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# On the optimal regulator problem

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## On the optimal regulator problem

Consider the linear multivariable system

$$\dot{x} = Ax + Bu \tag{1}$$

$$y = Cx \tag{2}$$

where x is the n-state vector, u is the r-input vector and y is the m-output vector, and A, B and C are nxn, nxr and mxn constant coefficient matrices respectively, where it is required to choose the input u(t), $0 \leqslant t \leqslant \infty$, such that,

$$J = \int_0^\infty \{(y-r)' Q(y-r) + u'Ru\}dt \tag{3}$$

is a minimum. r is the m-reference vector and Q and R are mxm and rxr weighting matrices respectively.

Initially augment the system to include the reference vector by defining the new (n+m)-state vector z where

$$z = \begin{bmatrix} x \\ r \end{bmatrix} \tag{4}$$

and, hence,

$$\dot{z} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} z + \begin{bmatrix} B \\ 0 \end{bmatrix} u \tag{5}$$

$$y = \begin{bmatrix} C & 0 \end{bmatrix} z , \tag{6}$$

and,

$$J = \int_0^\infty \{[(C \;\; 0)z - (0 \;\; I)z]' Q[(C \;\; 0)z - (0 \;\; I)z] + u'Ru\}dt \tag{7}$$

$$\text{i.e. } J = \int_0^\infty \left\{ z' \begin{bmatrix} C'QC & -C'Q \\ -QC & Q \end{bmatrix} z + u'Ru \right\} dt \tag{8}$$

Now substitute the relevant matrices into the matrix Riccati equation to give

$$-\begin{bmatrix} \dot{K}_1 & \dot{K}_2 \\ \dot{K}'_2 & \dot{K}_3 \end{bmatrix} = \begin{bmatrix} C'QC & -C'Q \\ -QC & Q \end{bmatrix} + \begin{bmatrix} K_1 & K_2 \\ K'_2 & K_3 \end{bmatrix}\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} A' & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} K_1 & K_2 \\ K'_2 & K_3 \end{bmatrix}$$

$$- \begin{bmatrix} K_1 & K_2 \\ K'_2 & K_3 \end{bmatrix}\begin{bmatrix} B \\ 0 \end{bmatrix} R^{-1} \begin{bmatrix} B' & 0 \end{bmatrix} \begin{bmatrix} K_1 & K_2 \\ K'_2 & K_3 \end{bmatrix} \tag{9}$$

where $K_1$ and $K_3$ are nxn and mxm symmetrical matrices respectively and $K_2$ is an nxm matrix.

Equation (9) then yields the expressions,

$$-\dot{K}_1 = C'QC + K_1A + A'K_1 - K_1BR^{-1}B'K_1 \quad (10)$$

$$-\dot{K}_2 = -C'Q + A'K_2 - K_1BR^{-1}B'K_2 \quad (11)$$

$$-\dot{K}_3 = Q - K'_2BR^{-1}B'K_2 \quad (12)$$

It is known that the optimal control law is given by

$$u = -R^{-1}\begin{bmatrix} B' & 0 \end{bmatrix}\begin{bmatrix} K_{10} & K_{20} \\ K'_{20} & K_{30} \end{bmatrix} z \quad (13)$$

$$\text{i.e. } u = -R^{-1}B'\begin{bmatrix} K_{10} & K_{20} \end{bmatrix} z \quad (14)$$

where $K_{10}$ and $K_{20}$ are the steady-state solutions of eqns (10) and (11) respectively. Equation (12) is always satisfied regardless of the value of $K_2$ and can be safely excluded from the set of equations.

Thus to solve the optimal regulator problem all that is required is to solve the <u>algebraic nxn matrix Riccati equation</u>

$$C'QC + K_{10}A + A'K_{10} - K_{10}BR^{-1}B'K_{10} = 0 \quad (15)$$

for $K_{10}$ using either the direct method or the eigenvector method or the transition matrix method, and then solve the <u>matrix equation</u>,

$$-C'Q + A'K_{20} - K_{10}BR^{-1}B'K_{20} = 0 \quad (16)$$

$$\text{i.e. } K_{20} = (A' - K_{10}BR^{-1}B')^{-1}C'Q \quad (17)$$

The optimal control law is then given by eqn (14).