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POLE AND ZERO RETENTION IN THE FACTORIZATION OF MULTIVARIABLE SYSTEMS, WITH APPLICATION TO MODEL REDUCTION

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Abstract

The problem of pole and zero retention in multivariable system reduction is regarded as the generation of a series factorization of the system transfer function matrix. Necessary and sufficient conditions for the existence of the factorization are derived in terms of the decomposition of the state space into $A$-invariant and $(A,B)$-invariant subspaces.

1. Introduction

The facility to retain dominant poles and zeros of an $m \times m$ linear time-invariant system $S(A,B,C)$ in $\mathbb{R}^n$ as poles and zeros of a reduced model $S(A_r,B_r,C_r)$ in $\mathbb{R}^r$ is an important part of model reduction methodology. In the case of a single-input, single-output system with strictly proper transfer function, $g(s)$, of order $n$, this can always be achieved by series factorization of $g(s)$ in the form

$$g(s) = g_1(s)h(s) \quad \cdots (1)$$

where $g_1(s)$ and $h(s)$ are proper transfer functions of order $n_1$ and $n_2$ respectively, $n_1 + n_2 = n$, and the dominant poles and zeros of $g(s)$ are subsets of the poles and zeros of $g_1(s)$. A reduced model $g_r(s)$ of $g(s)$ of the required form is then obtained by computing a reduced model $h_r(s)$ of $h(s)$ and setting

$$g_r(s) = g_1(s)h_r(s) \quad \cdots (2)$$

This paper considers the extension of this approach to the multi-input, multi-output case in the natural manner.
by considering the existence of the series factorization of the system transfer function matrix, G(s), in the form

\[ G(s) = G_1(s)H(s) \]  

...(3)

where \( G_1 \) and \( H \) are proper transfer function matrices possessing realizations of order \( n_1 \) and \( n_2 \) respectively, \( n_1 + n_2 = n \), and the dominant poles and zeros of \( G \) are subsets of the poles and zeros of \( G_1 \). The condition \( n = n_1 + n_2 \) is required to ensure that we do not introduce any extra system states.

A general solution of this problem is not presented. A useful solution is obtained however for the case when \( G_1(s) \) is strictly proper and \( H(s) \) is taken to have the form

\[ H(s) = I + G_2(s) \]  

...(4)

where \( G_2(s) \) is strictly proper. This decomposition is illustrated in Fig.1. It has the advantage that it can be generated by state feedback transformations and enables the problem to be examined using geometric methods.

The geometric formulation of the problem is described in section 2, where necessary and sufficient conditions for the existence of the decomposition are derived in terms of the existence of direct sum decompositions of the system state space. The results are illustrated by an example in section 3 and, in section 4, their application to model reduction is outlined.
2. **Factorization of the Transfer Function Matrix**

Consider the mxm system $S(A,B,C)$ and the factorization of $G(s) = C(sI-A)^{-1}B$ into the form defined by equations (3) and (4). The following lemmas establish the connection between this problem and geometric feedback theory.

---

**Lemma 1:** Let $F$ be an arbitrary mxn matrix, then

$$G(s) = C(sI_n-A+BF)^{-1}B[I_m + F(sI_n-A)^{-1}]$$

**Proof:** Follows directly from the identity

$$C(sI_n-A+BF)^{-1}B = C(sI_n-A)^{-1}(I_n+BF(sI_n-A)^{-1})^{-1}B$$

$$= C(sI_n-A)^{-1}B(I_m+F(sI_n-A)^{-1}B)^{-1}$$

---

**Lemma 2:** $G(s)$ has a factorization of the form $G(s) = G_1(s)(I+G_2(s))$ where each strictly proper transfer function matrix $G_i$ has a realization of dimension $n_i$ $(i = 1,2)$ with $n_1 + n_2 = n$ if, and only if, there exists an mxn matrix $F_o$ such that

$$G_1(s) = C(sI-A+BF_o)^{-1}B, \quad G_2(s) = F_o(sI-A)^{-1}B$$

...(7)

have a realization of dimension $n_i$ $(i = 1,2)$ and $n_1 + n_2 = n$. Furthermore, if $S(A,B,C)$ is controllable, then for a specific choice of $G_1(s)$ and $G_2(s)$ the matrix $F_o$ is unique.

**Proof:** Given $G_1$ and $G_2$ satisfying equation (7) with $n_1 + n_2 = n$ then sufficiency follows from Lemma 1 with $F = F_o$. Conversely, if $G(s)$ has a decomposition of the
required form, let $G_i(s)$ have a realization of dimension $n_i$ described by the triple $(A_i, B_i, C_i)$ \((i = 1, 2)\). A state representation of the system is hence

\[
\begin{align*}
\dot{x}_1(t) &= A_1 x_1(t) + B_1 v(t) \\
v(t) &= u(t) + z(t) \\
\dot{x}_2(t) &= A_2 x_2(t) + B_2 u(t) \\
y(t) &= C_1 x_1(t) \quad z(t) = C_2 x_2(t) \quad \ldots (8)
\end{align*}
\]

or, using the composite state $x(t) = (x_1^T(t), x_2^T(t))^T$,

\[
\begin{align*}
\dot{x}(t) &= \begin{bmatrix} A_1 & B_1 C_2 \\ 0 & A_2 \end{bmatrix} x(t) + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t) \\
y(t) &= \begin{bmatrix} C_1 & 0 \end{bmatrix} x(t) \quad \ldots (9)
\end{align*}
\]

Then the choice of $F_0 = [0, C_2]$ satisfies the desired conditions.

Finally if $S(A, B, C)$ is controllable and $G_2(s) = F_0 (sI - A)^{-1} B$

$= \tilde{F}_0 (sI - A)^{-1} B$ for some matrices $F_0, \tilde{F}_0$, it follows that

$F_0 A^{i-1} B = \tilde{F}_0 A^{i-1} B \quad (i = 1, 2, \ldots n)$

whence

$\left( F_0 - \tilde{F}_0 \right) [B, AB, \ldots, A^{n-1}B] = 0$

and controllability implies that $F_0 = \tilde{F}_0$.

The immediate interpretation of Lemma 2 is that if we drop (for the moment) the condition that the pole and zero sets of $G_1$ contain specified subsets of the pole and zero sets of $G$, then the factorization problem reduces to the generation of conditions for the existence of $F_0$. These are described in the following theorem.
Theorem 1:
Suppose that $S(A,B,C)$ is both controllable and observable. Then there exists an $F_o$ satisfying the conditions of Lemma 2 if, and only if, there exist subspaces $\eta_1$ and $\eta_2$ in $\mathbb{R}^n$ of dimension $n_1$ and $n_2$ such that

$$A\eta_2 \subset \eta_2 + \mathbb{R}(B) \ , \ C\eta_2 = \{0\} \quad \ldots(10)$$

$$A\eta_1 \subset \eta_1 \quad \ldots(11)$$

$$\eta_1 \oplus \eta_2 = \mathbb{R}^n \quad \ldots(12)$$

Proof
Given (10)-(12), there exists $F_o$ such that

$$(A-BF_o)\eta_2 \subset \eta_2 \ , \ F_o\eta_1 = \{0\} \quad \ldots(13)$$

so that $\eta_2$ is just the unobservable subspace of $S_1 = S(A-BF_o,B,C)$ and $\eta_1$ is just the unobservable subspace of $S_2 = S(A,B,F_o)$. It follows directly that $G_1$ and $G_2$ (as given by (7)) have realizations of the required dimensions.

Conversely, suppose there exists $F_o$ satisfying the conditions of Lemma 2. Note that $S_1$ and $S_2$ are controllable but not observable. Let $\eta_1$ (resp. $\eta_2$) be the unobservable subspace of $S_2$ (resp. $S_1$), then controllability and observability imply that $\eta_i$ has dimension $n_i$, $i = 1, 2$. It is easily verified that equations (10) and (11) are satisfied by this choice of $\eta_1$ and $\eta_2$. Equation (12) is proved by using the definitions to show that $\eta_1 \cap \eta_2$ is an $A$-invariant subspace in the kernel of $C$. Observability then implies that $\eta_1 \cap \eta_2 = \{0\}$ which proves (12).
Our original problem now reduces to the problem of choosing subspaces \( \eta_1 \) and \( \eta_2 \) which satisfy (10)-(12) and such that the pole and zero sets of \( G_1 \) contain specified subsets of the pole and zero sets of \( G \). Note that the conditions of Theorem 1 take a natural form for the analysis of this problem, as

(a) The \( A \)-invariant subspace \( \eta_1 \) can be associated with poles \( p_1, p_2, \ldots, p_{n_1} \) of \( S(A, B, C) \) obtained by computing the characteristic polynomial \( \rho_1(s) = (s - p_1) \ldots (s - p_{n_1}) \) of the restriction \( A|\eta_1 \) of \( A \) to \( \eta_1 \).

(b) The \( \{A, B\} \)-invariant subspace \( \eta_2 \) in the kernel of \( C \) can be associated (Wonham, 1974) with invariant zeros \( z_1, z_2, \ldots, z_{n_2} \) of \( S(A, B, C) \) obtained by computing the characteristic polynomial \( \rho_2(s) = (s - z_1) \ldots (s - z_{n_2}) \) of \( (A - BF_0)|\eta_2 \).

These considerations enable us to characterize the pole-zero structure of \( G_1 \) and \( H \) explicitly in terms of the choice of \( \eta_1 \) and \( \eta_2 \). Let \( \rho(M) \) denote the characteristic polynomial of a square matrix \( M \). Then it is trivially verified from (10)-(13) that

\[
\rho(A - BF_0) = \rho(A|\eta_1)\rho(A - BF_0|\eta_2) = \rho_1(s)\rho_2(s)
\]

...(14)

Using this expression we can prove the following corollary to Theorem 1.

**Corollary:**

With the preceding notation, if the conditions of Theorem 1 are satisfied then \( \rho_1(s) \) is the characteristic polynomial of any minimal realization of \( G_1(s) \), and \( \rho_2(s) \) is the zero polynomial of any minimal realization of \( H(s) \).
Proof:

Note first that a minimal realization of a controllable system is obtained by factoring out the unobservable subspace, or unobservable modes corresponding to cancelling poles and zeros.

If \( S(A,B,C) \) is controllable and observable then (c.f. proof of Thm. 1) \( \eta_2 \) is the unobservable subspace of \( S(A-BF_0,B,C) \). Hence by (14), \( \rho_1(s) \) is the characteristic polynomial of any minimal realization of \( G_1(s) \).

The zero polynomial (Owens, 1978) of \( S(A,B,F_0,I) \) is given by

\[
Z_H(s) \triangleq \begin{vmatrix} sI_n - A & -B \\ F_0 & I_m \end{vmatrix} = |sI_n - A + BF_0| = \rho(A-BF_0)
\]

by application of Schur's lemma. As \( \eta_1 \) is the unobservable subspace of \( S(A,B,F_0) \), (14) implies that the zero polynomial of any minimal realization of \( H(s) \) is just \( \rho_2(s) \).

This result could also be obtained by direct transformation to the basis \((\eta_1, \eta_2)\) for \( \mathbb{R}^n \) and using the system state equations in the form (9).

We are now in a position to solve our original problem. The following theorem summarizes the development so far and provides necessary and sufficient conditions.

Theorem 3:

Let the \( m \)-input, \( m \)-output system \( S(A,B,C) \) be invertible, controllable and observable and have \( n_z \) zeros. Let
$P_o = \{p_1, p_2, \ldots, p_\ell\}$ be a subset of the poles of $S(A, B, C)$ and let $Z_o = \{z_{n_z-r+1}, \ldots, z_{n_z}\}$ be a subset of its zeros. Then the system transfer function matrix $G(s)$ has a factorization of the required form where $P_o$ and $Z_o$ are subsets of the poles and zeros of a minimal realization of $G_1(s)$ if, and only if, there exists subspaces $\eta_1$ and $\eta_2$ satisfying the conditions of theorem one and such that

(a) $(s-p_1)(s-p_2)\ldots(s-p_\ell)$ divides $\rho_1(s)$
(b) $\rho_2(s)$ divides $(s-z_1)(s-z_2)\ldots(s-z_{n_z-r})$

(Note: in particular it is necessary that $n_1 \geq \ell$ and $n_2 \leq n_z - r$)

Proof

The proof follows from the previous development noting that, if $G_1(s)$ is to have $Z_o$ as a subset of its zeros, then the zeros of $H(s)$ are a subset of $\{z_1, z_2, \ldots, z_{n_z-r}\}$.

Given the subspaces $\eta_1$ and $\eta_2$ satisfying the conditions of theorem 3, the matrix $F_o$ can, in principle, be computed from equation (13). The procedure for choosing $\eta_1$ and $\eta_2$ is particularly simple if $S(A, B, C)$ has distinct eigenvalues and zeros. In this case let $\{w_1, w_2, \ldots, w_n\}$ be the linearly independent eigenvectors corresponding to the poles $\{p_1, p_2, \ldots, p_n\}$ of $A$ and let $\{v_1, v_2, \ldots, v_{n_z}\}$ be the linearly-independent zero-directions corresponding to the zeros $\{z_1, z_2, \ldots, z_{n_z}\}$. It is trivially verified that a subspace is $A$-invariant if, and only if, it is spanned by a finite subset of $\{w_1, w_2, \ldots, w_n\}$,
and \( \{A,B\} \)-invariant if, and only if, it is spanned by a finite subset of \( \{v_1,v_2,\ldots,v_{n_2}\} \). There are hence only a finite number of candidates for \( \hat{\eta}_1 \) and \( \hat{\eta}_2 \). The number of candidates is further reduced by noting that we must have

\[
\hat{\eta}_1 \triangleq \text{span}\{w_1,w_2,\ldots,w_k\} \subset \eta_1 \quad \ldots(15)
\]

and

\[
\eta_2 \subset \text{span}\{v_1,v_2,\ldots,v_{n_2-r}\} \triangleq \hat{\eta}_2 \quad \ldots(16)
\]

(If the system state-space is transformed to the basis \( \{w_1,\ldots,w_n\} \), and the zero directions recalculated, selection of suitable \( \eta_1 \) and \( \eta_2 \) becomes particularly simple as the eigenvectors are now just unit vectors \( e_1,e_2,\ldots,e_n \)).

Given suitably chosen \( \eta_1 \) and \( \eta_2 \), transformation to a basis matrix \([E_1,E_2]\) for \( \eta_1 \oplus \eta_2 \) yields the system \( S(\hat{A},\hat{B},\hat{C}) \) in the form of equation (9), i.e.

\[
\hat{A} = \begin{bmatrix} A_1 & B_1C_2 \\ 0 & A_2 \end{bmatrix} ; \quad \hat{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} ; \quad \hat{C} = \begin{bmatrix} C_1 & 0 \end{bmatrix} \quad \ldots(17)
\]

In this representation, \( B_1 \) is full rank (for \( B_1x = 0 \) implies that \( Bx \in \eta_2^* \), and invertibility then implies that \( x = 0 \)), so that \( C_2 \) may be calculated uniquely. Hence minimal realizations \( S(A_i,B_i,C_i) \) of \( G_i(s) \) \((i = 1,2)\) can be derived directly from the transformed system of eqn. (17).

The following simple numerical example has been constructed to illustrate the application of the preceding theory.
3. Example of Transfer Function Matrix Factorization

Consider the invertible system \(S(A,B,C)\) defined by

\[
A = \begin{bmatrix}
2 & 0 & 1 & 0 & 0 & 1 \\
2 & 1 & 3 & 1 & -3 & 1 \\
3 & 5 & 4 & 1 & -7 & 1 \\
-2 & -2 & -6 & -2 & 6 & 0 \\
2 & 4 & 3 & 1 & -6 & 1 \\
-6 & 0 & -1 & 0 & 0 & -5
\end{bmatrix}; \quad B = \begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & 0 & 1 & 0 \\
1 & 0 \\
0 & 2 \\
-0.9 & 0.1
\end{bmatrix}; \quad C = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0
\end{bmatrix}
\]

which is controllable and observable, with poles at \(\{0, 0.3, -2, -3, -4\}\), zeros at \(\{-1.378, -3.387, -4.4351, 1\}\), and transfer function matrix

\[
G(s) = \begin{bmatrix}
\frac{s^3+4.1s^2-2.9s-12.2}{s(s+2)(s-3)(s+4)} & \frac{2.1s^2+9.2s+3.7}{s(s-3)(s+3)(s+4)} \\
\frac{2.1s^2+8.1s-0.2}{s(s+2)(s-3)(s+4)} & \frac{s^3+5.1s^2-0.8s-20.3}{s(s-3)(s+3)(s+4)}
\end{bmatrix}
\]

We wish to factor \(G(s)\) into the form of Fig. 1 such that \(G_1(s)\) has \(\{0, 0, 3\}\) as a subset of its poles, and \(\{1\}\) as a subset of its zeros.

With the eigenvalues \(\{0, 0, 3\}\) and the zeros \(\{-1.378, -3.387, -4.4351, 1\}\) are associated the eigenvectors

\[
w_1 = \begin{bmatrix}
0 \\
-1 \\
0 \\
-2 \\
-1 \\
0
\end{bmatrix}, \quad w_2 = \begin{bmatrix}
1 \\
-1 \\
0 \\
0 \\
1 \\
-1
\end{bmatrix}, \quad w_3 = \begin{bmatrix}
1 \\
1 \\
2 \\
-2 \\
1 \\
-1
\end{bmatrix}
\]
and the zero directions

\[ v_1 = \begin{bmatrix} 0 \\ 0 \\ -0.608 \\ 1 \\ -0.608 \\ -0.14 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 0 \\ 1.721 \\ 1 \\ 1.721 \\ 0.334 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1.41 \\ 1 \\ 1.41 \\ 1.025 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 0 \\ 0 \\ -3.78 \\ 1 \\ .667 \\ .113 \end{bmatrix} \]

respectively. The subspaces \( \eta_1 \ni \text{span}\{w_1, w_2, w_3\} \) and
\( \eta_2 \ni \text{span}\{v_1, v_2, v_3\} \) do not intersect, and hence satisfy
the conditions of Theorem 3. Defining the more convenient
basis \( \{v_1', v_2', v_3'\} \) for \( \eta_2 \) as

\[ v_1' = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_2' = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad v_3' = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \]

then transformation of the system \( S(A, B, C) \) to the basis
\( \{w_1, w_2, w_3, v_1', v_2', v_3'\} \) yields \( S(\hat{A}, \hat{B}, \hat{C}) \) of the form (17)
where.

\[ \hat{A} = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & \frac{1}{3} & 2 \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 3 & \frac{1}{3} & \frac{1}{3} & 2 \frac{1}{3} \\ 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ 2 & 1 \frac{1}{3} \\ 3 & \frac{1}{3} \\ 3 & \frac{1}{3} \\ 1 & 1 \frac{1}{3} \\ 1 \frac{1}{3} & 3 \end{bmatrix} \]
\[
\hat{C} = \begin{bmatrix}
  0 & 1 & 1 & 0 & 0 & 0 \\
  -1 & -1 & 1 & 0 & 0 & 0
\end{bmatrix}
\]

The matrices \(A_1, B_1, C_1, A_2, B_2\) are obtained by inspection, and a simple calculation yields \(C_2\). Thus

\[
A_1 = \begin{bmatrix}
  0 & 0 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & 3
\end{bmatrix} ;
B_1 = \begin{bmatrix}
  \frac{1}{3} & -\frac{1}{3} \\
  \frac{2}{3} & \frac{1}{3} \\
  \frac{1}{3} & \frac{1}{3}
\end{bmatrix} ;
C_1 = \begin{bmatrix}
  0 & 1 & 1 \\
  -1 & -1 & 1
\end{bmatrix}
\]

and

\[
A_2 = \begin{bmatrix}
  -2 & 0 & 0 \\
  0 & -3 & 0 \\
  0 & 0 & -4
\end{bmatrix} ;
B_2 = \begin{bmatrix}
  1 & 0 \\
  0 & 1 \\
  0.1 & 0.1
\end{bmatrix} ;
C_2 = \begin{bmatrix}
  0 & 1 & 1 \\
  1 & 0 & 1
\end{bmatrix}
\]

It is simple to verify that \(S(A_1, B_1, C_1)\) does indeed have a zero \(z_1 = 1\), and the desired poles. We find that

\[
G_1(s) = \frac{1}{s(s-3)} \begin{bmatrix}
  s-2 & 1 \\
  1 & s-2
\end{bmatrix}
\]

\[
G_2(s) = \frac{1}{s+4} \begin{bmatrix}
  0.1 & \frac{1.1s+4.3}{s+3} \\
  \frac{1.1s+4.2}{s+2} & 0.1
\end{bmatrix}
\]

and \(G_1(s)(I+G_2(s)) = G(s)\).

The desired decomposition has thus been achieved!
4. Application to Model Reduction

Again, consider an m×m invertible system \( S(A,B,C) \) with transfer function matrix \( G(s) \), and suppose that the decomposition \( G(s) = G_1(s)(I+G_2(s)) \), as described above, exists, where \( G_1(s) \) contains specified subsets of the poles and zeros of \( G(s) \). These subsets are assumed to be of considerable importance in characterizing the system dynamics, and we wish to retain them in a reduced model. If \( G_2^*(s) \) is some reduced order model of \( G_2(s) \), then
\[
G^*(s) = G_1(s)(I + G_2^*(s)) \quad \text{...(18)}
\]
will be a reduced order model of \( G(s) \) which preserves the desired subsets of the poles and zeros of \( G(s) \).

Equivalently, in state-space form, if \( S(A,B,C) \) is in the form (17) (or (9)) corresponding to the transfer function matrix factorization (18), and \( S(A_2^*,B_2^*,C_2^*) \) is a state-space representation of \( G_2^*(s) \), then \( G^*(s) \) is described by \( S(A^*,B^*,C^*) \) where
\[
A^* = \begin{bmatrix} A_1 & B_1 C_2^* \\ 0 & A_2^* \end{bmatrix} ; \quad B^* = \begin{bmatrix} B_1 \\ B_2^* \end{bmatrix} ; \quad C^* = [C_1 \ 0] \quad \text{...(19)}
\]

Assuming \( S(A,B,C) \) and \( S(A_2^*,B_2^*,C_2^*) \) are controllable and observable, \( S(A^*,B^*,C^*) \) will be a minimal realization of \( G^*(s) \), provided that the choice of \( G_2^*(s) \) has not produced any pole-zero cancellation between \( G_1(s) \) and \((I+G_2^*(s))\).

The use of the factorization \( G(s) = G_1(s)(I+G_2(s)) \) as a means of preserving poles and zeros in model-reduction has the bonus that suitable choice of reduced model, \( G_2^*(s) \),
of $G_2(s)$ will ensure that $G^*(s)$, as described by equation (18), matches a desired number of moments of $G(s)$ both about $s = 0$ and $s = \infty$. To state this precisely (without proof, which is obvious):

Proposition:

If $G_2^*(s)$ is a reduced-order model of $G_2(s)$ which matches the first $m_0$ terms of the series expansion about $s = \infty$,

$$G_2(s) = \sum_{i=1}^{\infty} M_is^i$$

and the first $n_0$ terms of the series expansion about $s = 0$,

$$G_2(s) = \sum_{i=0}^{\infty} N_is^i$$

then $G^*(s)$, as defined by equation (18), will match the first $m_0+1$ and $n_0$ terms respectively of the series expansions of $G(s)$ about $s = \infty$ and $s = 0$.

The application of the above ideas can be illustrated by considering again the simple example of section (3). Here it is clear that the unstable poles 0, 0, 3 and the R.H.P. zero at $z = 1$ are of such importance that they must be retained explicitly in a reduced order model. We make use of the given factorization $G(s) = G_1(s)(I+G_2(s))$.

It can be seen that the residue of the pole of $G_2(s)$ at $-4$ is small, and, applying a well-established model reduction principle (Davison 1966, Marshall 1966), we neglect this mode to obtain
\[ A_2^* = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} ; \quad B_2^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} ; \quad C_2^* = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \]

The reduced system has transfer function matrix

\[
G^*(s) = \begin{bmatrix}
\frac{s^2-3}{s(s-3)(s+2)} & \frac{2s+1}{s(s-3)(s+3)} \\
\frac{2s}{s(s-3)(s+2)} & \frac{s^2+s-5}{s(s-3)(s+3)}
\end{bmatrix}
\]

which again has poles at 0, 0, 3, and a zero at 1.

5. Conclusions

This paper considers one possible approach to the problem of simultaneous retention of poles and zeros in the derivation of reduced-order models of linear multivariable systems. It is based on a series factorization of the system transfer function matrix into proper and strictly proper parts. Because of its demonstrated connection with state feedback, the factorization can be shown to exist precisely when the system state-space admits a direct sum decomposition in terms of \(A\)- and \(\{A,B\}\)-invariant subspaces. These invariant subspaces lend themselves naturally to the problem of ensuring that one component of the factorization retains desired pole and zero subsets of the original system; reduction of the remaining subsystem will then yield a reduced-order model with the desired properties.

Although consideration has only been given to the particular case in which \(G_1(s)\) is strictly proper and \(H(s)\) takes the form \(I+G_2(s)\), with \(G_2(s)\) strictly proper,
it may be useful to examine conditions under which the system admits a more general factorization, e.g. \( G_1(s), H_1(s) \) both strictly proper, or both proper but not strictly proper. This problem together with the computational problem presented by the result of this paper are the subject of future work.

REFERENCES


Figure 1