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Competition between Two Cities using Cordon Tolls: An Exploration of Response Surfaces and Possibilities for Collusion

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ABSTRACT

Extending the literature on competition in the presence of road tolls, this paper explores the implications of competition between two cities. We assume that the two city authorities wish to maximise the welfare of their own residents whilst taking advantage of tax export mechanisms available to them by charging traffic from the competing city. The problem is first posed as an Equilibrium Problem with Equilibrium Constraints (EPEC) which is a special form of a Nash game with a hierarchical structure. Due to the inherent non-convexity of EPECs, it is possible that only local solutions are found. Hence we introduce the concept of local Nash Equilibrium. Using a simple network and grid search to explore the response surfaces and to determine the Nash Equilibrium toll levels, we conduct two numerical examples with a simple test network but with different trip demand assumptions. Our numerical examples indicate the possibility that there may exist multiple local Nash solutions and that competition may lead to a sub-optimal outcome for one or both authorities depending on whether there exists a stronger player. We also consider the impact of elasticity of demand and other parameter assumptions on the potential number of Nash solutions. We then introduce the notion of collusion whereby cities share out some of the revenues collected and demonstrate that as collusion levels are increased then the Nash solutions tend to converge towards the global regulator solution when cities are assumed identical and both cities are incentivised to collude. However with asymmetric demand then the weaker city's residents remain worse off than in the no toll case and the stronger city has no incentive to collude.

Key words: Networks, pricing, competition, Nash Equilibrium, Tolls

1. INTRODUCTION

There has been a strong focus in recent years on road user charging, with economic theory suggesting benefits will accrue to a city from a combination of congestion relief and recycling of revenues within the city (Walters, 1961). Beyond the theoretical benchmark of full marginal cost pricing the design of practical charging schemes, such as those adopted by UK local authorities in recent Transport Innovation Funds (Department for Transport, 2005; Transport Select Committee, 2006) bids, have generally focused on pricing cordons around single, mono-centric cities (Shepherd *et al*, 2008). It is possible in such cases to design the location and level of charges for a cordon so as to systematically maximise the potential welfare gain to the city (Shepherd and Sumalee, 2004; Sumalee et al, 2005), yet there is an implicit premise here that the city acts *in isolation*.

In this paper we consider the implications of competition between cities when each considers the introduction of fiscal demand management measures by setting road tolls. In the context of toll roads, several authors have studied the welfare implications of competition between a public and private operator (e.g. Verhoef *et al*, 1996; De Palma and Lindsey, 2000; Yang and Woo, 2004; Zhang and Levinson, 2005; Yang et al, 2009). The focus in these studies is on the impacts of alternative ownership regimes, and of public versus private control in the form of either monopoly pricing or competitive Nash equilibria. Xiao *et al*, (2007) extended these works by bounding the inefficiency of private toll road competition for a network with parallel links.

In addition to the often discussed issue of competition among profit motivated organizations, there also exist competitive issues between public sector organisations. Proost and Sen (2006), used the TRENEN (Proost and Van Dender 2001) strategic model to investigate the outcome of a Nash Game between a local authority in control of parking charges and regional government in charge of a toll cordon. They found that the city was incentivised to over-charge for parking. Tax exporting behaviour is a concept from the public economics literature (e.g. Stiglitz, 2000). In the context of using tolls as fiscal instruments, the argument is that local governments wish to score political points with their residents and do so by laying the burden of paying the toll onto “foreign” (i.e. non-resident) users in the local area. This tax exporting behaviour is a theme which has continued to recur in the literature e.g. De Borger *et al* (2007), Ubbels and Verhoef (2008), Guhenmann *et al* (2011).

One limitation of the study by Proost and Sen (2006), which the authors recognized, was that the TRENEN strategic model does not embody a network and is thus unable to take the route choice considerations of users expliclity into account. The modelling framework in Gühnemann et al (2011) was a tolling game between two authorities, one controlling a cordon surrounding the city of Sheffield in the UK which was plagued by air quality problems and another surrounding the Peak district which had the problem of serious through traffic. One key conclusion of this study was that the Peak district tended to act as a net tax exporter because traffic had no alternative but to travel through the Peak district as alternative routes were even more costly in terms of travel time and distances. In addition, it was found that the Nash game tended to result in the transfer of environmental problems from one jurisdiction to affect other areas and this lent support to the argument that some form of global regulation was necessary since left to their own devices, authorities might be tempted to play “beggar thy neighbour” policies which would have a detrimental impact on global welfare. De Borger *et al* (2007) and Ubbels and Verhoef (2008) examined the issues of competition between countries/regions setting tolls and capacities, investigating the implications of players adopting two-stage games but using networks where route choice was also absent. In this paper we explicitly take into account route choice.

We build on our previous work on representing multi-actor systems through game-theoretic representations where the problem of toll competition between operators in a network was considered. Represented through a Nash network game, in Koh and Shepherd (2010) conditions were established under which the equilibria of such non-cooperative decision-making differed markedly from the solution that could arise from a more collusive game between operators. It was also shown that this latter, collusive solution could be determined based on the ‘particle swarm optimisation’ method (Koh, 2008). This work also demonstrated the potential for multiple Nash Equilibria to occur in games where players face an equilibrium constraint which parallels the results discussed in the context of bidding strategies of generators in deregulated transmission constrained electricity markets (Hu and Ralph, 2007;Son and Baldick, 2004).

The paper is structured as follows. Following this introduction, Section 2 sets up the problem of competition between two cities and provides the mathematical background. Using a small network, two scenarios for a single test network are studied in Section 3 utilising simple grid search (an exhaustive search over the parameter space) to identify potential Nash Equilibria. These two scenarios differ only with respect to the individual components of the trip matrix, the first representing identical cities and then the second where one city is more attractive than the other as a destination thus introducing the notion of strong and weak player. We also investigate the impacts of changes in the elasticity of demand on the existence of multiple Nash Equilibria in the network. Section 4 considers the situation when cities are able to share a proportion of the revenues collected from road pricing even though they continue to be in competition with each other. Section 5 wraps up the paper with some conclusions and directions for further research.

2. METHODOLOGY

A highway network is represented as a graph comprising links indexed by the set $L = \{1, 2, ..., |L|\}$. We assume that there are two regulatory authorities (labelled A and B), each authority having their own pre-defined subset of network links over which they may charge a toll: authority *i* being able to set a toll on links in set $L_i \subseteq L$ (for $i \in \{\mathrm{A}, \mathrm{B}\}$), with $L_\mathrm{A} \cap L_\mathrm{B} = \phi$. Although not necessary, for simplicity we make the restriction in this paper that each authority *i* has a *single*[1] toll level $\tau_i \geq 0$ that they may determine and levy on all links in their link sub-set L_i. Together, then, the two tolls to be determined can be collected in the vector $\boldsymbol{\tau} = \begin{pmatrix}\tau_\mathrm{A} & \tau_\mathrm{B}\end{pmatrix}^\#$ with # denoting the transpose. In practice, in addition to non-negativity constraints, we may wish to impose additional simple bounds on the tolls (e.g. upper bounds that are believed reasonable), and thus for each $i \in \{\mathrm{A}, \mathrm{B}\}$, we suppose that there is a pre-defined set $T_i \subseteq \{x \in \mathbb{R}^2 \; \& \; x \geq 0\}$ that defines the permissible tolls, so that we must have $\boldsymbol{\tau} \in T_\mathrm{A} \times T_\mathrm{B}$. The travellers in the network are all supposed to perceive these tolls in the same way, regardless of which authority levied the toll and regardless of their own socio-economic status. Aside from the tolls, travellers perceive other attributes that motivate their travel (e.g. travel time), and for each link these are collected together in a single generalized cost of travel, excluding tolls. This toll-excluding generalized cost typically will depend, through congestion, on the flow on the link, and so for each link $l = 1, 2, ..., |L|$ we represent it as a monotonically increasing, continuous function $c_l(v_l)$ of the flow v_l on link *l*. Taking the tolls together with the toll-excluding generalized cost gives us the complete generalized cost function, given any link flow or toll levels as:

$$g_l(v_l, \boldsymbol{\tau}) = \begin{cases} c_l(v_l) + \tau_\mathrm{A} & \text{if } l \in L_\mathrm{A} \\ c_l(v_l) + \tau_\mathrm{B} & \text{if } l \in L_\mathrm{B} \\ c_l(v_l) & \text{otherwise} \end{cases} \qquad (l = 1, 2, ..., |L|, v_l \geq 0, \boldsymbol{\tau} \in T_A \times T_B) \tag{1}$$

[1] The assumption of a single toll level is not restrictive. Firstly, cordon schemes currently in operation such as in Bergen (Norway), Milan (Italy) and Stockholm (Sweden) have a uniform charge levels over a given modelled period, at all points entering the cordon area. Secondly, we do not wish to allow the city to charge a different amount to non-residents as this would be seen as less acceptable.

Clearly, should we wish to represent it that way, $c_l(\bullet) = g_l(\bullet, 0)$ for all $l \in L$. The functions (1) may be collected together into a vector mapping $\mathbf{g}(\mathbf{v}, \boldsymbol{\tau})$ with l^{th} element $g_l(v_l, \boldsymbol{\tau})$ $(l = 1, 2, ..., |L|)$.

Our network also comprises Origin-Destination (OD) movements indexed by the set $K = \{1, 2, ..., |K|\}$, with d_k $(k \in K)$ denoting the travel demand for OD movement *k*. We suppose that in advance, we neither know the OD travel demand vector **d** nor the link flow vector **v**, but that they are contained in the demand-feasible set *D* given by:

$$D = \left\{ (\mathbf{v}, \mathbf{d}) : \mathbf{v} = \sum_{k \in K} \mathbf{x}^{(k)} \text{ where } \mathbf{A}\mathbf{x}^{(k)} = \mathbf{E}^{(k)} d_k,\ \mathbf{x}^{(k)} \geq 0,\ d_k \geq 0\ \forall k \in K \right\} \tag{2}$$

where $\mathbf{x}^{(k)}$ is the vector of link flows for OD movement *k*, where **A** is the node-link incidence matrix for the network, and where $\mathbf{E}^k$ is a vector that defines the origin and destination nodes for OD movement *k* (for more details the reader is referred to Lawphongpanich and Hearn, 2004).

We further suppose that for each origin-destination movement *k*, there exists a separable, bounded, continuous, monotonically increasing demand function that expresses the origin-destination demand level for that movement as a function of the generalized OD travel cost for that movement. In fact, we shall refer not to the function itself but to its inverse (which exists under the stated assumptions), namely the OD generalized cost $w_k(d_k)$ that would need to exist in order to generate a given level of OD travel demand d_k, for each $k \in K$. These functions are assumed to be continuous, bounded, and monotonically decreasing.

Given any particular toll vector $\boldsymbol{\tau}$, it is supposed that the resulting perceptions of generalized cost determine the OD travel demand and routing patterns through an elastic demand Wardrop equilibrium. Now, if the toll vector $\boldsymbol{\tau}$ was to be decided by a single regulatory authority, then we could define a Global Regulatory Problem in the form of a Mathematical Program with Equilibrium Constraints (MPEC), which (following Lawphongpanich and Hearn, 2004) is given by:

$$\begin{aligned} &\underset{\boldsymbol{\tau} \in T_{\mathrm{A}} \times T_{\mathrm{B}}, (\mathbf{v}, \mathbf{d}) \in D}{\text{Maximise}} \quad \sum_{k \in K} \int_0^{d_k} w_k(z)\,dz - \sum_{l \in L} v_l c_l(v_l) \\ &\text{s.t.} \qquad \mathbf{g}(\mathbf{v}, \boldsymbol{\tau})^{\#}(\mathbf{u} - \mathbf{v}) - \mathbf{w}(\mathbf{d})^{\#}(\mathbf{d} - \mathbf{e}) \geq 0 \quad \forall (\mathbf{u}, \mathbf{e}) \in D \end{aligned} \tag{3}$$

Note that the toll vector itself does not appear in the upper level (social welfare) objective function of (3), its role instead is in shaping behaviour as represented in the lower level constraint. In fact, since under the stated assumptions on the cost and demand functions, there is a *unique* solution in (**v**,**d**) for any given toll vector, then the variational inequality constraint determines a unique such demand/flow allocation given any toll vector. Then we may simplify (3) so that only the toll vector appears as the maximization variable (since for any given toll vector, a unique demand and flow vector is uniquely generated by the VI constraint):

$$\underset{\boldsymbol{\tau}\in T_{\mathrm{A}}\times T_{\mathrm{B}}}{\text{Maximise}} \quad \sum_{k\in K}\int_0^{d_k} w_k(z)dz - \sum_{l\in L} v_l c_l(v_l)$$
$$\text{s.t.} \qquad \mathbf{g}(\mathbf{v},\boldsymbol{\tau})^{\#}(\mathbf{u}-\mathbf{v}) - \mathbf{w}(\mathbf{d})^{\#}(\mathbf{d}-\mathbf{e}) \ge 0 \quad \forall(\mathbf{u},\mathbf{e})\in D \tag{4}$$

Problem (4) represents a situation in which a single regulator sets all the toll levels so as to maximize the benefit to the whole network. However, we shall also be specifically interested in the toll levels that arise if the two authorities *compete*. In this case, we assume that each authority has jurisdiction over setting tolls on its own set of links, but that its responsibility is only to trips that originate in its area. Thus, we partition the origin-destination movements into two mutually exclusive and exhaustive sets, such that K_i is the index set of OD movements originating in authority *i* (for $i\in\{\mathrm{A},\mathrm{B}\}$), with $K = K_{\mathrm{A}}\cup K_{\mathrm{B}}$ and $K_{\mathrm{A}}\cap K_{\mathrm{B}} = \phi$. In parallel, we also partition the link flow variable, such that $\tilde{v}_{li}$ denotes the flow on link *l* of demand originating from Authority *i*, clearly with $v_l = \tilde{v}_{l\mathrm{A}} + \tilde{v}_{l\mathrm{B}}\ (l = 1,2,...,|L|)$. In vector notation, if the authority link flows are collected in a $|L|\times 2$ matrix $\tilde{\mathbf{V}}$, then they are related to the aggregate link flow vector by $\mathbf{v} = \tilde{\mathbf{V}}\mathbf{1}$, where $\mathbf{1} = \begin{pmatrix}1 & 1\end{pmatrix}^T$.

Let us first consider Authority A. Authority A is assumed to maximise social welfare of its own residents by adjusting the toll level of links over which it has control, *anticipating* the impact of the toll on travellers' route and demand decisions, but *reacting* to the toll level levied by Authority B. That is to say, Authority A does not anticipate the effect that their own choice of toll will have on Authority B's response, but they simply react to the toll set by Authority B. Let us assume for the moment that Authority B has already decided its toll level $\tau_{\mathrm{B}} = \overline{\tau}_{\mathrm{B}} \in T_{\mathrm{B}}$, and that this is known to Authority A. Authority A is then supposed to determine its own toll level τ_{A} by solving an MPEC that is a variant of (3):

$$\underset{\substack{(\tau_{\mathrm{A}},\tilde{\mathbf{V}},\mathbf{d})\\ \text{such that}\\ \tau_{\mathrm{A}}\in T_{\mathrm{A}},(\tilde{\mathbf{V}}\mathbf{1},\mathbf{d})\in D}}{\text{Maximise}} \quad \sum_{k\in K_{\mathrm{A}}}\int_0^{d_k} w_k(z)dz - \sum_{l\in L}\tilde{v}_{l\mathrm{A}}c_l(\tilde{v}_{l\mathrm{A}}+\tilde{v}_{l\mathrm{B}}) - \alpha\sum_{l\in L_{\mathrm{B}}}\overline{\tau}_{\mathrm{B}}\tilde{v}_{l\mathrm{A}} + \alpha\sum_{l\in L_{\mathrm{A}}}\tau_{\mathrm{A}}\tilde{v}_{l\mathrm{B}}$$
$$\text{s.t.} \qquad \mathbf{g}\left(\tilde{\mathbf{V}}\mathbf{1},\begin{pmatrix}\tau_1\\ \overline{\tau}_2\end{pmatrix}\right)^{\#}(\mathbf{u}-\tilde{\mathbf{V}}\mathbf{1}) - \mathbf{w}(\mathbf{d})^{\#}(\mathbf{d}-\mathbf{e}) \ge 0 \quad \forall(\mathbf{u},\mathbf{e})\in D \tag{5}$$

The first term in the objective function of (5) is the Marshallian measure of the trips made from origins located within Authority A's jurisdiction. The second term represents the generalized cost of travel (excluding tolls) for traffic with origins in Authority A. The third term represents the toll revenue spent by residents from Authority A on links controlled by Authority B, i.e. those with origins in Authority A and travelling on tolled links in Authority B. This is a transfer payment and it increases the coffers of Authority B at the expense of Authority A. The fourth term represents the toll revenue spent by residents from Authority B within Authority A, this being a transfer payment that increases the coffers of Authority A at the expense of Authority B. The parameter α is a scalar tax exporting parameter, for which we shall assume a common value for both authorities $(0\le\alpha\le 1)$. Our main numerical examples in Section 3 focus on α taking the value of 1 and we consider varying alpha values in Section 4.

While problem (5) has a similar mathematical structure to problem (3) a key difference is that our problem is now defined in terms of link flows disaggregated by authority (the ‘authority link flows’). In general networks, for any given toll vector, we cannot guarantee uniqueness of the authority link flows, even though our assumptions guarantee uniqueness of the total link flows. Therefore, if applied in a general network, (5) maximizes social welfare in two ways: partly through the toll, but additionally by assuming that we can control the authority link flows over-and-above the toll effect. Another way to view this is that while we assume user equilibrium for the total link flows, we assume system optimization for the authority link flow splits, wherever there is ambiguity in these splits to exploit (the so-called ‘weak’ formulation of MPEC; see Červinka, 2008). However, at present our proposal is to restrict attention to applying (5) in special network structures in which the uniqueness of the total link flows automatically guarantees uniqueness of the authority link flows.

We later consider such a network example. Assuming then, that our network structure ensures uniqueness of the authority link flows, problem (5) may be simplified to:

$$\begin{aligned} &\underset{\tau_{\mathrm{A}} \in T_{\mathrm{A}}}{\text{Maximise}} \quad S_{\mathrm{A}}\left(\begin{pmatrix} \tau_{\mathrm{A}} \\ \bar{\tau}_{\mathrm{B}} \end{pmatrix}, \tilde{\mathbf{V}}, \mathbf{d}\right) \\ &\text{s.t.} \qquad \mathbf{g}\left(\tilde{\mathbf{V}}\mathbf{1}, \begin{pmatrix} \tau_{\mathrm{A}} \\ \bar{\tau}_{\mathrm{B}} \end{pmatrix}\right)^{\#} (\mathbf{u} - \tilde{\mathbf{V}}\mathbf{1}) - \mathbf{w}(\mathbf{d})^{\#}(\mathbf{d} - \mathbf{e}) \geq 0 \quad \forall (\mathbf{u}, \mathbf{e}) \in D \end{aligned} \tag{6}$$

Where

$$S_{\mathrm{A}}(\boldsymbol{\tau}, \tilde{\mathbf{V}}, \mathbf{d}) = \sum_{k \in K_{\mathrm{A}}} \int_0^{d_k} w_k(z)dz - \sum_{l \in L} \tilde{v}_{l\mathrm{A}} c_l(\tilde{v}_{l\mathrm{A}} + \tilde{v}_{l\mathrm{B}}) - \alpha \sum_{l \in L_{\mathrm{B}}} \tau_{\mathrm{B}} \tilde{v}_{l\mathrm{A}} + \alpha \sum_{l \in L_{\mathrm{A}}} \tau_{\mathrm{A}} \tilde{v}_{l\mathrm{B}} \tag{7}$$

As in our earlier problem (4), in problem (6) the flow variables $(\tilde{\mathbf{V}}, \mathbf{d})$ are uniquely determined by the variational inequality constraint at any given toll vector $\boldsymbol{\tau}$, under the restrictive assumptions we have made. In order to reflect this, introduce the following implicit functions:

For given $\boldsymbol{\tau} \in T_{\mathrm{A}} \times T_{\mathrm{B}}$, $(\tilde{\mathbf{V}}^*(\boldsymbol{\tau}), \mathbf{d}^*(\boldsymbol{\tau}))$ denotes the unique solution in $(\tilde{\mathbf{V}}, \mathbf{d})$ to

$$\mathbf{g}\left(\tilde{\mathbf{V}}\mathbf{1}, \boldsymbol{\tau}\right)^{\#} (\mathbf{u} - \tilde{\mathbf{V}}\mathbf{1}) - \mathbf{w}(\mathbf{d})^{\#}(\mathbf{d} - \mathbf{e}) \geq 0 \quad \forall (\mathbf{u}, \mathbf{e}) \in D \qquad (\tilde{\mathbf{V}}\mathbf{1}, \mathbf{d}) \in D \ . \tag{8}$$

Thus (6) may then be equivalently written in succinct form:

$$\underset{\tau_{\mathrm{A}} \in T_{\mathrm{A}}}{\text{Maximise}} \quad S_{\mathrm{A}}\left(\begin{pmatrix} \tau_{\mathrm{A}} \\ \bar{\tau}_{\mathrm{B}} \end{pmatrix}, \tilde{\mathbf{V}}^*\begin{pmatrix} \tau_{\mathrm{A}} \\ \bar{\tau}_{\mathrm{B}} \end{pmatrix}, \mathbf{d}^*\begin{pmatrix} \tau_{\mathrm{A}} \\ \bar{\tau}_{\mathrm{B}} \end{pmatrix}\right) \tag{9}$$

Now, in an analogous way to the behaviour of Authority A, Authority B determines its toll level τ_{B} conditional on the toll level of Authority A by considering its own counterpart to objective function (7) namely:

$$S_{\mathrm{B}}(\boldsymbol{\tau}, \tilde{\mathbf{V}}, \mathbf{d}) = \sum_{k \in K_{\mathrm{B}}} \int_0^{d_k} w_k(z)dz - \sum_{l \in L} \tilde{v}_{l\mathrm{B}} c_l(\tilde{v}_{l\mathrm{A}} + \tilde{v}_{l\mathrm{B}}) - \alpha \sum_{l \in L_{\mathrm{A}}} \tau_{\mathrm{A}} \tilde{v}_{l\mathrm{B}} + \alpha \sum_{l \in L_{\mathrm{B}}} \tau_{\mathrm{B}} \tilde{v}_{l\mathrm{A}} \tag{10}$$

The inter-play of the two authorities in each aiming to maximize its own welfare by setting a toll, conditional on the other authority's toll, while anticipating the impact on the travellers, leads us to an example of a so-called Equilibrium Problem with Equilibrium Constraints (EPEC) (Mordukhovich, 2005). This overall problem we may write, based on the functions defined in (7),(8) and (10), as follows:

$$\text{Find } \begin{pmatrix}\tau_{\mathrm{A}} & \tau_{\mathrm{B}}\end{pmatrix}^T \in T_{\mathrm{A}} \times T_{\mathrm{B}} \text{ such that simultaneously:}$$

$$\begin{aligned}
&S_{\mathrm{A}}\left(\begin{pmatrix}\tau_{\mathrm{A}}\\ \tau_{\mathrm{B}}\end{pmatrix}, \tilde{\mathbf{V}}^*\begin{pmatrix}\tau_{\mathrm{A}}\\ \tau_{\mathrm{B}}\end{pmatrix}, \mathbf{d}^*\begin{pmatrix}\tau_{\mathrm{A}}\\ \tau_{\mathrm{B}}\end{pmatrix}\right) \geq S_{\mathrm{A}}\left(\begin{pmatrix}h_{\mathrm{A}}\\ \tau_{\mathrm{B}}\end{pmatrix}, \tilde{\mathbf{V}}^*\begin{pmatrix}h_{\mathrm{A}}\\ \tau_{\mathrm{B}}\end{pmatrix}, \mathbf{d}^*\begin{pmatrix}h_{\mathrm{A}}\\ \tau_{\mathrm{B}}\end{pmatrix}\right) \quad \forall h_{\mathrm{A}} \in T_{\mathrm{A}}\\
&S_{\mathrm{B}}\left(\begin{pmatrix}\tau_{\mathrm{A}}\\ \tau_{\mathrm{B}}\end{pmatrix}, \tilde{\mathbf{V}}^*\begin{pmatrix}\tau_{\mathrm{A}}\\ \tau_{\mathrm{B}}\end{pmatrix}, \mathbf{d}^*\begin{pmatrix}\tau_{\mathrm{A}}\\ \tau_{\mathrm{B}}\end{pmatrix}\right) \geq S_{\mathrm{B}}\left(\begin{pmatrix}\tau_{\mathrm{A}}\\ h_{\mathrm{B}}\end{pmatrix}, \tilde{\mathbf{V}}^*\begin{pmatrix}\tau_{\mathrm{A}}\\ h_{\mathrm{B}}\end{pmatrix}, \mathbf{d}^*\begin{pmatrix}\tau_{\mathrm{A}}\\ h_{\mathrm{B}}\end{pmatrix}\right) \quad \forall h_{\mathrm{B}} \in T_{\mathrm{B}}
\end{aligned} \tag{11}$$

Problem (11) assumes that the authorities can only determine their own toll conditional on the other authority, but places no further restriction on the admissible tolls. That is to say, the conditions require that, as far as one authority is concerned, their toll gives (marginally, i.e. based only optimizing their own toll) a global optimum solution to their individual MPEC, conditional on the other authority's toll setting.

Taking the conditions for both authorities together, equation (11) defines a problem that we will henceforth simply refer to as a Nash Equilibrium (NE) (Nash,1950). However, we shall also be interested in Nash games that are variants of (11), Specifically if, rather than each authority determining a global optimum toll conditional on the other authority's toll choice, we consider the possibility that each authority only determines a local optimum to their individual MPEC. In this case we require conditions (11) only to hold within a local neighbourhood of the given toll vector. Following Son and Baldick (2004), we shall refer to an equilibrium of such a Nash game as a Local Nash Equilibrium (LNE). Thus for an LNE, each authority only needs establish optimality within a neighbourhood of the given solution (see Ye and Zhu, 2003, for such an example).

Since the LNE conditions are weaker, the solution set to the NE problem is contained within the solution set to the LNE. It is our proposal that both kinds of solution are relevant for investigation, since it is not clear which is a more realistic representation of the behaviour of authorities in setting their tolls. This is an issue we return to in the case studies.

3. CASE STUDIES

All our case studies use the same topological network as shown in Figure 1. The travel cost on all links in the network adopt the standard BPR functional form as given in (12). The free flow travel time parameter (t_0) is 450 seconds for all links except 2,5,8 and 11 which is 1000 seconds. The capacity parameter (κ_l) is 1500 pcus/hr for all links except for 2,5,8 and 11 which is 3000 pcus/hr. The links (2,5,8 and 11) therefore represent a high capacity bypass that avoids travel through the town centre.

$$c_l(v_l) = t_0(1 + 0.15(\frac{v_l}{\kappa_l}))^4 \tag{12}$$

<<INSERT FIGURE 1 APPROXIMATELY HERE >>

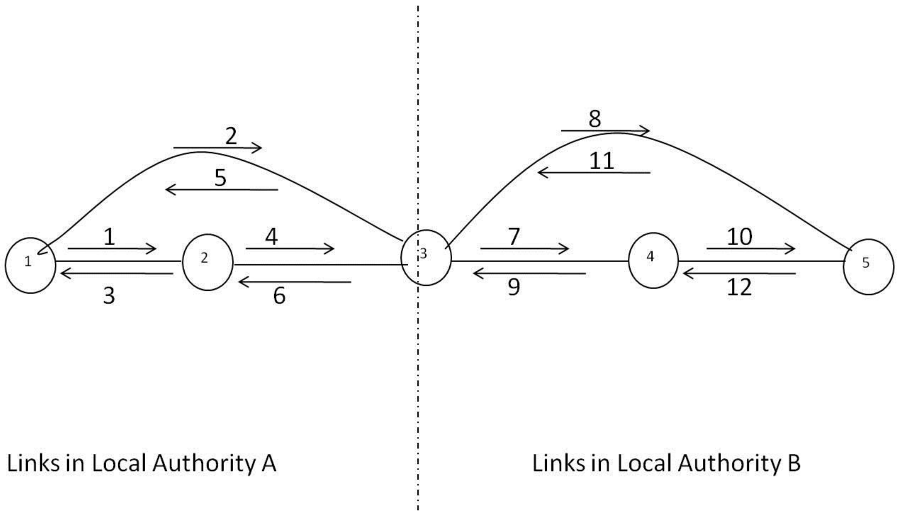

Figure 1:Network for Numerical Examples[2]

On the demand side there are 12 Origin-Destination pairs. All nodes excluding Node 3 are origin or destination zones. There are two Central Business Districts (CBD) (zone 2 and zone 4) located within Authority A and B respectively.

The dotted line through Node 3 on Figure 1 demarcates the boundary of jurisdiction between the two authorities. The base demand represents a typical morning peak with dominant flows to the CBDs from the suburb of each local authority (zones 1 and 5). However we also introduce demand to/from other zones which represent interaction between the authorities with associated problems of through traffic. We assume elastic demand and the demand function, which gives the trips as a function of the generalised costs of travel, adopts the power law specification:

$$d_k = d_{k,0}\left(\frac{b_k}{b_{k,0}}\right)^p, k \in K \tag{13}$$

In (13), $d_{k,0}, b_{k,0}, b_k$ refer to the base trips, base costs and costs for origin destination pair k and p is the power parameter with the restriction that $p > 0$. We assume that p does not vary by OD pair. Equation (13) implies an inverse demand function of the form (14)

$$w_k(d_k) = b_{k,0}\left(\frac{d_k}{d_{k,0}}\right)^{\frac{1}{p}}, k \in K \tag{14}$$

We assume that Authority A sets a uniform common toll on Links 1 and 6 to simulate a cordon into its CBD zone 2 while Authority B sets a uniform common toll on Links 7 and 12 to simulate a cordon for travel into its CBD zone 4. In this way we represent a situation which may arise in reality, namely that of cities who both wish to set up a cordon charge around their CBD with the idea of maximizing the welfare of their residents (as set out in (7)).

[2] The numbers indicated are link numbers referred to in the text and direction of travel is indicated by the arrows. The dotted line down node 3 demarcates the limits of jurisdiction of each authority.

As noted in section 2, a key property required of our formulation (in moving from (5) to (6)) is uniqueness of link flows disaggregated by authority, at any given toll vector. This is established for the particular network under consideration in Appendix A, requiring some mild additional assumptions that are readily verifiable during our numerical analysis, and indeed they have been verified to hold. Considering Authority A's network (by symmetry, analogous implications can be drawn for Authority B's network), uniqueness is established by a combination of (a) identifying routes that would never be efficient under Wardrop conditions; (b) applying conservation-of-flow at the authority level; and (c) noting where authority flows do and do not mix. In the case of Authority A's network (analogous properties hold for Authority B's network, by symmetry), we end up with mixing of the flows between authorities on links 1, 3 and 6 only, whereas links 2 and 4 only carry Authority A flow and link 5 only carries Authority B flow.

In our numerical experiments, we consider two different cases. In both cases the network remains as defined above and there is symmetry between the network within Authority A and that within Authority B. The only difference between the two cases concerns the individual trips within the trip matrix. In case study 1 (hereinafter 'case 1') the base demand in the no toll case is also symmetric which represents a case where cities are equal in terms of production and attraction and in terms of network supply. For case study 2 (hereinafter 'case 2') the same network is used but we adjusted the base demand so that the city in Authority A is seen as stronger in terms of its ability to attract users. Details of the matrix used in each case study are given in the relevant sections.

To solve the global regulator problem for each case study, we applied the Cutting Constraint algorithm of Hearn and Lawphongpanich (2004). We set out details of the CCA in Appendix B. In other cases, we carried out a grid search of the welfare obtained by each authority with tolls between 0 and 1000 in units of 10. In some cases we refined the grid search between units of 1 to "zoom in" on the potential solutions. For ease of exposition, we use the notation $\{\tau_A, \tau_B\}$ to indicate a particular combined toll strategy tuple denoting the tolls set by Authority A and B respectively.

3.1 Case 1: Symmetric Demand

Table 1 shows the details of the matrix that is used for Case 1.

<<INSERT TABLE 1 APPROXIMATELY HERE >>

Table 1: Base Trips ($d_{k,0}$) and (Base Costs, $b_{k,0}$) for Case 1.

Authority in Charge	From \ To	1 (Residential Zone in Authority A)	2 (CBD of Authority A)	4 (Residential Zone in Authority B)	5 (CBD of Authority B)
A	1	0	1000 (488.08)	200 (1389.75)	100 (1839.86)
A	2	100 (450.11)	0	100 (901.67)	100 (1351.77)
B	4	100 (1351.77)	100 (901.67)	0	100 (450.11)
B	5	100 (1839.86)	200 (1389.75)	1000 (488.08)	0

3.1.1 Case 1: Global Regulator Benchmark

As a benchmark, let us assume that a global regulator is in place to determine the uniform toll on both Links 1 and 6 and another uniform toll on Links 7 and 12. As mentioned, this problem is a standard Continuous Toll Pricing Problem and can be solved with the Cutting Constraint Algorithm. In this case the objective function for the "Global Regulator" is given by (4).

The welfare surface of the Global Regulator's problem for Case 1 is shown in Figure 2 with a contour plot on the right. Notice that for the global regulator we found in this problem that there exists only one optimum around a toll combination of (80,80). Beyond toll levels of around 90 seconds from either authority then there is a sudden drop off in benefits which continues to be the case as toll levels are increased to 1000 seconds (not shown).

<<INSERT FIGURE 2 APPROXIMATELY HERE >>

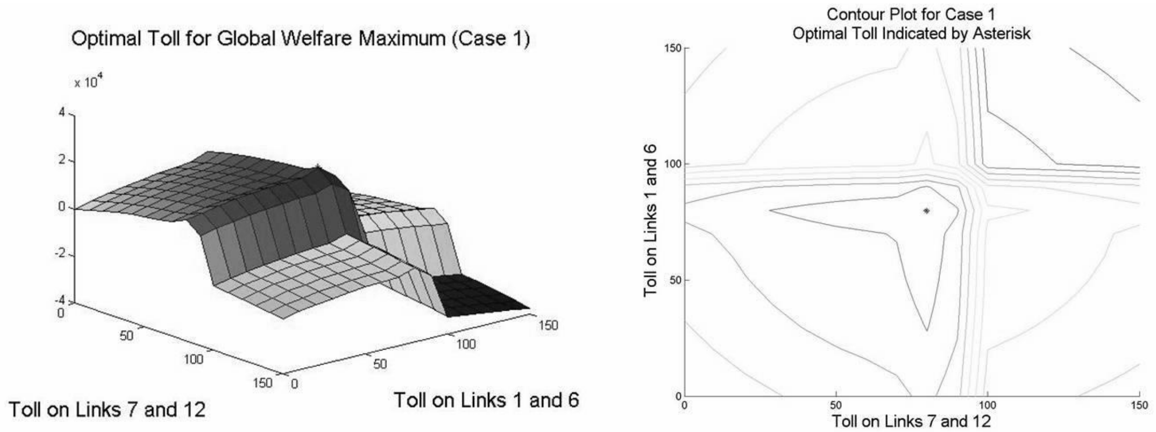

Figure 2: (Left Pane) Surface Plot of Global Welfare for Case 1 around region of the optimum; (Right Pane) Contour Plot of Global Welfare for Case 1 around region of the optimum.

Table 2 shows the solution and as expected due to symmetry both authorities' welfare increases by the same amount and tolls are set to the same value in both authorities.

<<INSERT TABLE 2 APPROXIMATELY HERE >>

Table 2: Results of the Global Regulator Problem for Case 1
(all units in seconds)

Scenario	Global Regulator
Authority A Toll on: Link 1 and Link 6	80
Authority B Toll on: Link 7 and Link 12	80
Welfare Gain (vs Do Nothing)	20,292
Welfare of Authority A	10,146
Welfare of Authority B	10,146

3.1.2 Case 1: Nash Game

First to explore the potential solutions we evaluated the welfare for each authority for a given toll pair with tolls ranging between 0-1000 seconds. Given that we have only two uniform tolls in our example then it is possible to visualize the welfare surfaces and to numerically estimate the gradients with respect to the authority's own toll at each point. Using a finite difference approach (Morton and Mayers, 2005) we were able to estimate these gradients and produce contour plots showing where the gradients are equal to zero. This is equivalent to finding where the "response surfaces" of the Nash game intersect, such intersections show where condition (11) could potentially be satisfied. Figure 3 shows the contour plots and points of intersection of the zero contours. In the figure the vertical lines show where the gradient of welfare for Authority B is zero and the horizontal lines show where the gradient of welfare for Authority A is zero as tolls set by B and A are varied respectively.

<<INSERT FIGURE 3 APPROXIMATELY HERE >>

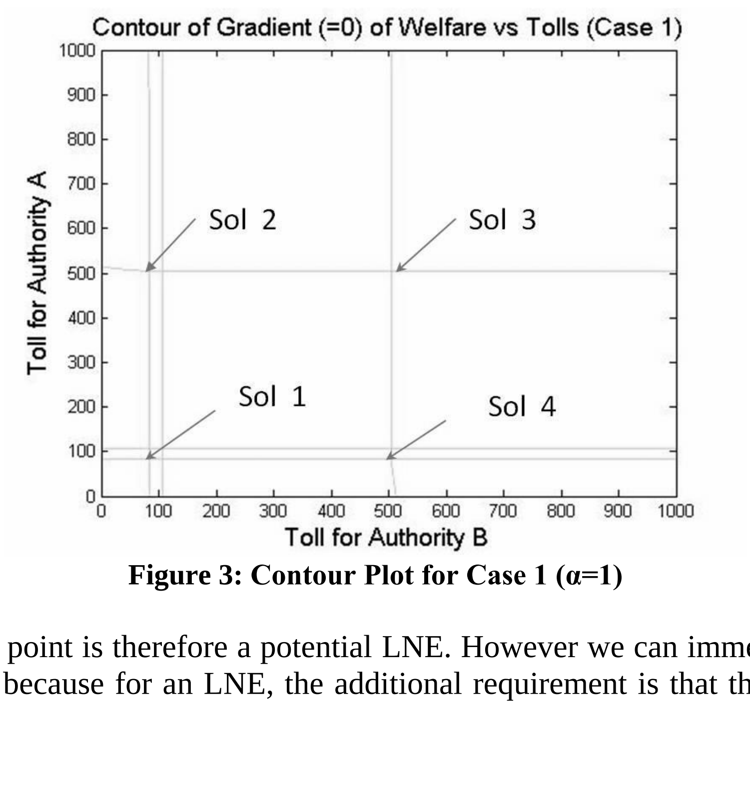

Figure 3: Contour Plot for Case 1 (α=1)

Each intersection point is therefore a potential LNE. However we can immediately disregard several solutions because for an LNE, the additional requirement is that they must intersect where both authorities' objectives are simultaneously maximized. We can identify 4 such solutions (marked on Figure 3) where both welfare surfaces are passing through a maximum by inspection of the welfare surfaces and by recognizing that there is a particular pattern to the welfare surfaces as tolls are increased which is maintained across the full range of tolls investigated. Figure 4 shows how welfare for Authority A varies with its own toll, for given values of tolls set by Authority B (85 or 505). Notice that there is a local maximum around a toll of 85 seconds followed by a minimum at a toll of 105 followed by a maximum around a toll of 505. It is worth noting here that the optimal toll for player A of 505 seconds does not appear to be affected by the toll played by player B. This suggests that there is little or no interaction between the players in the high toll regime. We come back to explore this and the number of potential LNE solutions later. This pattern is repeated for the other player due to symmetry, and we can then infer that the intersections between the first and third contours for each player in Figure 3 are where the simultaneous maxima resulting in an LNE may exist.

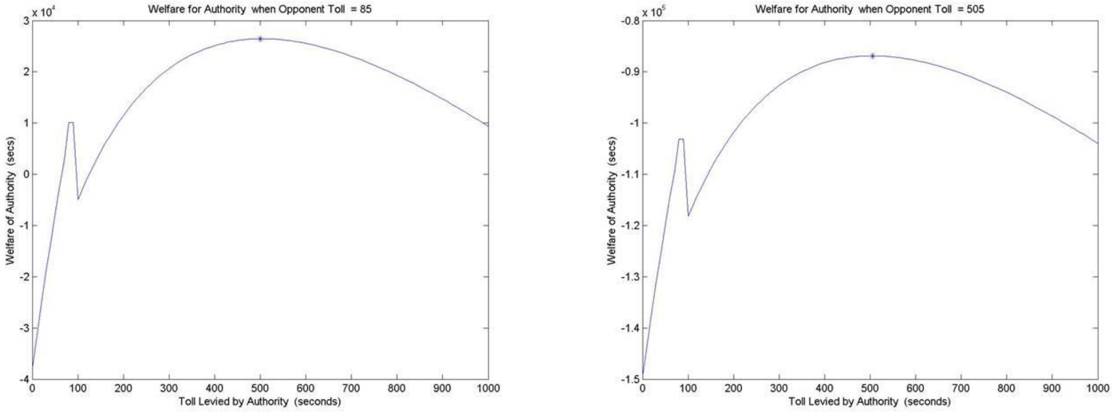

Figure 4: (Left Pane) Welfare Plot for Authority A Showing Optimum at around 505 when Authority B levies a toll of 85; (Right Pane) Welfare Plot for Authority A Showing Optimum at around 505 when Authority B levies a toll of 505.

<< INSERT FIGURE 4 APPROXIMATELY HERE >>

Using the welfare surfaces provided by the grid search we were able to confirm that four LNE exists as shown in Table 3 which all satisfied the condition in (11).

<<INSERT TABLE 3 APPROXIMATELY HERE >>

Table 3: Local Nash Equilibria for Case 1 (all units are seconds and α = 1)

Solution Number	Toll Set by Authority A	Toll Set by Authority B	Welfare of Authority A	Welfare of Authority B	Total Welfare
1	85	85	9,096 (2)	9,096 (2)	18,192
2	505	85	24,076 (1)	-101,839 (4)	-77,763
3	505	505	-86,872 (3)	-86,872 (3)	-173,744
4	85	505	-101,839 (4)	24,076 (1)	-77,763

Figures in parentheses show the preference ranking for each authority pertaining to a particular outcome

To explore the solutions further we calculated the vector field plot of the reaction functions at each point on the grid. The arrows in the vector force field plots in Figure 5 show the finite differenced approximations to the gradients of the welfare surfaces for each player with respect to their own toll and the direction a player should move when selecting their toll levels given the current tolls. The left hand pane of the figure shows the vector force field centred on the Nash Equilibrium labelled solution 1, while the right hand pane shows the vector force field centred on the Nash Equilibrium labelled solution 3. From inspection of the Vector Field Plots we can confirm that these are LNE. It is evident from the vector plots that the basin of attraction is far smaller around the first of these solutions and as a toll set by the other player moves beyond 100 seconds the players may well be attracted to solution 3.

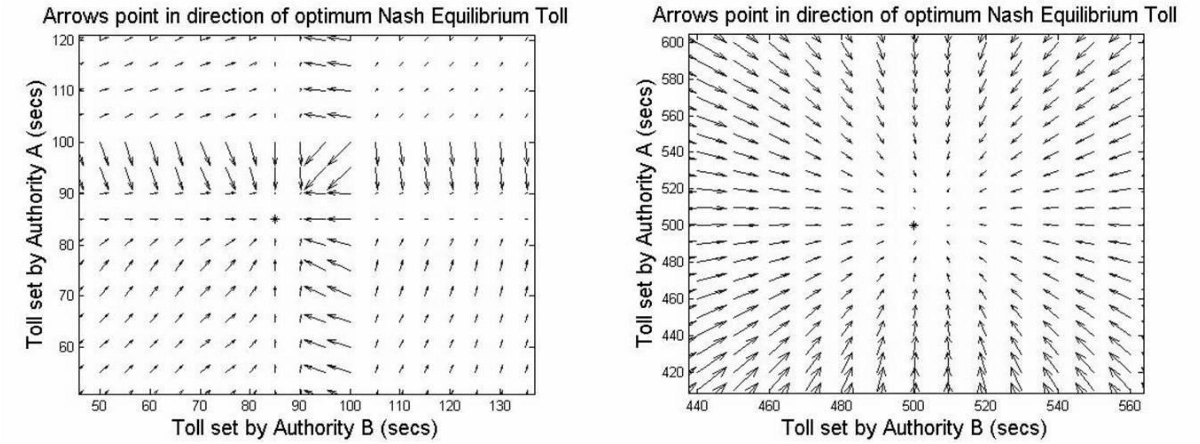

Figure 5: (Left Pane) Vector Field Plot of Reaction Functions around Toll Vector of 85,85 for Case 1 (α=1); (Right Pane) Vector Force Plot of Reaction Functions around Toll Vector of 505,505 for Case 1 (α=1)

<< INSERT FIGURE 5 APPROXIMATELY HERE >>

Similar plots show that the basin of attraction around solutions 2 and 4 are also relatively small and that solution 3 is the only solution which satisfies (11) in the global sense. Solutions 1, 2 and 4 are therefore only Nash solutions in a local neighbourhood i.e. LNE.

An alternative way to look at the outcome of the authorities' decision making and whether or not they act in a local neighbourhood or not when setting tolls is to use a simplified pay-off table as was done in Son and Baldick (2004).

Table 4 shows the pay-off matrix in terms of welfare changes for authorities A and B given the tolls can only be set at values of 0, 85 or 505 (taken from our knowledge of where the possible LNE occur).

<<INSERT TABLE 4 APPROXIMATELY HERE >>

Table 4: Case 1: Pay-off matrix (thousand seconds) near each LNE solution (Welfare A, Welfare B)

Toll A/B	0	85	505
0	(0,0)	(-40.4, 50.7)	(-150, 66.8)
85	(50.7, -40.4)	(9.1, 9.1)	(-103.0, 24.1)
505	(66.8, -150)	(24.1, -103.0)	(-88,-88)

The arrows in the table show the direction in which each authority would move in terms of toll set given the current tolls. Firstly we notice that both players have an incentive to move away from the no toll situation assuming that the other player does not charge. That is both have a first mover incentive. Then if we consider player A to move first, then player A has an incentive to move through to toll=85 and then to a toll of 505. Player B would then respond accordingly and with these limited decisions available to the players the outcome is always the NE solution which satisfies condition (11) i.e. it confirms the fact that solution 3 is in fact the NE rather than simply an LNE.

Next we widen the grid to include some more local decisions around the solution at {85,85} as shown in Table 5. Now we see that when the authority considers local moves only around tolls of 85 seconds then it is possible to remain in solutions 1,2 and 4 i.e. the {85,85} solution or one of the other {85,505} solutions. This can be seen for example by examining the local decision around the {85,85} pay-off cell. From this cell there is no benefit for either player to increase or decrease the toll and so this is an LNE. However we can also notice that as soon as one authority charges above 90 seconds then they are incentivised to move towards solution 3 the NE solution. The question of how authorities will set tolls in reality is obviously linked to the scale of the tolls and whether these are considered to be acceptable to the public. Whilst we have not defined how strategies are set in this paper (as we have simply explored the response surfaces to find solutions to the problem), our future research will investigate the dynamics of the toll setting strategies and how this may result in an LNE solution.

<<INSERT TABLE 5 APPROXIMATELY HERE >>

Table 5: Case 1: Pay-off matrix (thousand seconds) with local toll moves around (85, 85)

Toll A/B	0	80	85	90	505
0	(0,0)	(-38.0, 49.4)	(-40.4, 50.7)	(-42.7, 49.5)	(-150, 66.8)
80	(49.4, -38.0)	(10.1, 10.1)	(7.7, 11.5)	(5.4, 10.2)	(-104.4, 26.5)
85	(50.7, -40.4)	(11.5, 7.7)	(9.1, 9.1)	(6.7, 7.8)	(-103.0, 24.1)
90	(49.5, -42.7)	(10.2, 5.4)	(7.8, 6.7)	(5.4, 5.4)	(-104.4 , 21.7)
505	(66.8, -150)	(26.5, -104.4)	(24.1, -103.0)	(21.7, -104.4)	(-88,-88)

3.1.3 Case 1: Policy implications

Firstly we note from the welfare surfaces (not shown) that both players would have an incentive to begin charging given that the other player does not charge. Once both players begin to toll then, as shown in Table 3, Authority A would clearly prefer Solution 2 while Authority B would prefer the diametrically opposed solution in terms of tolls, solution 4. If we assume that the authorities then have full information about the expected change in welfare over the full range of tolls then for a given toll played by their opponent, they would move towards a toll of around 505 seconds. In response the second mover would also set a toll of around 505 seconds (as can be inferred by Figure 4 above) and the authorities would end up at solution 3 which is a classic Prisoner's dilemma whereby both authorities are worse off than in the no toll case.

It is also interesting from a policy point of view that solution 1 with tolls set at {85,85} is in the vicinity of the global regulator solution with both authorities receiving an increase in welfare. It could be argued that such a solution may be found if the upper bounds of the toll sets considered were somehow restricted to within the range 0-90 seconds. As this is only a toy network example we cannot say anything about the scale issue here but we can recognise that in reality there may well exist an upper bound on the toll set by some public acceptability limits. Otherwise as solutions 1, 2 and 4 are only NE in a local neighbourhood then these are unlikely to be obtained in a game with full information. Later we discuss the case where the authorities are assumed to collude by reducing the value of α, but next we examine the potential for multiple LNE.

3.1.4 Case 1: Exploring the potential for multiple LNE

As noted earlier when discussing Figures 3 and 4 the optimal toll for authority A does not appear to be affected by the toll set by authority B in the high toll regime. This section first of all explains how this comes about by focussing on flow regimes and then explores which other factors can influence whether or not multiple LNE may exist.

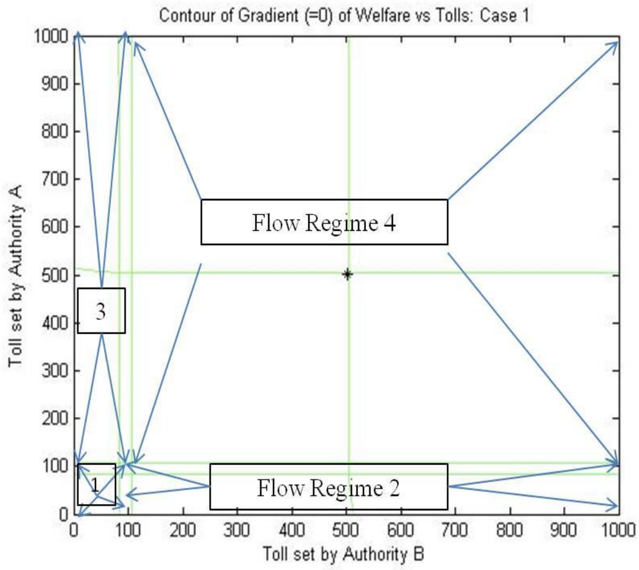

Figure 6: Flow regimes under alternative toll assumptions

Figure 6 shows where the flows on the network can be decomposed into 4 “regimes” depending on the toll tuple and that these flow regimes correspond to the contours from figure 3. We can draw the following insights regarding traffic flows in these 4 regimes.

1. When there are no tolls, the bypass links are not used at all. Hence all traffic regardless of destination utilise links through the town centre. This is due to the difference in free-flow costs for using the bypass compared with the town centre route. Within regime 1, as the tolls are increased then eventually some users begin to use the bypass links 2 and 11 and we hence obtain a “mixed traffic regime” i.e. flows on both the town centre route and flows on the bypass. Regime 1 is characterized by the set of tolls below 100 seconds.
2. In flow regime 2, once the tolls set by Authority B (on links 7 and 12) increase beyond 100 seconds, all through traffic in authority B’s area uses the bypass links. That is a toll greater than 100 seconds invokes the use of links 8 and 11 (the bypass routes in Authority B) but not links 2 or 5 which is still a function of tolls set by Authority A. The only traffic using the tolled links 7+12 are effectively captive (as in equilibrium they have no competitive alternative route across the range of feasible toll levels) to those links and we have a separated regime in B’s part of the network. By this we mean that sub-networks such as link 8 versus links 7+10 do not have the same

cost at equilibrium and this is obtained by segregation of OD demands. Note that by symmetry, flow regime 3 is similar to flow regime 2 but responds to tolls on links 1 and 6.

3. In Regime 4 all bypass links are used and the traffic using the tolled routes is only "effectively captive traffic" (traffic that have destinations within the tolled area i.e. zone 2 or zone 4 which do not have any competitive alternative route across the range of feasible toll levels). All other traffic uses the bypass links. Each sub-network is in equilibrium but with higher costs for through traffic.

These regimes all come about because of the extremely low delays experienced on the bypass links relative to those on the through links. With our base demands it seems that the delays which result on the bypass links are negligible compared to the free flow cost of 1000 seconds and that the assignment becomes an all-or-nothing assignment in regimes 2-4.

Understanding these regimes helps us explain why the optimal toll set by A does not appear to be affected by the toll set by B in the high toll regime. Solution 4 lies in the separated flow regime so that the toll is in effect only affecting captive users and no more re-routing in response to a toll is possible. This separated regime implies that the optimal toll for player A is dependent only on the demand towards the central zone (node 2) and that the welfare function can only be increased by affecting the consumer surplus of own residents heading towards node 2 and the congestion experienced on link 1 plus the amount of revenue collected on link 6 from those non-residents travelling to node 2. All other flows and link costs are fixed once the tolls exceed 100 seconds. This sub-problem faced by player A is not influenced by the toll set by player B as all those who enter A's network from authority B have not been charged a toll in B's network by definition. They have either come from zone 4 via link 9 without charge or have come from zone 5 via the bypass link 11 again with no charge. This explains why there is no interaction effect between players once we are in this separated regime. Next we investigate whether the number of LNE solutions varies with increased elasticity.

3.1.5 Case 1: Number of Potential LNE with changes in Elasticity of Demand

The power law demand function implies a constant elasticity demand assumption and this is reflected in the parameter p in (13). Specifically p represents the (absolute) percentage change in demand as a result of a percentage increase in generalized costs (inclusive of tolls). Thus with everything else (base demands and network link parameters) held constant, we can vary the parameter p to assess the impact of an (absolute) increase in elasticity on the number of potential LNE in the network.

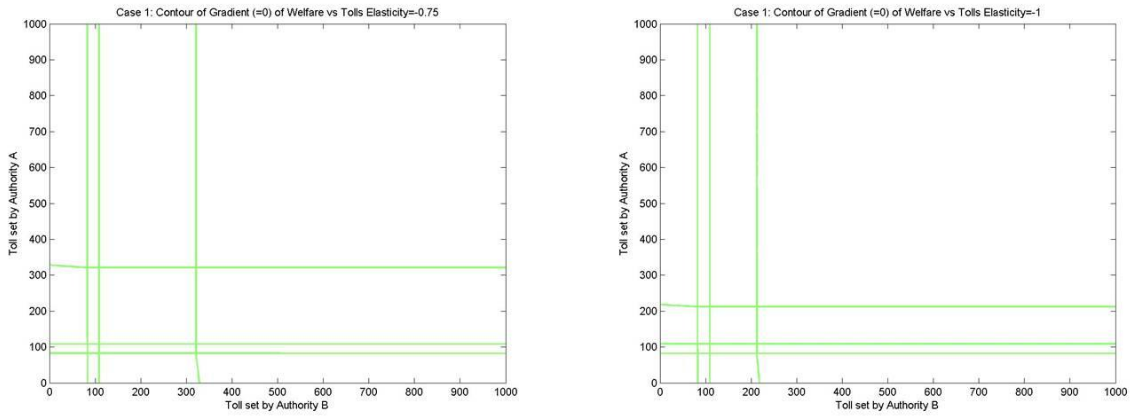

Figure 1a (Left Pane) Contours of Case 1 (α=1) with Elasticity = -0.75 (Right Pane) Contours of Case 1 (α=1) with Elasticity = -1

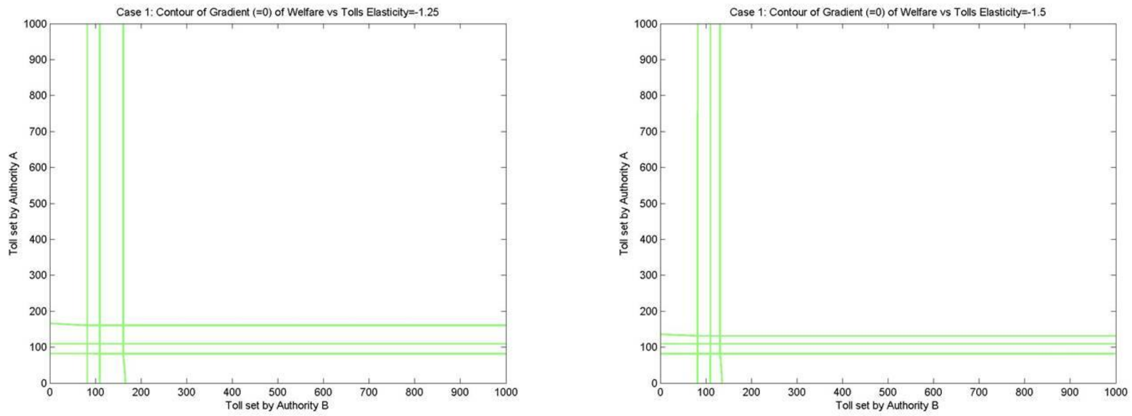

Figure 7b: (Left Pane) Contours of Case 1 (α=1) with Elasticity = -1.25 (Right Pane) Contours of Case 1 (α=1) with Elasticity = -1.5

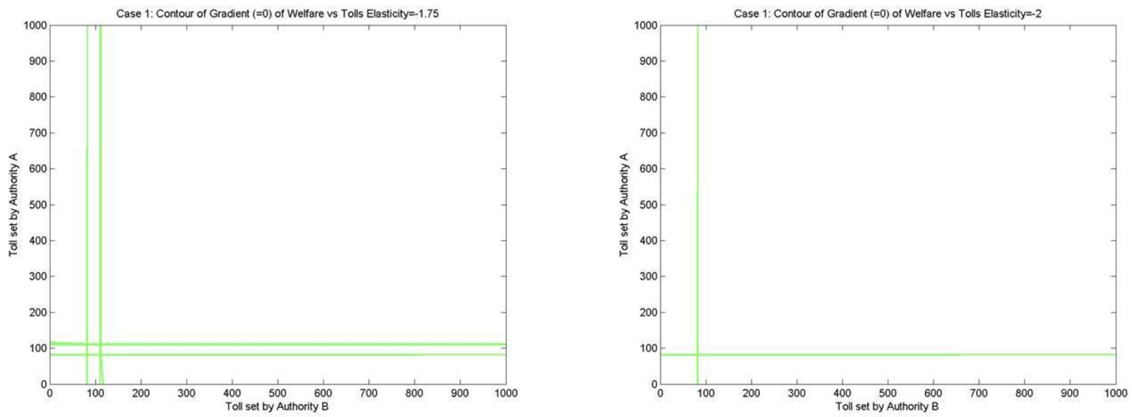

Figure 7c: (Left Pane) Contours of Case 1 (α=1) with Elasticity = -1.75 (Right Pane) Contours of Case 1 (α=1) with Elasticity = -2

<<INSERT TABLE 6 APPROXIMATELY HERE >>

Table 6: Case 1:Number of possible LNE as Elasticity Increases

p (Elasticity Parameter)	Number of Intersections	Number of LNE	Toll Level at Nash Equilibrium (seconds)
0.58 (Base Case)	9	4	{505,505}
0.75	9	4	{320,320}
1	9	4	{85,85}
1.25	9	4	{85,85}
1.5	9	4	{85,85}
1.75	4	4	{85,85}
2.0	1	1	{85,85}

Figures 7a to 7c show the contour plots and hence number of intersections as elasticity is increased from 0.58 to 2.0. As mentioned earlier, some intersections of the contours are eliminated from potential consideration as LNE because although the numerically estimated gradients are equal to 0, at least one of the Authority's objective functions attains a minimum at that point. This contradicts the requirements that for an LNE both objectives must be simultaneously maximized. Hence by process of inspection, we can eliminate some intersections from consideration. However as shown in Figures 7a-7c and in Table 6, it is still clear that with elasticities up to -1.75, there are 4 LNE. Somewhere between a value of -1.75 and -2.0 the number of LNE is reduced to one, as with an elasticity of -2, multiple NE are eliminated from this network. The one remaining solution is in the mixed flow regime. This demonstrates that there can exist networks which exhibit only one NE solution and that in this case there would not be a prisoner's dilemma.

Table 6 also shows that as elasticity increases, the NE solution tends towards the low toll regime rather than the high toll regime. The left pane of Figure 8 shows the graph of welfare for Authority A as the toll it sets varies (Authority B's toll held fixed) in the case when the parameter p is kept at the base value of 0.58. We note that the global optimum of welfare in this case occurs to the right of the local optimum and this is in the high toll regime. In contrast, the right pane of Figure 8 shows the same graph with absolute elasticity increased to $p = 1.25$. In this case, we note that the global optimum occurs to the left of the local optimum in the low toll regime. This demonstrates why, as elasticity is increased we see the NE solution move from a high toll regime to a low toll one. This has important policy implications in that if elasticity is higher then the authorities are less likely to end up in a Prisoner's dilemma, the users will face lower tolls and all residents will see an increase in total welfare.

It is also noticeable that the low toll Nash solution does not change as elasticity increases. This is again down to the specific parameters in our network and in particular it is related to the very small impact on delay on the bypass links as a small proportion of the flow is diverted from link 1 to link 2 for example. With low levels of through traffic, the congestion impact on the bypass links is only a fraction of a second so that the optimal toll is always in the same integer range. This is network specific and is not expected to be generalised.

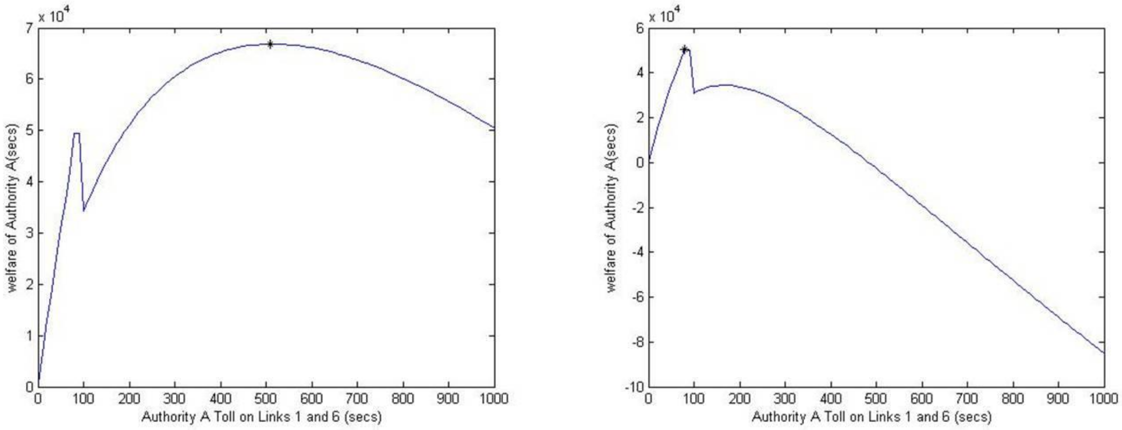

Figure 8. (Left Pane) Global optimum of own authority welfare is in the high toll regime and to the *right* of the local optimum at elasticity of -0.58 as own authority toll varies ; (Right Pane) Global optimum of own authority welfare is in the low toll regime and to the *left* of the local optimum at elasticity of -1.25 as own authority toll varies.

We did also investigate other changes to the network parameters and found that if we increase both the through demand and adapt the congestion function on the links – to increase the delays on the bypass links then this can also result in there being only one NE solution. Whilst we have therefore demonstrated that multiple NE may exist under certain conditions and that under other conditions only one NE solution may exist, we are not in a position to say whether for any general network there will be one or multiple NE solutions. This is something that should be investigated in further research.

3.2 Case 2: Asymmetric Demand

In Case 2, we modified the Demand Matrix used in Case 1 from that shown in Table to that as shown in Table 7. However the network remains exactly the same in both cases.

Table 7: Base Trips ($d_{k,0}$) and (Base Costs, $b_{k,0}$) for Case 2.

Authority in Charge	From \ To	1 (Residential Zone in Authority A)	2 (CBD of Authority A)	4 (Residential Zone in Authority B)	5 (CBD of Authority B)
A	1	0	1300 (488.08)	0	0
A	2	100 (450.34)	0	100 (900.04)	100 (1350.06)
B	4	100 (1361.26)	100 (910.92)	0	100 (450.02)
B	5	200 (1849.35)	400 (1399.00)	700 (488.08)	0

In constructing the asymmetric case we have maintained the number of trips originating from each zone, but have re-distributed them so that the CBD in Authority A is now more attractive relative to the CBD in authority B. Note that the total number of trips from A to B is reduced from 500 to 200 while the number from B to A increases from 500 to 800.

3.2.1 Case 2: Global Regulator Benchmark

The welfare surface of the Global Regulator's problem for Case 2 is shown in Figure 9 with a contour plot on the right. In addition, our search over the entire surface confirms that similar to Case 1, there exists only one optimum around a toll set of {90,80}. The results are summarised in Table 8.

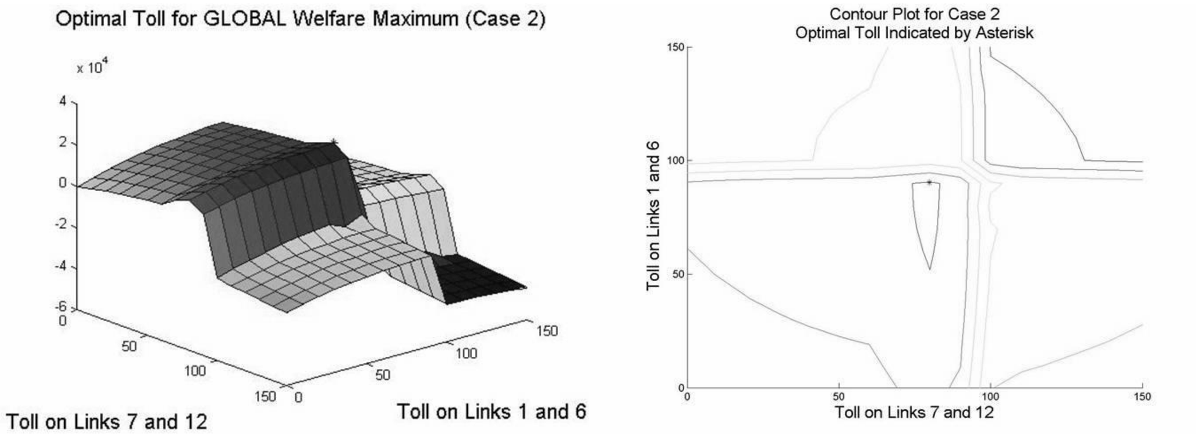

Figure 9: (Left Pane) Surface Plot of Global Welfare for Case 2 around region of the optimum; (Right Pane) Contour Plot of Global Welfare for Case 2 around region of the optimum.

Table 1: Results of the Global Regulator Problem for Case 2 (all units in seconds)

Scenario	Global Regulator
Authority A Toll on: Link 1 and Link 6	90
Authority B Toll on: Link 7 and Link 12	80
Welfare Gain (vs Do Nothing)	21418
Welfare of Authority A	62012
Welfare of Authority B	-40593

It is interesting that Authority B suffers from negative welfare even in the global regulator problem.

3.2.2 Case 2: Nash Game

For the case when $\alpha = 1$, i.e., full tax exporting between the authorities, we again used a finite grid search and contours of the gradients to explore the response surfaces for each authority to identify where potential local NE that may exist. Figure 10 shows the contour plots and points of intersection of the zero contours.

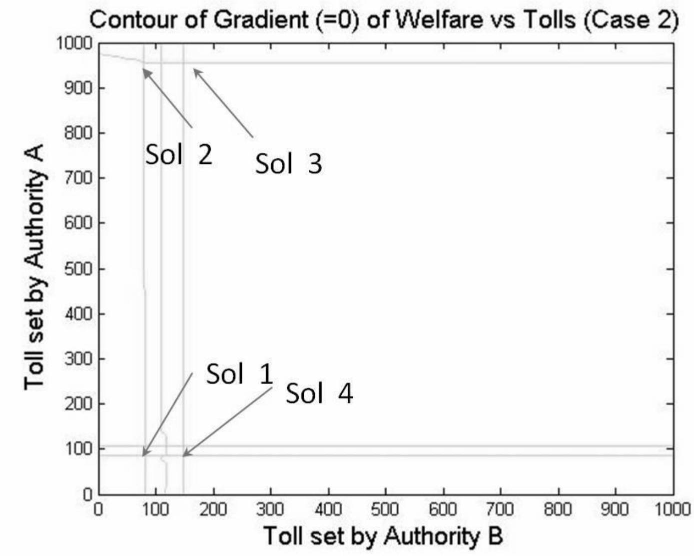

Figure 10: Contour Plot for Case 2 (α=1)

Once again we can identify 4 potential solutions where both welfare surfaces are passing through a maximum by inspection of the welfare surfaces and by recognizing that there is a particular pattern to the welfare surfaces as tolls are increased which is maintained across the full range of tolls investigated.

Using the welfare surfaces provided by the grid search we were able to confirm that once again there are four LNE solutions as shown in Table which all satisfied the condition in (11) .

Table 9: Local Nash Equilibria for Case 2 (all units are seconds and α = 1)

Solution Number	Toll Set by Authority A	Toll Set by Authority B	Welfare of Authority A	Welfare of Authority B	Total Welfare
1	85	81	58025 (3)	-36666 (1)	21359
2	955	81	150671 (1)	-392798 (3)	-242127
3	955	150	142737 (2)	-422454 (4)	-279717
4	85	150	49777 (4)	-65860 (2)	-16083

Figures in parentheses show the preference ranking for each authority pertaining to a particular outcome.

It is the case that in all 4 LNE, the impact on B's welfare is adverse, recall that B suffers from negative welfare even under the global regulator benchmark. Compared to case 1, it seems that the outcome will favour the stronger player. Solution 2 and Solution 3 are both highly favoured outcomes for Player A with the same toll level of 955 set by player A which demonstrates the power of Authority A. Comparing the preference ranking in Case 2 with that from Case 1 and with reference to Table 9, now Authority A gives Solution 1 {85,81} nearer to the global regulator outcome {90,80} (cf. Table 8) a lower ranking while Authority B actually prefers this. Note that similar to Case 1, we found that as the absolute elasticity increased, we move towards a low toll solution and only one NE.

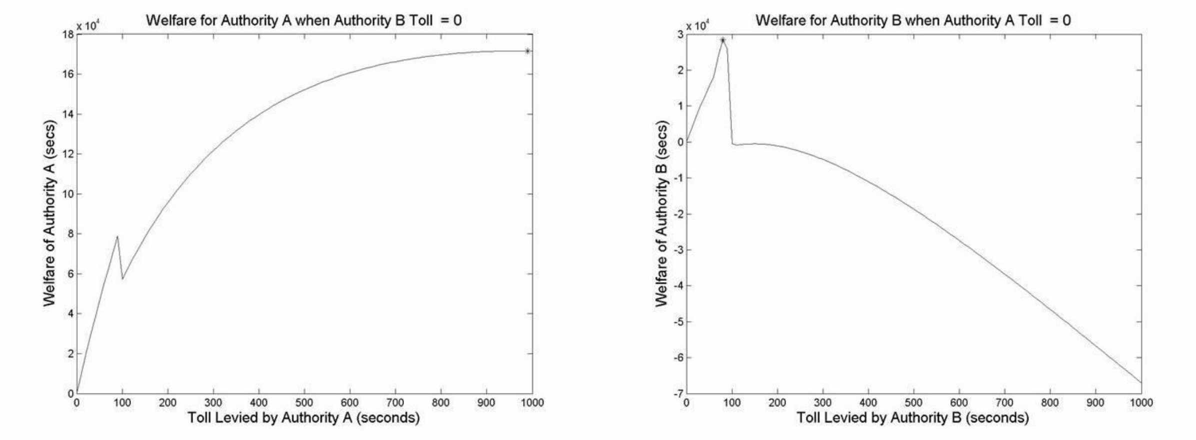

Figure 11: (Left Pane) Welfare Plot for Authority A Showing Optimum at around 980 when Authority B does not levy any toll ; (Right Pane) Welfare Plot for Authority B Showing Optimum at around 80 when Authority A does not levy any toll.

If B is always worse off why do they toll? The left pane of Figure 11 Figure illustrates the welfare of Authority A as it varies its toll when Authority B does not levy any toll. The right pane does the same for Authority B on the assumption that Authority A does not levy any toll. These show that both authorities have an incentive to enter the game since their individual welfares are higher compared to doing nothing. It is also evident that Authority A has a much larger incentive than Authority B.

When the game begins, and we have shown that there is indeed such an incentive for one authority to begin the game, Authority B always ends up in the equivalent of a prisoners' dilemma situation because it is always worse off under all the 4 LNE of Figure than if it had not done anything. Similarly A is always better off (cf. Table 9). This is in stark contrast to case 1 where both authorities ended up being worse off.

It is however possible to show that solution 2 is the NE as if A moves first then they set a toll of 955 and B responds with full information with a toll of 81 and vice versa.

3.2.3 Policy Implications

Our analysis offers a potential explanation for why large cities such as London can start the game and gain a first mover advantage while smaller authorities (when including set up and operating costs) decide that in fact the benefits of going alone are not even there – so this explains why there is a no-move case for the smaller towns – especially if they think that the other larger town will retaliate and they may end up being even worse off.

In addition, our findings also lend support to the findings of an econometric study by Levinson (2001). Levinson found that that the more non-resident workers a state (in the United States) has, the greater the likelihood of tolling. By way of analogy to this case study, Authority A has a larger number of non-resident workers (compared to Authority B since more commute to work in its jurisdiction compared to Case 1) and therefore has a stronger incentive to apply tolls.

4. CONSIDERING COLLUSION

Thus far we have assumed that α=1, i.e. there is full tax exporting behaviour. In the Edinburgh congestion charging proposals, authorities surrounding the city of Edinburgh were invited to share the revenues from the scheme so that they would lend support to the proposals (Saunders, 2005). This form of revenue sharing can be modelled with the parameter α. When α=0 then we have full recycling of revenues back to those who paid the tolls. For values in between there is some sharing of revenues collected i.e. some proportion of revenues are returned to the relevant authority.

To find toll levels for each Authority that satisfy (11), we carried out a grid search of welfares for each authority varying the toll levels between 0 and 1000 second and carried out the contour plots of based on finite difference to approximate the gradients. As we have found from results in Section 3, there may be more than 1 NE that will satisfy (11) even within the range of tolls considered. Hence we also carried out a Gauss Jacobi diagonalization type algorithm (see Appendix C for details) for the purposes of locating the Nash Equilibrium toll solution within the locality of the grid search solution.

4.1 Collusion – Case 1

For Case 1, Table 10 shows the results of the Gauss Jacobi Algorithm for values of α between 0 and 1 inclusive. We also carried out a grid search of the welfare surfaces similar to the previous case studies mentioned above.

Table 10: Results of Gauss Jacobi Algorithm for Case 1 for different α (all units are seconds)

α	Toll on Links 1 and 6 Set by Authority A	Toll on Links 7 and 12 Set by Authority B	Welfare of Authority A	Welfare of Authority B	Global Welfare
0	80.00	80.00	10146	10146	20292
0.2	81.30	81.40	10057	10067	20125
0.4	159.27	159.32	-19791	-19785	-39575
0.6	240.24	242.72	-32099	-31680	-63779
0.8	356.08	356.08	-54030	-54030	-108059
1	504.70	504.70	-88006	-88005	-176011

Table 10 shows that as α is reduced i.e. increasing the revenue recycling back to those who paid, then there is a tendency for the solution to move towards the lower toll regime. In fact in the extreme case when α=0 we obtain the exact same solution as under the global regulator problem for Case 1. In this case there is an incentive for both authorities to collude which also brings greater benefits to society. However, this is no longer true when the demand is asymmetric as will be shown later.

4.2.2 Collusion – Case 2

For Case 2, Table 11 shows the results of the Gauss Jacobi Algorithm with different starting points for selected values of α between 0 and 1.

Table 11: Results of Gauss Jacobi Algorithm for Case 2 for different α (all units are seconds)

α	Toll on Links 1 and 6 Set by Authority A	Toll on Links 7 and 12 Set by Authority B	Welfare of Authority A	Welfare of Authority B	Global Welfare
0	102.73	80.00	9064	-13286	-4222
0.2	171.61	80.60	18584	-27629	-9045
0.4	263.59	81.20	34843	-58788	-23945
0.6	395.04	81.78	59781	-115609	-55828
0.8	609.77	82.03	96634	-218552	-121918
1	953.17	83.05	150585	-392325	-241740

The result for the Global Regulator Problem for Case 2 was presented in Table 8. A single regulator would set a toll 90 and 80 on links (1 and 6) and links (7 and 12) respectively to maximise welfare. However the result with α=0 is not in fact the same as the global regulator problem. It seems that in this case the stronger authority is still able to charge more (102 compared to 90 in the GRP). So in this case while Authority B would prefer to collude, Authority A would obviously gain more by not colluding and with tax exporting behaviour society would be worse off as a whole. The implication when comparing the asymmetric case to the symmetric case is that there may be a greater need for regulation when there exists a stronger player (as is the case in other sectors). For society to be better off as a whole in case 2 there needs to be a regulator in place which could also offset any disbenefits to those residents from Authority B by re-distribution of the revenues collected.

5. SUMMARY AND CONCLUSIONS

In this paper we have explored the implications of competition between cities when setting toll charges. First we have set up the problem as an Equilibrium Problem with Equilibrium Constraints (EPEC) which is a special form of a Nash game with a hierarchical structure. Using a simple network we then applied simple grid search methods to determine the Nash Equilibrium toll levels, finding both local (LNE) and global NE solutions. We then investigated the policy implications for a symmetric and asymmetric case with and without collusion.

In our first case study using a symmetric trip matrix, it is interesting that either authority should in principle wish to move first, but that once a move is made then the Nash game takes them both to a sub-optimal position due to the larger basin of attraction. They both end up worse off in a prisoner's dilemma. For the asymmetric case, we find that the outcome of the game tended to be in favour of the stronger player with the weaker player being worse off than in the no toll case despite there being an initial incentive for the weaker player to set a toll.

In our example we can see that if cities were to set tolls using a simple game or pay-off approach with limited step size (to represent a cautious decision maker), then it would be feasible for them to arrive at welfare improving LNE Solutions in the symmetric case. Whilst we have not defined how strategies are set in this paper (as we have simply explored the response surfaces to find solutions to the problem), our future research will investigate the dynamics of the toll setting strategies and how this may result in an LNE solution. We will investigate how cities compete, which indicators can be used for decision-making and which kind of update strategies are likely.

We also investigated which factors affected whether there exists only one or multiple NE solutions. We found that for the network studied here, increasing elasticity not only results in a shift from multiple NE to one NE solution but that the global NE solution also moved towards the low toll regime where both cities' residents are better off.

We also reported that changing the amount of through demand and the congestion function used also results in only one NE solution. Whilst we demonstrated that multiple NE may exist under certain conditions and that under other conditions only one NE solution may exist, we are not in a position to say whether for any general network there will be one or multiple NE solutions. Further research should consider more general networks, where mixed flow regimes are more likely, and the number of Nash Equilibria that may arise in such general networks.

Finally, we also demonstrated that some signalling or collusion as may be expected in reality could in this case work to benefit all residents should cities act to maximise welfare under the symmetric case, which is in contrast to our previous work on toll competition between private operators where profit maximising behaviour coupled with collusion led to a decrease in welfare for residents (though increased profits for the operators). However we also showed that with the asymmetric case the opposite is true and where there exists a player with market power then there could in fact be a stronger case for regulation. In modelling the collusion between authorities, we introduced a collusion parameter, α and assume that the parameter was common to both authorities. Further research could possibly investigate the impact of different values of this collusion parameter and how it would ultimately impact the conclusions presented in this paper.

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APPENDIX A: Uniqueness of Equilibrium Link Flows Disaggregated by Authority

We establish uniqueness of the equilibrium link flows disaggregated by authority, at any given toll vector, for the network shown in Figure 1 and assumptions specified in section 4. In order to do so, we shall make some mild additional assumptions. Let g_l^* denote the equilibrium generalized cost on link l corresponding to a given solution to (11). Formally, for any given toll vector solution $\boldsymbol{\tau}$ to (11) these are given uniquely by the elements of vector $\mathbf{g}^*$:

$$\mathbf{g}^* = \mathbf{g}\left(\tilde{\mathbf{V}}^*(\boldsymbol{\tau})\ \mathbf{1},\ \boldsymbol{\tau}\right) \tag{15}$$

Specifically we make the assumptions:

$$g_3^* + g_2^* > g_4^* \tag{16}$$

$$g_4^* + g_5^* > g_3^* \tag{17}$$

$$g_{10}^* + g_{11}^* > g_9^* \tag{18}$$

$$g_{10}^* + g_{11}^* > g_9^* \tag{19}$$

$$g_9^* + g_8^* > g_{10}^* \tag{20}$$

With these assumptions, then we are able to establish uniqueness of authority flows through the following steps:

i. Our assumptions on the cost functions and demand functions (stated in section 2) are well-known to be sufficient to guarantee uniqueness of the equilibrium *total* link flows and OD demands, so our question can be equivalently posed: in the given network structure, is this uniqueness sufficient to also guarantee uniqueness of the link flows disaggregated by authority?

ii. At equilibrium, intra-authority OD movements will never use the links of the other authority. For example, one possible route form node 1 to node 2 is to follow the route given by the link sequence {2,7,10,11,6}, but since link costs are strictly positive it follows that such a route will always have higher cost than the route following links {2,6}, and so this earlier route can never appear in an equilibrium solution at any toll vector. An analogous argument may be made for all intra-authority OD movements, so for such movements we need only consider the routes that use links strictly within that authority's jurisdiction.

iii. The network structure is entirely equivalent to one in which an additional bi-directional, dummy link is added by dividing node 3 in two and inserting the link between the two nodes resulting from the divided node 3. The only flow on the left-pointing direction of this link will be (all of) that demand travelling from Authority B (node 4 or 5) to Authority A (node 1 or 2), there will be no intra-authority demand using it given the remarks in point ii. above. Returning now to the original network definition, we may thus (if we are thinking just from the viewpoint of Authority A's network) represent the demand from Authority B as if it were from an origin at node 3 with OD flow to nodes 1 and 2 equal to the relevant OD flows from the sum of nodes 4 and 5 (noting that such sums are unique since the individual demands are unique by remark i.). By symmetry, the same argument may be made regarding demand from Authority A to B, if we are thinking from the perspective of Authority B's network.

iv. Considering Authority A's network, links 2 and 4 take traffic into node 3. In view of the comments in remark ii., such links could never be part of an equilibrium route for traffic from Authority B. Therefore links 2 and 4 only carry Authority A's demand, and these flows are unique since the total link flows are unique by remark i.

v. Assumption (17) above means that for demand travelling from node 2 to nodes 4 or 5, it is more costly (at equilibrium) to travel on the indirect route to node 3 (via links 3 and 2) than via the direct route via link 4, and so such demand will never use the indirect route. This implies that the only Authority A flow on link 2 is that demand from node 1 (destined for nodes 2, 4 or 5). All the remaining demand from node 1 to these other nodes must use link 1. Since at equilibrium we uniquely determine the total demand from node 1 (as the sum of demands to nodes 2, 4 and 5), and since in step iv. we have uniquely determined the Authority A flow on link 2, and since by the argument just made this flow on link 2 can only be from node 1, then by subtracting the (unique) link 2 flow from the (unique) total demand from node 1, then we have uniquely determined the flow on link 1 that is due to demand from node 1. Now we can note that no demand from node 2 would ever use link 1, so that the *only* Authority A demand on link 1 is that from node 1, and this is something we have just uniquely determined. Thus the Authority A flow on link 1 is unique, and by subtraction from the total link 1 flow (which is unique by remark i.) then the Authority B flow on link 1 is also unique.

vi. Assumption (17)implies that it is never efficient for demand from node 2 to travel to node 1 via the indirect route of links 4 and 5, in preference to the direct route via link 3. In particular, it means that link 5 is not used by demand from node 2; neither is this link on a route from node 1. Therefore no Authority A flow uses link 5, only Authority B flow and so this must equal the total flow on link 5, which is unique by remark i.

vii. Since by remark iii., the total Authority B demand arriving at node 3 (and destined for nodes 1 and 2) is uniquely determined, and since links 5 and 6 are the only exit nodes from node 3, and since by remark vi. the Authority B flow on link 5 is unique, then it follows that the Authority B flow on link 6 can be uniquely determined by conservation of Authority B flow at node 3. By subtraction from the total link 6 flow, the Authority A flow on link 6 is then also unique.

viii. Consider node 2. By remarks iv., v. and vii., the Authority B flow on links 1, 4 and 6 is uniquely determined. By remark i., the total Authority B OD flow that is destined for node 2 is uniquely determined, and by definition there is no Authority B OD flow originating at node 2. Therefore, applying conservation-of-flow at node 2 to the Authority B flow, then the Authority B flow on link 3 may be uniquely determined, as it is then the only unknown in the conservation equation. By subtraction from the total link 3 flow, the Authority A flow on link 3 is then also unique.

ix. Remarks iv.–viii. establish uniqueness of the authority flows on links 1–6, i.e. those under Authority A’s jurisdiction. By symmetry, equivalent arguments can be made about links 7–12 (under Authority B’s jurisdiction), exploiting assumptions (18)and (19) in place of (16) and (17) .

APPENDIX B: Cutting Constraint Algorithm

As mentioned in the main text, the global regulator sets the tolls to optimize the welfare for the entire network (irregardless of authority jurisdiction). This is effectively a Mathematical Program with Equilirium Constraints (MPEC). The economic paradigm for a generic MPEC is based on the setting of a Stackleberg game where the leader sets his strategic decision variables and the road users on the network take the leader's decision variables as given and optimize their route choice according to Wardrop's Equilibrium Condition. A large amount of development has occurred in this branch of mathematical optimisation (Luo *et al* 1996) which has applications in e.g. mechanics, robotics and transportation analysis. The primary difficulty with the MPEC is that they fail to satisfy certain technical conditions (known as constraint qualifications) at any feasible point (Chen and Florian, 1995; Scheel and Scholtes, 1995). In recent research, Koh *et al* (2009) investigated the use of the cutting constraint algorithm (CCA) (Lawphongpanich and Hearn, 2004) to solve an MPEC in the context of second best congestion pricing and capacity optimisation.

Reinterpretation of Variational Inequality Condition

Let us define the additional variable
$\bar{\tau}$: a pre-specified upper bound on tolls, $\bar{\tau} = [\bar{\tau}_i]$

As we have defined in the main paper (see equation (2)), the feasible region of flow vectors or "demand-feasible set" D, is a linear equation system of flow conservation constraints. From convex set theory, e.g. (Bazaraa *et al* 2008, Theorem 2.1.6 p.43), $(\mathbf{v},\mathbf{d}) \in D$ can be defined as a convex combination of a set of extreme points. Hence we can write Wardrop's equilibrium condition of route choice as follows:

$$\mathbf{g}(\mathbf{v},\boldsymbol{\tau})^{\#}(\mathbf{u}^e - \mathbf{v}) - \mathbf{w}(\mathbf{d})^{\#}(\mathbf{q}^e - \mathbf{e}) \geq 0 \quad \forall(\mathbf{u},\mathbf{e}) \in D$$

Where $(\mathbf{u}^e, \mathbf{q}^e)$ is the vector of extreme link flow and demand flow indexed by the superscript *e*, and *E* is the set of all extreme points of the demand-feasible set *D*

A Cutting Constraint Algorithm for the MPEC

The Cutting Constraint Algorithm redefines the variational inequality using the extreme points. Together with the initial extreme point, generated by an initial shortest path problem, and the constraints defining feasible flows, the master problem is solved to find the optimal tolls and capacities at each iteration. Subsequently new extreme points ("cuts") are found by solving a sub problem using the results for the current iteration.

The CCA Algorithm is as follows:

Step 0: Initialise the problem by finding the shortest paths for each O-D pair; set *l* (iteration counter) = 0; define the aggregated link flow and demand flow $(\mathbf{u}^l, \mathbf{q}^l)$; and include $(\mathbf{u}^l, \mathbf{q}^l)$ into E.

Step 1: Set $l = l + 1$ Solve the Master Problem with all extreme points in *E* and obtain the solution vector $(\mathbf{v},\mathbf{d},\tau)$;then set $(\mathbf{v}^l,\mathbf{d}^l,\tau^l)$.

Step 2: Solve the Sub Problem with $\left(\mathbf{v}^l, \mathbf{d}^l, \tau^l\right)$ and obtain the new extreme point $(\mathbf{u}^l,\mathbf{q}^l)$;

Step 3: Convergence Check:

If $\mathbf{g}(\mathbf{v}^l, \tau^l)^{\#}(\mathbf{u}^l - \mathbf{v}^l) - \mathbf{w}(\mathbf{d}^l)^{\#}(\mathbf{q}^l - \mathbf{d}^l) \geq 0$, terminate and $\left(\mathbf{v}^l, \mathbf{d}^l, \tau^l\right)$ is the solution, otherwise include $(\mathbf{u}^l, \mathbf{q}^l)$ into E and return to Step 1.

The Master Problem in Step 1 is defined as follows:

$$
\begin{aligned}
&\min_{(\tau,\beta,\mathbf{v},\mathbf{d})} \psi_1\left(\tau, \beta, \mathbf{v}, \mathbf{d}\right) \\
&s.t. \\
&0 \leq \tau_i \leq \bar{\tau}_i \\
&\left(\mathbf{v}, \mathbf{d}\right) \in D \\
&\mathbf{g}(\mathbf{v},\boldsymbol{\tau})^{\#}(\mathbf{u}^e - \mathbf{v}) - \mathbf{w}(\mathbf{d})^{\#}(\mathbf{q}^e - \mathbf{e}) \geq 0 \quad \forall e \in E
\end{aligned}
$$

The sub problem of Step 2 is a shortest path problem which is formulated as follows:

$$
\begin{aligned}
&\min_{(\mathbf{u},\mathbf{q})} \mathbf{c}\left(\mathbf{v}, \tau\right)^T \cdot \mathbf{u} - \left(\mathbf{D}^{-1}\left(\mathbf{d}, \tau\right)\right)^T \cdot \mathbf{q} \\
&s.t. \\
&\left(\mathbf{u}, \mathbf{q}\right) \in D
\end{aligned}
$$

Further details of our implementation of the algorithm can be found in Koh *et al* (2009).

APPENDIX C: Gauss Jacobi Diagonalization Algorithm

The Gauss Jacobi/Diagonalization Algorithm (Harker, 1984) which was used find the toll tuple when the collusion parameter α was varied, as discussed in Section 4, operates as follows:

Gauss Jacobi/Diagonalisation Algorithm:

Step 0: Set iteration counter $k = 0$. Select a convergence tolerance parameter, ε, (ε>0). Choose a toll level for each authority. Let the initial toll set be $\boldsymbol{\tau}^k = \left(\tau_A^k \;\; \tau_B^k\right)^{\#}$. Set $k = k + 1$ and go to Step 1,

Step 1: Utilise the Cutting Constraint Algorithm (see Appendix B) of Hearn and Lawphongpanich (1984) to solve each authority's individual welfare optimization problem i.e. the equivalent of (5) , assuming that the opponent's toll is held fixed.

Step 2: If $\left|\tau_A^{k+1} - \tau_A^k\right|$ and $\left|\tau_B^{k+1} - \tau_B^k\right|$ are both less than ε terminate, else set $k = k + 1$ and return to Step 1 where $|\bullet|$ refers to the Euclidean Norm.

REFERENCES

Bazaraa M. S., Sherali H.D., and Shetty C.M.(2006) *Nonlinear Programming: Theory and Algorithms*, Wiley, Hoboken, New Jersey.

Červinka, M. (2008), Hierarchical Structures in Equilibrium Problems, Doctoral Thesis, Charles University, Prague, Czech Republic.

Chen Y. and Florian M.,(1995) The nonlinear bilevel programming problem: Formulations, regularity and optimality conditions, *Optimization*,32(3),pp.193-209.

De Borger, B. Dunkerly, F. and Proost, S. (2007) Strategic investment and pricing decisions in a congested transport corridor. *Journal of Urban Economics*, **62(2)**, 294-316.

De Palma, A. and Lindsey R., (2000) Private Roads: competition under various ownership regimes. *Annals of Regional Science*, **34(1)**, 13-35.

Deparment for Transport (2005) The Transport Innovation Fund available at: *http://webarchive.nationalarchives.gov.uk/20090104005813/http://www.dft.gov.uk/pgr/regional/tif/thetransportinnovationfund*, accessed 31 March 2011.

Gühnemann A., Koh, A., Shepherd S., Lawler M, (2011) Implications of interdependencies between charging strategies of local authorities for the protection of sensitive areas in the Trans-Pennine Corridor. *Utilities Policy* **19**(1):42-52

Harker, P.T. (1984) A variational inequality approach for the determination of oligopolistic market equilibrium, *Mathematical Programming*, **30(1)**, 105-111.

Hu X., Ralph, D., (2007) Using EPECs to model bilevel games in restructured electricity markets with locational prices. *Operations Research*, **55(5)**, 809-827.

Koh, A. (2008) Nash Equilibrium, Collusion in Games and Co-evolutionary Particle Swarm Algorithm. In: Filipič B, Šilc J (eds) Bioinspired Optimization Methods and their Applications, Institut Jožef Stefan , Slovenia, 19-28.

Koh, A. Shepherd, S.P. and Sumalee, A. (2009). Second Best Toll And Capacity Optimisation in Networks: Solution Algorithm And Policy Implications. *Transportation*, **36(2)**, 147-165.

Koh, A. and Shepherd S.P. (2010) Tolling, Collustion and Equilibrium Problems with Equilibrium Constraints. *Trasporti Europei/ European Transport* **44**,3-22.

Lawphongpanich, S. and Hearn, D. (2004) An MPEC Approach to Second Best Toll Pricing. *Mathematical Programming B*, **101(1)**, 33-55

Levinson, D. (2001) Why States Toll: An Empirical Model of Finance Choice *Journal of Transport Economics and Policy*, **35(2)**, 223-238

Luo Z. Q., Pang J. S. and Ralph. D.,(1996) *Mathematical Programs with Equilibrium Constraints*, Cambridge University Press, Cambridge, England.

Mordukhovich B.S. (2005) Optimization and equilibrium problems with equilibrium constraints. *OMEGA International Journal of Management Science*, **33(5)**, 379-384.

Morton K. W., and Mayers D. F., (2005) Numerical Solution of Partial Differential Equations, An Introduction. Cambridge University Press, Cambridge, England.

Proost, S. and Sen, A. (2006) Urban transport pricing reform with two levels of government: a case study of Brussels, Transport Policy, **13(2)** 127-139.

Proost, S. and Van Dender, K. (2001) "Methodology and structure of the urban model" In De Borger B, Proost S (eds) Reforming Transport Pricing in the European Union- A Modelling Approach. Edward Elgar , Cheltenham, England, 65-92.

Saunders, J. (2005) The rise and fall of Edinburgh's congestion charging plans *Proceedings of the Institution of Civil Engineers*, *Transport* **158(4)**, 193-201.

Stiglitz, J., 2000. *Economics of the Public Sector*. WW Norton, London.

Scheel H., and Scholtes S.,(1995) Mathematical Programs with Equilibrium Constraints: stationarity, optimality and sensitivity, *Mathematics of Operations Research* **25(1)**, 1-22.

Shepherd, S.P., May, A.D. and Koh, A. (2008) How to design effective road pricing cordons." *Proceedings of the Institution of Civil Engineers, Transport* **161(3)** 155-165.

Son Y.S., and Baldick R., (2004) Hybrid Coevolutionary Programming for Nash Equilibrium Search in Games with Local Optima. *IEEE Transactions on Evolutionary Computation* **8(4)**, 305-315.

Sumalee, A. May, A.D. and Shepherd, S.P. (2005) Comparison of judgmental and optimal road pricing cordons. *Transport Policy*, **12(5)**, 384-390.

Transport Select Committee (2006) "The draft Local Transport Bill and the Transport Innovation Fund: Government Response to the Committee's Ninth Report of Session 2006-07" 13th Special Report of the House of Commons Transport Select Committee, HC1056 available at: http://www.publications.parliament.uk/pa/cm200607/cmselect/cmtran/1053/105302.htm, accessed 31 March 2011.

Ubbels, B. and Verhoef, E.T. (2008) Governmental competition in road charging and capacity choice. *Regional Science and Urban Economics*, **38(2)**, 174-190.

Verhoef, E.T, Nijkamp,P., Rietveld, P. (1996) Second-best congestion pricing : the case of an untolled alternative. *Journal of Urban Economics* **40(3)** 279-302.

Walters, A. (1961) Theory and Measurement of private & social cost of highway congestion. *Econometrica*, **29(4)**, 676-699.

Wardrop, J.G. (1952) Some theoretical aspects of road traffic research, *Proceedings of the Institution of Civil Engineers*, Part II, **1(1)**, 325-378.

Xiao, F., Yang, H. and Han, D. (2007) Competition and efficiency of private toll roads. *Transportation Research B* **(41)3**, 292-308.

Yang, H. Xiao F. and Huang H. (2009) Private Road Competition and Equilibrium with Traffic Equilibrium Constraints. *Journal of Advanced Transportation*, **43(1)**, 21-45.

Ye, J.J., and Zhu, Q.J. (2003) Multiobjective optimization problem with variational inequality constraints. Mathematical Programming, **96(1)**, 139-160.

Zhang, L. and Levinson, D. (2005) Road pricing on autonomous links. *Transportation Research Record* **1932**, 147-155.