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ASYMPTOTIC ROOT-LOCI OF LINEAR MULTIVARIABLE
SYSTEMS: A GEOMETRIC ANALYSIS

by

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Abstract

Recent results on the asymptotic behaviour of the root-loci of a linear time-invariant system $S(A,B,C)$ are formulated in geometric terms and equivalent results obtained in two cases of practical interest in terms of the matrix coefficients in the expansion of $(A-pBC)^\ell$, $\ell \geq 1$.

1. Introduction

A recent paper (Shaked and Kouvaritakis, 1976) presented a theoretical analysis of the asymptotic behaviour of the eigenvalues of the linear, time-invariant system $S(A,B,C)$

$$\begin{aligned} \dot{x} &= Ax(t) + Bu(t) \quad , \quad u(t) \in R^m \quad , \quad x(t) \in R^n \\ y(t) &= Cx(t) \quad , \quad y(t) \in R^m \end{aligned} \quad \dots(1)$$

when subject to unity negative feedback with scalar gain $p \geq 0$. The closed-loop system takes the form

$$\begin{aligned} \dot{x}(t) &= \{A - pBC\}x(t) + Br(t) \\ y(t) &= Cx(t) \end{aligned} \quad \dots(2)$$

Previous work (Shaked and Kouvaritakis, 1976) used determinantal manipulation techniques to obtain explicit formula for the asymptotic directions and pivots of the root-locus. This paper describes some solutions of this problem by geometric analysis of the closed-loop eigenvalue equation

$$\{s(p)I_n - A + pBC\}x(p) = 0 \quad , \quad \|x(p)\| = 1 \quad , \quad p \geq 0 \quad \dots(3)$$

and the identification of the asymptotic directions and pivots in terms of the structural properties of the matrix coefficients in the expansion of $(A - pBC)^\ell$, $\ell \geq 1$.

2. Asymptotic Behaviour of Closed-loop Eigenvectors

Let $\{p_j\}_{j \geq 1}$ be an unbounded sequence of positive real numbers and $\{s(p_j)\}_{j \geq 1}$, $\{x(p_j)\}_{j \geq 1}$ a corresponding sequence of closed-loop eigenvalues and eigenvectors respectively. By extraction of a suitable subsequence, it is possible to assume that

$$\lim_{j \rightarrow \infty} x(p_j) = x_\infty \quad , \quad \|x_\infty\| = 1 \quad \dots(4)$$

If $R(Q)$, $N(Q)$ denote the range and null space of a matrix Q , then

Theorem 1

(a) If the sequence $\{|s(p_j)|\}_{j \geq 1}$ is unbounded, then $x_\infty \in R(B)$.

- (b) If the sequence $\{s(p_j)\}_{j \geq 1}$ has a finite cluster point λ , then $x_\infty \in W_B$, where (Owens, 1975) W_B is the maximal subspace of $N(C)$ satisfying the relation $AW_B \subset W_B + R(B)$.

Proof

To prove (a), divide equation (3) by $s(p)$, from which

$$x_\infty = \lim_{j \rightarrow \infty} (s(p_j))^{-1} p_j B C x(p_j) \in R(B).$$

To prove (b), equation (3) implies that $(\lambda I_n - A)x_\infty = \lim_{j \rightarrow \infty} p_j B C x(p_j) \in R(B)$ ie $x_\infty \in W_B$.

Q.E.D.

3. Asymptotic Root-loci for Uniform-rank Multivariable Systems

To illustrate the general structure of the geometric relationships defining the asymptotic form of the system root-locus, consider the case of an open-loop system (equation (1)) satisfying the relations

$$\begin{aligned} CA^{j-1}B &= 0 & j < k \\ |CA^{k-1}B| &\neq 0 \end{aligned} \quad \dots (5)$$

Equivalently, if $G(s) = C(sI_n - A)^{-1}B$ is the open-loop system transfer function matrix, then

$$G_\infty^{(k)} \triangleq \lim_{s \rightarrow \infty} s^k G(s) = CA^{k-1}B \quad \dots (6)$$

exists, is nonsingular and $|G(s)| \neq 0$. On intuitive grounds, the above relations imply that each loop has a dynamic behaviour analogous to a classical rank k transfer function and, as such, $G(s)$ will be termed a uniform rank transfer function matrix and $S(A, B, C)$ a uniform rank system.

The following theorem defines the asymptotic form of the root-locus plot in terms of the expansion of the matrix $(A - pBC)^\ell$. For convenience, define

$$\Gamma_0 = 0, \quad \Gamma_\ell = \sum_{j=1}^{\ell} A^{j-1} B C A^{\ell-j}, \quad \ell \geq 1 \quad \dots (7)$$

and note from condition (5) that

$$(A-pBC)^\ell = A^\ell - p\Gamma_\ell, \quad 1 \leq \ell \leq k \quad \dots(8)$$

and, from equation (7), by induction,

$$\Gamma_0 = 0 \quad \dots(9)$$

$$\begin{aligned} \Gamma_{j+1} &= A\Gamma_j + BCA^j \\ &= \Gamma_j A + A^j BC, \quad j \geq 0 \end{aligned} \quad \dots(10)$$

Theorem 2

With the above notation, and $S(A,B,C)$ of uniform rank, the closed-loop system $S(A-pBC,B,C)$ has km unbounded poles of the form, $1 \leq j \leq m$, $1 \leq \ell \leq k$

$$\mu_{j\ell}(p) = p^{\frac{1}{k}} \eta_{j\ell} + \alpha_j + \varepsilon_{j\ell}(p) \quad \dots(11)$$

where $\eta_{j\ell}$, $1 \leq \ell \leq k$ are the k th roots of λ_j where λ_j is a non-zero solution of

$$\{\lambda_j I_n + BCA^{k-1}\} x_\infty = 0, \quad \|x_\infty\| = 1 \quad \dots(12)$$

Also,

$$\lim_{p \rightarrow \infty} \varepsilon_{j\ell}(p) = 0$$

and if,

$$N(\lambda_j I_n + BCA^{k-1}) \cap R(\lambda_j I_n + BCA^{k-1}) = \{0\}, \quad \dots(13)$$

then, the pivot α_j is a solution of the relation

$$\{k\alpha_j I_n - A\} x_\infty \in R(\lambda_j I_n + BCA^{k-1}) \quad \dots(14)$$

The remaining $n-km$ poles tend to the zeros of $S(A,B,C)$.

Proof

Equation (3) implies that

$$\{(s(p))^\ell I_n - (A-pBC)^\ell\} x(p) = 0, \quad \ell \geq 1 \quad \dots(15)$$

or, by equation (8), for $1 \leq \ell \leq k$,

$$\left\{ \frac{(s(p))^\ell}{p} I_n - p^{-1} A^\ell + \Gamma_\ell \right\} x(p) = 0 \quad \dots(16)$$

In an analogous manner to section 2, suppose that the family $\{x(p)\}$ has a cluster point x_∞ ($\|x_\infty\| = 1$) then

$$\lim_{p \rightarrow \infty} \frac{s(p)^\ell}{p} x_\infty = \Gamma_\ell x_\infty, \quad 1 \leq \ell \leq k \quad \dots(17)$$

Considering only unbounded eigenvalues, then (Theorem 1) $x_\infty \in R(B)$ so that (equations (5), (7))

$$\lim_{p \rightarrow \infty} \frac{s(p)^\ell}{p} = 0, \quad \ell < k \quad \dots(18)$$

and,

$$\Gamma_k x_\infty = BCA^{k-1} x_\infty \neq 0 \quad \dots(19)$$

ie (equation (17)) $\lambda_j \triangleq \lim_{p \rightarrow \infty} p^{-1} (s(p))^k$ exists, is non-zero and is a solution of the eigenvalue equation (12). Write $s(p) = p^{\frac{1}{k}} \eta_{j\ell} + \psi_{j\ell}(p)$ where $\eta_{j\ell}^k = \lambda_j$ and

$$\lim_{p \rightarrow \infty} p^{\frac{1}{k}} \psi_{j\ell}(p) = 0 \quad \dots(20)$$

It follows that

$$\lim_{p \rightarrow \infty} p^{\frac{1}{k}} \left\{ \frac{s(p)^k}{p} - \lambda_j \right\} x(p) = - \lim_{p \rightarrow \infty} p^{\frac{1}{k}} \{ \lambda_j + \Gamma_\ell \} x(p) \quad \dots(21)$$

Writing (equation (10)) $\Gamma_k = A\Gamma_{k-1} + BCA^{k-1}$ and noting (equation (16)) that

$$\lim_{p \rightarrow \infty} p^{\frac{1}{k}} A\Gamma_{k-1} x(p) = -\lambda_j \frac{k-1}{k} Ax_\infty \quad \dots(22)$$

then

$$\left\{ \lim_{p \rightarrow \infty} p^{\frac{1}{k}} \left\{ \frac{s(p)^k}{p} - \lambda_j \right\} \Gamma_{k-1} \right\} Ax_\infty = \lim_{p \rightarrow \infty} p^{\frac{1}{k}} \{ \lambda_j + BCA^{k-1} \} x(p) \quad \dots(23)$$

Using equation (13) and noting that $x_\infty \in N(\lambda_j I_n + BCA^{k-1})$, it follows that $\lim_{p \rightarrow \infty} p^{\frac{1}{k}} \left(\frac{s(p)^k}{p} - \lambda_j \right)$ exists, so that (equation (20))

$$\begin{aligned} \lim_{p \rightarrow \infty} p^{\frac{1}{k}} \left(\frac{s(p)^k}{p} - \lambda_j \right) &= \lim_{p \rightarrow \infty} k \lambda^{\frac{k-1}{k}} \mu_{j\ell}(p) \\ &= k \lambda^{\frac{k-1}{k}} \alpha_{j\ell} \end{aligned} \quad \dots(24)$$

for some finite scalar $\alpha_{j\ell}$. It is easily seen that $\alpha_{j\ell}$ is independent of ℓ and writing $\alpha_{j\ell} = \alpha_j$, equation (23) implies that

$$(k\alpha_j I_n - A)x_\infty \in R(\lambda_j I_n + BCA^{k-1}) \quad \dots(25)$$

as required.

Finally, it can be shown (Owens, 1975) that $S(A,B,C)$ has $n-km$ zeros, each of which (Shaked and Kouvaritakis, 1976) attracts a pole at high gain. Q.E.D.

The above theorem provides explicit geometrical conditions for the construction of the asymptotes of the root locus plot. For purposes of calculation, write $x_\infty = Bz$, then (equation (12)) as $\text{rank } B = m$ (equation (5)), λ_j is the solution of the eigenvalue problem,

$$\begin{aligned} 0 &= \{\lambda_j I_m + CA^{k-1}B\}\alpha_j \\ &= \{\lambda_j I_m + G_\infty^{(k)}\}\alpha_j, \quad \alpha_j \neq 0 \end{aligned} \quad \dots(26)$$

Condition (13) is equivalent to the requirement that $G_\infty^{(k)}$ has a complete set of eigenvectors. To calculate the pivot, suppose that $u_1, \dots, u_\ell$ are linearly independent eigenvectors of BCA^{k-1} spanning the eigenspace corresponding to the eigenvalue λ_j , and let $v_1^+, \dots, v_\ell^+$ be the corresponding dual eigenvectors satisfying $v_j^+ u_\ell = \delta_{j,k}$, then, if M_j is the $\ell \times \ell$ matrix with elements

$$(M_j)_{rq} = k^{-1} v_r^+ A u_q, \quad 1 < r, q < \ell \quad \dots(27)$$

it follows that α_j is a solution of the eigenvalue equation

$$\{\alpha_j I_\ell - M\}\beta_j = 0 \quad \beta_j \neq 0 \quad \dots(28)$$

4. Asymptotic Root-loci for Non-uniform-rank Multivariable Systems

In more general situations (Shaked and Kouvaritakis, 1976) $S(A-pBC, B, C)$ will have unbounded poles of various orders as $p \rightarrow +\infty$. The main result of this section (Theorem 3) provides a geometric characterization of the asymptotes of the root-locus in certain situations of practical interest, using relations analogous to those of Theorem 2.

Write, $\ell \geq 1$

$$(A-pBC)^\ell = \sum_{j=0}^{\ell} (-1)^j p^j B_{j,\ell} \quad \dots (29)$$

from which, by induction,

$$B_{0,\ell} = A, \quad B_{1,\ell} = BC \quad \dots (30)$$

and, for $\ell \geq 1$,

$$\begin{aligned} B_{0,\ell+1} &= AB_{0,\ell} \\ B_{j,\ell+1} &= AB_{j,\ell} + BCB_{j-1,\ell}, \quad 1 \leq j \leq \ell \\ B_{\ell+1,\ell+1} &= BCB_{\ell,\ell} \end{aligned} \quad \dots (31)$$

so that (equations (9), (10), (30)), for $\ell \geq 1$,

$$B_{0,\ell} = A^\ell, \quad B_{1,\ell} = \Gamma_\ell, \quad B_{\ell,\ell} = (BC)^\ell \quad \dots (32)$$

In a similar manner to equation (5), let k be the uniquely defined integer such that

$$CA^{j-1}B = 0, \quad j < k, \quad CA^{k-1}B \neq 0 \quad \dots (33)$$

then (equation (8))

$$BC(A-pBC)^\ell = \begin{cases} BCA^\ell, & 0 \leq \ell < k \\ BCA^{k-1}(A-pBC), & \ell = k \end{cases} \quad \dots (34)$$

so that, $k \leq \ell \leq 2k-1$,

$$BC(A-pBC)^\ell = BCA^{k-1} \{ A^{\ell+1-k} - p \Gamma_{\ell+1-k} \} \quad \dots (35)$$

Defining

$$V_0 \triangleq R(B) , \quad V_\ell \triangleq R(B) \cap \bigcap_{j=1}^{\ell} N(BCA^{j-1}) \equiv \bigcap_{j=1}^{\ell} N(\Gamma_j) \cap R(B) , \quad \ell \geq 1 \quad \dots(36)$$

then, if $|G(s)| \neq 0$, there exists an integer $\hat{k}$ such that

$$V_\ell \neq \{0\} \quad (\ell < \hat{k}) , \quad V_\ell = \{0\} \quad (\ell \geq \hat{k}) \quad \dots(37)$$

for, if $x \in V_\ell$ for all $\ell \geq 1$, then $A^\ell x \in N(c)$ for all $\ell \geq 0$ ie (theorem 1)

$x \in W_B$ and the proposition is proved by noting (Owens, 1975) that $|G(s)| \neq 0$ implies that $W_B \cap R(B) = \{0\}$. It is easily shown that

$$k \leq \hat{k} \quad \dots(38)$$

Defining, $k-1 \leq \ell \leq 2k-1$,

$$W_\ell \triangleq R(BCA^{k-1} \Gamma_{\ell+1-k}) , \quad X_\ell \triangleq \bigcap_{j=k-1}^{\ell} N(BCA^{k-1} \Gamma_{j+1-k}) \quad \dots(39)$$

the following theorem is proved below,

Theorem 3

With the above notation, $|G(s)| \neq 0$, and $k \leq \hat{k}$, then, if

$$\begin{aligned} V_\ell \cap W_\ell &= \{0\} , & k-1 \leq \ell \leq \hat{k}-1 \\ \{\Gamma_{\ell+1} V_\ell\} \cap W_\ell &= \{0\} , & k-1 \leq \ell \leq \hat{k}-1 \end{aligned} \quad \dots(40)$$

the closed-loop system $S(A-pBC, B, C)$ possesses unbounded poles of the form,
 $k \leq \ell \leq \hat{k}$,

$$s(p) = p^{\frac{1}{\ell}} \eta + f(p) \quad \dots(41)$$

where, if λ is a non-zero solution of the relation

$$\{\lambda I_n + BCA^{\ell-1}\} x_\infty \in W_{\ell-1} \quad \dots(42)$$

$$0 \neq x_\infty \in V_{\frac{1}{\ell}-1}$$

then η is an ℓ th root of λ , $p^{\frac{1}{\ell}}$ is the positive real ℓ th root of p and

$$\lim_{p \rightarrow \infty} p^{\frac{1}{\ell}} f(p) = 0 \quad \dots(43)$$

Moreover, if, for $k \leq \ell \leq \hat{k}$,

$$\{\{\lambda I_n + BCA^{\ell-1}\} x_{\ell-1} + W_{\ell-1}\} \cap V_{\ell-1} \cap \{\lambda I_n + BCA^{\ell-1}\}^{-1} W_{\ell-1} = \{0\} \quad \dots(44)$$

then $f(p)$ takes the form

$$f(p) = \alpha + \varepsilon(p) \quad \dots(45)$$

where $\lim_{p \rightarrow \infty} \varepsilon(p) = 0$, and the 'pivot' α is a constant, finite solution of the relation

$$\{\lambda \alpha I_n - A\} x_\infty \in \{\lambda I_n + BCA^{\ell-1}\} x_{\ell-1} + W_{\ell-1} \quad \dots(46)$$

Proof

Equation (3) implies that

$$\{(s(p))^\ell I_n - (A - pBC)^\ell\} x(p) = 0, \quad \|x(p)\| = 1, \quad \ell \geq 1 \quad \dots(47)$$

Using equations (8), (29), (32), this takes the form

$$\{p^{-1}(s(p))^\ell - p^{-1}A^\ell + \Gamma_\ell\} x(p) = \begin{cases} 0 & ; \quad \ell \leq k \\ \sum_{j=2}^{\ell} (-1)^j p^{j-1} B_{j,\ell} x(p) & ; \quad \ell > k \end{cases} \quad \dots(48)$$

Taking the case of $\ell = k$, it follows that $p^{-1}s(p)$ is bounded ($p \rightarrow \infty$) from which $\lim_{p \rightarrow \infty} p^{-1}(s(p))^k$ exists for every closed-loop pole. If $\lim_{p \rightarrow \infty} p^{-1}(s(p))^k = \lambda \neq 0$, then, if $\lim_{p \rightarrow \infty} x(p) = x_\infty$ ($\|x_\infty\| = 1$), $\Gamma_j x_\infty = 0$, $j < k$, so that $x_\infty \in V_{k-1}$ and

$$\{\lambda I_n + BCA^{k-1}\} x_\infty = 0 \in W_{k-1} \quad \dots(49)$$

proving (42) in the case of $\ell = k$. Using induction, suppose that $\lim_{p \rightarrow \infty} p^{-1}(s(p))^r = 0$, $k \leq r < \ell$, and $x_\infty \in V_{\ell-1}$, so that (equation (40)),

$$\lim_{p \rightarrow \infty} \sum_{j=2}^r (-1)^j p^{j-1} B_{j,r} x(p) = 0, \quad k \leq r < \ell \quad \dots(50)$$

Using equation (31), (32), (35), equation (50) becomes

$$\begin{aligned} \lim_{p \rightarrow \infty} \{A \sum_{j=2}^{r-1} (-1)^j p^{j-1} B_{j,r-1} x(p) - BC \sum_{j=1}^{r-1} (-1)^j p^j B_{j,r-1} x(p)\} \\ = \lim_{p \rightarrow \infty} pBCA^{k-1} \Gamma_{r-k} x(p) = 0, \quad k \leq r < \ell \end{aligned} \quad \dots(51)$$

In a similar manner, it can be shown that

$$\lim_{p \rightarrow \infty} \{p^{-1}(s(p))^{\ell} I_n + \Gamma_{\ell}\} x(p) = \lim_{p \rightarrow \infty} pBCA^{k-1} \Gamma_{\ell-k} x(p) \in W_{\ell-1} \quad \dots(52)$$

Equation (40) implies that $p^{-1}(s(p))^{\ell}$ can only have a finite cluster point λ .

If $\lambda = 0$, then $\Gamma_{\ell} x_{\infty} \in W_{\ell-1}$ or (equation (40)) $x_{\infty} \in N(\Gamma_{\ell}) \cap V_{\ell-1} = V_{\ell}$ and

$\lim_{p \rightarrow \infty} pBCA^{k-1} \Gamma_{\ell-k} x(p) = 0$. Alternatively, if $\lambda \neq 0$, it is a solution of the relation

$$\{\lambda I_n + BCA^{\ell-1}\} x_{\infty} = BCA^{k-1} \Gamma_{\ell-k} z \in W_{\ell-1} \quad \dots(53)$$

for some vector $z \in R^n$, proving equation (42). Note that, if

$\lim_{p \rightarrow \infty} p^{-1}(s(p))^{\ell} = 0$, $1 \leq \ell \leq \hat{k}$, then, from the definition of $\hat{k}$, $x_{\infty} \notin R(B)$ ie (theorem 1) $s(p)$ has a finite limit.

Finally, if $p^{-1}(s(p))^{\ell} \rightarrow \lambda \neq 0$ ($p \rightarrow \infty$) and $\ell = k$, equation (40) follows directly from theorem 2. Alternatively, if $\ell > k$, rewrite equation (48) in the form,

$$\begin{aligned} & \frac{1}{p^{\ell}} \{p^{-1}(s(p))^{\ell-\lambda}\} x(p) \\ &= p^{\frac{1}{\ell}} \{-\{\lambda I_n + \Gamma_{\ell}\} + p^{-1} A^{\ell}\} + \sum_{j=2}^{\ell} (-1)^j p^{j-1} B_{j,\ell} \} x(p) \\ &= p^{\frac{1}{\ell}} \{-\{\lambda I_n + \Gamma_{\ell}\} + p^{-1} A^{\ell} + A \sum_{j=2}^{\ell-1} p^{j-1} (-1)^j B_{j,\ell-1} - BC \sum_{j=1}^{\ell-1} (-1)^j p^j B_{j,\ell-1}\} x(p) \end{aligned} \quad \dots(54)$$

From (48) replacing ℓ , by $\ell-1$,

$$\lim_{p \rightarrow \infty} p^{\frac{1}{\ell}} A \left\{ \sum_{j=2}^{\ell-1} p^{j-1} (-1)^j B_{j,\ell-1} - \Gamma_{\ell-1} \right\} x(p) = \eta^{\frac{\ell-1}{\ell}} A x_{\infty} \quad \dots(55)$$

so that, using equation (10), equation (54) takes the form

$$\begin{aligned} & \lim_{p \rightarrow \infty} \{p^{\frac{1}{\ell}} \{p^{-1}(s(p))^{\ell-\lambda}\} I_n - \eta^{\frac{\ell-1}{\ell}} A\} x_{\infty} \\ &= -\lim_{p \rightarrow \infty} p^{\frac{1}{\ell}} \{\lambda I_n + BCA^{\ell-1} + BC \sum_{j=1}^{\ell-1} (-1)^j p^j B_{j,\ell-1}\} x(p) \\ &= -\lim_{p \rightarrow \infty} p^{\frac{1}{\ell}} \{\lambda I_n + BCA^{\ell-1} - pBCA^{k-1} \Gamma_{\ell-k}\} x(p) \end{aligned} \quad \dots(56)$$

Write $x(p) = x_1(p) + x_2(p)$, $x_1(p) \in X_{\ell-1}$, $x_2(p) \in X_{\ell-1}^{\perp}$, then the relations (equations (51)-(53))

$$\begin{aligned} \lim_{p \rightarrow \infty} p \text{BCA}^{k-1} \Gamma_{r-k} x(p) &= 0, & k \leq r < \ell \\ \lim_{p \rightarrow \infty} p \text{BCA}^{k-1} \Gamma_{\ell-k} x(p) &= \text{BCA}^{k-1} \Gamma_{\ell-k} z & (\text{finite}) \end{aligned} \quad \dots(57)$$

imply that $\lim_{p \rightarrow \infty} p^{\frac{1}{\ell}} x_2(p) = 0$ ie

$$\begin{aligned} \lim_{p \rightarrow \infty} \{p^{\frac{1}{\ell}} \{p^{-1} (s(p))^{\ell-\lambda} I_n - \eta^{\frac{\ell-1}{\ell}} A\} x_{\infty} \\ = -\lim_{p \rightarrow \infty} p^{\frac{1}{\ell}} \{ \{\lambda I_n + \text{BCA}^{\ell-1}\} x_1(p) - p \text{BCA}^{k-1} \Gamma_{\ell-k} x(p) \} \end{aligned} \quad \dots(58)$$

Condition (44) implies that $\lim_{p \rightarrow \infty} p^{\frac{1}{\ell}} \{(s(p))^{\ell-1-\lambda} p^{-1-\lambda}\}$ exists or (equation (41), (43))

$$\lim_{p \rightarrow \infty} p^{\frac{1}{\ell}} \{p^{-1} (p^{\frac{1}{\ell}} \eta + f(p))^{\ell-\lambda}\} = \lim_{p \rightarrow \infty} \ell \eta^{\frac{\ell-1}{\ell}} f(p) \triangleq \ell \eta^{\frac{\ell-1}{\ell}} \alpha \quad \dots(59)$$

for one finite constant α . Substituting into (58) yields the relation

$$\{\ell \alpha I_n - A\} x_{\infty} \in \{\lambda I_n + \text{BCA}^{\ell-1}\} X_{\ell-1} + W_{\ell-1} \quad \dots(60)$$

which prove the result.

Q.E.D.

It is easily shown that theorem 3 reduces to theorem 2 if $S(A, B, C)$ is of uniform rank.

5. Illustrative Example

Consider the non-uniform-rank system defined by the matrices

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

from which, $|G(s)| \neq 0$, $k = 1$

$$\Gamma_0 = 0, \quad \Gamma_1 = BC = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{bmatrix}, \quad (BC)^2 = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$BCA = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 1 & 1 & 2 \end{bmatrix} \quad \dots(62)$$

so that

$$V_0 = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}, \quad V_1 = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}, \quad V_2 = \{0\} \quad \dots(63)$$

ie $\hat{k} = 2 = 2k$. Also,

$$W_0 = \{0\}, \quad W_1 = \text{span}\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}, \quad X_0 = \mathbb{R}^3, \quad X_1 = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\} \dots(64)$$

so that $W_0 \cap V_0 = W_1 \cap V_1 = \{0\}$, $\Gamma_1 V_0 \cap W_0 = \{0\}$, $\Gamma_2 V_1 \cap W_1 = \{0\}$.

To calculate the first order asymptote, solve the equation

$$\{\lambda I_3 + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{bmatrix}\} x_\infty \in W_0, \quad x_\infty \in V_0 \quad \dots(65)$$

ie $\lambda = -1$ and $x_\infty = \{0, -1, 1\}^T \in R(B)$. The corresponding pivot is the solution of the relation

$$\{\alpha I_3 - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & -1 \\ 1 & 0 & 0 \end{bmatrix} z, \quad z \in \mathbb{R}^3 \quad \dots(66)$$

ie $\alpha = 0$, and the asymptote takes the form $-p$; and passes through the origin of the complex plane. Considering now the second order type asymptote,

$$\{\lambda I_3 + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}\} x_\infty \in W_1, \quad x_\infty \in V_1 \quad \dots(67)$$

ie $\lambda = -2$ and $x_\infty = \{0, 1, 0\}^T$. The corresponding pivot is the solution of the relation

$$\begin{aligned} \{2\alpha I_3 - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} &= \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + z_3 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 2 & 0 \\ -1 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \quad \dots(68) \end{aligned}$$

ie $\alpha = 1$ and the second order asymptote takes the form $\pm p^{\frac{1}{2}} \sqrt{-2} + 1$.

6. Conclusions

The geometric characterization of the asymptotes of multivariable root-loci of a linear system has been discussed for two cases of practical interest. The work augments the analysis of Shaked and Kouvaritakis (1976) and illustrates the fact that (i) two integer parameters $k, \hat{k}$, derived from geometric considerations, play a fundamental role in the description of the root-locus, and (ii) both the asymptotic directions and pivots are described by inclusion relationships in the state space of the form

$$\{\xi I_n + F\}x \in Q, \quad z \in P \quad \dots(69)$$

where F is a $n \times n$ matrix and P, Q are well-defined subspaces of the state space. A glance at theorems 2,3 will indicate that the matrices $A, B C A^{j-1}, \Gamma_j, k \leq j \leq \hat{k}$ play a fundamental role in the root-locus theory. Writing,

$$\Gamma_j = [B, AB, \dots, A^{j-1}B] \begin{bmatrix} CA^{j-1} \\ CA^{j-2} \\ \vdots \\ CA \\ C \end{bmatrix} \quad \dots(70)$$

it is seen that the controllability and observability matrices play an important role in determining the structure of the root-locus. Further work could relate the structure of the root-loci to parameters defining controllability and observability, provide valuable insight into difficulties occurring in pole allocation and suggest new algorithms for the calculation of the system asymptotes.

References

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