ANALYSIS AND DESIGN OF VARIABLE STRUCTURE SYSTEMS

USING A GEOMETRIC APPROACH

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Abstract

Multivariable variable structure systems in the sliding mode are studied using a geometric approach. The properties of system order reduction and disturbance rejection are proved using projector theory. New design methods for the choice of switching hyperplanes are derived for the closed-loop eigenvalue/eigenvector assignment problem.

## 1. Introduction

Variable structure systems (VSS) are a special class of nonlinear systems characterized by a discontinuous control action which changes structure upon reaching a set of switching hyperplanes $s(x) = 0$. A fundamental property of VSS is the sliding motion of the state point on the intersection of the switching hyperplanes. During the sliding mode the system has invariance properties, yielding motion which is independent of certain system parameters and disturbances, and the system behaves like a linear system.

Order reduction is a fundamental property of VSS in the sliding mode. This is due to the motion of the state which is constrained to lie on the intersection of the m switching hyperplanes. During the sliding mode the order of the system is reduced because the motion of the state is governed by n-m "slow" modes. The remaining m modes are the "fast" modes (see Young et al., 1977).

A new method of analysing VSS in the sliding mode is developed in this paper. The study has been motivated by the observation that a basic operator associated with the dynamics qualifies as a projector. Projector theory provides a neat method for the analysis and design of VSS. Using projector theory certain VSS features are explained and others are expanded. A simple explanation of order reduction is given together with a re-examination of the invariance prinicple of Draženović (1969). It is shown that Draženović's invariance conditions are a special case although the two resulting conditions are the same when CB is nonsingular. The physical interpretation of invariance is also given. The invariance of the system zeros in the sliding mode is investigated. It is found that certain interrelations exist between the n-m closed-loop eigenvectors $w_i$ (associated with the n-m assigned eigenvalues) and the matrices of the system S(A, B, C). These relationships are exploited further when formulating new methods for constructing the matrix C specifying the switching hyperplanes and hence specifying the n-m closed-loop eigenvalues.

Design methods are proposed for the design of the switching hyperplanes. The methods given have the ability to assign the matrix CB arbitrarily. This may be useful as a design option since it has already been established that a diagonally dominant CB matrix ensures the convergence of the fast motion to the switching hyperplanes (Utkin, 1978b). Another advantage is the ability to exercise partial control over the closed-loop eigenvectors associated with the n-m assigned eigenvalues. The freedom in selecting these eigenvectors increases with the number of inputs, or equivalently, with increasing range space of B.

## 2. Variable structure systems in the sliding mode

Throughout we consider the time-invariant system S(A, B, C)

$$\dot{x} = Ax + Bu \tag{2.1}$$

$$s = Cx \tag{2.2}$$

where $x \in R^n$, $u \in R^m$, $s \in R^m$. The matrices B and C are assumed to have full rank m and $|CB| \neq 0$.

Variable structure systems (Utkin, 1977 and 1978a) are characterized by a discontinuous control action which changes structure upon reaching a set of switching surfaces. The control has the form

$$u_i = \begin{cases} u_i^+(x) & s_i(x) > 0 \\ u_i^-(x) & s_i(x) < 0 \end{cases} \tag{2.3}$$

where $u_i$ is the ith component of u and $s_i(x)$ is the ith of the m switching hyperplanes which satisfy

$$s(x) = Cx = 0. \tag{2.4}$$

The above system with discontinuous control is termed a variable structure system (VSS) since the effect of the switching hyperplanes is to alter the feedback structure of the system.

Sliding motion occurs, if at a point on a switching surface $s_i(x) = 0$, the directions of motion along the state trajectories on either side of the surface are not away from the switching surface. The state then slides and remains for some finite time on the surface $s_i(x) = 0$ (Utkin, 1977 and 1978a). The conditions for sliding motion to occur on the ith hyperplane may be stated in numerous ways. We need

$$\lim_{s_i \to 0^+} \dot{s}_i < 0 \quad \text{and} \quad \lim_{s_i \to 0^-} \dot{s}_i > 0 \tag{2.5}$$

or equivalently

$$s_i \dot{s}_i < 0 \tag{2.6}$$

in the neighbourhood of $s_i(x) = 0$. In the sliding mode the system satisfies the equations

$$s_i(x) = 0 \text{ and } \dot{s}_i(x) = 0 \tag{2.7}$$

and the system has invariance properties, yielding motion which is independent of certain parameter variations and disturbances. Thus variable structure systems are usefully employed in systems with uncertain and time-varying parameters.

Consider the behaviour of the system dynamics during sliding when the sliding mode exists on all the hyperplanes assuming that the non-unique control u has been suitably chosen. During sliding equation (2.4) and its derivative

$$\dot{s} = C\dot{x} = 0 \tag{2.8}$$

hold and the equations governing the system dynamics may be obtained by substituting an equivalent control $u_{eq}$ for the original control u. From (2.1) and (2.8) the linear equivalent control is

$$u_{eq} = -(CB)^{-1} CAx \tag{2.9}$$

and substitution in equation (2.1) yields the equations governing the system dynamics in the sliding mode

$$\dot{x} = [I - B(CB)^{-1}C]Ax = A_{eq}x. \tag{2.10}$$

Notice that during sliding m state variables can be expressed in terms of the remaining (n-m) state variables using (2.4). This allows a reduction in the order of the system matrix.

## 3. Projectors

3.1 Definition: Given a decomposition of space S into subspaces $S_1$ and $S_2$ so that for any $x \in S$

$$x = x_1 + x_2 \; ; \quad x_1 \in S_1, \quad x_2 \in S_2 \tag{3.1}$$

the linear operator P that maps x into $x_1$ is called a projector on $S_1$ along $S_2$, i.e.

$$Px_1 = x_1 \; ; \quad Px_2 = 0. \tag{3.2}$$

### 3.2 Properties of Projectors

Some useful properties of projectors are listed below (Pease, 1965):

1) A linear operator P is a projector if and only if it is idempotent, i.e. if

$$P^2 = P. \tag{3.3}$$

2) If P is the projector on $S_1$ along $S_2$ then (I-P) is the projector on $S_2$ along $S_1$.

3) If P is the projector on R(P) (Range of P) along N(P) (Null space of P then (I-P) is the projector on N(P) along R(P).

4) For any $x \in R(P)$

$$Px = x, \tag{3.4}$$

and

$$(I-P)x = 0. \tag{3.5}$$

5) $$\text{rank } (P) = \text{trace } (P) \tag{3.6}$$

and

$$\text{rank } (I-P) = n\text{-rank } (P). \tag{3.7}$$

6) $$R(P) = N(I-P) \tag{3.8}$$

and

$$N(P) = R(I-P). \tag{3.9}$$

### 3.3 Relevant Projectors

Certain matrix operators encountered in variable structure systems (VSS) are projectors.

1) $B(CB)^{-1}C$ is a projector.

Proof: Since

$$[B(CB)^{-1}C]^2 = B(CB)^{-1}CB(CB)^{-1}C = B(CB)^{-1}C$$

$B(CB)^{-1}$ is therefore idempotent and consequently a projector. $B(CB)^{-1}C$ projects $R^n$ on $R(B)$ along $N(C)$ because

$$R[B(CB)^{-1}C] = R(B) \tag{3.10}$$

since $R(BK) = R(B)$ if B and K are full rank. In our case $K = (CB)^{-1}C$ which is full rank since B and CB are full rank. Similarly $N[B(CB)^{-1}C] = N(C)$.

Since nullity (HC) = nullity (C) if H and C are full rank where $H = B(CB)^{-1}$ which is full rank.

2) $[I - B(CB)^{-1}C]$ is a projector.

Proof: Either from Property 2 or by expanding $[I - B(CB)^{-1}C]^2$ and showing that it is equal to $[I - B(CB)^{-1}C]$.

$[I - B(CB)^{-1}C]$ projects $R^n$ on $N(C)$ along $R(B)$. Since the rank of a matrix is the dimension of its range space, by letting $P = B(CB)^{-1}C$ we obtain from (3.10)

$$\text{rank }(P) = \text{rank }(B) = m \tag{3.11}$$

and from (3.7)

$$\begin{aligned} \text{rank }(I-P) &= n - \text{rank }(P) \\ &= n - m. \end{aligned} \tag{3.12}$$

Therefore any $n \times n$ matrix pre-multiplied by $[I - B(CB)^{-1}C]$ will have at most rank n-m.

Both of the above projectors turn out to be of invaluable help in exploring the basic features of Variable Structure Systems.

3) If A is an $m \times n$ matrix and $A^g$ is a generalized inverse of A then

$$AA^g,\ A^gA,\ I_m - AA^g,\ I_n - A^gA$$

are all idempotent and therefore projectors (Graybill, 1969). The proof follows immediately from the definition of the generalized inverse of a matrix, i.e.

$$(AA^g)^2 = AA^gAA^g = AA^g$$

$AA^g$ and $A^gA$ are projectors on $R(A)$ and $R(A^g)$ respectively and $(I_m - AA^g)$, $(I_n - A^gA)$ are projectors on $N(A^g)$ and $N(A)$ respectively.

Further properties of $B(CB)^{-1}C$ and $[I_n - B(CB)^{-1}C]$ include

a) The matrix $B(CB)^{-1}$ qualifies as a right inverse of C.

Proof:

Since $CB\,(CB)^{-1} = I_m$

it follows that

$$C^g = B(CB)^{-1}.$$

Also $(CB)^{-1}C$ qualifies as a left inverse of B.

b) $C^gC$ projects $R^n$ on $R(C^g)$ or $R(B)$ along $N(C)$.

c) $[I - C^gC]$ projects $R^n$ on $N(C)$ along $R(B)$. In other words the column space of $[I - C^gC]$ is the same as that of $N(C)$.

## 4. Projector Theory and VSS in the Sliding Mode

We shall now apply the above theory to VSS.

### 4.1 Order Reduction

In the sliding mode the equation describing the system is given by

$$\dot{x} = [I - B(CB)^{-1}C]Ax = A_{eq}x \,. \tag{2.10}$$

$[I - B(CB)^{-1}C]$ is a projector which maps all the columns of A on $N(C)$. The order of the system has therefore been reduced because the state vector is now constrained to lie in $N(C)$ which is an $(n-m)$th dimensional subspace (see (3.14)).

### 4.2 The Invariance Principle Revisited

The invariance principle formulated by Draženović (1969) states that for the system given by

$$\dot{x} = Ax + Bu + Df \tag{4.2}$$

$$s = Cx \tag{4.3}$$

to be invariant to disturbance $f \in R^{\ell}$ in the sliding mode, the columns of the matrix D should belong to the range space of B, i.e. $\mathrm{col}(D) \in R(B)$. This principle will now be re-examined and a more general version derived. This generalization extends the theory to the case where CB is singular (assuming sliding exists).

Theorem 4.1: The system given by (4.2) and (4.3) is invariant with respect to the disturbance f in the sliding mode if

$$\mathrm{col}(D) \in R[B(CB)^{-1}C] \qquad (4.4)$$

or

$$\mathrm{col}(D) \in R[B(CB)^{g}C] \quad \text{if CB is singular} \qquad (4.5)$$

where col ( ) stands for columns of ( ).

Proof: The system in the sliding mode satisfies

$$\dot{x} = [I - B(CB)^{-1}C]Ax + [I - B(CB)^{-1}C]Df. \qquad (4.6)$$

For the system to be invariant to f, $[I - B(CB)^{-1}C]D$ should be zero.

Suppose $|CB| \neq 0$. If

$$\mathrm{col}\ (D) \in R[B(CB)^{-1}C]$$

then (3.5) gives

$$[I - B(CB)^{-1}C]D = 0$$

as required.

Conversely, if

$$[I - B(CB)^{-1}C]D = 0$$

then

$$\mathrm{col}(D) \in N[I - B(CB)^{-1}C]$$

and from (3.8)

$$\mathrm{col}(D) \in R[B(CB)^{-1}C]\ .$$

For $|CB| = 0$ we can replace $(CB)^{-1}$ by $(CB)^{g}$ in the above proof.

The condition (4.4) is identical to that given by Draženović for the case $|CB| \neq 0$ because

$$R[B(CB)^{-1}C] = R(B)$$

and the condition (4.4) is equivalent to rank $(B \;\; D)$ = rank B (Draženović, 1969).

Remark 4.1 $B(CB)^{g}C$ can easily be shown to be a projector.

Remark 4.2 When CB is singular the invariancy is weakened since

$$R[B(CB)^{g}C] \subset R(B).$$

Therefore in this case there will be no rejection in the sliding mode to any disturbance that belongs to R(B) but not to $R(B(CB)^g C)$.

Remark 4.3 Substituting h(x, t) for Df in equation (4.2) allows the effect of parameter variations in A and B to be incorporated into the system equations. An analysis similar to that in the proof of Theorem 4.1 yields the conditions for invariance to parameter variations in A and B.

Remark 4.4 It is well known that the scalar system

$$\dot{x}_i = x_{i+1} \qquad i = 1, \ldots, n-1$$

$$\dot{x}_n = -\sum_{i=1}^{n} a_i x_i + bu$$

is invariant to parameter variations when it is in the sliding mode. This is because all variations in the $a_i$ and b belong to R(B) where $B = [0, 0, \ldots, b]^T$.

## 4.3 Physical Interpretation of Disturbance Invariance

Physical insight into the invariance principle is achieved using projector theory. Let $P = B(CB)^{-1}C$.

From the previous definitions the projector P decomposes the state space X into the direct sum

$$X = R(P) + N(P) \tag{4.7}$$

or

$$X = R(P) + R(I-P). \tag{4.8}$$

Alternatively

$$R(P) \cap R(I-P) = \{\phi\}. \tag{4.9}$$

Since $x \in R(I-P)$ during sliding, for x not to be affected by any disturbance f, the disturbance should lie in the complementary subspace of (I-P), i.e. $f \in R(P)$ which is the condition of invariance.

## 4.4 Effect of Sliding on the System Zeros

Young (1977) has shown for scalar variable structure systems that the system zeros are unaffected by the sliding mode. This is to be expected since the sliding mode results from state feedback, and it is well known that state feedback cannot affect the system zeros (Kouvaritakis et al., 1976). However, it is instructive to demonstrate that this is indeed the case here.

Given the system S(A, B, C) we wish to show that the system zeros are not affected by the organization of a sliding mode on the intersection of the hyperplane

$$s = Gx = 0 . \tag{4.10}$$

We note that sliding results from the application of state feedback

$$u_{eq} = -(GB)^{-1} GAx \tag{4.11}$$

which yields the closed-loop system

$$\dot{x} = [I - B(GB)^{-1}G]Ax = A_G x. \tag{4.12}$$

Let us prove that the zeros of $S(A_G, B, C)$ are identical to the zeros S(A, B, C). The zeros of $S(A_G, B, C)$ are given (see El-Ghezawi et al., 1982) by the n-m eigenvalues of

$$\begin{aligned} M^g[I - B(CB)^{-1}C]A_G M &= M^g[I - B(CB)^{-1}C][I - B(GB)^{-1}G]AM \\ &= M^g[I - B(GB)^{-1}G - B(CB)^{-1}C + B(CB)^{-1}CB(GB)^{-1}G]AM \\ &= M^g[I - B(CB)^{-1}C]AM. \end{aligned} \tag{4.13}$$

But the eigenvalues of (4.13) are the zeros of the systems S(A, B, C) and $S(A_G, B, C)$. Therefore sliding does not alter the system zeros.

## 5. Further Insight into Variable Structure Systems

It is now apparent that projector theory provides a neat method of studying many properties of VSS in the sliding mode. It also exposes the relationships between recurring themes associated with VSS in the sliding mode. These themes involve the closed-loop eigenvector matrix $W = (w_1 w_2 \ldots w_{n-m})$ of $A_{eq}$, the input matrix B and the projector matrix $P = B(CB)^{-1}C$ together with the generalized inverses of W and B. The relationships obtained in this section will be invaluable when formulating new methods for constructing the switching hyperplanes matrix C in Section 8.

## 5.1 The Relationship between B, W and P.

Lemma 5.1 The closed-loop eigenvector matrix W of $A_{eq}$ is independent of the columns of B, i.e.

$$R(W) \cap R(B) = \{\phi\} . \qquad (5.1)$$

Proof: The nonsingularity of CB implies that the columns of B are independent of N(C) and since col(W) $\in$ N(C) equation (5.1) is established.

Theorem 5.1: In VSS the selected generalized inverses of B and W should satisfy

$$B^g W = 0 \qquad (5.2)$$

and

$$W^g B = 0. \qquad (5.3)$$

Proof: Since R(B) = R(P) and col (W) lies in the null space of P then (Hohn, 1973),

$$P[B \quad W] = [B \quad 0] . \qquad (5.4)$$

From (5.1) the inverse of [B W] always exists.

Thus P is given by

$$P = [B \quad 0]\,[B \quad W]^{-1} \qquad (5.5)$$

Let

$$T = [B \quad W] \qquad (5.6)$$

and

$$T^{-1} = \begin{bmatrix} F \\ G \end{bmatrix} \qquad (5.7)$$

then using $T^{-1}T = I_n$ it can be shown that

$$F = B^g \qquad (5.8)$$

and

$$G = W^g \qquad (5.9)$$

such that

$$B^g W = 0 \qquad (5.2)$$

$$W^g B = 0 \qquad (5.3)$$

Q.E.D.

Substituting F (5.8) and G (5.9) in (5.7) and then in (5.5) we get

$$P = BB^g. \tag{5.10}$$

If the calculation of the inverse of the matrix T is to be avoided, P should be obtained either from (5.10) subject to condition (5.2) or from (5.4) as the solution of

$$PW = 0. \tag{5.11}$$

The solution of (5.11) is

$$P = H(I - WW^g) \tag{5.12}$$

where H is an $n \times n$ arbitrary matrix and $W^g$ satisfies (5.3).

## 5.2 The Relationship Between P and $A_{eq}$

Using

$$A_{eq} = (I - P)A \tag{2.10}$$

and multiplying both sides by (I-P) gives

$$(I-P)A_{eq} = (I-P)^2A = (I-P)A = A_{eq}. \tag{5.13}$$

Therefore $A_{eq}$ is (I-P)-invariant or equivalently the columns of $A_{eq}$ belong to R(I-P). This implies that

$$PA_{eq} = 0. \tag{5.14}$$

Equation (5.14) can also be obtained by multiplying both sides of (2.8) by P.

From (2.10) assuming $|A| \neq 0$,

$$I - P = A_{eq}A^{-1} \tag{5.15}$$

which establishes that $A_{eq}A^{-1}$ is a projector since (I-P) is a projector. From (5.15)

$$P = I - A_{eq}A^{-1}. \tag{5.16}$$

By multiplying both sides of (2.10) by $B^g$ and noting from (5.10) that $P = BB^g$ we get

$$B^gA_{eq} = 0 . \tag{5.17}$$

# 6. Projector Theory and the Design of VSS in the Sliding Mode

The utilization of projector theory in the design of VSS in the sliding mode appears promising since it leads to new methods for constructing the switching hyperplanes.

The problem of selecting the switching hyperplanes with desired design objectives can be easily solved using projector theory. Desired design objectives may encompass

(i) Arbitrary eigenvalue assignment.

(ii) Arbitrary specification of CB.

(iii) The choice (partially) of the eigenvectors of $A_{eq}$.

The freedom in choosing the assigned eigenvectors is partial and the degree of freedom increases with increasing R(B).

Existing design methods (Young et al., 1977) and (Utkin and Yang, 1978) cater for case (i) above. All our proposed methods require the availability of the closed-loop eigenvectors W. The determination of these eigenvectors will be described in the following section.

## 7. The determination of the Eigenvector Matrix W

The design methods to be described in section 8 for the construction of the switching hyperplanes require the availability of the closed-loop eigenvectors W. A well known fact related to the linear feedback systems eigenvalue-eigenvector assignment question is that

$$(A + BK)W = WJ \qquad (7.1)$$

where K is an $m \times n$ feedback matrix chosen to yield the desired closed-loop poles specified by the eigenvalues of J (Sinswat and Fallside, 1977). The $(n-m) \times (n-m)$ matrix J may be diagonal or have Jordan block form. If rank (K) = m then equation (7.1) implies that

$$\mathrm{col}(AW - WJ) \in R(B) \ . \qquad (7.2)$$

The problem of arbitrary eigenvector assignment has been tackled by Shah et al. (1975) where it has been shown that, in general, it is impossible to specify all components of any one eigenvector arbitrarily using state feedback. In matrix form (7.2) is equivalent to

$$AW - WJ = BL \qquad (7.3)$$

where L is an arbitrary $m \times (n-m)$ matrix chosen to provide linear combinations of the columns of B. This influences the solution of W and provides partial control over the n-m eigenvectors $w_i$. The eigenvectors should be independent of B, i.e. they satisfy

$$R(W) \cap R(B) = \{\phi\}. \qquad (7.4)$$

The solution of equation (7.3) which also satisfies (7.4) may be determined algebraically utilizing the structure of the given system.

## 8. The Construction of the Switching Hyperplanes C

The problem of constructing the switching hyperplanes constitutes a special case in the more general problem of pole-assignment. Some design methods select the switching hyperplanes which minimize quadratic functionals (Utkin and Yang, 1978). Here the switching hyperplanes matrix C is to be chosen so that $A_{eq}$ has m zero-valued eigenvalues and n-m eigenvalues specified by the designer. Any eigenvalue assignment method can be used. However, a reduction in the computational effort involved especially in the case when $m \simeq n$ can be obtained using properly adapted eigenvalue placement algorithms (Young et al., 1977) and Utkin and Yang (1978). In addition methods available for zero assignment given matrices A, B can also be used to obtain the matrix C (Kouvaritakis and MacFarlane, 1976, part 2).

Method I: The "$B^g$" method.

Let the matrix C satisfy

$$CB = S \qquad (8.1)$$

where S is an arbitrary $m \times m$ nonsingular matrix and

$$CW = 0 . \qquad (8.2)$$

A solution to (8.1) always exists, since B is full rank, giving the particular solution

$$C = SB^g . \qquad (8.3)$$

This solution also satisfies (8.2) since it is required from (5.2) that $B^gW = 0$. A systematic method of finding $B^g$ which will always satisfy $B^gW = 0$ is by constructing $[B \;\; W]^{-1}$. The first m rows of this inverse gives $B^g$ satisfying $B^gW = 0$ (see (5.5) - (5.9)).

Remark  It can be shown that the direct calculation of P is not necessary for the determination of C. We have

$$B(CB)^{-1}C = BS^{-1}C = P. \qquad (8.4)$$

A solution for C always exists since $BS^{-1}$ and P in (8.4) have the same range (see (3.10)). A particular solution is

$$C = SB^gP. \qquad (8.5)$$

From (5.10) $P = BB^g$ and therefore

$$C = SB^gBB^g \qquad (8.6)$$

Since

$$B^gB = I_m \qquad (8.7)$$

it follows that

$$C = SB^g \qquad (8.8)$$

which is independent of P.

Method II: The "W" method.

Here C is determined directly from the $n \times (n-m)$ eigenvector matrix W.

Since

$$\text{col}\,(W) \in N(C) \qquad (8.9)$$

it follows that

$$C = \Gamma W^{\perp}$$

where $\Gamma$ is an arbitrary nonsingular $m \times m$ matrix and $W^{\perp}$ is the annihilator of W (i.e. $W^{\perp}W = 0$). If the value of CB is immaterial, $\Gamma$ can be chosen arbitrarily. However, if CB is required to assume a certain value S, $\Gamma$ must be determined from

$$CB = S = \Gamma W^{\perp}B \qquad (8.11)$$

$$\Gamma = S(W^{\perp}B)^{-1}. \qquad (8.12)$$

The inverse of $(W^{\perp}B)$ always exists since $R(W) \cap R(B) = \{\phi\}$.

From (8.11) and (8.12)

$$C = S(W^{\perp}B)^{-1}W^{\perp}. \qquad (8.13)$$

The C calculated using this method is also equal to $SB^{g}$. This is because $(W^{\perp}B)^{-1}W^{\perp}$ qualifies as a generalized inverse of B and satisfies

$$(W^{\perp}B)^{-1}W^{\perp}W = B^{g}W = 0. \qquad (8.14)$$

The matrix $(W^{\perp}B)^{-1}W^{\perp}$ will always qualify as $B^{g}$ that satisfies $B^{g}W = 0$ irrespective of the choice of $W^{\perp}$.

## 9. Examples

In all these examples A and B are given. It is required to find the switching hyperplanes matrix C which will assign the specified eigenvalues of J (7.1) to $A_{eq}$.

### Example 1

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

We wish to assign two repeated eigenvalues at -4. Since the eigenvalues are repeated and $n-m > m$ we should use the Jordan block form

$$J = \begin{bmatrix} -4 & 1 \\ 0 & -4 \end{bmatrix}.$$

We next calculate the eigenvector matrix using condition (7.2)

$$W = \frac{1}{6}\begin{bmatrix} 1 & -5 \\ -4 & 26 \\ 16 & -128 \end{bmatrix}.$$

Using the $B^{g}$ method (Method I) we get $B^{g} = [16 \quad 8 \quad 1]$ and $C = SB^{g} = S[16 \quad 8 \quad 1]$.

Using the W method (Method II) with $CB = S$ we get

$$C = S[W_1 W_2]^{\perp} = \frac{1}{6}\begin{bmatrix} 1 & -5 \\ -4 & 26 \\ 16 & -128 \end{bmatrix}^{\perp}$$

$$= S[16 \quad 8 \quad 1]$$

where S is an arbitrary constant.

As a check the resulting eigenvalues of $A_{eq}$ are given by

$$sp(A_{eq}) = sp\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -16 & -8 \end{bmatrix} = \{0, -4, -4\} \quad .$$

Two eigenvectors associated with the two eigenvalues at -4 can be easily shown to be equal to $w_1$ and $w_2$.

<u>Example 2</u>

$$A = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} .$$

It is required to assign a single eigenvalue at $\lambda = -1$ so $J = -1$. Note that A has three eigenvalues at -1

$$A - \lambda I = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = A_s \; .$$

By choosing $\ell_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ where $\ell_1$ is the first column of the matrix L eqn (7.3) we obtain the single eigenvector satisfying (5.1)

$$W = \begin{bmatrix} h \\ -1 \\ 1 \end{bmatrix}$$

where h is arbitrary but nonzero.

Using Method II with $CB = SI_2$ and $h = 1$

$$C = SI_2(W^{\perp}B)^{-1}W^{\perp}$$

$$= S\begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= S\begin{bmatrix} -2 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Note that if Method I had been used then a $B^g$ which ensures $B^gW = 0$ will be obtained as

$$B^g = \begin{bmatrix} -2 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

giving once again

$$C = SB^g = SI_2B^g = S\begin{bmatrix} -2 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

<u>Example 3</u>

$$A = \begin{bmatrix} 1.38 & -0.2077 & 6.715 & -5.676 \\ 10.1952 & -4.29 & 9.998 & -13.802 \\ 1 & 0 & -5 & 2 \\ 2.2037 & 4.273 & 3.343 & -5 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 5.679 & 0 \\ 1.136 & -3.146 \\ 1.136 & 0 \end{bmatrix}.$$

The solution of (7.3) with

$$J = \begin{bmatrix} -10 & 0 \\ 0 & -10 \end{bmatrix} \text{ and } L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

gives

$$W = \begin{bmatrix} -0.3235 & 0.3891 \\ 0.3891 & 0.3936 \\ 0.3781 & -0.6878 \\ -0.2155 & -0.0480 \end{bmatrix}.$$

Since the two eigenvectors W are independent and $R(B) \cap R(W) = \{\phi\}$, using (5.5) the matrix P can be determined together with $B^g$ which is given by the first two rows of the inverse of $[B \;\; W]^{-1}$. Let the matrix CB be assigned the diagonally dominant value

$$CB = S = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} .$$

Using Method I, $B^g$ is found as the first two rows of

$$[B \;\; W]^{-1} = \begin{bmatrix} -0.0751 & 0.1128 & 0 & 0.3163 \\ -0.5386 & 0.0098 & -0.3178 & 0.2686 \\ -0.8167 & 0.5016 & 0 & -2.5077 \\ 1.8908 & 0.4171 & 0 & -2.085 \end{bmatrix}$$

and $\qquad C = SB^g$

$$= \begin{bmatrix} 0.0751 & -0.1128 & 0 & -0.3163 \\ 1.0773 & -0.0196 & 0.6357 & -0.5373 \end{bmatrix} .$$

## Conclusion

A new treatment of VSS in the sliding mode has been developed using projector theory. The employment of projectors in the study of VSS has been shown to provide further insight into their operation. Furthermore new methods for constructing the switching hyperplanes have been formulated. In addition to solving the eigenvalue placement problem these methods allow CB to be specified arbitrarily and allow partial control over the choice of the closed-loop eigenvectors. The examples included have illustrated the feasibility of the proposed methods.

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