Yang-Baxter maps and finite reduction groups with degenerated orbits

S. Konstantinou-Rizos and A. V. Mikhailov
Department of Applied Mathematics, University of Leeds, Leeds
mmskr@leeds.ac.uk, A.V.Mikhailov@leeds.ac.uk

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Abstract

In this paper we construct Yang-Baxter maps using Darboux-Lax representations, which are invariant under the action of finite reduction groups. We present 4 and 6-dimensional YB maps corresponding to all sl_2 automorphic Lie algebras with degenerated orbits. We also consider vector generalisations of these Yang-Baxter maps.

1 Introduction

The Yang-Baxter (YB) equation

$$Y^{12} \circ Y^{13} \circ Y^{23} = Y^{23} \circ Y^{13} \circ Y^{12}, \tag{1}$$

originates in the works of Yang [37] and Baxter [6]. Here Y^{ij} denotes the action of a linear operator $Y : U \otimes U \to U \otimes U$ on the ij factor of the triple tensor product $U \otimes U \otimes U$, where U is a vector space. In this form, equation (1) is known in the literature as the *quantum YB equation.*

Drinfel'd in 1992 [12] proposed to replace U by an arbitrary set A and, therefore, the tensor product $U \otimes U$ by the Cartesian product $A \times A$. In our paper A is an algebraic variety $\mathbb{K}^N$, where K is any field of zero characteristic, such us $\mathbb{C}$ or $\mathbb{Q}$.

In [35] Veselov proposed the term *Yang Baxter map* for the set-theoretical solutions of the quantum YB equation. Specifically, we consider the map $Y : A \times A \to A \times A$,

$$Y : (x, y) \mapsto (u(x, y), v(x, y)). \tag{2}$$

Furthermore, we define the functions $Y^{i,j} : A \times A \times A \to A \times A \times A$ for $i, j = 1, 2, 3,\ i \neq j$, which appear in equation (1), by the following relations

$$\begin{aligned} Y^{12}(x, y, z) &= (u(x, y), v(x, y), z), && (3)\\ Y^{13}(x, y, z) &= (u(x, z), y, v(x, z)), && (4)\\ Y^{23}(x, y, z) &= (x, u(y, z), v(y, z)), && (5) \end{aligned}$$

where $x, y, z \in A$. The variety A, in general, can be of any dimension. Thus, elements $x \in A$ are points in $\mathbb{K}^N$. The map Y^{ji}, $i < j$, is defined as Y^{ij} where we swap $u(k,l) \leftrightarrow v(l,k)$, $k, l = x, y, z$. For example, $Y^{21}(x,y,z) = (v(y,x), u(y,x), z)$.

The map (2) is a YB map, if it satisfies the YB equation (1). Moreover, it is called *reversible* if the composition of Y^{ij} and Y^{ji} is the identity map,

$$Y^{ij} \circ Y^{ji} = Id. \tag{6}$$

We use the term *parametric YB map* when u and v are attached with parameters $a, b \in K^n$, $K = \mathbb{R}, \mathbb{C}$, namely $u = u(x, y; a, b)$ and $v = v(x, y; a, b)$, meaning that the following map

$$Y_{a,b} : (x, y; a, b) \mapsto (u(x, y; a, b), v(x, y; a, b)), \tag{7}$$

satisfies the *parametric YB equation*

$$Y^{12}_{a,b} \circ Y^{13}_{a,c} \circ Y^{23}_{b,c} = Y^{23}_{b,c} \circ Y^{13}_{a,c} \circ Y^{12}_{a,b}. \tag{8}$$

Following Suris and Veselov in [33], we call a *Lax matrix for a parametric YB map*, a matrix $L = L(x; c; \lambda)$ depending on a variable x, a parameter c and a *spectral parameter* λ, which

1. satisfies the Lax equation

$$L(u; a, \lambda)L(v; b, \lambda) = L(y; b, \lambda)L(x; a, \lambda), \quad \text{for any } \lambda \in K \text{ and} \tag{9}$$

2. $\frac{\partial}{\partial x} \det(L) = 0$.

In what follows, the Lax matrix L in (9) is a Darboux Matrix for a Lax operator.

It is obvious that the Lax-equation (9) does not always have a unique solution, which motivated Kouloukas and Papageorgiou in [19] to propose the term *strong Lax matrix* for a YB map. This is when the Lax-equation is equivalent to

$$(u, v) = Y_{a,b}(x, y). \tag{10}$$

They also proved a sufficient condition for the solutions of the Lax-equation to define YB maps [17, 19].

Equations like the Lax-equation (9) are being met quite often in the area of integrable systems as, for instance, in the case of the Darboux transformations, where it represents the compatibility condition of the Darboux transformation around the square. In this case, it can be interpreted as a system of discrete equations.

One of the most famous parametric YB maps is the Adler's map [1]

$$(x_1, x_2) \longrightarrow (u, v) = \left(x_2 + \frac{a-b}{x_1 + x_2}, x_1 - \frac{a-b}{x_1 + x_2} \right), \tag{11}$$

which occurs from the 3-D consistent discrete potential KDV equation [25, 29]. In terms of Lax matrices, Adler's map (11) is obtained from the following strong Lax matrix [33, 36]

$$L(x; a, \lambda) = \begin{pmatrix} x & 1 \\ x^2 + a - \lambda & x \end{pmatrix}. \tag{12}$$

In [30, 31] a variety of YB maps is constructed using the symmetries of multi-filed equations on quad graphs.

It follows from the structure of the Lax-equation (9) that we can extract *invariants* of the YB map, which we denote as $I_i(x_1, x_2)$. The invariants are useful if one is interested in the dynamics of such maps. In terms of dynamics, the most interesting maps are those which are not involutive. Although, involutive maps have also useful applications [14]. In all the cases presented in the next sections the YB maps are not involutive. The dynamics of YB maps is discussed in [36].

Now, following [13, 34] we define integrability for YB maps.

Definition 1.1. *A $N-$ dimensional YB map, Y, is said to be completely integrable or Liouville integrable if*

1. *there is a Poisson matrix, J, of rank $2n$, which is invariant under Y,*

2. *map Y has $r-$functionally independent invariants, which are in involution with respect to the corresponding Poisson bracket, i.e. $\{I_i, I_j\} = 0$, $i, j = 1, \ldots, r$, $i \neq j$,*

3. *there are $k = N - 2r$ in the number Casimir functions, C_i, $i = 1, \ldots, k$, which are invariant under Y, namely $C_i \circ Y = C_i$.*

In what follows, we explain what is a Darboux transformation for a given Lax operator, introduce the Darboux matrices for the NLS equation, the $\mathbb{Z}_2$ reduction and the dihedral reduction group and construct parametric YB maps.

2 Darboux Transformations

Darboux transformations and their relations to the theory of *integrable systems* have been extensively studied [22, 32]. Such transformations can be derived from Lax pairs as, for instance, in [32], or in a more systematic algebraic manner in [16, 11].

We are interested in Darboux transformations corresponding to Lax operators of the following form

$$\mathfrak{L} = \mathfrak{L}(\mathbf{p}(x); \lambda) = D_x + U(\mathbf{p}(x); \lambda), \tag{13}$$

where U belongs to an automorphic Lie algebra.

Darboux transformations can be viewed as gauge transformations which depend rationally on a spectral parameter, λ. They map fundamental solutions, Ψ, of the equation $\mathfrak{L}(\mathbf{p}(x); \lambda)\Psi = 0$ to other fundamental solutions, $\widetilde{\Psi} = M\Psi$, of the equation $\mathfrak{L}(\widetilde{\mathbf{p}}(x); \lambda)\widetilde{\Psi} = 0$.

In this context, we say that a matrix M is a *Darboux matrix* for a given Lax operator of the form (13) if

1. $\mathfrak{L}(\widetilde{\mathbf{p}}; \lambda) = M\mathfrak{L}(\mathbf{p}; \lambda)M^{-1}$,

2. $\frac{\partial}{\partial x} \det M = 0$.

The first condition means that the resulting operator $\widetilde{\mathfrak{L}}$ has exactly the same form with $\mathfrak{L}$, but is evaluated on new potential, $\widetilde{\mathbf{p}}(x)$. The second condition results from Abel's theorem, namely that the Wronskian of a fundamental solution is $x-$independent, since U is traceless.

The structure of Lax operators has a natural Lie algebraic interpretation in terms of Kac-Moody algebras and automorphic Lie algebras [20, 21, 8, 9]. While a Kac-Moody algebra is associated with an automorphism of finite order, automorphic Lie algebras correspond to a finite group of automorphisms, which is called the *reduction group* [23].

In the case of 2×2 matrices, which we study in this paper, the essentially different reduction groups are the trivial group (with no reduction), the cyclic group $\mathbb{Z}_2$ (leading to the Kac-Moody algebra A_1^1) and the Klein group $\mathbb{Z}_2 \times \mathbb{Z}_2$ [24, 21].

We shall present 4 and $6-$dimensional YB maps for all the following cases. The trivial group associated with the nonlinear Schrödinger equation (NLS) equation [38]

$$p_t = p_{xx} + 4p^2 q, \qquad q_t = -q_{xx} - 4pq^2. \tag{14}$$

The $\mathbb{Z}_2$ group associated to the derivative nonlinear Schrödinger equation (DNLS) equation

$$p_t = p_{xx} - 4(p^2 q)_x, \qquad q_t = -q_{xx} - 4(pq^2)_x. \tag{15}$$

and the $\mathbb{Z}_2 \times \mathbb{Z}_2$ group associated to the deformation of the DNLS equation

$$p_t = p_{xx} + 4(p^2 q)_x + 4q_x, \qquad q_t = -q_{xx} - 4(pq^2)_x - 4p_x. \tag{16}$$

In [16] we used Darboux transformations to construct integrable sustems of discrete equations, which have the multidimensional consistency property [3, 4, 7, 26, 27, 28]. The compatibility condition of Darboux transformations around the square is exactly the same with the Lax equation (9). Therefore, in this paper, we use Darboux transformations to contruct YB maps.

We start with the well known example of the Darboux transformation for the nonlinear Schrödinger equation and construct its associated YB map.

2.1 The Nonlinear Schrödinger equation

In this case, $U(p, q; \lambda) = U(\lambda)$ is a matrix of the form

$$U(\lambda) = \lambda U_1 + U_0, \qquad \text{where} \qquad U_1 = \sigma_3 = \operatorname{diag}(1, -1), \qquad U_0 = \begin{pmatrix} 0 & 2p \\ 2q & 0 \end{pmatrix}. \tag{17}$$

The Darboux Transformation, M, of $\mathfrak{L}$ is given by [16, 32]

$$M = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a + \widetilde{p} q & \widetilde{p} \\ q & 1 \end{pmatrix}. \tag{18}$$

Thus, we define the matrix

$$M(\mathbf{x}; a, \lambda) = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a + x_1 x_2 & x_1 \\ x_2 & 1 \end{pmatrix}. \tag{19}$$

and substitute it in the Lax equation (9),

$$M(\mathbf{u};a,\lambda)M(\mathbf{v};b,\lambda)=M(\mathbf{y};b,\lambda)M(\mathbf{x};a,\lambda). \tag{20}$$

Equation (20) has a unique solution $\mathbf{u}=\mathbf{u}(\mathbf{x},\mathbf{y})$, $\mathbf{v}=\mathbf{v}(\mathbf{x},\mathbf{y})$ which define a map $\mathbf{x}\to\mathbf{u}(\mathbf{x},\mathbf{y})$, $\mathbf{y}\to\mathbf{v}(\mathbf{x},\mathbf{y})$, given by

$$(\mathbf{x},\mathbf{y})\xrightarrow{Y_{a,b}}\left(y_1-\frac{a-b}{1+x_1y_2}x_1,y_2,x_1,x_2+\frac{a-b}{1+x_1y_2}y_2\right). \tag{21}$$

One can verify that the above map with parameters a,b satisfies the YB equation (8), i.e. it is a parametric YB map with strong Darboux-Lax matrix (19). Moreover, according to definition (6), this is a reversible map but not an involution.

It follows from (20) that the trace of $M(\mathbf{y};b,\lambda)M(\mathbf{x};a,\lambda)$, is a polynomial in λ whise coefficients are

$$\mathrm{Tr}(M(\mathbf{y};b,\lambda)M(\mathbf{x};a,\lambda))=\lambda^2+\lambda I_1(\mathbf{x},\mathbf{y})+I_2(\mathbf{x},\mathbf{y}),$$

where

$$I_1(\mathbf{x},\mathbf{y})=x_1x_2+y_1y_2+a+b, \tag{22}$$
$$I_2(\mathbf{x},\mathbf{y})=bx_1x_2+ay_1y_2+x_1y_2+x_2y_1+x_1x_2y_1y_2+ab+1. \tag{23}$$

The constant terms in I_1,I_2 can be omitted. It is easy to check that I_1,I_2 are in involution with respect to invariant Poisson brackets defined as

$$\{x_1,x_2\}=\{y_1,y_2\}=1,\qquad \text{and all the rest}\qquad \{x_i,y_j\}=0, \tag{24}$$

and the corresponding Poisson matrix is invariant under the YB map (21). Therefore the map (21) is completely integrable.

The map (21) first appear in the work of Adler Yamilov [5]. Its interpretation as a YB map was given in [18].

2.1.1 Nonlinear Schrödinger equation: A 6-dimensional YB map

We consider a more general matrix whose entries depend on three variables x_1,x_2 and X, namely

$$M(\mathbf{x},X;\lambda)=\lambda\begin{pmatrix}1&0\\0&0\end{pmatrix}+\begin{pmatrix}X&x_1\\x_2&1\end{pmatrix}. \tag{25}$$

It follows from the Lax equation (9)

$$M(\mathbf{x},X;\lambda)M(\mathbf{y},Y;\lambda)=M(\mathbf{v},V;\lambda)M(\mathbf{u},U;\lambda) \tag{26}$$

that

$$v_1=x_1,\ u_2=y_2,\ U+V=X+Y,\ u_2v_1=x_1y_2,$$
$$u_1+Uv_1=y_1+x_1Y,\ u_1v_2+UV=x_2y_1+XY,\ v_2+u_2V=x_2+Xy_2.$$

The corresponding algebraic variety is a union of two six-dimensional components. The first one is obvious from the Lax equation (26), it corresponds to the permutation map

$$\mathbf{x} \mapsto \mathbf{u} = \mathbf{y}, \quad \mathbf{y} \mapsto \mathbf{v} = \mathbf{x}, \quad X \mapsto U = Y, \quad Y \mapsto V = X,$$

which is a trivial YB map. The second one can be represented as a rational 6-dimensional non-involutive YB map of $\mathbb{K}^3 \times \mathbb{K}^3 \to \mathbb{K}^3 \times \mathbb{K}^3$

$$\begin{array}{ll} x_1 \mapsto u_1 = \frac{y_1 + x_1^2 x_2 - x_1 X + x_1 Y}{1 + x_1 y_2}, & y_1 \mapsto v_1 = x_1, \\ x_2 \mapsto u_2 = y_2, & y_2 \mapsto v_2 = \frac{x_2 + y_1 y_2^2 + y_2 X - y_2 Y}{1 + x_1 y_2}, \\ X \mapsto U = \frac{y_1 y_2 - x_1 x_2 + X + x_1 y_2 Y}{1 + x_1 y_2}, & Y \mapsto V = \frac{x_1 x_2 - y_1 y_2 + x_1 y_2 X + Y}{1 + x_1 y_2}. \end{array} \tag{27}$$

From the trace of $M(\mathbf{y}; b, \lambda) M(\mathbf{x}; a, \lambda)$ we obtain the following invariants of (27)

$$I_1(\mathbf{x}, \mathbf{y}, X, Y) = X + Y \qquad \text{and} \qquad I_2(\mathbf{x}, \mathbf{y}, X, Y) = x_2 y_1 + x_1 y_2 + XY. \tag{28}$$

As stated in the definition of a Darboux matrix, the determinant of the matrix (25) must be constant. Therefore, $\det M = c(\lambda)$, from which follows that

$$X - x_1 x_2 = a = constant. \tag{29}$$

A substitution $X \to a + x_1 x_2$ in the Darboux matrix (25) leads to (19). The Adler-Yamilov map is a restriction of the YB map (27) on the invariant leaves

$$A_a = \{(x_1, x_2, X) \in \mathbb{R}^3; X = a + x_1 x_2\}, \quad B_b = \{(y_1, y_2, Y) \in \mathbb{R}^3; Y = a + y_1 y_2\}. \tag{30}$$

2.2 $\mathbb{Z}_2$ reduction

In this case U is given by

$$U(p, q; \lambda) = \lambda^2 U_2 + \lambda U_1, \qquad \text{where} \qquad U_2 = \sigma_3, \qquad U_1 = \begin{pmatrix} 0 & 2p \\ 2q & 0 \end{pmatrix}. \tag{31}$$

The corresponding Lax operator (13) is invariant with respect to the following involution

$$L(\lambda) = \sigma_3 L(-\lambda) \sigma_3, \tag{32}$$

The involution (32) generates the so-called Reduction group [23, 21] and it is isomorphic to $\mathbb{Z}_2$. The Lax operator in this case is known as the spatial part of the Lax pair for the derivative-Schrödinger equation [10, 15].

The Darboux matrix in this case is given by [16]

$$M := \lambda^2 \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 & fp \\ f\widetilde{q} & 0 \end{pmatrix} + \begin{pmatrix} c_1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{33}$$

From the constant determinant property of M follows that f satisfies the equation

$$f^2 p \widetilde{q} - f + c_2 = 0, \tag{34}$$

where c_2 a non-zero arbitrary constant.

Replacing $(fp, f\widetilde{q}; c_1, c_2) \to (x_1, x_2; 1, k)$, the Darboux matrix becomes

$$M(\mathbf{x}; k; \lambda) = \lambda^2 \begin{pmatrix} k + x_1 x_2 & 0 \\ 0 & 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 & x_1 \\ x_2 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{35}$$

The Lax-equation for M is equivalent to the following

$$(\mathbf{x}, \mathbf{y}) \xrightarrow{Y_{a,b}} \left(y_1 + \frac{a-b}{a - x_1 y_2} x_1, \frac{a - x_1 y_2}{b - x_1 y_2} y_2, \frac{b - x_1 y_2}{a - x_1 y_2} x_1, x_2 + \frac{b-a}{b - x_1 y_2} y_2 \right). \tag{36}$$

One can easily verify that the above map satisfies parametric YB equation (8) and it is reversible. Therefore, it is a parametric YB map with strong Darboux-Lax matrix (35). Moreover, map (36) is not involutive.

The invariants of map (36) are given by

$$I_1(\mathbf{x}, \mathbf{y}) = b x_1 x_2 + a y_1 y_2 + x_1 x_2 y_1 y_2 + ab, \qquad I_2(\mathbf{x}, \mathbf{y}) = (x_1 + y_1)(x_2 + y_2) + a + b. \tag{37}$$

The constant terms in I_1 and I_2 can be omitted. Those are the invariants we retrieve from the trace of $M(\mathbf{y}; b, \lambda) M(\mathbf{x}; a, \lambda)$. However, the quantities $x_1 + y_1$ and $x_2 + y_2$ in I_1 are invariants themselves. The Poisson bracket in this case is given by

$$\{x_1, x_2\} = \{y_1, y_2\} = \{x_2, y_1\} = 1, \qquad \text{and all the rest} \qquad \{x_i, y_j\} = 0. \tag{38}$$

The rank of the Poisson matrix is 2, I_1 is one invariant and $I_2 = C_1 C_2 + a + b$, where $C_1 = x_1 + y_1$ and $C_2 = x_2 + y_2$ are Casimir functions. The latter are preserved by (36), namely $C_i \circ Y_{a,b} = C_i$, $i = 1, 2$. Therefore, map (36) is completely integrable.

Moreover, the map (36) can be expressed as a map of two variables on the symplectic leaf

$$x_1 + y_1 = c_1, \qquad x_2 + y_2 = c_2. \tag{39}$$

2.2.1 $\mathbb{Z}_2$ reduction: 6-dimensional YB map

We now consider a more general map than (35) with entries depending on the variables x_1, x_2 and X, given by

$$M(\mathbf{x}, X; k, \lambda) = \lambda^2 \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 & x_1 \\ x_2 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{40}$$

In this case, from the Lax equation we obtain the following equations

$$\begin{gathered} u_2 v_1 = x_1 y_2, \quad u_2 v_3 = x_3 y_2, \quad u_3 v_1 = x_1 y_3, \quad u_3 v_3 = x_3 y_3 \\ u_1 + v_1 = x_1 + y_1, \quad u_3 + u_1 v_2 + v_3 = x_3 + x_2 y_1 + y_3, \quad u_2 + v_2 = x_2 + y_2. \end{gathered}$$

As in the case of nonlinear Schrödinger equation, the algebraic variety consists of two components. The first 6-dimensional component corresponds to the permutation map

$$\mathbf{x} \mapsto \mathbf{u} = \mathbf{y}, \quad \mathbf{y} \mapsto \mathbf{v} = \mathbf{x}, \quad X \mapsto U = Y, \quad Y \mapsto V = X,$$

and the second corresponds to the following 6-dimensional YB map

$$\begin{aligned} x_1 &\mapsto u_1 = f_1(\mathbf{x},\mathbf{y},X,Y), & y_1 &\mapsto v_1 = f_2(\pi\mathbf{y},\pi\mathbf{x},Y,X),\\ x_2 &\mapsto u_2 = f_2(\mathbf{x},\mathbf{y},X,Y), & y_2 &\mapsto v_2 = f_1(\pi\mathbf{y},\pi\mathbf{x},Y,X),\\ X &\mapsto U = f_3(\mathbf{x},\mathbf{y},X,Y), & Y &\mapsto V = f_3(\pi\mathbf{y},\pi\mathbf{x},Y,X), \end{aligned} \tag{41}$$

where π is the *permutation function*, $\pi(x_1,x_2)=(x_2,x_1)$, $\pi^2=1$ and f_1, f_2 and f_3 are given by

$$f_1(\mathbf{x},\mathbf{y},X,Y) = \frac{(x_1+y_1)X - x_1Y - x_1x_2(x_1+y_1)}{X - x_1(x_2+y_2)}, \tag{42}$$

$$f_2(\mathbf{x},\mathbf{y},X,Y) = \frac{X - x_1(x_2+y_2)}{Y - y_2(x_1+y_1)}y_2, \tag{43}$$

$$f_3(\mathbf{x},\mathbf{y},X,Y) = \frac{X - x_1(x_2+y_2)}{Y - y_2(x_1+y_1)}Y. \tag{44}$$

This map has the following invariants

$$I_1(\mathbf{x},\mathbf{y},X,Y) = XY, \qquad I_2(\mathbf{x},\mathbf{y},X,Y) = \mathbf{x}\cdot\pi\mathbf{y} + X + Y, \tag{45}$$

$$I_3(\mathbf{x},\mathbf{y},X,Y) = x_1+y_1, \qquad I_4(\mathbf{x},\mathbf{y},X,Y) = x_2+y_2. \tag{46}$$

By definition, the Darboux-Lax matrix (40) must have constant determinant, from which

$$X - x_1x_2 = a = constant. \tag{47}$$

Changing $X \to a + x_1x_2$ in (40) we obtain matrix (35). Furthermore, using the transformation

$$X = a + x_1x_2, \qquad Y = b + y_1y_2, \qquad U = a + u_1u_2 \qquad \text{and} \qquad V = b + v_1v_2, \tag{48}$$

we obtain from (41) and (42)-(44) the YB map (36).

2.2.2 $\mathbb{Z}_2$ reduction: Another 6-dimensional YB map

Now, lets go back to the Darboux matrix (33) and replace $(p,\widetilde{q},f;c_1) \to (x_1,x_2,X;1)$, namely

$$M(\mathbf{x},X;\lambda) = \lambda^2\begin{pmatrix} X & 0\\ 0 & 0\end{pmatrix} + \lambda\begin{pmatrix} 0 & x_1X\\ x_2X & 0\end{pmatrix} + \begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}, \tag{49}$$

where, according to (34)), X obeys the following equation

$$x_1x_2X^2 - X + c_2 = 0. \tag{50}$$

The Lax equation implies the following equations

$$\begin{gathered} u_1u_3 + v_1v_3 = x_1x_3 + y_1y_3, \quad u_2u_3 + v_2v_3 = x_2x_3 + y_2y_3,\\ u_3v_3 = x_3y_3, \quad u_3v_1v_3 = x_1x_3y_3, \quad u_2u_3v_3 = x_3y_2y_3, \quad u_2u_3v_1v_3 = x_1x_3y_2y_3,\\ u_3 + v_3 + u_1u_3v_2v_3 = x_3 + y_3 + x_2x_3y_1y_3. \end{gathered} \tag{51}$$

Now, the first 6-dimensional component of the algebraic variety corresponds to the trivial map (41) and the second component corresponds to a map of the form (41), with f_1, f_2 and f_3 now given by

$$
\begin{aligned}
f_1(\mathbf{x},\mathbf{y},X,Y) &= \frac{-1}{f_3(\mathbf{x},\mathbf{y})}\frac{x_1X+(y_1-x_1)Y-x_1x_2y_1XY-x_1^2x_2X^2}{x_1x_2X+x_1y_2Y-1}, && (52)\\
f_2(\mathbf{x},\mathbf{y},X,Y) &= y_2, && (53)\\
f_3(\mathbf{x},\mathbf{y},X,Y) &= \frac{x_1x_2X+x_1y_2Y-1}{x_1y_2X+y_1y_2Y-1}X. && (54)
\end{aligned}
$$

One can verify that the above map is a non-involutive YB map. The invariants of this map are given by

$$I_1(\mathbf{x}\cdot\pi\mathbf{y},X,Y)=XY \qquad \text{and} \qquad I_2(\mathbf{x},\mathbf{y},X,Y)=(\mathbf{x}\cdot\pi\mathbf{y})XY+X+Y. \tag{55}$$

2.3 Dihedral Group

In the case of dihedral group, U is given by

$$
\begin{gathered}
U(p,q;\lambda)=\lambda^2U_2+\lambda U_1+\lambda^{-1}U_{-1}-\lambda^{-2}U_{-2}, \qquad \text{where}\\
U_2\equiv U_{-2}=\sigma_3, \qquad U_1=\begin{pmatrix}0 & 2p\\ 2q & 0\end{pmatrix} \qquad \text{and} \qquad U_{-1}=\sigma_1U_1\sigma_1.
\end{gathered} \tag{56}
$$

Here, the reduction group consists of the following set of transformations acting on (13),

$$L(\lambda)=\sigma_3L(-\lambda)\sigma_3 \qquad \text{and} \qquad L(\lambda)=\sigma_1L(\lambda^{-1})\sigma_1, \tag{57}$$

and it is isomorphic to $\mathbb{Z}_2\times\mathbb{Z}_2\cong D_2$, [21].

In this case, the Darboux matrix is given by [16]

$$M=\lambda^2\begin{pmatrix}f & 0\\ 0 & 0\end{pmatrix}+\lambda\begin{pmatrix}0 & fp\\ f\widetilde{q} & 0\end{pmatrix}+\lambda^{-1}\begin{pmatrix}0 & f\widetilde{q}\\ fp & 0\end{pmatrix}+\lambda^{-2}\begin{pmatrix}0 & 0\\ 0 & f\end{pmatrix}+gI, \tag{58}$$

where the entries f and g obey the following equations

$$fg-f^2p\widetilde{q}=c_1 \qquad \text{and} \qquad f^2+g^2-f^2\widetilde{q}^2-f^2p^2=c_2. \tag{59}$$

It follows from (59), that functions f and g can be expressed in terms of p and $\widetilde{q}$, as solutions of quadratic equations. Then, the Darboux matrix depends only on two variables and then we construct a $4-$dimensional parametric YB map. Althought, we have omitted these expressions because of their length. However, we have seen in the previous sections that the $6-$dimensional YB maps can reduce to $4-$dimensional maps using invariants.

In the next section we construct a $6-$dimensional map from (58).

2.3.1 Dihedral group: A 6-dimensional YB map

We now consider the matrix $N := fM$, where M is given by (58), and we change $(p, \widetilde{q}, f^2) \to (x_1, x_2, X)$. Then,

$$N(\mathbf{x}, X; c_1, \lambda) = \begin{pmatrix} \lambda^2 X + x_1x_2X + c_1 & \lambda x_1 X + \lambda^{-1} x_2 X \\ \lambda x_2 X + \lambda^{-1} x_1 X & \lambda^{-2} X + x_1x_2X + c_1 \end{pmatrix}, \tag{60}$$

where we have substituted the product fg by

$$fg = c_1 + x_1x_2X, \tag{61}$$

from the first equation of (59).

The Lax equation for the Darboux-Lax matrix (60) reads

$$N(\mathbf{u}, U; a, \lambda)N(\mathbf{v}, V; b, \lambda) = N(\mathbf{y}, Y; b, \lambda)N(\mathbf{x}, X; a, \lambda), \tag{62}$$

from where we obtain an algebraic system of equations, omitted because of its length.

The first 6-dimensional component of the corresponding algebraic variety corresponds to the trivial YB map

$$\mathbf{x} \mapsto \mathbf{u} = \mathbf{y}, \quad \mathbf{y} \mapsto \mathbf{v} = \mathbf{x}, \quad X \mapsto U = \frac{a}{b}Y, \quad Y \mapsto V = \frac{b}{a}X,$$

and the second component corresponds to the following map

$$\begin{aligned} x_1 &\mapsto u_1 = \frac{f(\mathbf{x}, \mathbf{y}, X, Y; a, b)}{g(\mathbf{x}, \mathbf{y}, X, Y; a, b)}, & y_1 &\mapsto v_1 = x_1 \\ x_2 &\mapsto u_2 = y_2, & y_2 &\mapsto v_2 = \frac{f(\pi\mathbf{y}, \pi\mathbf{x}, Y, X; b, a)}{g(\pi\mathbf{y}, \pi\mathbf{x}, Y, X; b, a)} \\ X &\mapsto U = \frac{g(\mathbf{x}, \mathbf{y}, X, Y; a, b)}{h(\mathbf{x}, \mathbf{y}, X, Y; a, b)}, & Y &\mapsto V = \frac{g(\pi\mathbf{y}, \pi\mathbf{x}, Y, X; b, a)}{h(\pi\mathbf{y}, \pi\mathbf{x}, Y, X; b, a)}, \end{aligned} \tag{63}$$

where f, g and h are given by

$$\begin{aligned} f(\mathbf{x}, \mathbf{y}, X, Y; a, b) = {}& a^2b^2x_1X + a^2b[x_2 - y_2 + 2x_1x_2y_1 + x_1^2(y_2 - 3x_2)]XY + \\ & a^2(y_2^2 - 1)[y_1(1 + x_1^2) - x_1(1 + y_1^2)]XY^2 - ab^2(x_1^2 - 1)(y_2 - x_2)X^2 - \\ & ab(x_1^2 - 1)[x_2^2(3x_1 - y_1) - x_1 - y_1 + 2y_2(y_1y_2 - x_1x_2)]X^2Y - \\ & a(x_1^2 - 1)(y_2^2 - 1)[y_2(y_1^2 - 1) + x_2(y_1^2 - 2x_1y_1 + 1)]X^2Y^2 + \\ & y_1(x_1^2 - 1)^2(x_2^2 - 1)(y_2^2 - 1)X^3Y^2 + b(x_1^2 - 1)^2(x_2^2 - 1)(y_2 - x_2)X^3Y + \\ & a^3b(y_1 - x_1)Y, \\[1em] g(\mathbf{x}, \mathbf{y}, X, Y; a, b) = {}& a^2b^2X + 2a^2by_2(y_1 - x_1)XY + a^2(y_2^2 - 1)(x_1 - y_1)^2XY^2 + \\ & 2ab(x_1^2 - 1)(1 - x_2y_2)X^2Y + 2ax_2(x_1^2 - 1)(y_2^2 - 1)(x_1 - y_1)X^2Y^2 + \\ & (x_1^2 - 1)^2(x_2^2 - 1)(y_2^2 - 1)X^3Y^2, \\[1em] h(\mathbf{x}, \mathbf{y}, X, Y; a, b) = {}& a^2b^2 - 2ab^2x_1(y_2 - x_2)X - 2ab(x_1y_1 - 1)(y_2^2 - 1)XY \\ & b^2(x_1^2 - 1)(x_2 - y_2)^2X^2 - 2by_1(x_2 - y_2)(x_1^2 - 1)(y_2^2 - 1)X^2Y + \\ & (x_1^2 - 1)(y_1^2 - 1)(y_2^2 - 1)^2X^2Y^2. \end{aligned} \tag{64}$$

It can be verified that this is a parametric YB map. From $\mathrm{Tr}(N(\mathbf{x},X;\lambda)N(\mathbf{y},Y;\lambda))$ we extract the following invariants for the above map

$$\begin{aligned}
I_1(\mathbf{x},\mathbf{y},X,Y;a,b) &= XY, && (65)\\
I_2(\mathbf{x},\mathbf{y},X,Y;a,b) &= bX+aY+(x_1+y_1)(x_2+y_2)XY, && (66)\\
I_3(\mathbf{x},\mathbf{y},X,Y;a,b) &= 2bx_1x_2X+2ay_1y_2Y+2(\mathbf{x}\cdot\mathbf{y}+x_1x_2y_1y_2)XY+2ab. && (67)
\end{aligned}$$

As stated earlier from the above 6−dimensional map we can construct a 4−dimensional YB map. In particular, from equations (59) and (61) one can obtain

$$(1-x_1^2-x_2^2+x_1^2x_2^2)X^2+(2x_1x_2-c_2)X+1=0, \tag{68}$$

where we have rescaled $c_1\to 1$.

Therefore the 4−dimensional map is given by

$$(u_1,u_2,v_1,v_2)=\left(\frac{f(\mathbf{x},\mathbf{y};a,b)}{g(\mathbf{x},\mathbf{y};a,b)},y_2,x_1,\frac{f(\pi\mathbf{y},\pi\mathbf{x};b,a)}{g(\pi\mathbf{y},\pi\mathbf{x};b,a)}\right), \tag{69}$$

where f, g and h are given by the above relations and X and Y are given implicitly by

$$\begin{aligned}
(1-x_1^2-x_2^2+x_1^2x_2^2)X^2+(2x_1x_2-a)X+1 &= 0, && (70)\\
(1-y_1^2-y_2^2+y_1^2y_2^2)Y^2+(2y_1y_2-b)Y+1 &= 0. && (71)
\end{aligned}$$

2.4 Dihedral group: A linearised YB map

We replace $(f\widetilde{q},fp)\to(x_1,x_2)$ and $(c_1,c_2)\to(\frac{1-k^2}{2},\frac{1+k^2}{2})$ in the Darboux matrix (58) to become

$$M(\mathbf{x};k,\lambda)=\lambda^2\begin{pmatrix}f&0\\0&0\end{pmatrix}+\lambda\begin{pmatrix}0&x_1\\x_2&0\end{pmatrix}+\lambda^{-1}\begin{pmatrix}0&x_2\\x_1&0\end{pmatrix}+\lambda^{-2}\begin{pmatrix}0&0\\0&f\end{pmatrix}+gI, \tag{72}$$

where f and g are given by

$$\begin{aligned}
f &= \frac{1}{2}\sqrt{k^2+(x_1-x_2)^2}+\frac{1}{2}\sqrt{1+(x_1+x_2)^2}, && (73)\\
g &= \frac{1}{2}\sqrt{1+(x_1+x_2)^2}-\frac{1}{2}\sqrt{k^2+(x_1-x_2)^2}. && (74)
\end{aligned}$$

The linear approximation to the YB map is given by

$$\begin{pmatrix}x_1\\x_2\\y_1\\y_2\end{pmatrix}\xrightarrow{U_0}\begin{pmatrix}u_1\\u_2\\v_1\\v_2\end{pmatrix}=\left(\begin{array}{cc|cc}
\frac{(a-1)(a-b)}{(a+1)(a+b)} & \frac{a-b}{a+b} & \frac{2a}{a+b} & \frac{(a+1)(b-a)}{(b+1)(a+b)}\\
0 & 0 & 0 & \frac{a+1}{b+1}\\ \hline
\frac{b+1}{a+1} & 0 & 0 & 0\\
\frac{(a-b)(b+1)}{(a+1)(a+b)} & \frac{2b}{a+b} & \frac{b-a}{a+b} & \frac{(b-1)(b-a)}{(b+1)(a+b)}
\end{array}\right)\begin{pmatrix}x_1\\x_2\\y_1\\y_2\end{pmatrix} \tag{75}$$

which is a linear parametric YB map and it is not involutive.

3 $2N \times 2N-$dimensional YB maps

We now replace the variables, x_1 and x_2, in the Lax matrices with $N-$vectors $\mathbf{w}_1$ and $\mathbf{w}_2^T$ to obtain $2N \times 2N$ YB maps. In what follows we use the following notation for a $n-$vector $\mathbf{w} = (w_1, ..., w_n)$

$$\mathbf{w} = (\mathbf{w}_1, \mathbf{w}_2), \qquad \text{where} \qquad \mathbf{w}_1 = (w_1, ..., w_N), \qquad \mathbf{w}_2 = (w_{N+1}, ..., w_{2N}) \tag{76}$$

and also

$$< u_i| := \mathbf{u}_i, \qquad |w_i >:= \mathbf{w}_i^T \qquad \text{and their dot product with} \qquad < u_i, w_i > . \tag{77}$$

3.1 NLS equation

Replacing the variables in (19) with $N-$vectors, namely

$$M(\mathbf{w}; a, \lambda) = \begin{pmatrix} \lambda + a + < w_1, w_2 > & < w_1| \\ |w_2 > & I \end{pmatrix}, \tag{78}$$

we obtain a unique solution of the Lax-Equation given by the following $2N \times 2N$ map

$$\begin{cases} < u_1| =< y_1| + f(z; a, b) < x_1|, \\ < u_2| =< y_2|, \end{cases} \tag{79}$$

and

$$\begin{cases} < v_1| =< x_1|, \\ < v_2| =< x_2 + f(z; b, a) < y_2|, \end{cases} \tag{80}$$

where f is given by

$$f(z; b, a) = \frac{b-a}{1+z}, \qquad z :=< x_1, y_2 > . \tag{81}$$

The above is a non-involutive parametric $2N \times 2N$ YB map with strong Lax matrix given by (78). As a YB map it appears in [30], but it is originally introduced by Adler [2]. Moreover, one can construct the above $2N \times 2N$ map for the $N \times N$ Darboux-Lax matrix (78) by taking the limit of the solution of the refactorisation problem in [19].

The invariants of this map are given by

$$I_1(\mathbf{x}, \mathbf{y}; a, b) =< x_1, x_2 > + < y_1, y_2 >, \tag{82}$$

$$\begin{aligned} I_2(\mathbf{x}, \mathbf{y}; a, b) = b < x_1, x_2 > + a < y_1, y_2 > + < x_1, y_2 > + < x_2, y_1 > + \\ < x_1, x_2 >< y_1, y_2 > . \end{aligned} \tag{83}$$

3.2 $\mathbb{Z}_2$ reduction

In the case of Z_2 we consider, instead of (35), the following matrix

$$M(\mathbf{w}; a, \lambda) = \begin{pmatrix} \lambda^2(a + < w_1, w_2 >) & \lambda < w_1| \\ \lambda |w_2 > & I \end{pmatrix}, \tag{84}$$

we obtain a unique solution for the Lax-Equation given by the following $2N \times 2N$ map

$$\begin{cases} < u_1| =< y_1| + f(z;a,b) < x_1|, \\ < u_2| = g(z;a,b) < y_2|, \end{cases} \tag{85}$$

and

$$\begin{cases} < v_1| = g(z;b,a) < x_1|, \\ < v_2| =< x_2| + f(z;b,a) < y_2|, \end{cases} \tag{86}$$

where f and g are given by

$$f(z;a,b) = \frac{a-b}{a-z}, \qquad g(z;a,b) = \frac{a-z}{b-z}, \qquad z :=< x_1, y_2 > . \tag{87}$$

The above map is a non-involutive parametric $2N \times 2N$ YB map with strong Lax matrix given by (84).

The invariants of the above map are given by

$$I_1(\mathbf{x},\mathbf{y};a,b) \quad = \quad < x_1 + y_1, x_2 + y_2 >, \tag{88}$$

$$I_2(\mathbf{x},\mathbf{y};a,b) \quad = \quad b < x_1, x_2 > +a < y_1, y_2 > + < x_1, x_2 >< y_1, y_2 > . \tag{89}$$

In fact, all the terms $x_i + y_i$ in I_1 are invariants.

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