ consists of all analytic functions $f : \mathbb{C}_+ \to Z$ which satisfy the condition that $f(s)/(1+s)$ lies in $H^2(\mathbb{C}_+, Z)$ and that the boundary function lies in $BMO(i\mathbb{R}, Z)$.*

Following Koosis [7, page 294] we get the following result. Let f be a locally integrable Z-valued function on $i\mathbb{R}$ and I an interval of finite nonzero measure $|I|$. Then

$$\sup_I \frac{1}{|I|} \int_I \|f(i\omega) - f_I\|\, d\omega \leq \sup_I \frac{2}{|I|} \int_I \|f(i\omega) - K\|\, d\omega \qquad (3)$$

for every $K \in Z$.
In order to analyse the function c_2, we shall need the following result, which is of interest in its own right. It says that a Laurent (multiplication) operator on $L^2(i\mathbb{R})$ is bounded if and only if it is uniformly bounded on the set of normalized rational functions of degree 1 (Cauchy kernels) in $H^2(\mathbb{C}_+)$. The same result follows for Toeplitz operators on $H^2(\mathbb{C}_+)$ with analytic symbol. The fact that a similar result holds for Hankel operators is Bonsall's theorem [2, 3] restated for the half-plane, as was noted in [10].

Lemma 2.8 *Let $f : i\mathbb{R} \to E$ be measurable and assume that there exists a constant $K > 0$ such that*

$$\left\| \frac{f(i\omega)}{i\omega - z} \right\|_{L^2(i\mathbb{R},E)} \leq \frac{K}{\sqrt{\operatorname{Re} z}}, \qquad z \in \mathbb{C}_+. \tag{4}$$

Then $f \in L^\infty(i\mathbb{R}, E)$.

Proof: We define $g : i\mathbb{R} \to \mathbb{R}$ by $g(i\omega) := \|f(i\omega)\|^2$. Of course, g is measurable and equation (4) reads

$$\int_{-\infty}^{\infty} g(i\omega) \frac{x}{x^2 + (\omega - y)^2}\, d\omega \leq K^2, \qquad x > 0, y \in \mathbb{R}.$$

Defining

$$\tilde{g}(x + iy) := \int_{-\infty}^{\infty} g(i\omega) \frac{x}{x^2 + (\omega - y)^2}\, d\omega, \qquad x > 0, y \in \mathbb{R},$$

we get that $\tilde{g}$ is a bounded function on $\mathbb{C}_+$. Now Fatou's theorem (see for example Hoffman [6, page 123]) implies that $\tilde{g}$ is harmonic and that $\tilde{g}$ has non-tangential limits which exist and agree with g almost everywhere on $i\mathbb{R}$. Thus g is in $L^\infty(i\mathbb{R}, \mathbb{R})$ and the lemma is proved. ■

Proposition 2.9 *If there exists a constant $M > 0$ such that*

$$\|C(sI - A)^{-1}\| \leq \frac{M}{\sqrt{\operatorname{Re} s}}, \qquad s \in \mathbb{C}_+,$$

then c is in $BMOA(F) \oplus L^\infty(i\mathbb{R}, E)$, i.e., $c_1 \in BMOA(F)$ and $c_2 \in L^\infty(i\mathbb{R}, E)$.

Proof: Using Lemma 2.5, we get that

$$\|\phi_{1,z}\|, \|\phi_{2,z}\| \leq \frac{M}{\sqrt{\operatorname{Re} z}}, \qquad z \in \mathbb{C}_+,$$

where $\phi_1(i\omega) := \frac{c_1(\bar{z}) - c_1(i\omega)}{\bar{z} - i\omega}$ and $\phi_2(i\omega) := -\frac{c_2(i\omega)}{\bar{z} - i\omega}$. Thus Lemma 2.8 shows $c_2 \in L^\infty(i\mathbb{R}, E)$.
For any $h > 0$ and $\sigma \in \mathbb{R}$ we define

$$E_{h,\sigma} := \frac{1}{2h} \int_{\sigma - h}^{\sigma + h} \|c_1(i\omega) - c_1(h + i\sigma)\|\, d\omega.$$

In order to show that c_1 is in $BMO(i\mathbb{R}, F)$, it is enough to show that $E_{h,\sigma}$ is bounded independently of h and σ. Here we have chosen $K = c_1(h + i\sigma)$ in (3).

Thus for any $h > 0$ and $\sigma \in \mathbb{R}$ we get

$$\begin{aligned} E_{h,\sigma} &= \int_{\sigma-h}^{\sigma+h} \left\| \frac{c_1(i\omega) - c_1(h+i\sigma)}{i\omega - (h+i\sigma)} \right\| \left| \frac{i\omega - (h+i\sigma)}{2h} \right| d\omega \\ &\leq \|\phi_{1,h+i\sigma}\| \left(\int_{-h}^{h} \frac{h^2+\mu^2}{4h^2}\, d\mu \right)^{1/2} \leq \frac{M}{\sqrt{h}} \sqrt{\frac{2h}{3}} = M\sqrt{\frac{2}{3}}. \end{aligned}$$

Thus c_1 is in $BMO(i\mathbb{R}, F)$. Since $c_1(s)/(1+s)$ is in $H^2(\mathbb{C}_+, F)$, we have $c_1 \in BMOA(\mathbb{C}_+, F)$, which completes the proof. ∎

Theorem 2.10 *If there exists a constant $M > 0$ such that*

$$\|C(sI - A)^{-1}\| \leq \frac{M}{\sqrt{\operatorname{Re} s}}, \qquad s \in \mathbb{C}_+,$$

then C is infinite-time admissible.

Proof: The previous proposition shows $c_1 \in BMOA(\mathbb{C}_+, F)$ and $c_2 \in L^\infty(i\mathbb{R}, E)$. In order to prove that C is infinite-time admissible, by Proposition 2.3, it is enough to show that Γ_c, as given in Proposition 2.3, is a bounded operator from U to $H^2(\mathbb{C}_-)$. Fefferman's theorem (see Meyer [8, Chapter 5] and Blasco [1]) states that

$$BMOA(\mathbb{C}_+, F) = (H^1(\mathbb{C}_+, F))^*.$$

Let $u \in U$. Writing $u = u_1 \oplus u_2$, where $u_1 : i\mathbb{R} \to F$ and $u_2 : i\mathbb{R} \to E$, we get

$$\begin{aligned} &\|\Gamma_c u\| \\ &= \sup_{\substack{g\in H^2(\mathbb{C}_-)\\ \|g\|=1}} |\langle g, \Gamma_c u\rangle| \\ &= \sup_{\substack{g\in H^2(\mathbb{C}_-)\\ \|g\|=1}} \left| \int_{-\infty}^{\infty} g(i\omega) \overline{\langle u(i\omega), c(i\omega)\rangle}\, d\omega \right| \\ &= \sup_{\substack{g\in H^2(\mathbb{C}_-)\\ \|g\|=1}} \left| \int_{-\infty}^{\infty} \langle \overline{g(i\omega)} u(i\omega), c(i\omega)\rangle\, d\omega \right| \\ &\leq \sup_{\substack{g\in H^2(\mathbb{C}_-)\\ \|g\|=1}} \left(\left| \int_{-\infty}^{\infty} \langle \overline{g(i\omega)} u_1(i\omega), c_1(i\omega)\rangle\, d\omega \right| + \left| \int_{-\infty}^{\infty} \langle \overline{g(i\omega)} u_2(i\omega), c_2(i\omega)\rangle\, d\omega \right| \right) \\ &\leq k_1 \sup_{\substack{g\in H^2(\mathbb{C}_-)\\ \|g\|=1}} \left(\|\bar{g} u_1\|_{H^1(\mathbb{C}_+,F)} \|c_1\|_{BMOA(\mathbb{C}_+,F)} + \|\bar{g} u_2\|_{L^1(i\mathbb{R},E)} \|c_2\|_{L^\infty(i\mathbb{R},E)} \right) \\ &\leq k_1 (\|u_1\|_{H^2(\mathbb{C}_+,F)} \|c_1\|_{BMOA(\mathbb{C}_+,F)} + \|u_2\|_{L^2(i\mathbb{R},E)} \|c_2\|_{L^\infty(i\mathbb{R},E)}) \\ &\leq k_2 \|u\|_U \end{aligned}$$

for some constants k_1, $k_2 > 0$ independent of u. ∎

3 Proof of the main results

In this section, we prove our main results, Theorem 1.3 and Corollary 1.4. Moreover, in Example 3.4 we show that the Carleson measure theorem is an easy corollary of Theorem 1.3.

Definition 3.1 *Let $T(t)$ be a C_0-semigroup of contractions on H. We say a subspace U of H is* unitary *if and only if $T(t)$ maps U isometrically onto U for every $t \geq 0$. Moreover, $T(t)$ is called* completely non-unitary *if the only unitary subspace of H is $\{0\}$.*

Proof of Theorem 1.3 Since $T(t)$ is a contraction semigroup on H, there is a unique orthogonal decomposition $H = H_1 \oplus H_2$, where H_1 and H_2 are $T(t)$-invariant, and $T(t)$ is unitary on H_1 and completely non-unitary on H_2. This result can be found in Davies [4, Theorem 6.6 on page 154].
Let $T_1(t)$ be the restriction of $T(t)$ from H to H_1 with generator A_1, and let $T_2(t)$ be the restriction of $T(t)$ from H to H_2 with generator A_2. Furthermore, let C_1 be the restriction of C from $D(A)$ to $D(A_1)$, and let C_2 be the restriction of C from $D(A)$ to $D(A_2)$. It is easy to see that $C_1 \in \mathcal{L}(D(A_1), \mathbb{C})$, and that $C_2 \in \mathcal{L}(D(A_2), \mathbb{C})$. Moreover, we get

$$\|C_1(sI - A_1)^{-1}\| \leq \frac{M}{\sqrt{\operatorname{Re} s}}, \qquad s \in \mathbb{C}_+,$$

and

$$\|C_2(sI - A_2)^{-1}\| \leq \frac{M}{\sqrt{\operatorname{Re} s}}, \qquad s \in \mathbb{C}_+.$$

Now $T_1(t)$ is a unitary C_0-semigroup (and thus normal), and the results of Weiss [14, 15] prove that C_1 is an infinite-time admissible observation operator for $T_1(t)$.
Moreover, $T_2(t)$ is a completely non-unitary C_0-semigroup of contractions, so by the model theorem (which can be found in Sz. Nagy and Foiaş [12, Chapter 6, Page 280]), there exists a C_0-semigroup $\tilde{T}(t)$ with generator $\tilde{A}$ on U, given by

$$\begin{aligned}
\tilde{T}(t)u &:= P_U[e^{-i\omega t}u(i\omega)], \qquad x \in U, \\
X &:= H^2(\mathbb{C}_+, F) \oplus \overline{\Delta L^2(i\mathbb{R}, E)}, \\
U &:= X \ominus \{\theta f \oplus \Delta f \mid f \in H^2(\mathbb{C}_+, E)\}, \\
\tilde{A}u &:= P_U[-i\omega u(i\omega)], \qquad u \in D(\tilde{A}), \\
D(\tilde{A}) &:= \{u \in U \mid i\omega u(i\omega) \in X\},
\end{aligned}$$

where E and F are closed subspaces of H, $\theta : \mathbb{C}_+ \to \mathcal{L}(E, F)$ is a holomorphic function satisfying $\|\theta(s)\| \leq 1$ on $\mathbb{C}_+$, $\Delta : i\mathbb{R} \to \mathcal{L}(E)$ is defined by

$$\Delta(i\omega) := [I_E - \theta(i\omega)^*\theta(i\omega)]^{1/2}, \qquad \omega \in \mathbb{R},$$

and P_U denotes the orthogonal projection from X onto U, such that $T_2(t)$ is unitary equivalent to $\tilde{T}(t)$. Now we can apply the results of the previous section to the semigroup $\tilde{T}(t)$, and thus to $T_2(t)$. Theorem 2.10 shows that C_2 is an infinite-time admissible observation operator for $T_2(t)$. Thus C is an infinite-time admissible observation operator, which completes the proof. ■

Proof of Corollary 1.4 Define $\gamma := \max\{\alpha, \sigma\}$. Defining $\tilde{T}(t) := e^{-\gamma t}T(t)$, we get that $\tilde{T}(t)$ is a contraction C_0-semigroup on H with infinitesimal generator $\tilde{A} := A - \gamma I$. Clearly, $C \in \mathcal{L}(D(\tilde{A}), \mathbb{C})$, and an easy calculation shows

$$\|C(sI - \tilde{A})^{-1}\| \leq \frac{M}{\sqrt{\operatorname{Re} s}}, \qquad s \in \mathbb{C}_+.$$

Now Theorem 1.3 implies that C is infinite-time admissible for $\tilde{T}(t)$, which implies that C is admissible for $\tilde{T}(t)$. Finally, this implies that C is admissible for $T(t)$. ■

In the following two examples, we show which choices for θ, E and F we need to make in order to obtain a diagonal completely non-unitary contraction semigroup or the right-shift semigroup.

Example 3.2 One special case of Theorem 1.3 is the case when $H = \ell^2$ and $T(t)$ is the diagonal contraction semigroup with entries $(e^{-\lambda_n t})_{n=1}^\infty$ where $\operatorname{Re} \lambda_n > 0$ for $n = 1, 2, \ldots$. To see how we may obtain this contraction semigroup, we take $E = F = \ell^2$ and

$$\theta(s) = \operatorname{diag}\left(\frac{s - \lambda_n}{s + \overline{\lambda_n}}\right) \in \mathcal{L}(E, F).$$

and $\Delta = 0$. It is easily checked that

$$\theta H^2(\mathbb{C}_+, E) = \{h = (h_n) \in H^2(\mathbb{C}_+, F) : h_n(\lambda_n) = 0 \quad \text{for } n = 1, 2, \ldots\}.$$

Then U consists of those $H^2(\mathbb{C}_+, F)$ functions whose nth component has the form $u_n/(s+\overline{\lambda_n})$, for all $n \geq 1$, with $u_n \in \mathbb{C}$. It is now straightforward to verify that

$$\tilde{T}(t)(u_n/(s + \overline{\lambda_n}))_{n=1}^\infty = (e^{-\lambda_n t}u_n/(s + \overline{\lambda_n}))_{n=1}^\infty,$$

and thus $\tilde{T}(t)$ is unitarily equivalent to the diagonal semigroup $T(t)$, as asserted.

Example 3.3 Another important special case arises when we take $E = 0$, $F = \mathbb{C}$, and so $\theta = 0$, $\Delta = 0$, $U = H^2(\mathbb{C}_+)$ and $T(t)u(s) = e^{-st}u(s)$ for $u \in U$. Here we obtain the right-shift semigroup, which was analysed in [10].

We conclude this section with the following example, which shows that the Carleson measure theorem is an easy corollary of Theorem 1.3.

Example 3.4 Let us now consider the semigroup $T(t)$ defined by multiplication by the function $\lambda \mapsto e^{-\lambda t}$ on $U = L^2(\mathbb{C}_+, \mu)$ for some measure μ on $\mathbb{C}_+$. For this semigroup the integration functional $Cf = \int_{\mathbb{C}_+} f(\lambda)d\mu(\lambda)$ satisfies Condition (1) of Theorem 1.3 if and only if there is a constant $M > 0$ such that

$$\int_{\mathbb{C}_+} \frac{d\mu(\lambda)}{|s + \lambda|^2} \leq \frac{M}{\operatorname{Re} s} \quad \text{for all } s \in \mathbb{C}_+. \tag{5}$$

The functional C will be infinite-time admissible if and only if there is a constant $K > 0$ such that

$$\left| \int_0^\infty \int_{\mathbb{C}_+} v(t) e^{-\lambda t} u(\lambda) d\mu(\lambda)\, dt \right| \leq K \|v\|_{L^2(0,\infty)} \|u\|_U$$

for all $v \in L^2(0,\infty)$ and $u \in U$. This is equivalent to the condition that

$$\left| \int_{\mathbb{C}_+} \hat{v}(\lambda) u(\lambda)\, d\mu(\lambda) \right| \leq \frac{K}{\sqrt{2\pi}} \|\hat{v}\|_{H^2} \|u\|_U$$

or

$$\|\hat{v}\|_{L^2(\mathbb{C}_+,\mu)} \leq \frac{K}{\sqrt{2\pi}} \|\hat{v}\|_{H^2} \quad \text{for all } \hat{v} \in H^2(\mathbb{C}_+). \tag{6}$$

The fact that (5) implies (6) is one of the standard forms of the Carleson measure theorem, and its deduction from Theorem 1.3 is based on reversing certain arguments in [15].

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