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ITERATIVE SOLUTION OF CONSTRAINED DIFFERENTIAL/ALGEBRAIC SYSTEMS

by


and

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1. **INTRODUCTION**

The dynamics of a large class of engineering systems can be approximately described by coupled algebraic and differential equations of the form (e.g. see appendix A.1)

\[ \dot{x}(t) = f(x(t), u(t), t), \quad x(0) = x_0 \quad (1) \]

\[ g(x(t), u(t), t) = 0 \quad (2) \]

where \( x \in \mathbb{R}^n \), \( u \in \mathbb{R}^l \), \( g : \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^m \rightarrow \mathbb{R}^m \) and \( f : \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^m \rightarrow \mathbb{R}^n \), e.g. the dynamics of a thermal nuclear reactor (Owens, 1973). A discrete representation of such systems takes the form

\[ x'(k+1) = f'(x'(k), u'(k), k), \quad k = 0, 1, \ldots, N-1 \quad (1') \]

\[ g'(x'(k), u'(k), k) = 0 \quad , \quad k = 0, 1, \ldots, N \quad (2') \]

where, now, \( g' : \mathbb{R}^n \times \mathbb{R}^l \rightarrow \mathbb{R}^m \), \( f' : \mathbb{R}^n \times \mathbb{R}^l \rightarrow \mathbb{R}^n \), \( x'(k) \in \mathbb{R}^n \), \( k = 0, 1, \ldots, N \), and \( u(k) \in \mathbb{R}^l \), \( k = 0, 1, \ldots, N \). The algebraic equations (2) (and equivalently (2')) are, typically, formed by neglecting fast stable time constants and may also incorporate other algebraic constraints. An important feature of such systems is that the equations may have no solution, a unique solution or an infinite number of solutions each of which is not necessarily isolated. The problem discussed in this paper is, given state and control constraints of the form

\[ x(t) \in \Omega_x(t), \quad u(t) \in \Omega_u(t), \quad t \in [0, T] \quad (3) \]

in the continuous case, or

\[ x'(k) \in \Omega'_x(k), \quad u'(k) \in \Omega'_u(k), \quad k = 0, 1, \ldots, N \quad (3') \]

for the discrete case, find a solution pair \((x, u)\) of (1)-(3) (or (1')-(3')) if one exists. It is assumed that any solution satisfying the equations and constraints given above is acceptable as a solution to the engineering problem. In general, it is not possible to achieve an
analytic solution and so iterative techniques are required to generate
a sequence \((x_j, u_j)\) tending to a limit \((x, u)\), where \((x, u)\) lies in the
solution set defined by (1)-(3) (or (1')-(3')).

One method of solution is to embed the above problem in an optimal
control setting, i.e. to solve the system of equations (1)-(3)
(respectively (1')-(3')) whilst minimising the cost criterion
\[
J(u) = \int_0^T L(x, u, t) \, dt \quad \text{(respectively, } J'(u'_0, u'_1, \ldots, u'_{N-1}) = P(x'_0, \ldots, x'_N, u'_0, \ldots, u'_{N-1})\text{)}
\]

An approach along these lines has been suggested (Owens, 1973) where the
minimum of \(J(u) = \int_0^T <g(x, u, t), Q(t)g(x, u, t)> \, dt\), \(Q(t) > 0\) for all \(t \in [0, T]\),
is sought subject to the constraints (1) and (3). This approach is
somewhat artificial and numerical problems with the minimisation algorithm
can arise. It is important to realise that the problem is not in itself
an optimal control problem, although optimisation techniques may help in
its solution.

This paper presents a method for the systematic iterative solution of
a linearised form of the above system (1)-(3)
\[
\dot{x}(t) = Ax(t) + Bu(t) \quad , \quad x(0) = x_0 \quad (4)
\]
\[
Eu(t) + Fx(t) = 0 \quad (5)
\]
\[
u(t) \in \Omega_u(t) \quad , \quad x(t) \in \Omega_x(t) \quad , \quad 0 \leq t \leq T \quad (6)
\]
(and the equivalent linearised form of the discrete system (1')-(3')), which
has guaranteed convergence in a well-defined computational sense,
and requires only standard Riccati and minimisation routines for
implementation. The formulation is quite general and can also be
applied to linear, constrained algebraic problems, continuous problems with
integral constraints and, in fact, any problem where the solution set is
the intersection of two closed convex sets in a suitable real Hilbert
space.
2. PROBLEM FORMULATION

Let $H$ be a real Hilbert space with $K_1 \subset H$, $K_2 \subset H$ two closed convex sets representing the system constraints and consider the general problem of finding a point $y \in K_1 \cap K_2$. A sequence $(y_j) \subset H$ is sought having one of the following properties:

(i) $y_j \to y^*$, in the sense of the norm, for some $y^* \in K_1 \cap K_2$

(ii) for each real number $\varepsilon > 0$, there exists an integer $N$ such that

whenever $j \geq N$, max \{inf $\|y_j - z\|$, inf $\|y_j - z\|$ \} < $\varepsilon$.

Property (i) represents the case of strong convergence where it is possible to construct a sequence $(y_j)$ whose strong limit lies in $K_1 \cap K_2$. This paper considers the case where convergence, in the strong sense, to an element of $K_1 \cap K_2$ cannot necessarily be guaranteed, although it is possible to construct a sequence which will generate a point arbitrarily close to both $K_1$ and $K_2$. This case is represented by property (ii) and is an acceptable form of convergence in the engineering problem in the sense that the system constraints are satisfied to an arbitrary accuracy.

Specific examples of the general problem defined above are:

(1) The linear system defined by (4)-(6), with $\Omega_x$, $\Omega_u$ closed convex sets, and $H = L_2^n[0,T] \times L_2^\infty[0,T]$, with associated norm

$$\| (x,u) \| = \left\{ \int_0^T (x^T Q(t)x + u^T R(t)u) \, dt \right\}^{\frac{1}{2}} , \quad Q(t) > 0, \quad R(t) > 0 \quad \forall \ t \in [0,T].$$

Here $K_1 = \{(x(t), u(t)) \in H : x(t) \in \Omega_x(t), u(t) \in \Omega_u(t), \ a.e., \ t \in [0,T]\}$, which is a closed convex set, and

$$K_2 = \{ (x(t), u(t)) \in H : x(t) = e^{A t} x(o) + \int_0^t e^{A(t-s)} Bu(s) \, ds \} \quad \text{Eu}(t) + Fx(t) \equiv 0 \},$$

which is a closed linear variety in $H$. 
(2) The problem of choosing \( u(t) \) such that \( x(t) \) satisfies
\[
\dot{x}(t) = ax(t) + bu(t), \quad x(o) = x_o, \quad x(T) = x_f
\]
subject to the constraint \( |u(t)| \leq 1 \forall t \in [0,T] \). Noting that
\[
x(T) - \exp(aT)x_o = \int_0^T \exp(-a\tau)bu(\tau)d\tau
\]
and defining \( H = L_2[0,T] \), then
\[
K_1 = \{ u \in H : \int_0^T \exp(-a\tau)bu(\tau)d\tau = x_f - \exp(aT)x_o \} \text{ and}
\]
\[
K_2 = \{ u \in H : |u(t)| < 1 \forall t \in [0,T] \}
\]

(3) The problem of obtaining a solution of the linear algebraic equation \( Ax = d \), where \( d \in \mathbb{R}^q, x \in \mathbb{R}^n \), \( q \leq n \), satisfying the constraints
\[
|x_j^\ast - x_j| \leq r_j, \quad 1 \leq j \leq q,
\]
for some \( q \leq n \). Here, \( H = \mathbb{R}^n \),

\[
K_1 = \{ x \in H : Ax = d \} \quad \text{and} \quad K_2 = \{ x \in H : |x_j^\ast - x_j| \leq r_j, \quad 1 \leq j \leq q \}.
\]

3. **Iterative Solution via Sequential Projection**

This Hilbert space formulation of the problem enables the simple geometric ideas of orthogonal projection (see, for example, Luenberger, 1969) to be utilised in the development of an algorithm for its solution. In this section an iterative scheme, based upon sequential application of the Projection Theorem, is developed.

The general problem outlined in section 2 is first considered, the results being presented in the following theorem.

**Theorem 1** Let \( K_1 \subset H, K_2 \subset H \), be two closed convex sets in a real Hilbert space \( H \) with \( K_1 \cap K_2 \) nonempty. Define
\[
K_j = \begin{cases} 
K_1 & , \text{j odd} \\
K_2 & , \text{j even}
\end{cases}
\]

Then, given the initial guess \( k_0 \subset H \), the sequence \( (k_j) \), \( j = 0,1,2,\ldots \), given by
\[ \|k_j - k_{j-1}\| = \inf_{k \in K_j} \|k - k_j\|, \quad j \geq 1 \quad (7) \]

with \( k_j \in K_j \), \( j \geq 1 \), is uniquely defined for each \( k_0 \in H \) and satisfies

\[ \|k_{j+1} - k_j\| < \|k_j - k_{j-1}\|, \quad j \geq 2 \quad (8) \]

Furthermore, for any \( x \in K_1 \cap K_2 \),

\[ \sum_{j=1}^{\infty} \|k_{j+1} - k_j\|^2 \leq \|x - k_1\|^2 \quad (9) \]

and, hence, for each \( \varepsilon > 0 \), there exists an integer \( N \) such that for \( j \geq N \)

\[ \inf_{k \in K_{j+1}} \|k - k_j\| < \varepsilon \quad (10) \]

**Proof**

Since \( K_j \) is a closed convex set in a Hilbert space, then, given \( k_j \in K_j \), the Projection Theorem guarantees the existence of a well-defined and unique \( k_{j+1} \in K_{j+1} \) such that

\[ \|k_{j+1} - k_j\| \leq \|x - k_j\| \]

for all \( x \in K_{j+1} \) proving uniqueness. Moreover, for any \( x \in K_{j+1} \), \( \langle x - k_{j+1}, k_j - k_{j+1} \rangle \leq 0 \) and, in particular, \( \langle k_j - k_{j+1}, k_{j+1} - k_j \rangle \geq 0 \) and, hence,

\[ \|k_j - k_{j-1}\|^2 = \|k_j - k_{j+1}\|^2 + \|k_{j+1} - k_j\|^2 + 2\langle k_j - k_{j+1}, k_{j+1} - k_j \rangle \]

\[ > \|k_j - k_{j+1}\|^2, \]

which verifies (8).

If \( x \in K_1 \cap K_2 \), then \( \langle x - k_{j+1}, k_{j+1} - k_j \rangle \geq 0 \) for all \( j \), and so

\[ \|x - k_j\|^2 = \|x - k_{j+1}\|^2 + \|k_{j+1} - k_j\|^2 + 2\langle x - k_{j+1}, k_{j+1} - k_j \rangle \]

\[ \geq \|x - k_{j+1}\|^2 + \|k_{j+1} - k_j\|^2 \]
An induction argument then gives
\[ \|x-k_1\|^2 \geq \|x-k_j\|^2 + \sum_{\ell=1}^{j-1} \|k_{j+1-\ell} - k_\ell\|^2 \]
for all \(j\), so that (in the limit)
\[ \|x-k_1\|^2 \geq \sum_{\ell=1}^{\infty} \|k_{j+1-\ell} - k_\ell\|^2 \]
as required. (10) now follows from this result.

Q.E.D.

Theorem 1 presents an iterative scheme satisfying the convergence criterion, property (ii) of section 2, and, using equation (8), each iteration is a better approximation to the solution of the problem. This scheme, based on the basic geometrical concept of orthogonal projection, is outlined in Figure 1.

Figure 2 describes the case where the tangent hyperplanes to \(K_1\) and \(K_2\) at the points \(k_i\) and \(k_{i+1}\) are nearly parallel and suggests that in this case convergence will be slow. The question of whether the scheme can be modified to incorporate some form of extrapolation parameter to speed up convergence is investigated in the next theorem. Attention is restricted to the case where \(K_2\) is a closed linear variety and a modified scheme is outlined in Figure 3. Note that the result reduce to theorem 1 if \(\lambda_i = 1, i \geq 1\).

**Theorem 2** Let \(K_1 \subseteq H\) be a closed convex set and \(K_2 = a + M, K_2 \subseteq H\), a closed linear variety in a real Hilbert space \(H\) such that \(K_1 \cap K_2\) is nonempty. \((M \subseteq H\) is a closed subspace and \(a \in H\)). Then, given \(r_1 \in K_2\), a sequence \(\{r_1, k_1, s_1, r_2, k_2, s_2, \ldots\}\) given by
\[ \|k_i - r_i\| = \inf_{y \in K_1} \|y - r_i\|, \quad k_i \in K_1 \] (11)
\[ \|s_i - r_i\| = \inf_{y \in K_2} \|y - k_i\|, \quad s_i \in K_2, \quad (12) \]

and
\[ r_{i+1} = r_i + \lambda_i (s_i - r_i), \quad (13) \]

with
\[ 1 \leq \lambda_i \leq \frac{\|k_i - r_i\|^2}{\|s_i - r_i\|^2}, \quad (14) \]

is well-defined for each \( r_i \in K_2 \). Furthermore,
\[ \|r_i - x\|^2 \geq \sum_{j=1}^{\infty} \|k_j - r_i\|^2, \quad (15) \]

and, hence, for each \( \varepsilon > 0 \) there is an integer \( N \) such that for \( j \geq N \)
\[ \inf_{y \in K_1} \|y - r_j\| < \varepsilon, \quad (16) \]

**Proof**

Given \( r_i \in K_2 \), then since \( K_1 \) is a closed convex set and \( K_2 \) a closed linear variety in a Hilbert space, the Projection Theorem guarantees the existence and uniqueness of a \( k_i \) and \( s_i \) satisfying (11) and (12), respectively. Furthermore, \( \langle x - k_i, x - k_i \rangle \leq 0 \) for all \( x \in K_1 \) and \( k_i - s_i \perp M \).

It is therefore only necessary to show that \( \lambda_i \), as given in (14), is well defined, i.e. that \( \|s_i - r_i\|^2 > 0 \) and \( \|k_i - r_i\|^2 \geq \|s_i - r_i\|^2 \). It is assumed that \( \|k_i - r_i\|^2 > 0 \) since otherwise \( k_i = r_i \) and the algorithm has converged. For this case, suppose that \( \|s_i - r_i\|^2 = 0 \), i.e. \( s_i = r_i \).

Then, for all \( x \in K_1 \), \( \langle x - k_i, k_i - s_i \rangle \geq 0 \) and, for all \( x \in K_2 \), \( \langle x - s_i, s_i - k_i \rangle = 0 \) or, equivalently, \( \langle x - k_i, k_i - s_i \rangle = -\|k_i - s_i\|^2 < 0 \) i.e. \( K_1 \cap K_2 \) is empty contrary to assumption. Noting that
\[ \|k_i - r_i\|^2 = \|k_i - s_i\|^2 + \|s_i - r_i\|^2 + 2\langle k_i - s_i, s_i - r_i \rangle \\
= \|k_i - s_i\|^2 + \|s_i - r_i\|^2, \]

and
\[ r_{i+1} = r_i + \lambda_i (s_i - r_i), \quad (13) \]

with
\[ 1 \leq \lambda_i \leq \frac{\|k_i - r_i\|^2}{\|s_i - r_i\|^2}, \quad (14) \]

is well-defined for each \( r_i \in K_2 \). Furthermore,
\[ \|r_i - x\|^2 \geq \sum_{j=1}^{\infty} \|k_j - r_i\|^2, \quad (15) \]

and, hence, for each \( \varepsilon > 0 \) there is an integer \( N \) such that for \( j \geq N \)
\[ \inf_{y \in K_1} \|y - r_j\| < \varepsilon, \quad (16) \]

**Proof**

Given \( r_i \in K_2 \), then since \( K_1 \) is a closed convex set and \( K_2 \) a closed linear variety in a Hilbert space, the Projection Theorem guarantees the existence and uniqueness of a \( k_i \) and \( s_i \) satisfying (11) and (12), respectively. Furthermore, \( \langle x - k_i, x - k_i \rangle \leq 0 \) for all \( x \in K_1 \) and \( k_i - s_i \perp M \).

It is therefore only necessary to show that \( \lambda_i \), as given in (14), is well defined, i.e. that \( \|s_i - r_i\|^2 > 0 \) and \( \|k_i - r_i\|^2 \geq \|s_i - r_i\|^2 \). It is assumed that \( \|k_i - r_i\|^2 > 0 \) since otherwise \( k_i = r_i \) and the algorithm has converged. For this case, suppose that \( \|s_i - r_i\|^2 = 0 \), i.e. \( s_i = r_i \).

Then, for all \( x \in K_1 \), \( \langle x - k_i, k_i - s_i \rangle \geq 0 \) and, for all \( x \in K_2 \), \( \langle x - s_i, s_i - k_i \rangle = 0 \) or, equivalently, \( \langle x - k_i, k_i - s_i \rangle = -\|k_i - s_i\|^2 < 0 \) i.e. \( K_1 \cap K_2 \) is empty contrary to assumption. Noting that
\[ \|k_i - r_i\|^2 = \|k_i - s_i\|^2 + \|s_i - r_i\|^2 + 2\langle k_i - s_i, s_i - r_i \rangle \\
= \|k_i - s_i\|^2 + \|s_i - r_i\|^2, \]
since \( s_i - r_i \subseteq M \) and \( k_i - s_i \perp M \), and so, if the algorithm has not converged,
\[ ||k_i - r_i||^2 > ||s_i - r_i||^2. \]

Now let \( x \in K_i \cap K_2 \) and consider
\[ \langle r_{i+1} - r_i, r_i - x \rangle = \lambda_i \langle s_i - r_i, r_i - x \rangle = \lambda_i \langle s_i - k_i, k_i - r_i, r_i - x \rangle \]
\[ = \lambda_i \langle k_i - r_i, r_i - x \rangle, \]

since \( r_i - x \subseteq M \) and \( s_i - k_i \perp M \). Then
\[ \langle r_{i+1} - r_i, r_i - x \rangle = \lambda_i \langle k_i - r_i, r_i - k_i, k_i - x \rangle \]
\[ = -\lambda_i ||k_i - r_i||^2 + \lambda_i \langle k_i - r_i, k_i - x \rangle \]
\[ \leq -\lambda_i ||k_i - r_i||^2, \]

by the definition of \( k_i \). Also, for \( \lambda_i \) satisfying (14),
\[ \lambda_i ||k_i - r_i||^2 = \frac{||k_i - r_i||^2}{||s_i - r_i||^2} \cdot ||s_i - r_i||^2 \geq \lambda_i ||s_i - r_i||^2 = ||s_i - r_i||^2. \]

Hence, for \( x \in K_i \cap K_2 \) and for any \( i \),
\[ ||r_{i+1} - x||^2 = ||r_i - x||^2 + ||r_{i+1} - r_i||^2 + 2\langle r_{i+1} - r_i, r_i - x \rangle \]
\[ \leq ||r_i - x||^2 + ||r_{i+1} - r_i||^2 - 2\lambda_i ||k_i - r_i||^2 \]

and, rearranging,
\[ ||r_i - x||^2 \geq ||r_{i+1} - x||^2 + (\lambda_i ||k_i - r_i||^2 - ||r_{i+1} - r_i||^2) + \lambda_i ||k_i - r_i||^2 \]
\[ \geq ||r_{i+1} - x||^2 + \lambda_i ||k_i - r_i||^2, \]

and, since \( \lambda_i \geq 1 \),
\[ ||r_i - x||^2 \geq ||r_{i+1} - x||^2 + ||k_i - r_i||^2. \]
An induction argument now gives,
\[ \| r_{1} - x \|^2 \geq \| r_{i+1} - x \|^2 + \sum_{j=1}^{i} \| k_j - r_j \|^2, \] for all \( i \), and so
\[ \| r_{1} - x \|^2 \geq \sum_{j=1}^{\infty} \| k_j - r_j \|^2, \] as required. (16) now follows immediately.
Q.E.D.

The iterative schemes presented in Theorems 1 and 2 will not, in a general Hilbert space, converge to a solution in a finite number of iterations. It can be shown, however, that, for the case where \( K_1 \) is a closed hyperplane and \( K_2 \) a closed linear variety convergence can be obtained in one iteration. In this case it is, in fact, possible to obtain a minimum norm solution. These results are formalised in the following theorem.

**Theorem 3.** Let \( K_1 = \{ x \in H: \langle \alpha, x - \alpha \rangle = 0, \alpha \in H, \| \alpha \| > 1 \} \) be a closed hyperplane in a Hilbert space \( H \) and define \( K_2 \) as in Theorem 2. Given \( r_1 \in K_2 \), then for \( k_1 \) and \( s_1 \) as defined in (11) and (12) of Theorem 2, respectively,
\[ r_2 = r_1 + \frac{\| k_1 - r_1 \|^2}{\| s_1 - r_1 \|^2} (s_1 - r_1) \in K_1 \cap K_2. \]

Furthermore, if \( \| r_1 \| < \| y \| \) for all \( y \in K_2 \), then \( \| r_2 \| < \| x \| \) for all \( x \in K_1 \cap K_2 \).

**Proof**

By translation, and without loss of generality, it is assumed that \( r_1 = 0 \) so that, from the definition of \( k_1 \), \( \langle k_1, x - k_1 \rangle = 0 \) for all \( x \in K_1 \) and, hence, \( \langle k_1, x - k_1 \rangle = 0 \) is an alternative definition of \( K_1 \). Since, by construction, \( \langle s_1, k_1 - s_1 \rangle = 0 \), it follows that \( \langle s_1, k_1 \rangle = \| s_1 \|^2 \) and so
\[ \langle k_1, r_2 - k_1 \rangle = \langle k_1, \frac{\|k_1\|^2}{\|s_1\|^2} s_1 - k_1 \rangle = \frac{\|k_1\|^2}{\|s_1\|^2} \langle k_1, s_1 \rangle - \|k_1\|^2 = 0 \]

which implies that \( r_2 \in K_1 \). By definition, \( r_2 \in K_2 \) and hence \( r_2 \in K_1 \cap K_2 \).

If \( y \in K_1 \cap K_2 \), then
\[
\langle y-r_2, r_2 \rangle = \frac{\|k_1\|^2}{\|s_1\|^2} \langle y - \frac{\|k_1\|^2}{\|s_1\|^2} s_1, s_1 \rangle = \frac{\|k_1\|^2}{\|s_1\|^2} \langle y, s_1 \rangle - \|k_1\|^2 \]
\[
= \frac{\|k_1\|^2}{\|s_1\|^2} \{\langle y, s_1 - k_1 \rangle + \langle y, k_1 \rangle - \|k_1\|^2 \}
\]

and, since \( s_1 - k_1 \perp K_2 \),
\[
\langle y-r_2, r_2 \rangle = \frac{\|k_1\|^2}{\|s_1\|^2} \{\langle y, k_1 \rangle - \|k_1\|^2 \} = \frac{\|k_1\|^2}{\|s_1\|^2} \langle y-k_1, k_1 \rangle = 0 ,
\]

by definition of \( k_1 \). It now follows that
\[
\|y\|^2 = \|y-r_2\|^2 + \|r_2\|^2 + 2\langle y-r_2, r_2 \rangle = \|y-r_2\|^2 + \|r_2\|^2 > \|r_2\|^2
\]
as required.

In the general case with \( r_1 \neq 0 \),
\[
\|y-r_1\|^2 = \|y-r_2\|^2 + \|r_2-r_1\|^2 ,
\]
and if \( \|r_1\| < \|y\| \) for all \( y \in K_2 \) it follows that \( r_1 \perp M \). Therefore, since \( y-r_r, \in M \),
\[
\|y\|^2 = \|y-r_1\|^2 + \|r_1\|^2 = \|y-r_2\|^2 + \|r_2-r_1\|^2 + \|r_1\|^2
\]
\[
\geq \|r_2-r_1\|^2 + \|r_1\|^2 = \|r_2\|^2 ,
\]
as \( r_2-r_1 \in M \).
Comments

1. In general, the existence of a strong limit point to the sequence generated in Theorems 2 and 3 has not been established. This is of interest theoretically but is of little consequence from a practical point of view since it has been demonstrated that the convergence property (ii) of section 2 is satisfied. However, for the case where $H$ is a finite dimensional space, a proof along the following lines can be obtained.

For any $K_1 \cap K_2$ and $r_i$ defined by Theorem 2, $\|r_i - x\|^2 \geq \|r_i - x\|^2$ for all $i$. Furthermore $\|r_i\| \leq \|r_i - x\| + \|x\| \leq \|r_i - x\| + \|x\|$, for all $i$, and so the sequence $(r_i)$ is bounded. Then, as $H$ is a finite dimensional space, $(r_i)$ is relatively compact and has at least one cluster value $r \in K_1 \cap K_2$. If $r$ and $\hat{r}$ are distinct cluster values of the sequence $(r_i)$ then there are subsequences $(r_{i_k})$ and $(r_{i_{\ell}})$ of $(r_i)$ such that $(r_{i_k}) \to r$ and $(r_{i_{\ell}}) \to \hat{r}$. Defining $\varepsilon = \frac{1}{2} \|r - \hat{r}\|$, there exists an integer $N$ such that for $k, \ell > N$, $r_{i_k} \in B(r, \varepsilon)$ and $r_{i_{\ell}} \in B(\hat{r}, \varepsilon)$, where $B(x, \varepsilon)$ is the open ball centred on $x$ with radius $\varepsilon$. Taking $i_k > i_{\ell}$, for some $k, \ell > N$, it follows that $\|r_{i_k} - \hat{r}\| > \|r_{i_{\ell}} - r\|$, since $r_{i_k} \in B(r, \varepsilon)$, $r_{i_{\ell}} \in B(\hat{r}, \varepsilon)$ and $\|r - \hat{r}\| = 2\varepsilon$, contradicting the result that $\|r_{i+1} - x\| < \|r_i - x\|$ for all $x \in K_1 \cap K_2$ (see proof of Theorem 2). Then $(r_i)$ must have a unique cluster value $r$ and hence $(r_i) \to r \in K_1 \cap K_2$.

2. If $K_1$ and $K_2$ are disjoint, the algorithm defined by Theorem 2 with $\lambda_i > 1$ may exhibit wild oscillation, as illustrated in Figure 4. With $\lambda_i = 1$, however, the algorithm is well behaved and, intuitively, converges to points $r_1 \in K_1$, $r_2 \in K_2$ defining the minimum distance between the two sets. A proof of this observation for the case where $H$ is finite dimensional now follows.

Taking $\lambda_i = 1$ for all $i$, Theorem 2 gives $\|r_{i+1} - k_{i+1}\| \leq \|r_{i+1} - k_i\| \leq \|r_i - k_i\|$ which implies that $\beta_i = \|r_i - k_i\|$ has limit $\beta$ and, since $H$ is finite dimensional, the sequence $(r_i - k_i)$ has cluster values $r \in K_2$, $k \in K_1$. 


with $\beta = \|k-r\| > 0$, for each $k_i, r_i, \langle k'-k, k_i-r_i \rangle > 0$ for all $k \in K$, giving $\|k-r\| = \inf_{y \in K} \|y-r\|$ and, by similar reasoning, $\|k-r\| = \inf_{y \in K} \|k-y\|$. Then, for all $k \in K_1$, $r \in K_2$,

$$
\|k'-r'\|^2 = \|k'-k+k-r+r\|^2 = \|k'-k\|^2 + \|k-r\|^2 + \|r-r'\|^2 + 2\langle k'-k, k-r \rangle + \langle k'-r', r-r' \rangle \geq \|k-r\|^2 + \|k'-k\|^2 + \|r-r'\|^2 + 2\langle k'-k, r-r' \rangle
$$

$$
= \|k-r\|^2 + \|k'-k+r-r'\|^2 \geq \|k-r\|^2
$$
as required.

3. The introduction of an extrapolation parameter $\lambda_i > 1$ can cause numerical errors introduced into the calculation at each iteration to be magnified at successive iterations. For, if errors $\varepsilon_i^r, \varepsilon_i^s$ are introduced at the $i$-th iteration in the calculation of $r_i$ and $s_i$, respectively, then (14) gives

$$
\varepsilon_{i+1}^r = (1-\lambda_i)\varepsilon_i^r + \lambda_i \varepsilon_i^s
$$

and, for $\lambda_i > 1$ and under worst-case conditions (i.e. $\varepsilon_i^r = -\varepsilon_i^s$),

$$
\|\varepsilon_{i+1}^r\| = (2\lambda_i^{-1}) \|\varepsilon_i^s\| \gg \|\varepsilon_i^s\| \quad \text{if} \quad \lambda_i > 1
$$

In practice this problem can be removed by setting $\lambda_i = 1$ every few iterations to reset the magnitude of the computational errors.

4. In general, the sequence $(r_i, k_i)$ does not converge to a minimum norm solution as a simple three dimensional example will testify. For the case where $K_i$ is a closed hyperplane, a minimum norm solution is obtained, however, (Theorem 3) and the algorithm converges in one iteration.

4. **EXAMPLES**

1. Consider the system $\dot{x}(t) = u(t), x(0) = 0, x(1) = 1$, where $u(t)$ is constrained to satisfy $\int_0^1 tu(t)dt = 1$. Defining $H = L_2[0,1]$ with
\[ \|f\| = \frac{1}{2} \int_0^1 f^2(t) \, dt, \quad f \in H, \quad K_1 = \{ u \in H : \int_0^1 tu(t) \, dt = 1 \} \] and
\[ K_2 = \{ u \in H : \int_0^1 u(t) \, dt = 1 \}, \] then \( K_1 \) is a closed hyperplane and \( K_2 \) a closed linear variety in \( H \). Application of Theorem 3 and Pontryagin's Minimum Principle therefore gives:

\[ r_1(t) = 1 \text{ for all } t \in [0, 1], \quad k_1(t) = 1 + \frac{3}{2} t \text{ and } s_1(t) = \frac{1}{4} + \frac{3}{2} t. \]

Then \( \lambda_1 = \frac{3}{4} / \left( \frac{3}{16} \right) \) and so \( r_2(t) = 1 + 4 \left( \frac{1}{4} + \frac{3}{2} t - 1 \right) = -2 + 6 t. \)

In this case, \( r_2 \in K_1 \cap K_2 \) and is of minimum norm, i.e., \( u = r_2 \) is also a solution of the associated minimum energy problem.

2. Consider, now, the discrete form of the problem outlined in equations (4)-(6) of section 2. Given a system described by equations of the form

\[ x(k+1) = \phi x(k) + \Delta u(k), \quad x(0) = x_0, \quad k = 0, 1, \ldots, N-1, \quad (17) \]

\[ Eu(k) + Fx(k) = 0, \quad k = 0, 1, \ldots, N \quad (18) \]

where \( x(k) \in \mathbb{R}^n \), \( u(k) \in \mathbb{R}^l \), \( k = 0, 1, \ldots, N \), and \( \phi \in \mathbb{R}^{n \times n} \), \( \Delta \in \mathbb{R}^{n \times l} \), \( E \in \mathbb{R}^{m \times n} \), \( F \in \mathbb{R}^{m \times l} \) \((m \times n)\) are real matrices, a solution \((x, u)\) of (17) and (18) is sought satisfying the constants \( x(k) \in \Omega_x \), \( u(k) \in \Omega_u \), \( k = 0, 1, \ldots, N \), where \( \Omega_x \) and \( \Omega_u \) are two closed convex sets. In the example discussed here, \( x \) is unconstrained (i.e., \( \Omega_x = \mathbb{R}^n \)) and \( \Omega_u \) is defined by

\[ \Omega_u = \{ u \in \mathbb{R}^l : u_{i \min} \leq u_i \leq u_{i \max}, \quad i = 1, 2, \ldots, l \} \quad (19) \]

However, the ideas are trivially extended to include convex state constraints.

The problem formulation is analogous to that outlined in example (1) of section 2 and is not repeated here. The spaces are, of course, now finite-dimensional and a suitable norm defined as
\[
\| (x,u) \|^2 = \frac{1}{Q} \sum_{k=0}^{N} \{ x^T(k)Qx(k) + u^T(k)Ru(k) \}, \quad Q > 0, \ R > 0 .
\]

The results of Theorem 2 are employed:

For each \( i \), given \( k_i = (x_{i}^{\text{ref}}, u_{i}^{\text{ref}}) \in K_1 \), \( s_i = (x,u) \in K_2 \) is calculated as
\[
\min_{y \in K_2} \| y - k_i \|, \quad \text{i.e.}
\]
\[
\min_{(x,u)} \frac{1}{2} \sum_{k=0}^{N} \left\{ \left[ x(k) - x_{i}^{\text{ref}}(k) \right]^TQ \left[ x(k) - x_{i}^{\text{ref}}(k) \right] + \left[ u(k) - u_{i}^{\text{ref}}(k) \right]^TR \left[ u(k) - u_{i}^{\text{ref}}(k) \right] \right\}
\]

where \((x,u)\) satisfies equations (17) and (18). This is a linear quadratic optimal control problem and has solution
\[
u(N-k) = -K(k)x(N-k) + g(k), \quad k = 0,1,\ldots,N, \quad (20)
\]
with \(x(k)\) given by equation (17). Expressions for the 'Riccati matrix' \(K(k)\) and 'tracking vector' \(g(k)\) are given in appendix A2. An initial estimate \(r_1 = (x,u)\) can be obtained setting \(x_{i}^{\text{ref}} = 0, u_{i}^{\text{ref}} = 0\).

Given \( r_1 = (x_{i}^{\text{ref}}, u_{i}^{\text{ref}}) \in K_2 \), \( k_i \) is calculated as
\[
\min_{y \in K_1} \| y - r_1 \|, \quad \text{i.e.}
\]
\[
\min_{(x,u)} \frac{1}{2} \sum_{k=0}^{N} \left\{ \left[ x(k) - x_{i}^{\text{ref}}(k) \right]^TQ \left[ x(k) - x_{i}^{\text{ref}}(k) \right] + \left[ u(k) - u_{i}^{\text{ref}}(k) \right]^TR \left[ u(k) - u_{i}^{\text{ref}}(k) \right] \right\}
\]

with \(u\) satisfying the constraint (19), which, if \( R \) is diagonal, has solution
\[
u_i = \begin{cases} 
u_{i}^{\text{max}}, & \text{for } u_{i}^{\text{ref}} \geq u_{i}^{\text{max}} \\
u_{i}^{\text{ref}}, & \text{for } u_{i}^{\min} \leq u_{i}^{\text{ref}} \leq u_{i}^{\text{max}}, \quad i = 1,\ldots,\ell, \text{ and } x = x_{i}^{\text{ref}} \\
u_{i}^{\text{min}}, & \text{for } u_{i}^{\text{ref}} \leq u_{i}^{\text{min}}
\end{cases}
\]

Hence computation of \(k_i\) simply involves 'clipping off' the components of \( r_1 \) where they violate the constraint set \( \Omega_u \).

\( r_{i+1} \) is now generated by \( r_{i+1} = r_i + \lambda_i(s_i - r_i) \), for suitable \( \lambda_i \),

\( 1 \leq \lambda_i \leq \| k_i - r_i \| / \| s_i - r_i \| \) and the iterative process repeated until numerical convergence is obtained.
A solution to equations (17) and (18) satisfying constraints of
the form (19) can therefore be generated by iterative application of
equations (17) and (20) and 'clipping off' the resulting control
trajectories where they violate the constraints. The 'Riccati matrix'
\( K \) is independent of \((x^{ref},u^{ref})\) and need only be calculated once whereas
the 'tracking vector' \( g \) has to be updated at each iteration. Two
numerical examples of the application of this algorithm are given below.

(a) \textbf{4-th order integrator plant}. It is desired to find a solution \((x,u)\)
of the algebraic/differential system
\[
\begin{align*}
\dot{x}_1 &= x_2 , & x_1(0) &= -0.5 , \\
\dot{x}_2 &= x_3 + u_1 , & x_2(0) &= 0.5 , \\
\dot{x}_3 &= x_4 + u_2 , & x_3(0) &= -0.5 , \\
\dot{x}_4 &= u_3 , & x_4(0) &= 0.5 ,
\end{align*}
\]
\[x_1 + x_2 + u_1 + u_2 + u_3 = 0 ,
\]
\[x_3 + x_4 + u_1 - u_2 + u_3 = 0 ,
\]
defined on the time interval \([0,1]\), subject to the constraint \(u_3 \geq 0\).
The time interval is divided into \(N\) steps of length \(h = 1/N\) and the problem
put into discrete form with \( \phi = I + hA + \frac{h^2A^2}{2!} + \ldots \), \( \Delta = hI + \frac{h^2A}{2!} + \frac{h^3A^2}{3!} \ldots \)

20 time steps were employed and the weighting matrices \(Q,R\) in the
norm were taken to be \(Q = I_4\), \(R = I_3\). The extrapolation factor \(\lambda_1\) was
set at \(\lambda_1 = \frac{\|k_1 - r_1\|^2}{\|s_1 - r_1\|^2}\), throughout, and convergence to an accurate
solution in 8 iterations is shown in Table 1 in terms of variation in \(\lambda_1\)
and distance between \(K_1\) and \(K_2\) with iteration. The initial and final
trajectories of \(u_3\) are given in Figure 5, Figure 6 describing the
corresponding plant outputs \(x_1\).

(b) \textbf{Nuclear Reactor Control Problem}. Large thermal nuclear power
reactors can exhibit unstable or underdamped oscillations in the power
distribution, with periods of 30-40 hours, due to the effects of the
fission product poison xenon-135 (Owens, 1973). The dynamics of such systems can be approximated by equations of the form

\[
\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \quad Eu(t) + Fx(t) = 0,
\]

where \(x(t) \in \mathbb{R}^{2m}, u(t) \in \mathbb{R}^{m+n}, t \in [0, T]\), and \(A_{2m \times 2m}, B_{2m \times (m+n)}, E_{(m+1) \times (m+n)}, F_{(m+1) \times 2m}\) are real matrices. \(x(t)\) represents the internal states inherent in the system, namely, xenon and its precursor iodine, and the power distribution and control rod reactivity are lumped together in \(u(t)\).

A typical one-dimensional model has system matrices

\[
A = \begin{bmatrix}
-0.29 \times 10^{-4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.29 \times 10^{-4} & -0.119 \times 10^{-3} & 0 & 0 & 0 & 0.195 \times 10^{-4} & 0 & 0 \\
0 & 0 & -0.29 \times 10^{-4} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.29 \times 10^{-4} & -0.991 \times 10^{-4} & 0 & 0 & 0 & 0.223 \times 10^{-4} \\
0 & 0 & 0 & 0 & -0.29 \times 10^{-4} & 0 & 0 & 0 \\
0.195 \times 10^{-4} & 0 & 0 & 0.29 \times 10^{-4} & -0.963 \times 10^{-4} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -0.29 \times 10^{-4} & 0 & 0 \\
0 & 0 & 0 & 0.223 \times 10^{-4} & 0 & 0 & 0.29 \times 10^{-4} & -0.954 \times 10^{-4}
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
0.112 \times 10^{-3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.902 \times 10^{-4} & 0 & 0.503 \times 10^{-5} & 0 & 0 & 0 & 0 & 0 \\
0 & 0.112 \times 10^{-3} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -0.852 \times 10^{-4} & 0 & 0.762 \times 10^{-5} & 0 & 0 & 0 & 0 \\
0 & 0 & 0.112 \times 10^{-3} & 0 & 0 & 0 & 0 & 0 \\
0.503 \times 10^{-5} & 0 & -0.83 \times 10^{-4} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.112 \times 10^{-3} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.762 \times 10^{-5} & 0 & -0.817 \times 10^{-4} & 0 & 0
\end{bmatrix}
\]
\[
E = \\
\begin{bmatrix}
-0.297 \times 10^{-4} & 0 & 0.594 \times 10^{-5} & 0 & 0.461 \times 10^{-1} & 0.128 \times 10^{-1} & 0.128 \times 10^{-1} \\
0 & -0.443 \times 10^{-4} & 0 & 0.679 \times 10^{-5} & 0 & 0.156 \times 10^{-1} & -0.156 \times 10^{-1} \\
0.594 \times 10^{-5} & 0 & -0.777 \times 10^{-4} & 0 & -0.802 \times 10^{-2} & 0.62 \times 10^{-2} & 0.62 \times 10^{-2} \\
0 & 0.679 \times 10^{-5} & 0 & -0.125 \times 10^{-3} & 0 & -0.81 \times 10^{-2} & 0.81 \times 10^{-2} \\
1.0 & 0 & 0.333 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
F = \\
\begin{bmatrix}
0 & -0.102 \times 10^{-3} & 0 & 0 & 0 & 0.204 \times 10^{-4} & 0 & 0 \\
0 & 0 & 0 & -0.815 \times 10^{-4} & 0 & 0 & 0 & 0.233 \times 10^{-4} \\
0 & 0.204 \times 10^{-4} & 0 & 0 & 0 & -0.786 \times 10^{-4} & 0 & 0 \\
0 & 0 & 0 & 0.233 \times 10^{-4} & 0 & 0 & 0 & -0.776 \times 10^{-4} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

In this case \( u_2(t) \) represents the dominant first spatial mode of oscillation, \( u_5(t) \) a bulk control action and \( u_6(t), u_7(t) \) two trimming controllers. In the absence of trimming controls \( u_6, u_7 \), the mode \( u_2(t) \) is unstable, and with initial condition

\[
x(0) = (10^{14}, -10^{14}, -10^{14}, 10^{14}, 0, 0, 0, 0)^T
\]

(21)

exhibits the transient shown in Figure 7, with peak magnitude \( \max_{t} |u_2(t)| = 5.39 \times 10^{14} \) over a time interval of 40 hours. The practical problem considered here is the choice of trimming control action \( u_6(t), u_7(t) \) to ensure that the power mode \( u_2(t) \), resulting from initial conditions (21), is adequately damped and satisfies the constraint

\[
|u_2(t)| \leq 0.4 \times 10^{14}, \quad 0 \leq t \leq 40
\]

which would obviously be a considerable improvement on the open loop behaviour.
The problem was solved in discrete time as in example (a). In order to offset the difference in magnitude in the power and control components of \( u(t) \), the weighting matrices \( Q, R \) in the norm were taken to be \( Q = I_8, R = \text{diag}(1, 1, 1, 1, 10^5, 10^8, 10^8) \). 20 time steps were employed and the extrapolation factor was initially set at

\[
\lambda_i = \frac{||k_i - r_i||^2}{||s_i - r_i||^2}
\]

for each iteration \( i \). In this case, an error growth, as predicted in section 2, was observed and the algorithm broke down. A choice of \( \lambda_k = 1 \) whenever \( \prod_{j=\ell}^{k} \lambda_j > 10^3 \), where \( \ell \) was the previous iteration at which \( \lambda_i \) was set to unity, had the effect of introducing an acceptable upper bound on the growth in errors and the algorithm now converged rapidly. To ensure the highest accuracy in the solution of the equations, \( \lambda_i \) was also set to unity on the final (converged) iteration.

Table 2 shows the rate of convergence and variation in \( \lambda_i \) with iteration. The initial and final iterates for the power mode \( u_2(t) \) are given in Figure 8 and Figure 9 describes the final trimming control trajectories \( u_6(t), u_7(t) \). For comparison purposes, the convergence rate for the case where \( \lambda_i \) was set to unity throughout, is indicated in Table 3. It is noted that, for the case where extrapolation was employed, the algorithm converged to an acceptable solution in 4 iterations, whereas, with no extrapolation, convergence has not been achieved after 50 iterations.

5. CONCLUSIONS

An iterative scheme for the solution of constrained algebraic/differential systems, based upon sequential application of optimisation techniques, has been presented. The algorithm has been derived in a Hilbert space setting and the formulation is quite general. Attention has been restricted to linear systems and for this case a Riccati-type solution is obtained. In this context, it is important to realise that although optimisation procedures have been used in the solution of this
problem, in general, the resulting solution is not optimal. The use of an extrapolation factor has been incorporated in the algorithm and it has been demonstrated, with the aid of a numerical example, that this can have a highly significant improvement on the convergence rate. Since this extrapolation parameter is always greater than unity, numerical errors can propagate but the scheme is easily adapted to contain such an error growth. Two illustrative control problems of moderate state dimension have been investigated and accurate solutions to both problems were obtained in a small number of iterations. Finally, it is noted that the norms used for the solution of the problem are unspecified and hence, intuitively, can be used to improve the conditioning of the algorithm.

REFERENCES


LUENBERGER, D.G., 1969, Optimisation by Vector Space Methods, Wiley.
APPENDICES

A1. Consider a control problem governed by equations of the form

\[ \dot{x}_1 = Q(x_1, x_2, t) \quad (22) \]
\[ \dot{x}_2 = \psi(x_1, x_2, u_c, t) \quad (23) \]

where it is required to control the state \( x_2 \). If equation (23) is stable
and has a fast acting time constant it can be reduced to the algebraic
equation

\[ \psi(x_1, x_2, u_c, t) = 0 \quad (24) \]

Then, if \( x_2 \) and \( u_c \) are lumped together as a pseudo 'control vector' \( u \),
i.e. \( u = (x_2, u_c)^T \), and taking \( x = x_1 \), equations (22) and (24) can be
rewritten as

\[ \dot{x} = f(x, u, t) \]
\[ g(x, u, t) = 0 \]

A2. The 'Riccati matrix' \( K \) and 'tracking vector' \( g \) of equation (20)
are given by the following recurrence relations

\[ R(k) = A^T Q(k-1) A + R \]
\[ S(k) = A^T Q(k-1) \Phi \]
\[ h(k) = Ru_{\text{ref}} (N-k) - A^T p(k-1) \]
\[ K(k) = R^{-1}(k) S(k) + R^{-1}(k) E^T [E R^{-1}(k) E^T]^{-1} [E F - E R^{-1}(k) S(k)] \]
\[ g(k) = R^{-1}(k) h(k) - R^{-1}(k) E^T [E R^{-1}(k) E^T]^{-1} E R^{-1}(k) h(k) \]
\[ k = 1, \ldots, N \]
\[ K(o) = R^{-1} E^T [E R^{-1} E^T]^{-1} E u_{\text{ref}} (N) + u_{\text{ref}} (N) \]

where

\[ Q(k) = Q + [\Phi - \Delta K(k)]^T Q(k-1) [\Phi - \Delta K(k)] + K^T(k) R K(k) \]
\[
\begin{align*}
  p^T(k) &= -x_{\text{ref}(N-k)}^T Q + g^T(k) \Delta^T Q(k-1) [\phi - \Delta K(k)] \\
  &\quad + p^T(k-1) [\phi - \Delta K(k)] - [g(k) - u_{\text{ref}(N-k)}]^{T} T_{RK}(k), \\
  k &= 1, \ldots, N-1,
\end{align*}
\]

and
\[
\begin{align*}
  Q(o) &= Q + K^T(o) R_{K}(o), \\
  p^T(o) &= -[g(o) - u_{\text{ref}(N)}]^{T} R_{K}(o) - x_{\text{ref}(N)}^{T} Q.
\end{align*}
\]
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<th>ITERATION (i)</th>
<th>$\lambda_i$</th>
<th>$|k_i - s_i|$</th>
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<td>5</td>
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<td>8</td>
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**Table 1**

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<th>ITERATION (i)</th>
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<th>$|k_i - s_i|$</th>
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**Table 2**

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<tr>
<td>50</td>
<td>1</td>
<td>16.9</td>
</tr>
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</table>

**Table 3**
Figure 3

Figure 4
Figure 5

Figure 6