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On the optimal regulator problem

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On the optimal regulator problem

Consider the linear multivariable system

\[ \dot{x} = Ax + Bu \]  \hspace{1cm} (1) \\
\[ y = Cx \]  \hspace{1cm} (2)

where \( x \) is the n-state vector, \( u \) is the r-input vector and \( y \) is the m-output vector, and \( A, B \) and \( C \) are mxn, nxr and mnx constant coefficient matrices respectively, where it is required to choose the input \( u(t) \), \( 0 \leq t \leq \infty \), such that,

\[ J = \int_0^\infty \{(y-r)^TQ(y-r) + u^TRu\} \, dt \]  \hspace{1cm} (3)

is a minimum. \( r \) is the m-reference vector and \( Q \) and \( R \) are mnx and nxn weighting matrices respectively.

Initially augment the system to include the reference vector by defining the new \((n+m)\)-state vector \( z \) where

\[ z = \begin{bmatrix} x \\ r \end{bmatrix} \]  \hspace{1cm} (4)

and, hence,

\[ \dot{z} = \begin{bmatrix} A & C \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ r \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u \]  \hspace{1cm} (5)

\[ y = \begin{bmatrix} C & 0 \end{bmatrix} z , \]  \hspace{1cm} (6)

and,

\[ J = \int_0^{\infty} \left[ \begin{bmatrix} (C & 0)z - (0 & I)z' \end{bmatrix} Q \left[ (C & 0)z - (0 & I)z \right] + u^TRu \right] \, dt \]  \hspace{1cm} (7)

i.e. \[ J = \int_0^{\infty} \left\{ z' \begin{bmatrix} C'C & -C'Q \\ -QC & Q \end{bmatrix} z + u^TRu \right\} \, dt \]  \hspace{1cm} (8)

Now substitute the relevant matrices into the matrix Riccati equation to give

\[ \begin{bmatrix} \dot{K}_1 & \dot{K}_2 \\ \dot{K}_2 & \dot{K}_3 \end{bmatrix} = \begin{bmatrix} C'C & -C'Q \\ -QC & Q \end{bmatrix} + \begin{bmatrix} K_1 & K_2 \\ R_2 & K_3 \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} A' & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} K_1 & K_2 \\ R_1 & K_3 \end{bmatrix} \]

\[ -\begin{bmatrix} K_1 & K_2 \\ K_2' & K_3' \end{bmatrix} \begin{bmatrix} B' \\ R^{-1}B' \end{bmatrix} \begin{bmatrix} \dot{K}_1 \\ \dot{K}_2 \end{bmatrix} \]  \hspace{1cm} (9)
where $K_1$ and $K_2$ are $n \times n$ and $m \times m$ symmetrical matrices respectively and $K_3$ is an $n \times n$ matrix.

Equation (9) then yields the expressions,

\begin{align*}
\dot{K}_1 &= C'QC + K_1 A + A'K_1 - K_1 B R^{-1} B'K_1 \\
\dot{K}_2 &= -C'Q + A'K_2 - K_1 B R^{-1} B'K_2 \\
\dot{K}_3 &= Q - K_2 B R^{-1} B'K_2
\end{align*}

(10)  (11)  (12)

It is known that the optimal control law is given by

\begin{equation}
\begin{bmatrix}
\dot{K}_{10} \\
\dot{K}_{20}
\end{bmatrix} = -R^{-1} \begin{bmatrix}
B' & 0 \\
K_1 & 0
\end{bmatrix} \begin{bmatrix}
K_{10} \\
K_{20}
\end{bmatrix} z
\end{equation}

(13)

\begin{equation}
i.e. \quad u = -R^{-1}B' \begin{bmatrix}
K_{10} \\
K_{20}
\end{bmatrix} z
\end{equation}

(14)

where $K_{10}$ and $K_{20}$ are the steady-state solutions of eqns (10) and (11) respectively. Equation (12) is always satisfied regardless of the value of $K_2$ and can be safely excluded from the set of equations.

Thus to solve the optimal regulator problem all that is required is to solve the algebraic $n \times n$ matrix Riccati equation

\begin{equation}
C'QC + K_{10} A + A'K_{10} - K_{10} B R^{-1} B'K_{10} = 0
\end{equation}

(15)

for $K_{10}$ using either the direct method or the eigenvector method or the transition matrix method, and then solve the matrix equation,

\begin{equation}
-C'Q + A'K_{20} - K_{10} B R^{-1} B'K_{20} = 0
\end{equation}

(16)

\begin{equation}
i.e. \quad K_{20} = (A' - K_{10} B R^{-1} B')^{-1} C'Q
\end{equation}

(17)

The optimal control law is then given by eqn (14).