Monograph:
Owens, D.H. (1975) Root-Loci Concepts for kth Order Type Multivariable Structures. Research Report. ACSE Report 39 . Department of Control Engineering, University of Sheffield, Mappin Street, Sheffield

ROOT-LOCI CONCEPTS FOR kth ORDER TYPE

MULTIVARIABLE STRUCTURES

by

D. H. Owens, B.Sc.,A.R.C.S.,Ph.D.,AFIMA

Lecturer in the Department of Control Engineering
University of Sheffield
Mappin Street
Sheffield S1 3JD

Research Report No. 39

December 1975

Abstract

With the philosophy that many physical multivariable systems can, for the purposes of control systems design be approximately represented by much simpler models and that the theoretical analysis of such models can provide valuable insight into the time and frequency domain characteristics of the system, this paper extends previous work by providing an analysis of a class of multivariable structures analogous to the classical kth order lag. It is shown that the system can be decoupled by state feedback using parameters defined by the inverse transfer function matrix. In more general situations, it is shown that an analysis of the asymptotic form of the system root-locus provides valuable insight into the desirable controller structures, and the analysis of the sensitivity of the root locus to controller parameters provides analytic solutions to the feedback control problem in the cases $k = 1$, $k = 2$.

List of Symbols

$G(s)$ = mxm plant transfer function matrix

$K(s,p)$ = mxm forward path controller transfer function matrix

$H(s)$ = mxm minor loop compensation transfer function matrix

A^+ = conjugate transpose of the matrix A

$(e_j)_{1 \leqslant j \leqslant m}$ = natural basis in R^m

$R(A)$ = range of the matrix A

$N(A)$ = null-space of the matrix A

$\mathrm{diag}\{\lambda_1, \ldots, \lambda_m\} \equiv \mathrm{diag}\{\lambda_j\}_{1 \leqslant j \leqslant m}$ = diagonal matrix with $\lambda_1, \ldots, \lambda_m$ along the diagonal

List of Figures

1. Introduction

Recent papers[1,2] have introduced the concept of multivariable first and second order type structures and derived closed-form solutions for high performance proportional and proportional plus integral unity negative feedback controllers. Motivation for the analysis lies

(i) in the analogy with classical theory where the analysis of first and second order system models provides direct insight into general design principles (eg the effect of compensation elements on closed-loop performance) and suggests rule of thumb approaches to the design and analysis of more complex structures.

and (ii) in the observation that many physical systems can be approximated by such low order models.

As presented[1], previous results cannot be easily extended to the analysis of more complex structures. This paper investigates the feasibility and usefulness of using a multivariable equivalent to the classical root-locus method as a tool for the general analysis of multivariable feedback problems. In section 2 the concept of multivariable root-locus is described. A complete analysis is presented in section 3 of a new class of multivariable kth order-type systems analogous to the classical kth order lag

$$g^{-1}(s) = a_o s^k + a_1 s^{k-1} + \ldots + a_{k-1}s + a_k \quad , \quad a_o \neq 0 \qquad \ldots(1)$$

As in the classical case, it is not possible, in general, to obtain an explicit formula for the closed-loop poles as a function of controller parameters. The results take the form of a derivation of the asymptotes of the root-locus diagram, an investigation of the sensitivity of the asymptotes to controller parameters and the derivation of necessary and sufficient conditions for the closed-loop system to be stable at high gains.

Although the results do not, in themselves, represent complete solutions to the design problem, it is shown in sections 4,5 that the analysis suggests analytic solutions to certain design problems, enables the systematic design of minor loop compensation elements and provides a direct link between time domain and frequency domain behaviour. Examples illustrating the application of the results are included in relevant sections.

2. Multivariable Root-loci, Feedback Design and Controller Structure

Consider a unity negative feedback configuration for the control of a system described by the mxm transfer function matrix (TFM) $G(s)$ and let $K(s,p)$ be the mxm forward path controller TFM, where p is a scalar parameter. The root-locus of the closed-loop system is defined to be the locus in the complex plane of the closed-loop poles in the region $p>0$.

Extending the classical approach, take

$$K(s,p) = pK_1(s)+K_2 \qquad \ldots(2)$$

where $K_1(s)$ is an mxm TFM and K_2 is a constant mxm matrix. In the classical case $K_2 = 0$, but, in the multivariable case, it will be seen that the choice of suitable non-zero K_2 can lead to useful simplifications of the root-locus plot with consequent benefits at the design stage.

In classical design theory, an analysis of the pole-zero configuration of the open-loop system $G(s)$ can be used to suggest a suitable compensator $K_1(s)$. In contrast, there is no known relation in the multivariable case between the pole-zero configuration of $G(s)$, $K_1(s)$ and the transient performance of the closed-loop system, so that the choice of a suitable $K_1(s)$ is a non-trivial task. A major objective of this paper is to illustrate how theoretical analysis of multivariable root-loci can yield valuable information on a suitable controller structure $K_1(s)$. Such results may suggest trial structures to initiate, guide and simplify the analysis of more complex systems using general design techniques[3-6].

Defining, for a TFM L(s),

$$L_{\infty}^{(j)} = \lim_{s\to\infty} s^{j}L(s) \qquad \ldots(3)$$

(whenever the limit exists), then it is assumed that

$$|K_{1\infty}^{(o)}| \neq 0 \qquad \ldots(4)$$

This is equivalent, intuitively, to the assumption that a stable, minimum-phase controller introduces no overall extra phase lag into the system. Finally, write

$$K_1(s) = K_{1\infty}^{(o)} + K_3(s) \qquad \ldots(5)$$

where $K_3(s)$ is a strictly proper TFM possessing a minimal realization of dimension q.

3. Root-loci for Multivariable kth-order Type Systems

Generalizing previous definitions [1,2], a multivariable first-order type system is represented by the mxm TFM

$$G(s) = C(sI_m-A)^{-1}B \qquad \ldots(6)$$

where A,B,C are constant mxm matrices and $|G(s)| \not\equiv 0$. Such a TFM arises in the analysis of systems whose input-output relations can be approximated[1] by a completely controllable and observable state space model with m-inputs, m-outputs and state dimension m. It follows that $|B| \neq 0$, $|C| \neq 0$ so that $G_{\infty}^{(1)} = CB$ is invertible and

$$\begin{aligned} G^{-1}(s) &= B^{-1}(sI_m-A)C^{-1} \\ &= (G_{\infty}^{(1)})^{-1}s+G^{-1}(s)\big|_{s=o} \end{aligned} \qquad \ldots(7)$$

which is consistent with the classical definition of a first order lag obtained from equation (1) with k = 1. The natural definition of a multivariable kth order lag is obtained from equation (1) to be an mxm TFM of the form

$$\overset{-1}{G}(s) = A_o s^k + A_1 s^{k-1} + \dots + A_{k-1} s + A_k$$
$$|A_o| \neq 0 \qquad \dots(8)$$

For example, if $k = 1$, $A_o = (G_\infty^{(1)})^{-1}$ and $A_1 = G^{-1}(s)|_{s=o}$. It is easily shown in general that

$$A_o^{-1} = G_\infty^{(k)} \qquad \dots(9)$$
$$A_k = G^{-1}(s)|_{s=o} \qquad \dots(10)$$
$$|G(s)| \not\equiv 0 \qquad \dots(11)$$

Such a system arises in the analysis of a completely controllable and observable system whose input-output dynamics can be approximated by m coupled kth order ordinary differential equations of the form, $|A_o| \neq 0$,

$$\left\{ A_o \frac{d^k}{dt^k} + A_1 \frac{d^{k-1}}{dt^{k-1}} + \dots + A_{k-1}\frac{d}{dt} + A_k \right\} y(t) = u(t) \qquad \dots(12)$$

For example, consider the simple mechanical system of Fig.1, where m_1, m_2 are masses, k_1, k_2 are spring constants and c is the dashpot damping constant. If y_1, y_2, u_1, u_2 are the mass and spring displacements from an equilibrium condition, the linearized equations of motion take the form

$$m_1\ddot{y}_1 + c(\dot{y}_1 - \dot{y}_2) + k_1 y_1 = k_1 u_1$$
$$m_2\ddot{y}_2 + c(\dot{y}_2 - \dot{y}_1) + k_2 y_2 = k_2 u_2 \qquad \dots(13)$$

so that $k = 2$,

$$A_o = \begin{bmatrix} \frac{m_1}{k_1} & 0 \\ 0 & \frac{m_2}{k_2} \end{bmatrix}, \quad A_1 = c\begin{bmatrix} \frac{1}{k_1} & \frac{-1}{k_1} \\ \frac{-1}{k_2} & \frac{1}{k_2} \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \dots(14)$$

and the system is a multivariable second order lag.

3.1. Decoupling and Asymptotic Root-loci

The control configuration for the system of equation (8) is indicated in Fig.2 where K(s,p) is defined by equations (2),(4),(5) and the minor loop compensation TFM takes the form

$$H(s) = \sum_{j=1}^{k} H_j s^{k-j} \qquad \dots(15)$$

representing inner loop state feedback. If complete state feedback is used, an analytic solution of the control problem can be derived as follows (Appendix 8.1).

Result 1 (Decoupling of kth order multivariable systems)

Choosing, $1\leqslant j\leqslant k$,

$$H_j = A_o \operatorname{diag}\{d_{ij}\}_{1\leqslant i\leqslant m} - A_j \qquad \ldots(16)$$

where d_{ij}, $1\leqslant i\leqslant m$, $1\leqslant j\leqslant k$ are real scalars, $K_2 = 0$ and

$$K_1(s) = A_o \operatorname{diag}\{k_j(s)\}_{1\leqslant j\leqslant m} \qquad \ldots(17)$$

where $\{k_j(s)\}_{1\leqslant j\leqslant m}$ are scalar transfer functions, then the closed-loop system transfer function matrix takes the form

$$H_c(s) = \operatorname{diag}\left\{ \frac{pg_j(s)k_j(s)}{1+pg_j(s)k_j(s)} \right\}_{1\leqslant j\leqslant m} \qquad \ldots(18)$$

where, $1\leqslant j\leqslant m$,

$$g_j(s) = \frac{1}{s^k+d_{j1}s^{k-1}+\ldots+d_{jk-1}s+d_{jk}} \qquad \ldots(19)$$

Equation (18) indicates that the closed-loop system is non-interacting, the forward path transmitances being characterized by the classical kth order lags $g_j(s)$, $1\leqslant j\leqslant m$, whose pole configuration can be specified by suitable choice of parameters d_{ij}, $1\leqslant i\leqslant m$, $1\leqslant j\leqslant k$. System compensation is achieved by suitable choice of compensation networks $k_j(s)$, $1\leqslant j\leqslant m$.

Although of theoretical interest, Result 1 requires direct measurement of the complete system state vector. In practice, for $k\geqslant 2$, this may be unacceptable or impossible. In such cases, a satisfactory design must be attempted by use of the degrees of freedom available in $K(s,p)$ together, possibly, with limited minor loop compensation. The following result (proved in Appendix 8.2) characterizes the closed-loop pole configuration in terms of fundamental structural properties of $K(s,p)$ and $H(s)$,

---

Result 2 (Asymptotic root-loci)

If $A_o^{-1}K_{1\infty}^{(o)}$ has a complete set of eigenvectors, with eigenvalues λ_j, $1 \leq j \leq m$, then the closed-loop system has km poles of the form, $1 \leq j \leq m$, $1 \leq \ell \leq k$,

$$\mu_{j\ell}(p) = p^{\frac{1}{k}}\eta_{j\ell} + \alpha_j + \varepsilon_{j\ell}(p) \qquad \ldots(20)$$

where $\eta_{j\ell}$, $1 \leq \ell \leq k$, are the kth roots of $-\lambda_j$, $p^{\frac{1}{k}}$ is the positive real kth root of p,

$$\lim_{p\to\infty} \varepsilon_{j\ell}(p) = 0 \qquad \ldots(21)$$

and α_j, $1 \leq j \leq m$, are the solutions of the equation

$$\{k\alpha_j I_m + A_o^{-1}\{A_1 + H_1 + K_2\delta_{k1} - K_{3\infty}^{(1)}(K_{1\infty}^{(o)})^{-1}A_o\}\}x_\infty \in R(\lambda_j I_m - A_o^{-1}K_{1\infty}^{(o)})$$

$$0 \neq x_\infty \in N(\lambda_j I_m - A_o^{-1}K_{1\infty}^{(o)}) \qquad \ldots(22)$$

Also,

$$k\sum_{j=1}^{m}\alpha_j = -\mathrm{tr}\{A_o^{-1}\{A_1 + H_1 + K_2\delta_{k1} - K_{3\infty}^{(1)}(K_{1\infty}^{(o)})^{-1}A_o\}\} \qquad \ldots(23)$$

and the remaining q closed-loop poles tend to the zeros[7] of $K_1(s)$.

The evaluation of α_j from equation (22) is discussed in Appendix 8.3.

By analogy with the classical terminology, equations (20),(21) suggest that the loci

$$\hat{\mu}_{j\ell}(p) = p^{\frac{1}{k}}\eta_{j\ell} + \alpha_j \quad , \quad p>0 \ , \quad 1 \leq j \leq m, \ 1 \leq \ell \leq k \qquad \ldots(24)$$

play the role of asymptotes of the system root-locus. The sets $\{\lambda_j\}_{1\leq j\leq m}$ and $\{\alpha_j\}_{1\leq j\leq m}$ are seen to be invariant under complex conjugation so the asymptotes are symmetrical about the real axis in the complex plane. By analogy with classical theory, $\eta_{j\ell}$ will be termed the direction of the asymptote $\hat{\mu}_{j\ell}(p)$ and α_j its intercept. The following list compares and contrasts Result 2 with the equivalent result for the classical kth order lag:

(a) In both cases, the closed-loop poles tend to infinity as fast of $p^{\frac{1}{k}}$.

(b) In both cases, dynamic compensation $K_3(s)$ has the effect of moving intercepts.

(c) In the multivariable case, both λ_j and α_j may be complex whereas, in the classical case, both λ and α are real eg taking $k = 1$, a multivariable first order type system can oscillate.

(d) The asymptotes of multivariable root-loci can be manipulated by a suitable form of controller, even if $K_1(s)$ is a constant proportional controller (see section 3.3) and $K_2 = 0$. In the classical case, asymptotes are defined uniquely by plant parameters.

Observations (a), (b) strengthen the analogy between multivariable kth order lags (eqn (8)) and the classical kth order lag (eqn (1)). Observations (c), (d) indicate, however, that multivariable systems possess more degrees of freedom in the sense that both the directions and intercepts of the root-locus can be manipulated by control action and hence represent design parameters available to the analyst.

Finally, in this section, apart from its obvious application in rough sketching and manipulation of root-loci, analysis of system asymptotes can yield valuable information on suitable controller structures. The following result follows directly from Result 2,

---

<u>Result 3</u> (Stability at high gain)

The closed-loop system is asymptotically stable at high gains p if, and only if, $k \leq 2$ and

(i) if $k = 1$, $\quad \mathrm{Re}\, \lambda_j > 0$, $\quad 1 \leq j \leq m$

(ii) if $k = 2$, λ_j, $1 \leq j \leq m$ must be real, strictly positive and $\mathrm{Re}\alpha_j < 0$, $1 \leq j \leq m$

(iii) the zeros(7) of $K_1(s)$ lie in the open left-half complex plane.

This result could, in practice, be used to assess a given controller structure (derived from physical considerations or engineering constraints) and suggest controller improvements. This is discussed by example in section 4.

3.2. Root-loci and Transient Performance

The usefulness of multivariable root-loci as a design tool will depend upon the availability of a working correspondence between the closed-loop poles, the transient performance and, in particular, the interaction behaviour of the closed-loop system. Firstly note that, taking the case of $k = 1$ and $H(s) \equiv 0$, for simplicity, and using equation (9),

$$\lim_{p\to\infty} p^{-1} \lim_{s\to\infty} s\{I_m+G(s)K(s,p)\}^{-1}G(s)K(s,p)$$

$$= G_\infty^{(1)}K_{1\infty}^{(o)} = A_o^{-1}K_{1\infty}^{(o)} \qquad \ldots(25)$$

which represents the initial derivative of the closed-loop system to unit step demands in output ie the matrix $A_o^{-1}K_{1\infty}^{(o)}$ (and hence its eigenvalue and eigenvector structure) provides a rough assessment of interaction effects at high gain in the vicinity of $t = 0+$. It seems that an approximately diagonal or diagonally dominant $A_o^{-1}K_{1\infty}^{(o)}$ is a necessary condition for small interaction effects.

In more general situations, let $\{u_j\}_{1\leq j\leq m}$ be the eigenvectors of $A_o^{-1}K_{1\infty}^{(o)}$ with dual eigenvectors $\{v_j^+\}_{1\leq j\leq m}$ satisfying $v_j^+u_k = \delta_{jk}$ and, for simplicity, take the case of $k = 1$ and proportional control. Note that

$$A_o^{-1}K_{1\infty}^{(o)} = \sum_{j=1}^{m} \lambda_j u_j v_j^+ \qquad \ldots(26)$$

and define

$$N = \sum_{j=1}^{m} \alpha_j u_j v_j^+ \qquad \ldots(27)$$

then, at high gains p, the closed-loop TFM can be approximated as follows (equations (8),(2)),

$$\{I_m+G(s)K(s,p)\}^{-1}G(s)K(s,p) = \{G^{-1}(s)+K(s,p)\}^{-1}K(s,p)$$
$$= \{sI_m+A_o^{-1}A_1+pA_o^{-1}K_1-N+A_o^{-1}K_2+N\}^{-1}\{pA_o^{-1}K_1+A_o^{-1}K_2\}$$
$$\simeq \{sI_m+pA_o^{-1}K_1-N\}^{-1}pA_o^{-1}K_1$$

$$= \sum_{j=1}^{m} \frac{p\lambda_j}{(s+p\lambda_j-\alpha_j)} u_j v_j^+ \qquad \ldots(28)$$

Hence, for small interaction effects at all high gains, either $u_j = e_j$, $1\leq j\leq m$ (where $\{e_j\}_{1\leq j\leq m}$ is the natural basis in R^m) when $v_j = e_j$, $1\leq j\leq m$, and

$$\{I_m+G(s)K(s,p)\}^{-1}G(s)K(s,p) \simeq \text{diag}\left\{\frac{p\lambda_j}{(s+p\lambda_j-\alpha_j)}\right\}_{1\leq j\leq m} \qquad \ldots(29)$$

or the asymptotes of the system should coincide (ie $\lambda_j = \lambda_k$ and $\alpha_j = \alpha_k$, $1\leq j,\ k\leq m$) when, noting that $\sum_{j=1}^{m} u_j v_j^+ = I_m$,

$$\{I_m+G(s)K(s,p)\}^{-1}G(s)K(s,p) \simeq \frac{p\lambda_1}{s+p\lambda_1-\alpha_1} I_m \qquad \ldots(30)$$

The above analysis implies that the choice of controller structure and manipulation of system asymptotes could play an important role in the synthesis of high performance feedback systems. The asymptotic directions are easily specified by suitable choice of eigenvalues of $A_o^{-1}K_{1\infty}^{(o)}$. The manipulation of system intercepts is discussed in the next section.

3.3. Manipulation of Asymptotes and Sensitivity Problems

In this section, some techniques for the manipulation of system intercepts are illustrated and a discussion of sensitivity is presented.

Examination of equation (22) indicates that minor-loop compensation (represented by H_1) provide a degree of freedom for the manipulation of system intercepts and, in the case of $k = 1$, the matrix K_2 provides an equivalent alternative. For simplicity, take the case of $K_1(s)$ fixed, $K_3(s) \equiv 0$, $K_2 = 0$ and assume that $A_o^{-1}K_{1\infty}^{(o)}$ has distinct eigenvalues

$\lambda_j \neq \lambda_k$ $(j \neq k)$. Using the notation of section 3.2, the intercepts of the uncompensated system are the solutions, $1 \leq j \leq m$, of

$$\{k\alpha_j I_m + A_o^{-1}A_1\}u_j \in R(\lambda_j I_m - A_o^{-1}K_{1\infty}^{(o)}) \qquad \ldots(31)$$

Let $\hat{\alpha}_1,..,\hat{\alpha}_m$ be the desired intercepts satisfying the constraint $\lambda_j = \bar{\lambda}_\ell$ implies $\hat{\alpha}_j = \hat{\alpha}_\ell$, then define

$$H_1 = kA_o \sum_{j=1}^{m} (\alpha_j - \hat{\alpha}_j)u_j v_j^+ \qquad \ldots(32)$$

Note that $H_1 = \bar{H}_1$ (ie H_1 is real), $A_o^{-1}H_1u_j = k(\alpha_j - \hat{\alpha}_j)u_j$, so that (equation (31))

$$\{k\hat{\alpha}_j I_m + A_o^{-1}\{A_1 + H_1\}\}u_j = \{k\alpha_j + A_o^{-1}A_1\}u_j \in R(\lambda_j I_m - A_o^{-1}K_{1\infty}^{(o)}) \qquad \ldots(33)$$

and hence (Result 2), $\hat{\alpha}_j$, $1 \leq j \leq m$, are the desired intercepts.

If minor loop compensation is not acceptable, then a similar result can be obtained by forward path dynamic compensation $K_3(s)$. Take $K_{1\infty}^{(o)}$ to be fixed, $K_2 = 0$, $H_1 = 0$, and $\lambda_j \neq \lambda_k$ $(j \neq k)$ and let $K_1(s)$ be the dyadic TFM[5],

$$K_1(s) = A_o \sum_{j=1}^{m} \frac{(s+a_j)}{(s+b_j)} u_j v_j^+ A_o^{-1} K_{1\infty}^{(o)} \qquad \ldots(34)$$

so that $$K_3(s) = A_o \sum_{j=1}^{m} \frac{a_j - b_j}{s+b_j} u_j v_j^+ A_o^{-1} K_{1\infty}^{(o)} \qquad \ldots(35)$$

$$K_{3\infty}^{(1)} = A_o \sum_{j=1}^{m} (a_j - b_j) u_j v_j^+ A_o^{-1} K_{1\infty}^{(o)} \qquad \ldots(36)$$

Hence, $1 \leq j \leq m$, applying Result 2, equation (31) and noting that $K_{3\infty}^{(1)}(K_{1\infty}^{(o)})^{-1}A_o u_j = A_o(a_j - b_j)u_j$, $1 \leq j \leq m$, then

$$\{\{k\alpha_j + a_j - b_j\}I_m + A_o^{-1}\{A_1 - K_{3\infty}^{(1)}(K_{1\infty}^{(o)})^{-1}A_o\}\}u_j$$
$$= \{k\alpha_j I_m + A_o^{-1}A_1\}u_j \in R(\lambda_j I_m - A_o^{-1}K_{1\infty}^{(o)}) \qquad \ldots(37)$$

so that, the intercepts of the compensated system are $\alpha_j + k^{-1}(a_j - b_j)$, $1 \leq j \leq m$.

The use of forward path and minor loop compensation in the manipulation of asymptotes is well known in classical theory. The following result illustrates an unusual characteristic of multi-input, multi-output systems in that, if proportional control is used, the system intercepts can be manipulated by suitable choice of eigenvectors of $A_o^{-1}K_{1\infty}^{(o)}$.

Result 4 (see Appendix 8.4)

If K_2, H_1, are fixed, $K_3(s) \equiv 0$ and $\lambda_j \neq \lambda_k$ $(j \neq k)$ then, for any set of numbers $\{\hat{\alpha}_j\}_{1 \leq j \leq m}$ (invariant under complex conjugation) satisfying

$$k \sum_{j=1}^{m} \hat{\alpha}_j = -\mathrm{tr}\{A_o^{-1}\{A_1 + H_1 + K_1\delta_{k1}\}\} \qquad \ldots(38)$$

and the requirement $\lambda_j = \bar{\lambda}_\ell$ implies $\hat{\alpha}_j = \bar{\hat{\alpha}}_\ell$, then there exists a real constant matrix $K_1(s) = K_1$ such that the system intercepts are $\hat{\alpha}_1, \ldots, \hat{\alpha}_m$.

Care must be taken in the application of Result 4 in the manipulation of intercepts if any of the eigenvalues λ_j, $1 \leq j \leq m$, are distinct but close together. In such cases, in the practical range of gains, the root-locus may possess apparent intercepts very different to those predicted by Result 2 and hence, the asymptotic analysis provides little information on closed-loop pole positions. To illustrate this problem, consider the multivariable first-order type system

$$G^{-1}(s) = sI_2 + \mathrm{diag}\,\{2,1\} \qquad \ldots(39)$$

Take $K_2 = 0$, $H_1 = 0$ and

$$K_1(s) = \begin{bmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{bmatrix} \qquad \ldots(40)$$

so that $A_o^{-1}K_{1\infty}^{(o)}$ has eigenvalues $\lambda_1 = 1+\varepsilon$ and $\lambda_2 = 1-\varepsilon$. By direct calculation, the closed-loop poles are

$$\mu(p) = -\frac{3}{2} + p \pm \sqrt{\tfrac{1}{4} + \varepsilon^2 p^2} \qquad \ldots(41)$$

ie if $\varepsilon = 0$, the intercepts are $\alpha_1 = -1$, $\alpha_2 = -2$, whereas, if $\varepsilon \neq 0$, the intercepts are $\alpha_1 = \alpha_2 = -\frac{3}{2}$. If ε is small and non-zero, the root-locus in the range of practical gains will behave as if the asymptotes are -p-1, -p-2 and the drift towards the actual asymptotes will be slow. Note however, that, by suitable choice of K_2, the sensitivity problem can be removed in this case eg choosing

$$K_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad \ldots(42)$$

the intercepts are $\alpha_1 = \alpha_2 = -2$, independent of the value of ε.

4. Multivariable First-order Lags

Consider the application of the results of the previous section in the case of k = 1.

4.1. Decoupling of First Order Systems

Consider the unstable multivariable first order system,

$$G(s) = \frac{1}{(s^2-1)} \begin{bmatrix} -1 & s-2 \\ 2 & 2s-1 \end{bmatrix} \qquad \ldots(43)$$

$$G^{-1}(s) = s \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \qquad \ldots(44)$$

so that (equation (8))

$$A_o = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} \quad , \quad A_1 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \qquad \ldots(45)$$

Applying Result 1 with k = 1, then $H(s) = H_1$ (a constant matrix). Choosing $d_{i1} = 1$, i = 1,2 then (equation (16))

$$H(s) \equiv H_1 = A_o - A_1 = \begin{bmatrix} -3 & 3 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix} \qquad \ldots(46)$$

Choosing (equation (17)) $K_1(s)$ to be the proportional controller

$$K(s,p) = pK_1(s) = p \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \qquad \ldots(47)$$

then, by direct calculation, the closed-loop TFM takes the form

$$H_c(s) = \mathrm{diag}\left\{ \frac{pk_1}{s+1+pk_1} \ , \quad \frac{pk_2}{s+1+pk_2} \right\} \qquad \ldots(48)$$

ie the closed-loop system is noninteracting and the desired transient characteristics can easily be obtained by suitable choice of gains k_1, k_2 and p.

4.2. Controller Assessment

The results of Section 3 can be used to evaluate trial controllers obtained from physical reasoning or limited by practical engineering considerations and can suggest suitable controller structures by simplification of relations (20),(22) using the degrees of freedom available in $K_1(s)$ and K_2. For example, consider the case of proportional plus integral control ($H(s) \equiv 0$)

$$K(s,p) = p\{K_{1\infty}^{(o)} + \frac{K_{3\infty}^{(1)}}{s}\} + K_2 \qquad \ldots(49)$$

On intuitive grounds[4], the controller of the form $K_2 = 0$, $K_{1\infty}^{(o)} = A_o$, $K_{3\infty}^{(1)} = cA_1$, (see equation (7)) has been proposed. It follows directly from Result 2 that $A_o^{-1}K_{1\infty}^{(o)} = I_m$ so that $\lambda_j = 1$, $1 \leqslant j \leqslant m$, and the intercepts are the solutions of the eigenvalue equation

$$\{\alpha_j I_m + A_o^{-1}A_1(1-c)\}x_\infty = 0 \quad , \qquad x_\infty \neq 0 \qquad \ldots(50)$$

ie $\alpha_j = -(1-c)p_j$, where $\{p_j\}_{1 \leqslant j \leqslant m}$ are the eigenvalues of $A_o^{-1}A_1$. The closed-loop system is therefore stable at high gains if, and only if, the zeros[7] of $K_1(s)$ lie in the open left-half complex plane ie the roots of the polynomial

$$s^m|K_{1\infty}^{(o)} + s^{-1}K_{3\infty}^{(1)}| \equiv |A_o| \cdot |sI_m + A_o^{-1}A_1 c| \qquad \ldots(51)$$

have strictly negative real parts. Noting (equation (7)) that $A_o^{-1}A_1 = -CAC^{-1}$, this is satisfied for all open-loop stable systems if $c>0$.

In section 3.2, intuitive conditions for a high performance, low-interaction feedback system were derived. In particular, it was shown that a low interaction feedback system can be expected if all system asymptotes are identical. For example, using the controller of equation (49), and choosing $K_{1\infty}^{(o)} = A_o$ then (Result 2) the asymptotic directions are all equal to -1, $1 \leq j \leq m$, implying stability at high gain. The intercepts (eqn (22)) are the solution of the eigenvalue problem,

$$\{\alpha_j I_m + A_o^{-1}(A_1 + K_2 - K_{3\infty}^{(1)}(K_{1\infty}^{(o)})^{-1}A_o)\}x_\infty = 0 \quad , \qquad x_\infty \neq 0 \qquad \ldots(52)$$

Choosing K_2, $K_{3\infty}^{(1)}$ to be solutions of the equation (α a real scalar)

$$A_o^{-1}(A_1 + K_2 - K_{3\infty}^{(1)}) = \alpha I_m \qquad \ldots(53)$$

then the system asymptotes are $\rho_j(p) = -p-\alpha$, $1 \leq j \leq m$, implying small interaction effects at high gain. Equation (53) has an infinite number of solutions eg choosing $\alpha = 0$ and $K_{3\infty}^{(1)} = cA_1$, then, solving for K_2, the controller $K(s,p)$ becomes (equations (2),(5))

$$K(s,p) = \{p + c + \frac{pc}{s}\}A_o - A_1 \qquad \ldots(54)$$

Such a controller has been suggested previously[1] and is capable of producing a high performance feedback system with fast response speed, zero steady state error and arbitrarily small interaction effects.

4.3. Example of Regulator Design with Controller Constraints

Consider the unstable first order type multivariable system of equations (43)-(45), and suppose that, due to engineering constraints, a diagonal proportional controller is required of the form $H(s) \equiv 0$, and $K(s,p) = p\,\mathrm{diag}\{x_1, x_2\}$ where p is the gain parameter of sections 2,3 and x_1, x_2 are scalar gain factors. Choosing $x_2 = 1$, the regulator design can be regarded as two steps,

(a) An investigation of the effect of the relative gain factor x_1 on the structure of the root-locus plot and the choice of x_1 to ensure stability at high gain.

(b) Root-locus analysis of the resulting system and the choice of a suitable gain p.

Applying Result 2 with $K_{1\infty}^{(o)} = \text{diag}\{x_1,1\}$, the eigenvalues of $A_o^{-1}\text{diag}\{x_1,1\}$ are

$$\lambda = 1 \pm \sqrt{1+x_1} \qquad \ldots(55)$$

so the closed-loop system is asymptotically stable at high gains if, and only if, $x_1<0$. Moreover, the relative gain parameter x_1 can be used to choose the degree of oscillation desired in the closed-loop system eg. choosing $x = -2$, the asymptotic directions take the form $-1\pm i$. The intercept (Result 2) α_1 is the solution of the equation,

$$\{\alpha_1 I_2 + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\}\begin{bmatrix} 1 \\ 1+i \end{bmatrix} = \beta\begin{bmatrix} 1 \\ 1-i \end{bmatrix} \qquad \ldots(56)$$

for some complex scalar β ie. $\alpha_1 = -i/2$ and, in a similar manner, $\alpha_2 = \bar{\alpha}_1$. The asymptotes are hence

$$\hat{\rho}_1(p) = -p(1+i) - i/2$$
$$\hat{\rho}_2(p) = -p(1-i) + i/2 \qquad \ldots(57)$$

which are sketched in Fig.3, together with the actual root-locus plot. It is seen that the root-locus plots approach the asymptotes rapidly, and a stable design can be achieved by suitable choice of p.

In summary, the use and manipulation of root-loci asymptotes has led to a systematic regulator design procedure. The asymptotes are seen to be a good representation of the system poles and the design procedure can be decomposed into the choice of relative gain to ensure stability at high gain and the subsequent choice of overall controller gain p.

5. Multivariable Second-order Lags

Consider the application of the results of section 3 in the case of k = 2. In section 5.1, the Result 1 is applied to the second order multivariable lag of equations (12)-(14). In section 5.2, it is shown that the sensitivity problem discussed in section 3.3 leads directly to an analytic solution to the feedback design problem. Finally, in section 5.3, the results of 5.2 are illustrated by an example.

5.1. Decoupling of Second-order Systems

Consider the multivariable second order type system defined by equations (12)-(14) with $m_1 = 1$, $m_2 = 2$, $k_1 = k_2 = 1$, $c = 1$. Applying Result 1 with $k = 2$, then $H(s) = H_1 s + H_2$, where

$$H_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} d_{11} & 0 \\ 0 & d_{21} \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \ldots(58)$$

$$H_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} d_{12} & 0 \\ 0 & d_{22} \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \ldots(59)$$

Choosing $d_{11} = d_{21} = 3$, $d_{12} = d_{22} = 2$, then

$$H(s) = \begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix} s + \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \qquad \ldots(60)$$

so, if

$$K(s,p) = p \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} k_1(s) & 0 \\ 0 & k_2(s) \end{bmatrix} \qquad \ldots(61)$$

then, by direct calculation, the closed-loop TFM has the form,

$$H_c(s) = \mathrm{diag}\left\{ \frac{k_1(s)}{s^2+3s+2+k_1(s)} \, , \, \frac{k_2(s)}{s^2+3s+2+k_2(s)} \right\} \qquad \ldots(62)$$

which is a non-interacting system.

5.2. Analytic Solution to the Control Problem

An analytic solution to the second order design problem can be obtained by the use of non-zero K_2 and an analysis of the sensitivity of the asymptotes (section 3.3). Taking, for simplicity, $H(s) \equiv 0$, then (eqns (9),(5),(2)),

$$\lim_{p\to\infty} p^{-1} \lim_{s\to\infty} s^{a} \{I_m + G(s)K(s,p)\}^{-1} G(s)K(s,p) = G_\infty^{(2)} K_{1\infty}^{(o)} = A_o^{-1} K_{1\infty}^{(o)} \qquad \ldots(63)$$

so the initial acceleration of the closed-loop system in response to step demands is represented by $A_o^{-1}K_{1\infty}^{(o)}$. Intuitively, if a low interaction system and similar response speeds are required from each channel, $A_o^{-1}K_{1\infty}^{(o)}$ will differ only slightly from the unit matrix ie its eigenvalues will be almost equal. As illustrated in section 3.3, the intercepts of the root-locus will be sensitive functions of the eigenvector structure of $A_o^{-1}K_{1\infty}^{(o)}$ in the sense that the practical range of gains will produce a root-locus diagram with apparent asymptotes $\pm i+\alpha_j$ (Result 2) where

$$\{2\alpha_j I_m + A_o^{-1}\{A_1+H_1-K_{3\infty}^{(1)}(K_{1\infty}^{(o)})^{-1}A_o\}\}x_\infty = 0 \quad , \quad x_\infty \neq 0 \qquad \ldots(64)$$

Taking $K_3(s) \equiv 0$, the apparent intercepts are the eigenvalues of $-\frac{1}{2}A_o^{-1}\{A_1+H_1\}$ ie this matrix plays an important role in root-locus structure and hence it could be expected that an analysis of its eigenvalue/eigenvector structure may lead to a considerable simplification of the design process. The following result is proved in Appendix 8.5.

Result 5

If $A_o^{-1}(A_1+H_1)$ has a complete set of eigenvectors $\{u_j\}_{1\leq j\leq m}$ with eigenvalues $\{\gamma_j\}_{1\leq j\leq m}$ and dual row eigenvectors $\{v_j^+\}_{1\leq j\leq m}$ satisfying $v_j^+u_k = \delta_{jk}$, $1\leq j, k\leq m$, then choosing

$$H_2 + K_2 = A_o \sum_{j=1}^{m} \omega_j u_j v_j^+ - A_2 \qquad \ldots(65)$$

$$K_1(s) = A_o \sum_{j=1}^{m} k_j(s) u_j v_j^+ \qquad \ldots(66)$$

where $\{\omega_j\}_{1\leq j\leq m}$ is a set of scalars and $\{k_j(s)\}_{1\leq j\leq m}$ is a set of transfer functions satisfying the constraint that $u_j = \bar{u}_\ell$ implies $\omega_j = \bar{\omega}_\ell$ and $\overline{k_j(s)} = k_\ell(\bar{s})$, then

$$\overline{K(s,p)} = K(\bar{s},p) \qquad \ldots(67)$$

so that $K(s,p)$ is a physically realizable TFM. Moreover, the closed-loop poles are the zeros of the rational polynomial

$$\prod_{j=1}^{m} (s^2+\gamma_j s+\omega_j+pk_j(s)) \qquad \ldots(68)$$

and the closed-loop TFM takes the form

$$\sum_{j=1}^{m} \frac{pk_j(s)}{(s^2+\gamma_j s+\omega_j+pk_j(s))} u_j v_j^+ + E(s,p) \qquad \ldots(69)$$

where the contribution of $E(s,p)$ to the closed-loop step responses tends to zero as $p\to\infty$.

In practical terms, equation (68) states that the root-locus diagram of the closed-loop system can be investigated by an analysis of the root-loci plots of the m single-input, single-output systems

$$g_j(s) = \frac{k_j(s)}{s^2+\gamma_j s+\omega_j} \quad , \qquad 1\leqslant j\leqslant m \qquad \ldots(70)$$

ie the stability analysis of the system is reduced to the stability analysis of m non-interacting single-input, single-output systems, to which well-known classical compensation techniques can be applied. Equation (69) provides direct insight into the resultant closed-loop response so, if high gains are applied, a good working model of the closed-loop system is

$$H_c(s) \doteq \sum_{j=1}^{m} \frac{pg_j(s)}{1+pg_j(s)} u_j v_j^+ \qquad \ldots(71)$$

ie, for small closed-loop interaction effects,

either (a) choose H_1 so that $u_j = e_j$, $1\leqslant j\leqslant m$

or (b) choose H_1 and $\{k_j(s)\}_{1\leqslant j\leqslant m}$ so that the closed-loop step responses of the systems $pg_j(s)/(1+pg_j(s))$, $1\leqslant j\leqslant m$, are similar.

Condition (a) can be attained by choosing

$$H_1 = A_o \operatorname{diag}\{\gamma_j\}_{1\leq j\leq m} - A_1 \qquad \ldots(72)$$

Condition (b) can be attained by suitable choice of compensation networks $\{k_j(s)\}_{1\leq j\leq m}$ to make the dominant poles of the systems $pg_j(s)/(1+pg_j(s))$ approximately equal. This technique is illustrated in the next section using an example.

5.3. Illustrative Example

Consider the multivariable second order lag,

$$G^{-1}(s) = A_o s^2 + A_1 s + A_2$$

$$\triangleq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} s^2 + \begin{bmatrix} 7 & 4 \\ 4 & 7 \end{bmatrix} s + \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} \qquad \ldots(73)$$

and the design of a forward path compensator $K(s,p)$ to produce a high performance, low interaction feedback system. Assume that minor loop compensation is not acceptable so that $H(s) \equiv 0$.

Applying Result 5, the eigenvectors of $A_o^{-1}A_1$ are $u_1 = \{1\ 1\}^T$ and $u_2 = \{-1\ 1\}^T$ corresponding to the eigenvalues $\gamma_1 = 11$, $\gamma_2 = 3$ respectively and dual eigenvectors $v_1^+ = \{1\ 1\}/2$. $v_2^+ = \{-1\ 1\}/2$. Choosing $\omega_1 = 1$, $\omega_2 = 2$, then (equation (70))

$$g_1(s) = \frac{k_1(s)}{s^2+11s+1} \quad , \quad g_2(s) = \frac{k_2(s)}{s^2+3s+2} \qquad \ldots(74)$$

The subsystem $g_1(s)$ is stable and highly overdamped so that proportional control action $k_1(s) = k_1$ is quite adequate, and, using classical analysis, the value $k_1 = 59.5$ is assumed. With proportional control the subsystem $g_2(s)$ is much less responsive than $g_1(s)$, so, using the results of section 5.2, the design requirement of low interaction demands that phase advance be introduced into $g_2(s)$ to increase its response speed. Choosing $k_2(s) = k_2(s+2)/(s+10)$, then the intercepts of the root loci of $g_1(s)$ and

$g_2(s)$ are identical, and choosing $k_2 = 50.5$, the dominant closed-loop poles in both subsystems are identical, implying similar step responses.

The resulting controller is defined by (Result 5),

$$K_2 = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -5 \\ -1 & -5 \end{bmatrix} \qquad \ldots(75)$$

$$K_1(s) = 59.5\begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 0.5 & 0.5 \end{bmatrix} + 50.5\,\frac{(s+2)}{(s+10)}\begin{bmatrix} -1 \\ 1 \end{bmatrix}\begin{bmatrix} -0.5 & 0.5 \end{bmatrix} \qquad \ldots(76)$$

and equation (2) with $p = 1$. The closed-loop responses to unit step demand in output are illustrated in Fig.4, where it is seen that the system responds rapidly to step demands with acceptable steady state error and small interaction. Residual steady state error can be removed by the inclusion of integral action in $k_1(s)$ and $k_2(s)$.

6. Summary and Conclusions

The paper has extended previous work[1,2] by presenting a definition and complete analysis of a new class of multivariable systems analogous to the classical kth order lag. The techniques used are decoupling using minor loop (state) feedback and a multivariable generalization of the classical root-locus method. It has been shown that the closed-loop system can be decoupled (Result 1) by the use of a minor loop compensator $H(s)$ (Fig.2) whose structure is well-defined in terms of parameters of the inverse TFM $G^{-1}(s)$. In such a case the control analysis reduces to the analysis of m non-interacting single-input, single-output systems. In more general situations, if complete state feedback is not acceptable, analysis of the asymptotic behaviour of the system root-locus provides a valuable technique for:

(i) ascertaining (Result 3) necessary and sufficient conditions on controller structure to ensure closed-loop stability at high gain. In particular, it is seen that stability at high gain is only possible if $k \leq 2$ and that the conditions for the cases $k = 1$, $k = 2$ indicate that design difficulties increase as k increases.

(ii) assessing (section 4.2 and 4.3) controllers deduced from physical reasoning or engineering constraints.

(iii) provides valuable insight (section 3.2) into closed-loop behaviour, and

(iv) suggests (sections 4.2, 5.2) systematic analytic design techniques for the cases of $k = 1,2$.

Result 2 indicates that the root-locus structure of a multivariable kth order lag is a direct generalization of that for the corresponding classical system (eqn (1)). Asymptotes can be manipulated systematically by suitable forward path compensation networks or minor loop compensation and, in contrast to the classical case, system intercepts can be manipulated by choice of controller structure even if proportional control is used. Care must be taken if this is used as a design tool as (section 3.3), in well defined circumstances, the asymptotes are sensitive functions of controller parameters.

Finally, in conjunction with previous results[1,2], it is noted that the classical concept of a kth order lag can be extended to the multivariable case, indicating that the basic difference between classical and multivariable systems lies in complexity of detail rather than overall structure. Using the observation that many physical systems can, for the purpose of control systems design, be approximated by much simpler reduced order models, the analysis of more complex structures using root-locus concepts may provide valuable insight into general design problems and make available a 'case-book' of controller structures to initiate and simplify the analysis of more complex structures using general design techniques.

7. References

1. D.H.Owens: 'First and second-order-like structures in linear multivariable control systems design', Proc.IEE, 122(9), 1975, pp.935-941.

2. D.H.Owens: 'Compensation of multivariable first order type systems', Electronics Letters, 11(21), 1975, pp.511-512.

3. H.H.Rosenbrock: 'Design of multivariable control systems using the inverse Nyquist array', Proc.IEE, 116(11), 1969, pp.1929-1936.

4. A.G.J.MacFarlane and J.J.Belletrutti: 'The characteristic locus design method', Automatica, 1973, 9, pp.575-588.

5. D.H.Owens: 'Dyadic expansion, characteristic loci and multivariable control systems design', Proc.IEE, 122(3), 1975, pp.315-320.

6. D.Q.Mayne: 'The design of linear multivariable systems', Automatica, 1973, 9, pp.201-207.

7. E.J.Davison, S.H.Wang: 'Properties and calculation of transmission zeros of linear multivariable systems', Automatica, 1974, 10, pp.643-658.

8. Appendices

8.1. Proof of Result 1

The closed-loop transfer function matrix follows directly from Fig.2.

$$\begin{aligned} H_c(s) &= \{I_m+\{I_m+G(s)H(s)\}^{-1}G(s)K(s,p)\}^{-1}\{I_m+G(s)H(s)\}^{-1}G(s)K(s,p) \\ &= \{I_m+G(s)\{H(s)+K(s,p)\}\}^{-1}G(s)K(s,p) \\ &= \{G^{-1}(s)+H(s)+K(s,p)\}^{-1}K(s,p) \qquad \ldots(77) \end{aligned}$$

Substituting from equation (8),(15),(16),(17),(2),(19)

$$\begin{aligned} H_c(s) &= \{A_o + \sum_{j=1}^{k} A_o \operatorname{diag}\{d_{ij}\}_{1\leq i\leq m} s^{k-j} + p\ \operatorname{diag}\{k_j(s)\}_{1\leq j\leq m}\}^{-1} pA_o \operatorname{diag}\{k_j(s)\}_{1\leq j\leq m} \\ &= \operatorname{diag}\left\{\frac{pg_j(s)k_j(s)}{1+pg_j(s)k_j(s)}\right\}_{1\leq j\leq m} \qquad \ldots(78) \end{aligned}$$

as required.

8.2. Proof of Result 2

The ratio of the closed-loop characteristic polynomial to the open-loop characteristic polynomial is given by

$$|I_m+G(s)\{H(s)+K(s,p)\}| = |G(s)|\cdot|G^{-1}(s)+H(s)+K(s,p)| \qquad \ldots(79)$$

so that the unbounded closed-loop poles are the solution of the eigenvalue problem

$$p^{-1}\{\sum_{j=o}^{k}(s(p))^j A_{k-j} + \sum_{j=o}^{k-1}(s(p))^j H_{k-j} + p\{K_{1\infty}^{(o)}+K_3(s)\}+K_2\}x(p) = 0 \qquad \ldots(80)$$

where $x(p)$ has Euclidean norm $\|x(p)\| = 1$, $p>0$. Letting $p\to\infty$, then the relation $|A_o| \neq 0$ implies that the Familly $(s(p))^k/p$ has a cluster point s_∞ satisfying

$$\{s_\infty A_o + K_{1\infty}^{(o)}\}\ x_\infty = 0 \quad , \quad \|x_\infty\| = 1 \qquad \ldots(81)$$

so that $-s_\infty$ is an eigenvalue λ_j of $A_o^{-1}K_{1\infty}^{(o)}$ and $x_\infty \in N(\lambda_j I_m - A_o^{-1}K_{1\infty}^{(o)})$. Let

$$s(p) = p^{\frac{1}{k}}\eta_{j\ell} + \psi_{j\ell}(p) \qquad \ldots(82)$$

where

$$\lim_{p\to\infty} p^{-\frac{1}{k}}\,\psi_{j\ell}(p) \qquad \ldots(83)$$

and $x(p) = x_\infty + z(p)$ where $z(p)\to 0$ as $p\to\infty$. Using equation (81), equation (80) becomes

$$\{\sum_{j=o}^{k}(s(p))^{j}A_{k-j} - ps_\infty A_o + \sum_{j=o}^{k-1}(s(p))^{j}H_{k-j} + pK_3(s(p)) + K_2\}x(p) = -p(s_\infty A_o + K_{1\infty}^{(o)})z(p) \qquad \ldots(84)$$

Using equation (83),

$$\lim_{p\to\infty}\frac{\{(s(p))^{r} - ps_\infty\delta_{rk}\}}{p^{1-k^{-1}}} = \begin{cases} 0 & ; \quad r<k-1 \\ \eta_{j\ell}^{k-1} & ; \quad r=k-1 \\ k\alpha_{j\ell}\eta_{j\ell}^{k-1} & ; \quad r=k \end{cases} \qquad \ldots(85)$$

where $\alpha_{j\ell} = \lim_{p\to\infty}\psi_{j\ell}(p)$ (if it exists), and (equation (82)),

$$\lim_{p\to\infty} p^{\frac{1}{k}}K_3(s(p))x(p) = \frac{1}{\eta_{j\ell}}K_{3\infty}^{(1)}x_\infty \qquad \ldots(86)$$

Dividing equation (84) by $p^{1-\frac{1}{k}}$, then (equation (85),(86)), letting $p\to\infty$,

$$\{k\eta_{j\ell}^{k-1}\alpha_{j\ell}A_o + \eta_{j\ell}^{k-1}(A_1+H_1) + K_2\delta_{k1} + \eta_{j\ell}^{-1}K_{3\infty}^{(1)}\}x_\infty \in R(s_\infty A_o + K_{1\infty}^{(o)}) \qquad \ldots(87)$$

or, as $\eta_{j\ell}^{k} = -\lambda_j$, $s_\infty = -\lambda_j$,

$$\{k\alpha_{j\ell}I_m + A_o^{-1}\{A_1+H_1+K_2\delta_{k1} - \lambda_j^{-1}K_{3\infty}^{(1)}\}\}x_\infty \in R(\lambda_j I_m - A_o^{-1}K_{1\infty}^{(o)}) \qquad \ldots(88)$$

Note that $\alpha_{j\ell}$ is finite, for, if it were infinite then $x_\infty \in R(\lambda_j I_m - A_o^{-1}K_{1\infty}^{(o)})$ contradicting the assumption that (i) $x_\infty \in N(\lambda_j I_m - A_o^{-1}K_{1\infty}^{(o)})$ and (ii) $A_o^{-1}K_{1\infty}^{(o)})$ has a complete set of eigenvectors. Equation (22) follows directly from (88) by substituting for $\lambda_j^{-1}x_\infty$ from (81), and obviously $\alpha_{j\ell} = \alpha_j$ is independent of ℓ. Equation (22) follows directly from the analysis of Appendix 8.3.

Finally, noting that

$$p^{-m}\ |I_m+G(s)\{H(s)+K(s,p)|\ \to\ |G(s)K_1(s)| \qquad (p\to\infty) \qquad \ldots(89)$$

then[7], the closed-loop system has a set of finite limit poles equal to the zeros of $G(s)K_1(s)$. From equation (4), the controller $K_1(s)$ has q zeros[7]. As (equations (8),(12)) G(s) has state dimension km and (Result 2) km poles are unbounded as $p\to\infty$, it follows directly that G(s) has no zeros and hence the remaining q closed-loop poles tend to the zeros of $K_1(s)$. This completes the proof.

8.3. Calculation of the Intercepts

For simplicity, consider the case of j = 1, and let λ_1 be an eigenvalue $A_o^{-1}K_{1\infty}^{(o)}$ of multiplicity ℓ. Let $u_1 \ldots u_\ell$ be linearly independent eigenvectors of $A_o^{-1}K_{1\infty}^{(o)}$ spanning the eigenspace corresponding to the eigenvalue λ_1, and $v_1^+ \ldots v_\ell^+$ the corresponding dual row eigenvectors satisfying $v_j^+u_k = \delta_{jk}$, $1\leq j, k\leq\ell$. Then equation (22) is equivalent to the relations, $1\leq j\leq\ell$,

$$v_j^+\{k\alpha_1 I_m+A_o^{-1}\{A_1+H_1+K_2\delta_{k1}-K_{3\infty}^{(1)}(K_{1\infty}^{(o)})^{-1}A_o\}\}\sum_{i=1}^{\ell}\beta_i u_i = 0 \qquad \ldots(90)$$

for some set of complex scalars β_i, $1\leq i\leq m$. If M is the $\ell x\ell$ matrix with elements

$$M_{ij} = -k^{-1}v_i^+A_o^{-1}\{A_1+H_1+K_2\delta_{k1}-K_{3\infty}^{(1)}(K_{1\infty}^{(o)})^{-1}A_o\}u_j \qquad \ldots(91)$$

and letting $\beta = (\beta_1,\beta_2,\ldots,\beta_\ell)^T$, then equation (90) can be written in the form

$$\{\alpha_1 I_\ell - M\}\beta = 0 \qquad \ldots(92)$$

so that the intercepts are the eigenvalues of the $\ell x\ell$ matrix M. Equivalently, if V is the matrix of eigenvectors of $A_o^{-1}K_{1\infty}^{(o)}$ whose first ℓ columns are equal to $u_1,\ldots,u_\ell$, then M is the $\ell x\ell$ matrix generated by the first ℓ rows and first ℓ columns of $-k^{-1}V^{-1}A_o^{-1}\{A_1+H_1+K_2\delta_{k1}-K_{3\infty}^{(1)}(K_{1\infty}^{(o)})^{-1}A_o\}V$.

Finally note that equation (23) follows directly by allowing j to vary in the range, $1 \leq j \leq m$, and noting that the trace relation is invariant under similarity transformation.

8.4. Proof of Result 4

For notational convenience let $N = -k^{-1}A_o^{-1}\{A_1+H_1\}$ and, using the analysis of Appendix 8.3, note that the result is proved if there exists a nonsingularity transformation V such that the diagonal elements of $V^{-1}NV$ are $\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_m$. We prove the result by induction. Suppose that there exists a transformation V_j such that the first j diagonal elements of $V_j^{-1}NV_j$ are $\hat{\alpha}_1, \ldots, \hat{\alpha}_j$. Consider the 2x2 matrix P_{j+1} generated by the $j+1, j+2$ rows and columns of $V_j^{-1}NV_j$. It is easily seen that there exists a nonsingular matrix W_{j+1} so that

$$W_{j+1}^{-1} P_{j+1} W_{j+1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad , \; b \neq 0 \qquad \ldots(93)$$

If $\hat{\alpha}_{j+1}$ is real, let

$$V_{j+1} = V_j \begin{bmatrix} I_j & 0 & 0 \\ 0 & W_{j+1} R_{j+1} & 0 \\ 0 & 0 & I_{m-j-2} \end{bmatrix} \qquad \ldots(94)$$

where $$R_{j+1} = \begin{bmatrix} 1 & 0 \\ \dfrac{\hat{\alpha}_{j+1}-a}{b} & 1 \end{bmatrix} \qquad \ldots(95)$$

then, it is easily shown that the $(j+1,j+1)$ element of $V_{j+1}^{-1}NV_{j+1}$ is $\hat{\alpha}_{j+1}$.

If $\hat{\alpha}_{j+1}$ is complex and $\hat{\alpha}_{j+2} = \overline{\hat{\alpha}_{j+1}}$, then, by sequential application of the above procedure, it is possible to assume that $a+d = 2\mathrm{Re}\hat{\alpha}_{j+1}$. In this case, let

$$R_{j+1} = \begin{bmatrix} 1 & 1 \\ x & \bar{x} \end{bmatrix} \qquad \ldots(96)$$

where $\text{Im}x \neq 0$, then, by suitable choice of x, it is easily verified, that the $(j+1,j+1)$ and $(j+2,j+2)$ elements of $V_{j+1}^{-1}NV_{j+1}$ are $\hat{\alpha}_{j+1}, \overline{\hat{\alpha}_{j+1}}$ respectively.

The result follows by letting j run in the range $1 \leqslant j \leqslant m$, and defining,

$$K_1 = V_m \,\text{diag}\,\{\lambda_j\}_{1\leqslant j\leqslant m}\; V_m^{-1} \qquad \ldots(97)$$

8.5. Proof of Result 5

The closed-loop TFM $H_c(s)$ takes the form, after some elementary manipulation,

$$H_c(s) = \{G^{-1}(s)+H(s)+K(s,p)\}^{-1}K(s,p) \qquad \ldots(98)$$

which, using equations (2),(8),(65),(66), takes the form

$$\begin{aligned} H_c(s) &= \{A_o s^2+(A_1+H_1)s+A_o \sum_{j=1}^{m} (\omega_j+pk_j(s))u_j v_j^{+}\}^{-1}\; pA_o \sum_{j=1}^{m} k_j(s)u_j v_j^{+}+K_2\} \\ &= \{s^2+A_o^{-1}(A_1+H_1)s+\sum_{j=1}^{m}(\omega_j+pk_j(s))u_jv_j^{+}\}^{-1}\{p\sum_{j=1}^{m}k_j(s)u_jv_j^{+}+A_o^{-1}K_2\} \end{aligned} \qquad \ldots(99)$$

Writing $A_o^{-1}(A_1+H_1) = \sum_{j=1}^{m} \gamma_j u_j v_j^{+}$, then, using the orthonormality of the eigenvectors $\{u_j\}$ and the dual set $\{v_j\}$,

$$H_c(s) = \sum_{j=1}^{m} \frac{pk_j(s)}{(s^2+\gamma_j s+\omega_j+pk_j(s))}\, u_j v_j^{+} + \sum_{j=1}^{m} \frac{1}{(s^2+\gamma_j s+\omega_j+pk_j(s))}\, u_j v_j^{+} A_o^{-1} K_2 \qquad \ldots(100)$$

Identifying the second term with $E(s,p)$ of equation (69), classical considerations indicate that the contribution of $E(s,p)$ to the closed-loop step responses decrease to zero as $p\to+\infty$, if the closed-loop system is stable. Note also that, by examination of the denominator terms of $H_c(s)$ the closed-loop poles of the system are the zeros of

$$\prod_{j=1}^{m} (s^2+\gamma_j s+\omega_j+pk_j(s)) \qquad \ldots(101)$$

Finally, the result follows by noting that $K_2 = \bar{K}_2$ and $\overline{K_1(s)} = K_1(\bar{s})$ so that $\overline{K(s,p)} = K(\bar{s},p)$.

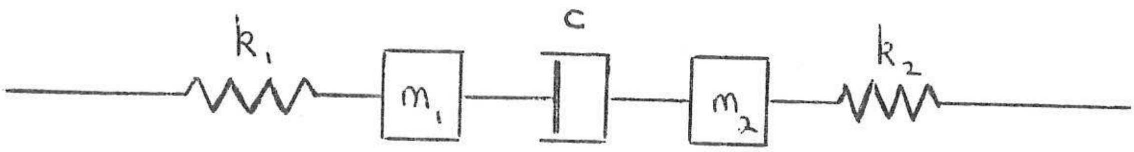

Fig 1.

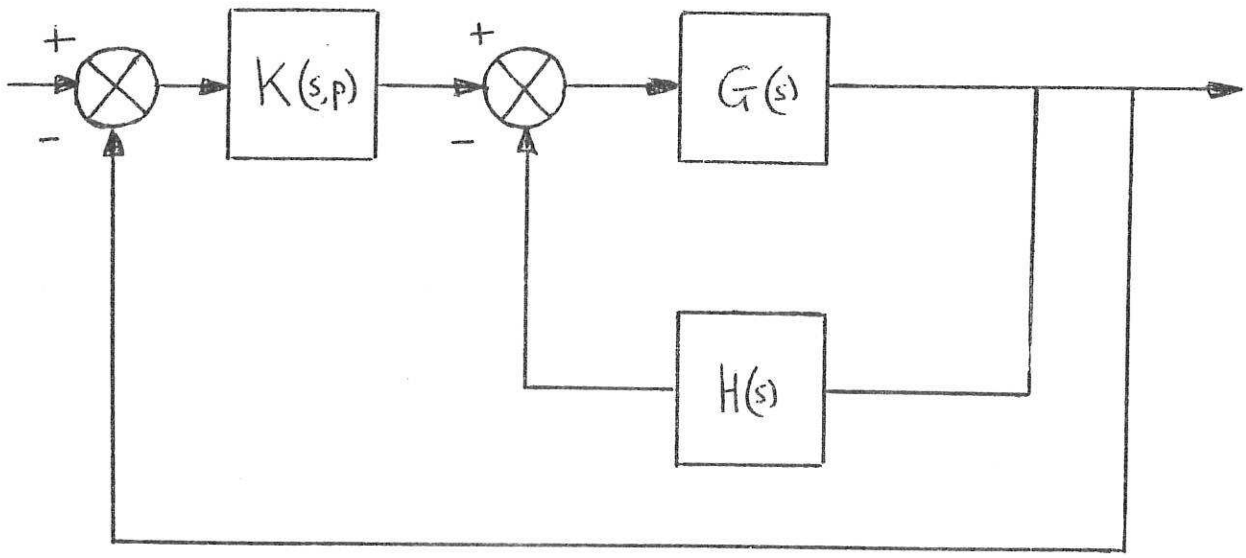

Fig. 2.

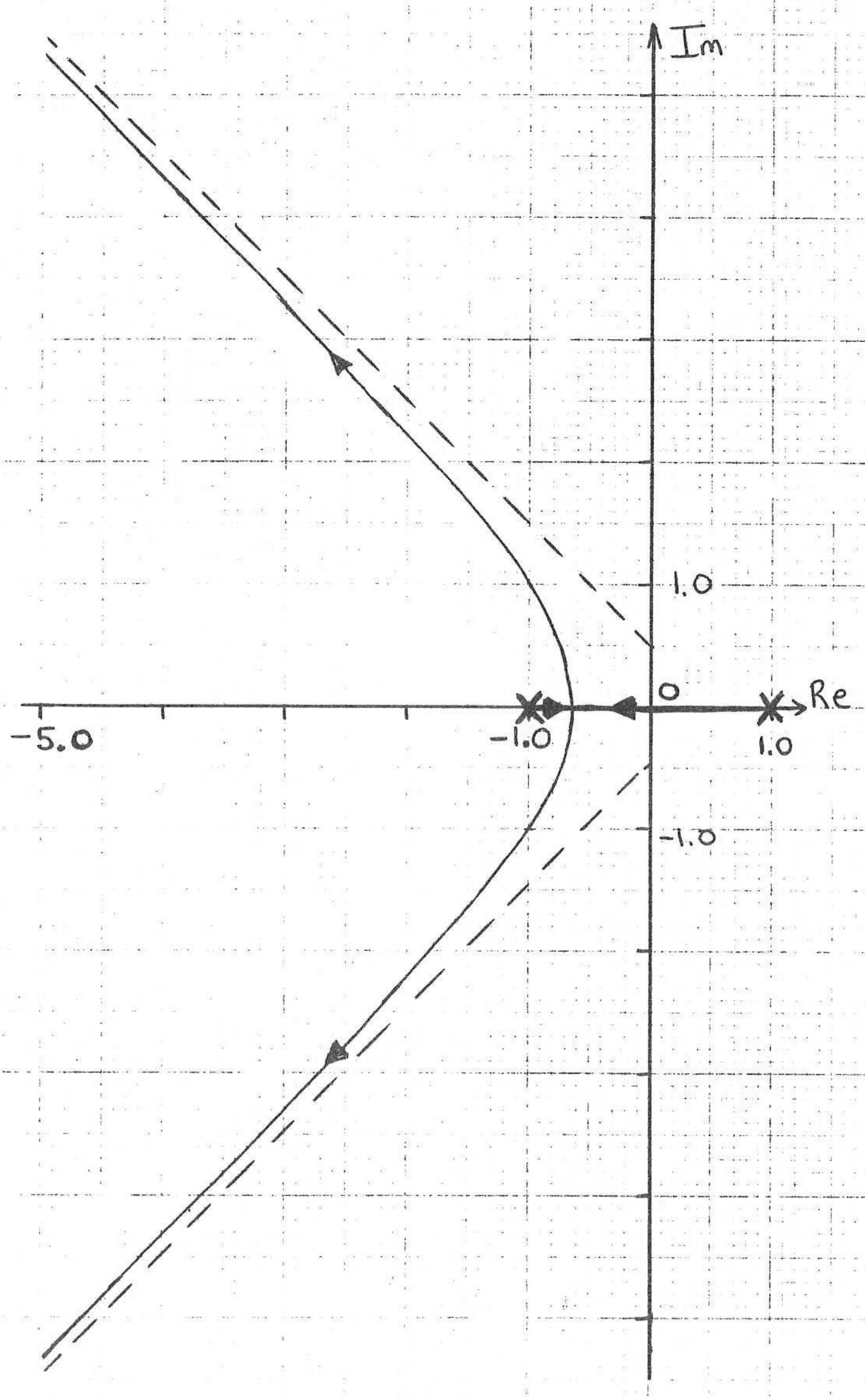

Fig. 3.

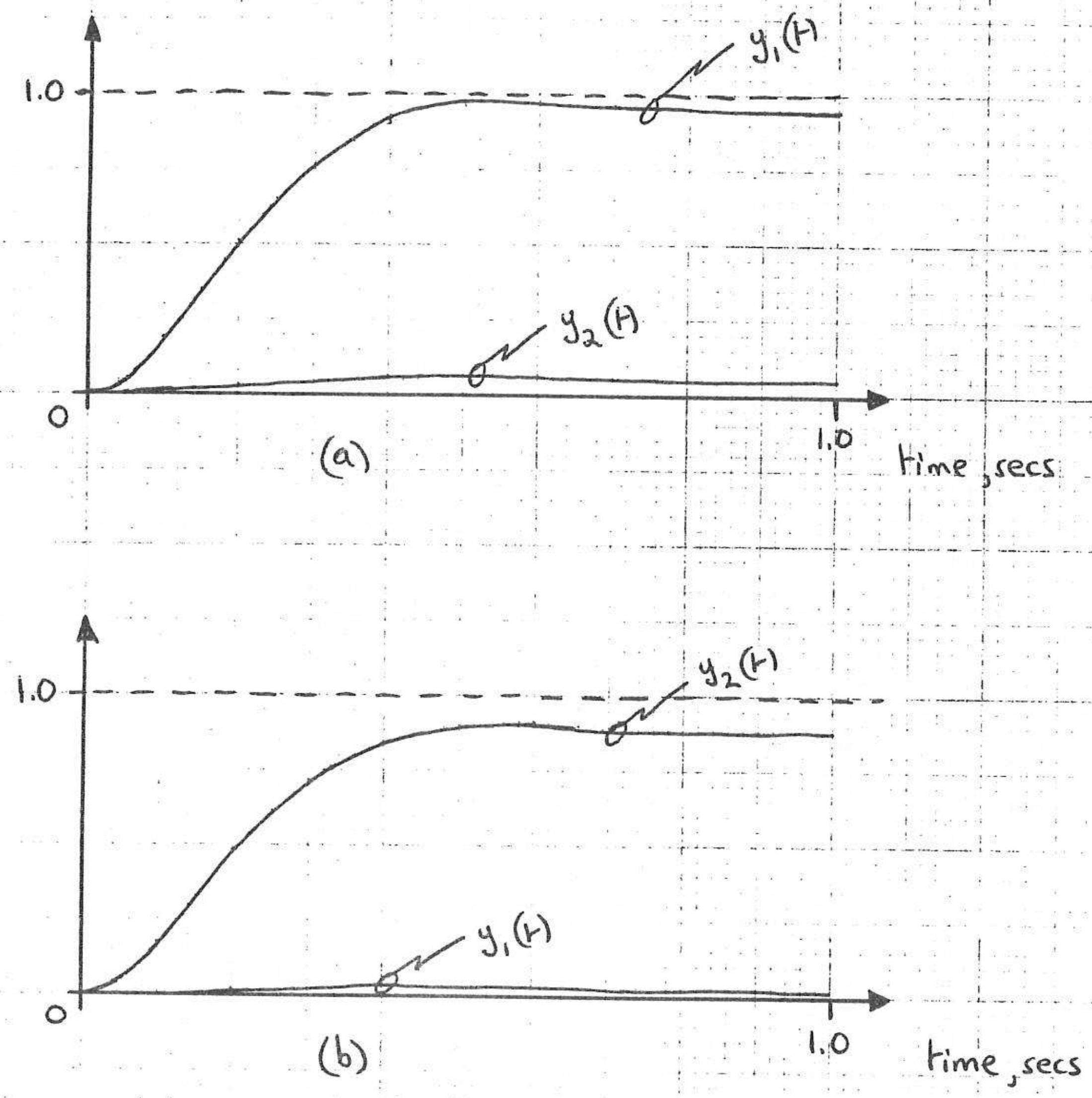

Fig. 4.