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BY

D.H. Owens

Department of Control Engineering,
University of Sheffield,
Mappin Street, Sheffield S1 3JD.

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by

D.H. Owens

Department of Control Engineering, University of Sheffield, UK

ABSTRACT

Although the root-loci of linear multivariable systems have many of the characteristics of there classical counterparts, there are a number of essentially multivariable possibilities that require careful identification. It is shown here that classical characteristics are generic in a carefully defined and very important sense and that non-generic behaviour only occurs in bad design conditions.

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INTRODUCTION

Recent years have seen the emergence of a considerable increase in understanding of the nature of the root-locus of a linear multivariable system $S(A,B,C)$ subjected to unity negative feedback with scalar gain $p \geq 0$ (1) - (4). A number of computational procedures based on algebraic (1),(5),(6) and state space (7) - (9) methods arenow available and a practical approach to their use in compensator design (1), (10) has emerged. Although it is now clear that multivariable root-loci possess many of the general properties of their classical counterparts, there are a number of essentially multivariable possibilities that require careful identification if root-loci are to be a viable design tool. It is the purpose of this paper to examine the notion that the structure of the unbounded/asymptotic branches of a multivariable root-locus is generic in a carefully defined sense, to identify what relevance this concept of genericity has to practice and to point out physical conditions corresponding to the non-generic case. Many of the results can be found in embryonic form throughout the literature and hence the major contribution of the paper is that of unification.

THE BASE STRUCTURE

The pseudo-classical base structure taken for the unbounded closed-loop poles of an m-input/m-output, invertible system $S(A,B,C)$ subjected to unity negative feedback is taken to be either

$$s_{j\ell}(p) = p^{1/\nu_j}\eta_{j\ell} + \mu_{j\ell}(p)$$

$$\lim_{p\to\infty} p^{-1/\nu_j}\mu_{j\ell}(p)=0 \ , \ 1 \leq \ell \leq \nu_j \ , \ 1 \leq j \leq m \qquad (1)$$

or, imposing a little more structure on $\mu_{j\ell}$,

$$s_{j\ell}(p)=p^{1/\nu_j}\eta_{j\ell} + \alpha_j + \varepsilon_{j\ell}(p)$$

$$\lim_{p\to\infty} \varepsilon_{j\ell}(p) = 0 \ , \ 1 \leq \ell \leq \nu_j \ , \ 1 \leq j \leq m \qquad (1a)$$

(Clearly the validity of the base structure (1a) implies the validity of (1)) where the real positive numbers $\nu_j \geq 1$ are integer and the $\eta_{j\ell}$, $1 \leq \ell \leq \nu_j$, are the distinct ν_jth roots of a non-zero, complex number $-\lambda_j$. In (now) standard jargon, $s_{j\ell}$ is said to be an <u>infinite zero</u> of <u>order</u> ν_j with <u>asymptotic directions</u> $\eta_{j\ell}$ and <u>pivot</u> α_j. The obvious questions to ask about such a base characterization are

(a) Is the base characterization (1) always valid for suitable choice of parameters? If not, then in what sense is it sometimes valid?

(b) Can the integers $\nu_1,\nu_2,\ldots,\nu_m$ be naturally identified with algebraic or geometric system structural invariants?

(c) What can be said about situations when (1) is not applicable?

(d) Do the parameters $\eta_{j\ell}$, α_j have any continuity/sensitivity characteristics?

(e) Is the characterization (1) invariant under state feedback and output injection?

To answer some of these questions, we introduce the following notion of 'genericity'.

<u>Definition</u>: A property $P(A,B,C)$ of the invertible, square system $S(A,B,C)$ is C^* - generic if the set

$$P^* \triangleq \{(F,K,N,M) \ : \ P(A+BF+KC,BM,NC), |N| \neq 0, |M| \neq 0\} \qquad (2)$$

has interior dense in $L \triangleq L(R^n,R^m) \times L(R^m,R^n) \times L(R^m) \times L(R^m)$ (regarded as a metric space). In other terms P^* is the collection of state feedback, output injection and constant post- and pre- compensators that ensure the validity of P. The notation is suggested by the work of Morse (11) and the authors work (12) connecting root-loci to the C^*- transformation group. Our main results will be that both base characterizations are C^*-generic but there are a number of other (related) results of significance to practical applications. As many of the proofs rely heavily on techniques already in the literature, proofs will frequently only be outlined.

ORDERS AND ASYMPTOTIC DIRECTIONS

We can say immediately that the base characteization (1) does not always hold. Consider, for example, the system with transfer function matrix

$$Q(s) = \begin{pmatrix} 0 & \frac{1}{s} \\ \frac{1}{s^2} & 0 \end{pmatrix} \tag{3}$$

which has infinite zeros $s = p^{2/3} \exp(k2\pi i/3), k=1,2,3,$ of non-integer order $3/2$ whilst, interchanging the loops leads to

$$Q(s) = \begin{pmatrix} \frac{1}{s} & 0 \\ 0 & \frac{1}{s^2} \end{pmatrix} \tag{4}$$

which has infinite zeros of integer orders one or two. Clearly the orders of the infinite zeros are not invariant under constant forward path compensation and may not even be integer. It is possible however to prove the following results (12):

Lemma 1: There exists a set of ordered integers $1 \leqslant n_1 \leqslant n_2 \leqslant \ldots \leqslant n_m$ (namely the C^* integer structural invariants of $S(A,B,C)$ (11)) such that infinite zeros of $S(A,BK_o,K_1C)$ have orders $\nu_j = n_j$ for all nonsingular mxm matrices (K_1,K_o) belonging to an open dense subset K^* of $L(R^m)xL(R^m)$ satisfying $K^* \cap (\{I_m\} x\ L(R^m)) \neq \emptyset$.

Proof: The first part of the result can be found in (12) and is based upon the existence of (F,K,N,M) with $|N| \neq 0$ and $|M| \neq 0$ satisfying

$$N\ C\ (sI_n - A - BF - KC)^{-1} B\ M = \mathrm{diag}\{\frac{1}{s^{n_1}}, \ldots, \frac{1}{s^{n_m}}\} \tag{5}$$

More precisely, taking $K_o = K_1 = I_m$ and defining $\Gamma = NM$, the invariants $\{n_j\}$ are taken to have q distinct entries $m_1, m_2, \ldots, m_q$ satisfying $m_1 < m_2 < \ldots < m_q$ where m_j has multiplicity d_j in $\{n_j\}$. Defining

$$\Gamma = \begin{pmatrix} \Gamma_{11} & \cdots & \Gamma_{1q} \\ \vdots & & \vdots \\ \Gamma_{q1} & \cdots & \Gamma_{qq} \end{pmatrix} \tag{6}$$

where Γ_{ij} has dimension $d_i x d_j$ and

$$P_i = \begin{pmatrix} \Gamma_{ii} & \cdots & \Gamma_{iq} \\ \vdots & & \vdots \\ \Gamma_{qi} & \cdots & \Gamma_{qq} \end{pmatrix} \tag{7}$$

then (12) $S(A,B,C)$ has $m_i d_i$ infinite zeros of order m_i, $1 \leqslant j \leqslant q$, if $|P_i| \neq 0$, $1 \leqslant i \leqslant q$. Moreover, under these conditions, the proof given in (12) indicates that both the orders and asymptotic directions of the roo-locus depend only upon the structure and numerical magnitudes of the elements of Γ. Considering now the system $S(A,BK_o,K_1C)$, it is clear that the corresponding (N,M) pair is just $(NK_1^{-1},K_o^{-1}M)$ so that the introduction of the constant compensators K_o and K_1 induces the transformation $\Gamma \to \Gamma^1 = N(K_oK_1)^{-1}M$. Let P_i^1, $1 \leqslant i \leqslant q$, be the submatrices of Γ^1 obtained by the same procedure as equations (6) and (7). It is easily seen that $|P_i^1| \neq 0$, $1 \leqslant i \leqslant q$, for K_oK_1 in an open dense subset $\tilde{K}$ of $L(R^m)$. The result now follows as $\{I_m\} \times \tilde{K} \subset K^* \triangleq \{(K_1K_o) : K_oK_1 \in \tilde{K}\}$ which is open and dense in $L(R^m)xL(R^m)$.

Lemma 2: If $(K_1,K_o) \in K^*$, the orders and asymptotic directions of the infinite zeros of the root-locus of $S(A,BK_o,K_1C)$ are invariant under state-feedback and output injection transformation of the form

$$(A,BK_o,K_1C) \to (A+BK_oF+KK_1C, BK_o, K_1C) \tag{8}$$

Proof: see reference (12).

Given these results, it is now possible to prove the following theorem:

Theorem 1: Given the invertible, square system $S(A,B,C)$, the property $P_1(A,B,C)$ that the infinite zeros have the base structure (1) with $\nu_j = n_j$, $1 \leqslant j \leqslant m$, is C^*-generic.

Proof: From lemmas 1 and 2 it is clear that $L(R^n,R^m) xL(R^m,R^n)xK^* \subset P_1^* \subset L$ and hence that P_1^* has interior dense in L as K^* is open and dense in $L(R^m)xL(R^m)$. This completes the proof of the result.

Despite the generality of this result, it is natural to focus attention on only those aspects of the plant $S(A,B,C)$ that can be affected by constant forward path and state feedback compensation. In these terms, Theorem 1 gives rise to the corrollaries.

Corollary 1.1: It is not necessarily the case that the base characterization (1) holds for $S(A,B,C)$ but $(0,0,I_m,I_m)$ lies in the closure of P_1^*.

Proof: The example of equation (3) proves the first point whilst the second follows as Theorem 1 indicates that P_1^* is dense in L.

Corollary 1.2: There is an open, dense subset $K_2 \subset L(R^m)$ such that

$$L(R^n,R^m) x \{0\} x \{I_m\} x K_2 \subset P_1^* \tag{9}$$

Proof: Write $K^* \cap (\{I_m\} \times L(R^m)) = \{I_m\} \times K_2$ where K_2 is dense in $L(R^m)$ and, noting that lemma 2 holds with $K = 0$, an argument almost identical to that used in proving theorem 1 can be used.

The practical implications of these results are clear, namely that we cannot expect in all cases that our plant S(A,B,C) will have a root-locus with the required asymptotic base characterization (1) but we can always find a forward path constant precompensator K_o to ensure that the compensated system $S(A,BK_o,C)$ has a root-locus with the base characterization (1). In particular, as K_2 is dense in $L(R^m)$, we can choose K_o on a 'random basis' with probability one of success. In fact, it appears that, from the point of view of applications studies, it is only necessary to consider this characterization! There is, of course, no general rigorous justification of this point of view but it has been noted in (13) that the non-generic cases appears to occur only in cases where 'loop-interchange' phenomena dominate at high gain and that loop-interchange intuitively corresponds to that undesirable design condition where the loop phases are additive, hence reducing gain margins in the closed-loop system. On this basis it appears to be best to steer clear of controllers that do not guarantee base characterization (1)!

Finally in this section, the following result has been conjectured (13) to be true in all cases:

Theorem 2: In all cases, the square, invertible system S(A,B,C) can only have infinite zeros of orders equal to arithmetic means of subsets of $\{n_1,n_2,\ldots,n_m\}$.

THE PIVOTS

Turning our attention now to the base characterization (1a) including the pivots, the analysis tends to increase in complexity but it can be unravelled using the techniques of dynamic transformation introduced in (1) and (5). The basic lemma is as follows:

Definition: Let K^{**} be the subset of nonsingular pairs $(K_1,K_o) \in L(R^m)xL(R^m)$ such that there exists a nonsingular matrix T and unimodular matrices L(s) and M(s) of the form

$$L(s) = \begin{pmatrix} I_{d_1} & 0 & \cdots & \cdots & 0 \\ 0(s^{-1}) & I_{d_2} & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & & 0 \\ 0(s^{-1}) & \cdots & \cdots & 0(s^{-1}) & I_{d_q} \end{pmatrix} \quad (10)$$

$$M(s) = \begin{pmatrix} I_{d_1} & 0(s^{-1}) & \cdots & \cdots & 0(s^{-1}) \\ 0 & I_{d_2} & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & & 0(s^{-1}) \\ 0 & \cdots & \cdots & 0 & I_{d_q} \end{pmatrix} \quad (11)$$

such that, if $Q(s) = C(sI_n - A)^{-1}B$, we have

$$L(s)T^{-1}K_1Q(s)\ K_oT\ M(s)$$
$$= \text{block diag}\,\{Q_j(s)\}_{1\le j\le q} + 0(s^{-(m_q+2)}) \quad (12)$$

where Q_j is a $d_j x d_j$ transfer function matrix of uniform rank m_j.

(Note: a function is $0(s^{-k})$ if $\lim_{|s|\to\infty} s^{k-1}0(s^{-k}) = 0$ and a TFM G(s) has uniform rank k if $\lim_{|s|\to\infty} s^k G(s)$ is finite and nonsingular (1)).

Lemma 3: (i) $(K_1,K_o) \in K^{**}$ iff $(I_m,K_oK_1) \in K^{**}$

(ii) $(V^{-1}K_1,K_oV) \in K^{**}$ iff $(K_1,K_o) \in K^{**}$ if $|V| \neq 0$

Proof: replace T in (12) by K_1T. to prove (i). (ii) follows from (i).

Lemma 4: K^{**} has interior dense in $L(R^m)xL(R^m)$ and contains a subset of the form $\{I_m\} \times K_3$ where K_3 is open and dense in $L(R^m)$. Moreover $K^{**} \subset K^*$.

Proof: The proof is given in outline only. It is sufficient to show that K_3 exists for then lemma 3 indicates that K^{**} contains the open, dense subset $\{(K_1,K_o) : K_oK_1 \in K_3\}$. Using the results of section 4 and, in particular, exercise 6.4.2 in (1), note that there exists a K_o such that QK_o has the required decomposition with $\{m_j\}$,$\{d_j\}$ and q replaced by $\{k_j\}$, $\{d_j^1\}$ and q^1. This decomposition is generated (1) by a finite number of operations on a generic structure for $[CBK_o, CABK_o,\ldots, CA^{n-1}B\ K_o]$ and it is clear that this decomposition exists in an open dense subset K_3 of $L(R^m)$. But $\{k_j\}$are the orders of the infinite zeros (1) for $S(A,BK_o,C)$ with $K_o \in K_3$ and hence, combining with lemma 1, we conclude that $q = q^1$, $m_j = k_j$ $(1\le j\le q)$ and $d_j = d_j^1$ $(1\le j\le q)$. This completes the proof of the result as $K^{**} \subset K^*$ follows trivially.

The main result of this section can now be stated:

Theorem 3: Given the square invertible system S(A,B,C) the property $P_2(A,B,C)$ that the infinite zeros have the base structure (1a) with $\nu_j=n_j$, $1 \le j \le m$, is C^*-generic.

Proof: Taking $K_1 = I_m$ in (12) let K_4 be the subset of K_3 such that $\lim_{s\to\infty} s^{m_j}Q_j(s)$ has distinct eigenvalues whenever $K_o \in K_4$. The results from root-locus compensation theory indicates that $K_4 \neq \emptyset$ and it is clearly open in K_3 (and hence $L(R^m)$) and also dense as small perturbations in eigenvalues can be generated by small perturbations to the input compensator K_o. Noting that $\{I_m\} \times K_4 \subset K^*$ and that the distinct eigenvalue assumption is sufficient to guarantee the validity of the base characterization (1a), the result is clearly independent of the presence state feedback or output injection by lemma 2. The theorem is therefore proven as the base characterization (1a) holds on the open dense subset $L(R^n,R^m)xL(R^m,R^n)x\{(K_1,K_o):K_oK_o \in K_4\} \subset L$.

Corollary 3.1: $(0,0,I_m,I_m)$ lies in the closure of P_2^*

Proof: As K_4 is dense in $L(R^m)$, we can choose a $K_o \in K_4$ arbitrarily close to I_m. The result follows as $(0,0,I_m,K_o)$ is arbitrarily close to $(0,0,I_m,I_m)$.

Corollary 3.2: There is an open, dense subset $K_5 \subset L(R^m)$ such that $L(R^n,R^m) \times \{o\} \times \{I_m\} \times K_5 \subset P_2^{*5}$.

The interpretation of the theorem and its corollaries is identical to that of theorem 1 and its corollaries. The inclusion of the pivot, however, does introduce an entirely multivariable phenomena, namely (1), the fact that the pivots of the m_jth order infinite zeros are discontinuous functions of input compensator data in the vicinity of points when the matrix

$\lim_{|s|\to\infty} s^{m_j} Q_j(s)$ has multiple eigenvalues. This behaviour is non-generic but (1, p. 300) it happens at what can be regarded as a common design condition and hence needs careful interpretation.

OPTIMAL ROOT-LOCI

The notion of root-locus can be carried over (see e.g. (14) - (17)) to the optimal linear state feedback controller for S(A,B,C) minimizing

$$J = \tfrac{1}{2} \int_0^\infty \{y^T(t)\, Q\, y(t) + p^{-1} u^T(t)\, R\, u(t)\}\, dt$$

$$Q = Q^T > 0 \;, \quad R = R^T > 0 \;, \quad p > 0 \qquad (13)$$

by plotting the variation of closed-loop poles as p increases from $p = 0+$ to $p = +\infty$. Clearly the idea of C*- genericity carries through to this case in the sense that it is possible to prove the following result:

Theorem 4: If S(A,B,C) is square and invertible, then the property $P_3(A,B,C)$ that the infinite zeros of the root-locus have a base characterization (1) with $\nu_j = 2n_j$, $1 \le j \le m$, is C*-generic with $P_3^* = L$.

Proof: In (15) the validity of the base characterization for S(A,B,C) with $F=K=0$ and $N=M=I_m$ is proved, whilst, in (18) it is shown that the infinite zeros always have orders $\nu_j = 2n_j$ and that the orders and asymptotic directions are entirely independent of any state feedback or output injection maps introduced into the system. P_3^* clearly is equal to L in this case and the result follows.

A similar result with base-characterization (1) replaced by base-characterization (1a) is proved in an identical manner. This is not considered here as pivots do not appear to play an important role in optimal root-loci as they do in the output feedback case.

CONCLUSIONS

The notion of a C*-generic property of a system S(A,B,C) has been shown to be the correct concept of genericity for the consideration of the asymptotic behaviour of the root-locus (and the optimal root-locus) of that system. In the optimal case the result turns out to be trivial in that every optimal root-locus is C*-generic in the sense that its asymptotic behaviour has the structure of base-characterizations (1) and (1a) independent of the choice of Q and R, the inclusion of state feedback and output injection loops and constant input/output transformations. In the non-optimal multivariable case, the notion is non-trivial as non-generic structures can be constructed in practice. It has been demonstrated here that base characterizations (1) and (1a) are still C*-generic with orders equal to the integer structural invariants of the C*-transformation group and that non-generic behaviour can always be removed by 'random' choice of constant forward path compensator. This last point is of particular importance as it indicates that non-generic behaviour can be ignored in practical terms provided that care is taken to avoid its creation by badly-designed control systems.

The proofs of the results have drawn heavily on different techniques and results in the literature. Further work should enable the construction of more 'uniformly based' proofs. This problem is under consideration.

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