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Marshall, S.A. (1973) *On the Optimal Regulator Problem*. Research Report. ACSE Report 12 . Department of Control Engineering, University of Sheffield, Mappin Street, Sheffield

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On the optimal regulator problem

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Research report No. 12

April 1973

On the optimal regulator problem

Consider the linear multivariable system

$$\dot{x} = Ax + Bu \quad (1)$$

$$y = Cx \quad (2)$$

where x is the n -state vector, u is the r -input vector and y is the m -output vector, and A , B and C are $n \times n$, $n \times r$ and $m \times n$ constant coefficient matrices respectively, where it is required to choose the input $u(t)$, $0 \leq t < \infty$, such that,

$$J = \int_0^{\infty} \{ (y-r)' Q (y-r) + u' R u \} dt \quad (3)$$

is a minimum. r is the m -reference vector and Q and R are $m \times m$ and $r \times r$ weighting matrices respectively.

Initially augment the system to include the reference vector by defining the new $(n+m)$ -state vector z where

$$z = \begin{bmatrix} x \\ r \end{bmatrix} \quad (4)$$

and, hence,

$$\dot{z} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} z + \begin{bmatrix} B \\ 0 \end{bmatrix} u \quad (5)$$

$$y = \begin{bmatrix} C & 0 \end{bmatrix} z, \quad (6)$$

and,

$$J = \int_0^{\infty} \{ [(C \ 0)z - (0 \ I)z]' Q [(C \ 0)z - (0 \ I)z] + u' R u \} dt \quad (7)$$

$$\text{i.e. } J = \int_0^{\infty} \left\{ z' \begin{bmatrix} C'QC & -C'Q \\ -QC & Q \end{bmatrix} z + u' R u \right\} dt \quad (8)$$

Now substitute the relevant matrices into the matrix Riccati equation to give

$$\begin{aligned} - \begin{bmatrix} \dot{K}_1 & \dot{K}_2 \\ \dot{K}'_2 & \dot{K}_3 \end{bmatrix} &= \begin{bmatrix} C'QC & -C'Q \\ -QC & Q \end{bmatrix} + \begin{bmatrix} K_1 & K_2 \\ K'_2 & K_3 \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} A' & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} K_1 & K_2 \\ K'_2 & K_3 \end{bmatrix} \\ &\quad - \begin{bmatrix} K_1 & K_2 \\ K'_2 & K_3 \end{bmatrix} \begin{bmatrix} B \\ 0 \end{bmatrix} R^{-1} \begin{bmatrix} B' & 0 \end{bmatrix} \begin{bmatrix} K_1 & K_2 \\ K'_2 & K_3 \end{bmatrix} \end{aligned} \quad (9)$$

where K_1 and K_3 are $n \times n$ and $m \times m$ symmetrical matrices respectively and K_2 is an $n \times m$ matrix.

Equation (9) then yields the expressions,

$$-\dot{K}_1 = C'QC + K_1A + A'K_1 - K_1BR^{-1}B'K_1 \quad (10)$$

$$-\dot{K}_2 = -C'Q + A'K_2 - K_1BR^{-1}B'K_2 \quad (11)$$

$$-\dot{K}_3 = Q - K_2'BR^{-1}B'K_2 \quad (12)$$

It is known that the optimal control law is given by

$$u = -R^{-1}[B' \ 0] \begin{bmatrix} K_{10} & K_{20} \\ K_{20}' & K_{30} \end{bmatrix} z \quad (13)$$

$$\text{i.e. } u = -R^{-1}B' [K_{10} \ K_{20}] z \quad (14)$$

where K_{10} and K_{20} are the steady-state solutions of eqns (10) and (11) respectively. Equation (12) is always satisfied regardless of the value of K_2 and can be safely excluded from the set of equations.

Thus to solve the optimal regulator problem all that is required is to solve the algebraic $n \times n$ matrix Riccati equation

$$C'QC + K_{10}A + A'K_{10} - K_{10}BR^{-1}B'K_{10} = 0 \quad (15)$$

for K_{10} , using either the direct method or the eigenvector method or the transition matrix method, and then solve the matrix equation,

$$-C'Q + A'K_{20} - K_{10}BR^{-1}B'K_{20} = 0 \quad (16)$$

$$\text{i.e. } K_{20} = (A' - K_{10}BR^{-1}B')^{-1}C'Q \quad (17)$$

The optimal control law is then given by eqn (14).