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The Friedman-Sheard programme in intuitionistic logic

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Abstract

This paper compares the roles classical and intuitionistic logic play in restricting the free use of truth principles in arithmetic. We consider fifteen of the most commonly used axiomatic principles of truth and classify every subset of them as either consistent or inconsistent over a weak purely intuitionistic theory of truth.

1 Introduction

There have been many proposals regarding how one may overcome the paradoxes induced by the liar sentence and its variants to obtain consistent axiomatic theories of truth. Feferman [1] observes three possible routes towards consistency. These involve restrictions of language, logic, or truth principles. An example of the first direction is Tarski's hierarchy of truth, formalised, for example, in Feferman [2] and Halbach [4]. Theories involving only the second restriction require the adoption of logics based on more than two truth values, partial logics or paraconsistent logics. Feferman, however, rejects the use of non-standard logics, i.e. logics other than classical or intuitionistic logic, as in these logics "nothing like sustained ordinary reasoning can be carried out" [1, p. 95]. The third direction is exemplified by the work of Friedman and Sheard [3]. The naive notion of truth is removed in favour of twelve principles, referred to as *Optional Axioms*, each conveying some desirable property of truth. These include direct weakenings of the Tarskian bi-conditional in the form of one direction of the equivalence or rules of inference, axioms for truth repetition and deletion, axioms ensuring commutation between quantifiers and the truth predicate, and axioms of truth completeness and consistency (see table 1 below for the complete list of axioms considered). All subsets of the twelve principles were characterised as either consistent or inconsistent over a classical base theory of truth, the upshot being nine maximal consistent subsets of the Optional Axioms (see theorem 3.2 below).

The aim of this paper is to investigate the role classical logic has on restricting the free use of these truth principles. In [3] a classical base theory, $\mathsf{Base}_{\mathtt{T}}$, is used incorporating a truth predicate whose underlying logic is also classical, i.e. $\mathsf{Base}_{\mathtt{T}} \vdash \mathtt{T}(\ulcorner A \vee \neg A \urcorner)$ for every sentence A where $\ulcorner A \urcorner$ denotes the Gödel number of A. We will carry out the Friedman-Sheard programme in a purely intuitionistic environment making use of a base theory in which neither the underlying logic nor the logic of the truth predicate is declared classical. Friedman and Sheard proved a number of inconsistency results regarding subsets of the Optional Axioms; however, the majority of these proofs make use of classical principles inherent in the base theory and it is not immediate that they can be eliminated. Furthermore, the following four principles are all equivalent over the classical base theory used by the authors, when stated for arbitrary sentences A and B of $\mathcal{L}_T$.

$$\mathtt{T}(\ulcorner A \urcorner) \vee \mathtt{T}(\ulcorner \neg A \urcorner),$$
$$\neg \mathtt{T}(\ulcorner A \urcorner) \to \mathtt{T}(\ulcorner \neg A \urcorner),$$
$$\mathtt{T}(\ulcorner A \vee B \urcorner) \to \mathtt{T}(\ulcorner A \urcorner) \vee \mathtt{T}(\ulcorner B \urcorner),$$
$$(\mathtt{T}(\ulcorner A \urcorner) \to \mathtt{T}(\ulcorner B \urcorner)) \to \mathtt{T}(\ulcorner A \to B \urcorner).$$

Using the intuitionistic base theory proposed here the first axiom implies the remaining three and the second is a consequence of the fourth, but these appear to be the only (non-classical) logical dependencies between them: to deduce the first from the second or fourth, one requires classical logic and to deduce the first from the third a classical truth predicate is required.

2 Intuitionistic logic

There are many ways to formulate first-order intuitionistic predicate calculus. We shall make use of the Hilbert-style formulation presented in, for example, [8, §2.4]. The basic logical connectives are $\wedge$, $\vee$ and $\to$; with $\perp$ a logical constant. Negation is considered defined: $\neg A$ abbreviates the implication $A \to \perp$. The rules of inference are *modus ponens* and *generalisation.*

Let $\mathcal{L}$ denote the basic language of arithmetic and $\mathcal{L}_{\mathtt{T}}$ the language $\mathcal{L}$ augmented with a unary predicate symbol $\mathtt{T}$. We will make use of models of intuitionistic logic, in particular intuitionistic Kripke ω-structures for $\mathcal{L}_{\mathtt{T}}$, which are introduced below. $\mathbb{N}$ denotes the standard model of arithmetic.

Definition 2.1 *A first-order intuitionistic Kripke ω-structure for $\mathcal{L}_{\mathtt{T}}$ is a triple $\mathfrak{M} = \langle W_{\mathfrak{M}}, \leq_{\mathfrak{M}}, \mathfrak{T}_{\mathfrak{M}} \rangle$ where $\langle W_{\mathfrak{M}}, \leq_{\mathfrak{M}} \rangle$ is a partially-ordered Kripke frame, $\mathfrak{T}_{\mathfrak{M}} \subseteq W_{\mathfrak{M}} \times \mathbb{N}$, and the following persistency requirement is satisfied: whenever $u \leq_{\mathfrak{M}} v \in W_{\mathfrak{M}}$,*

$$\langle u, m \rangle \in \mathfrak{T}_{\mathfrak{M}} \textit{ implies } \langle v, m \rangle \in \mathfrak{T}_{\mathfrak{M}}$$

for every $m \in \mathbb{N}$.

We write $\mathfrak{T}_u$ for the set $\{x : \langle u, x\rangle \in \mathfrak{T}_{\mathfrak{M}}\}$. $W_{\mathfrak{M}}$ is referred to as the carrier *of $\mathfrak{M}$, $w \in W_{\mathfrak{M}}$ as a* world *of $\mathfrak{M}$, and $\mathfrak{T}_w$ as the* interpretation of truth at w.

A Kripke ω-structure determines a satisfaction relation, $u \Vdash A$, for $u \in W_{\mathfrak{M}}$ and sentences A in $\mathcal{L}_{\mathtt{T}}$ defined as follows.

1. *$w \Vdash \bot$ does not hold for any $w \in W_{\mathfrak{M}}$,*
2. *$w \Vdash R(t_0, \ldots, t_{n-1})$ iff $R(t_0, \ldots, t_{n-1})$ is true in $\mathbb{N}$, where R is an n-ary predicate symbol in $\mathcal{L}$ for a primitive recursive relation and t_0, $\ldots$, t_{n-1} are closed terms of $\mathcal{L}$,*
3. *$w \Vdash \mathtt{T}(s)$ iff $s^{\mathbb{N}} \in \mathfrak{T}_w$,*
4. *$w \Vdash A \wedge B$ iff $w \Vdash A$ and $w \Vdash B$,*
5. *$w \Vdash A \vee B$ iff either $w \Vdash A$ or $w \Vdash B$,*
6. *$w \Vdash A \to B$ iff for every $u \geq w$ $u \Vdash A$ implies $u \Vdash B$,*
7. *$w \Vdash \exists x A(x)$ iff there is an $n \in \mathbb{N}$ such that $w \Vdash A(\bar{n})$,*
8. *$w \Vdash \forall x A(x)$ iff for every $u \geq w$ and every $n \in \mathbb{N}$, $u \Vdash A(\bar{n})$.*

We may write $\Vdash_{\mathfrak{M}}$ to emphasise the relation $\Vdash$ is defined with respect to $\mathfrak{M}$ and drop the subscript $\mathfrak{M}$ from $W_{\mathfrak{M}}$ and $\mathfrak{T}_{\mathfrak{M}}$ when $\mathfrak{M}$ is clear from the context. If $\mathfrak{M}$ is an intuitionistic Kripke ω-structure and A is a sentence of $\mathcal{L}_{\mathtt{T}}$, $\mathfrak{M}$ models *A, written $\mathfrak{M} \models A$, if $w \Vdash A$ for every $w \in W_{\mathfrak{M}}$.*

An intuitionistic Kripke ω-structure $\mathfrak{M} = \langle W_{\mathfrak{M}}, \leq_{\mathfrak{M}}, \mathfrak{T}_{\mathfrak{M}}\rangle$ is a classical *(ω-)model if its universe has at most one element, i.e. $|W_{\mathfrak{M}}| = 1$. In that case $\mathfrak{M}$ is determined by $\mathfrak{T}_w$ where $w \in W_{\mathfrak{M}}$ and $\mathfrak{T}_{\mathfrak{M}} = \{w\} \times \mathfrak{T}_w$.*

Intuitionistic first-order predicate calculus (IPC) is sound with respect to the class of intuitionistic Kripke ω-structures, since it is sound with respect to all Kripke structures (see, for example, [5, 8]).

We may assume $\mathcal{L}$ contains a function symbol for every primitive recursive function and that we have some primitive recursive Gödel coding, $\ulcorner . \urcorner$ of $\mathcal{L}_{\mathtt{T}}$-formulae. If f is a primitive recursive function we denote by $\underset{\cdot}{f}$ its corresponding symbol in $\mathcal{L}$. Let HA denote the theory of Heyting arithmetic, the intuitionistic theory with the usual Peano axioms for successor and multiplication, defining axioms for every primitive recursive function and the schema of induction for all formulae in its language. Denote by $\mathsf{HA}_{\mathtt{T}}$ the theory HA formulated in the language $\mathcal{L}_{\mathtt{T}}$, that is, with the induction schema extended to include all formulae of $\mathcal{L}_{\mathtt{T}}$. We say a theory S has the *disjunction property* if, whenever $\mathsf{S} \vdash A \vee B$, either $\mathsf{S} \vdash A$ or $\mathsf{S} \vdash B$ holds, and has the *existence property* if whenever $\mathsf{S} \vdash \exists x A(x)$ there is a term t

such that $\mathsf{S} \vdash A(t)$. It is well known that HA has both the disjunction and existence property (see, for example [8, chap. 3, thm. 5.10]). This also holds for $\mathsf{HA}_{\mathtt{T}}$; the presence of the truth predicate has no effect on the proof.

To each logical connective $*$ is associated a primitive recursive function symbol $\dot{*}$ in $\mathcal{L}$ representing it for sentences: that is, for all sentences A, B in $\mathcal{L}_{\mathtt{T}}$, $(\ulcorner A \urcorner \dot{*} \ulcorner B \urcorner) = \ulcorner A * B \urcorner$, for $*$ each of $\wedge$, $\vee$, $\rightarrow$; and that if either x or y is not the code of a sentence of $\mathcal{L}_{\mathtt{T}}$, $x \dot{*} y = \ulcorner \bar{0} = \bar{1} \urcorner$, where $\bar{n}$ denotes the n-th numeral. $\dot{\neg} x$ abbreviates $x \dot{\rightarrow} \ulcorner \bot \urcorner$.

It will be necessary to quantify over codes of $\mathcal{L}_{\mathtt{T}}$ sentences and formulae of $\mathcal{L}_{\mathtt{T}}$ with at most one free variable and thus we introduce the following notation. Let $\mathtt{Sent}_{\mathcal{L}'}(x)$ denote the formal predicate which expresses 'x is the code of a sentence of $\mathcal{L}'$' and let $subn(m, n)$ denote the primitive recursive function such that $subn(\ulcorner A(x) \urcorner, n) = \ulcorner A(\bar{n}) \urcorner$ if A is a formula of $\mathcal{L}_{\mathtt{T}}$ with at most x free. If m is not the code of a sentence with at most one free variable $subn(m, n) = \ulcorner \bar{0} = \bar{1} \urcorner$. We then introduce the following abbreviations.

- $\forall \ulcorner A \urcorner F(\ulcorner A \urcorner)$ abbreviates $\forall x(\mathtt{Sent}_{\mathcal{L}_T}(x) \rightarrow F(x))$,
- $\exists \ulcorner A \urcorner F(\ulcorner A \urcorner)$ abbreviates $\exists x(\mathtt{Sent}_{\mathcal{L}_T}(x) \wedge F(x))$,
- $\forall \ulcorner A(x) \urcorner F(\ulcorner A(x) \urcorner)$ abbreviates $\forall x(\mathtt{Sent}_{\mathcal{L}_T}(su\dot{b}n(x, \bar{0})) \rightarrow F(x))$,
- $\exists \ulcorner A(x) \urcorner F(\ulcorner A(x) \urcorner)$ abbreviates $\exists x(\mathtt{Sent}_{\mathcal{L}_T}(su\dot{b}n(x, \bar{0})) \wedge F(x))$.

To simplify uses of the function $subn$ we make use of the dot convention for variables, namely, by $\ulcorner A(\dot{x}) \urcorner$ we represent the term $subn(\ulcorner A(x) \urcorner, x)$.

3 A closer look at the Optional Axioms

We can now define the base theory $\mathsf{Base}^i_{\mathtt{T}}$ over which our analysis takes place.

Definition 3.1 *Let* $\mathsf{Base}^i_{\mathtt{T}}$ *denote the theory extending* $\mathsf{HA}_{\mathtt{T}}$ *with the additional axioms:*

1. *T-Imp:* $\forall x \forall y(\mathtt{T}(x) \wedge \mathtt{T}(x \dot{\rightarrow} y) \rightarrow \mathtt{T}(y))$,
2. $\forall x(\mathtt{val}^i(x) \rightarrow \mathtt{T}(u\dot{c}l(x)))$,
3. $\forall x(\mathtt{Ax}_{\mathsf{PRA}}(x) \rightarrow \mathtt{T}(x))$,

where $\mathtt{val}^i(x)$ *expresses that* x *is the Gödel number of an intuitionistically valid first-order* $\mathcal{L}_{\mathtt{T}}$*-formula and* $\mathtt{Ax}_{\mathsf{PRA}}(x)$ *expresses that* x *is the Gödel number of a non-logical axiom of primitive recursive arithmetic.* $\mathsf{Base}_{\mathtt{T}}$ *is the theory* $\mathsf{Base}^i_{\mathtt{T}}$ *formulated in classical logic plus the the principles* $\forall \ulcorner A \urcorner(\mathtt{T}(\ulcorner A \vee \neg A \urcorner))$ *stating that the underlying logic of the predicate* $\mathtt{T}$ *is classical.*

Table 1, below, lists the twelve principles of truth considered by Friedman and Sheard. The next theorem summarises the work of [3].

Name	Axiom Schema
T-In	$\forall x(A(x) \to \mathtt{T}(\ulcorner A(\dot{x})\urcorner))$
T-Out	$\forall x(\mathtt{T}(\ulcorner A(\dot{x})\urcorner) \to A(x))$

Name	Axiom
T-Rep	$\forall x(\mathtt{T}(x) \to \mathtt{T}(\ulcorner \mathtt{T}(\dot{x})\urcorner))$
T-Del	$\forall x(\mathtt{T}(\ulcorner \mathtt{T}(\dot{x})\urcorner) \to \mathtt{T}(x))$
T-Cons	$\forall x \neg(\mathtt{T}(x) \wedge \mathtt{T}(\underset{\cdot}{\neg} x))$
T-Comp	$\forall \ulcorner A \urcorner(\mathtt{T}(\ulcorner A\urcorner) \vee \mathtt{T}(\ulcorner \neg A\urcorner))$
$\forall$-Inf	$\forall \ulcorner A(x)\urcorner(\forall n \mathtt{T}(\ulcorner A(\dot{n})\urcorner) \to \mathtt{T}(\ulcorner \forall x\, A(x)\urcorner))$
$\exists$-Inf	$\forall \ulcorner A(x)\urcorner(\mathtt{T}(\ulcorner \exists x\, A(x)\urcorner) \to \exists n\, T(\ulcorner A(\dot{n})\urcorner))$

Name	Rule of inference
T-Intro	From $A(x)$ infer $\mathtt{T}(\ulcorner A(\dot{x})\urcorner)$
T-Elim	From $\mathtt{T}(\ulcorner A(\dot{x})\urcorner)$ infer $A(x)$
$\neg$T-Intro	From $\neg A(x)$ infer $\neg\mathtt{T}(\ulcorner A(\dot{x})\urcorner)$
$\neg$T-Elim	From $\neg\mathtt{T}(\ulcorner A(\dot{x})\urcorner)$ infer $\neg A(x)$.

Table 1: List of principles considered by Friedman and Sheard.

Theorem 3.2 *(Friedman and Sheard [3]) The following are the only maximal consistent subsets of the twelve Optional Axioms listed in table 1, over* $\mathsf{Base}_{\mathrm{T}}$.

A. *T-In, T-Del, T-Intro, T-Rep,* $\neg$*T-Elim, T-Comp,* $\forall$*-Inf,* $\exists$*-Inf.*

B. *T-Rep, T-Cons, T-Comp,* $\forall$*-Inf,* $\exists$*-Inf.*

C. *T-Del, T-Cons, T-Comp,* $\forall$*-Inf,* $\exists$*-Inf.*

D. *T-Intro, T-Elim, T-Cons, T-Comp,* $\neg$*T-Elim,* $\neg$*T-Intro,* $\forall$*-Inf,* $\exists$*-Inf.*

E. *T-Intro, T-Elim, T-Del, T-Cons,* $\neg$*T-Intro,* $\forall$*-Inf.*

F. *T-Intro, T-Elim, T-Del,* $\neg$*T-Elim,* $\forall$*-Inf.*

G. *T-Intro, T-Elim, T-Rep,* $\neg$*T-Elim,* $\forall$*-Inf.*

H. *T-Out, T-Rep, T-Elim, T-Del, T-Cons,* $\neg$*T-Intro,* $\forall$*-Inf.*

I. *T-Rep, T-Del, T-Elim,* $\neg$*T-Elim,* $\forall$*-Inf.*

The independence of the connectives under intuitionistic logic naturally provides three further principles of truth to consider, which are presented in table 2. We refer to the principles in tables 1 and 2 as *Optional Axioms.* The additional axioms listed in table 2 are all equivalent to T-Comp over $\mathsf{Base}_{\mathrm{T}}$. Over $\mathsf{Base}^{i}_{\mathrm{T}}$, however, T-Comp implies all three and T-Comp(w) is a consequence of $\rightarrow$-Inf, but these appear to be the only dependencies between them; to deduce T-Comp from either T-Comp(w) or $\rightarrow$-Inf, one requires classical logic and to deduce T-Comp from $\vee$-Inf a classical truth predicate is required (*cf.* propositions 3.3 and 3.4 below).

Name	Axiom
T-Comp(w)	$\forall \ulcorner B \urcorner [\neg \mathtt{T}(\ulcorner B \urcorner) \rightarrow \mathtt{T}(\ulcorner \neg B \urcorner)]$
$\vee$-Inf	$\forall \ulcorner A \urcorner \forall \ulcorner B \urcorner [\mathtt{T}(\ulcorner A \vee B \urcorner) \rightarrow \mathtt{T}(\ulcorner A \urcorner) \vee \mathtt{T}(\ulcorner B \urcorner)]$
$\rightarrow$-Inf	$\forall \ulcorner A \urcorner \forall \ulcorner B \urcorner [(\mathtt{T}(\ulcorner A \urcorner) \rightarrow \mathtt{T}(\ulcorner B \urcorner)) \rightarrow \mathtt{T}(\ulcorner A \rightarrow B \urcorner)]$

Table 2: Additional Optional Axioms inspired by intuitionistic logic.

The move to intuitionistic logic provides more freedom to express principles of truth without falling into inconsistency. For example, over $\mathsf{Base}_{\mathtt{T}}$ the principles of truth disjunction and truth existence, $\vee$-Inf and $\exists$-Inf respectively, both imply T-Comp and are consistent with a set of Optional Axioms only if T-Comp is. Over $\mathsf{Base}^{i}_{\mathtt{T}}$, however, the two principles are consistent with every consistent subset of the Optional Axioms.

It is worth remarking on the use of relativised quantifiers in these axioms, as compared with the unrelativised form that the other Optional Axioms. Stating T-Comp(w) in its unrelativised form, $\forall x(\neg \mathtt{T}(x) \rightarrow \mathtt{T}(\dot{\neg} x))$, and assuming the simple statement $\mathtt{T}(x) \rightarrow \mathtt{Sent}_{\mathcal{L}_{\mathtt{T}}}(x)$ (which one would want to be consistent with all sets of Optional Axioms) one could obtain $\neg \mathtt{T}(\bar{n})$ if n is not the code of an $\mathcal{L}_{\mathtt{T}}$-sentence, and hence

$$\mathtt{T}(\bar{n} \mathrel{\dot{\rightarrow}} \ulcorner \bot \urcorner). \tag{1}$$

However, we assumed $(x \mathrel{\dot{\rightarrow}} y) = \ulcorner \bar{0} = \bar{1} \urcorner$ if either x or y is not the code of an $\mathcal{L}_{\mathtt{T}}$-sentence, so eq. (1) yields $\mathtt{T}(\ulcorner \bot \urcorner)$, and hence, by T-Imp, $\mathtt{T}(\ulcorner A \urcorner)$ for every $\mathcal{L}_{\mathtt{T}}$-sentence A. Not only would this be inconsistent with a large portion of the Optional Axioms, it does not express the intuitive concept behind T-Comp(w), namely that whenever it is inconsistent to state a sentence A is true, $\neg A$ is true. The problem can be resolved by relativising the quantifier to only range over codes of $\mathcal{L}_{\mathtt{T}}$-sentences. Perhaps one may also fix the problem by defining $x \mathrel{\dot{\rightarrow}} y$ so that it is equal to $\ulcorner \bar{0} = \bar{1} \urcorner$ if y is not the code of a sentence, but equal to $\ulcorner \bar{0} = \bar{0} \urcorner$ if x is not the code of a sentence. If stated for non-sentences, T-Comp(w) and T-Comp would then hold vacuously, but this will only cause to complicate matters later where we must forever perform a case distinction in the back of our minds when utilising $\dot{\rightarrow}$. A similar situation arises when considering the axiom $\rightarrow$-Inf.

In the end, the change is purely cosmetic and so we pick the relativised form which provides less opportunity for problems in the long term. Before we continue it is worth noting that this issue does not arise for T-Cons (the only other principle making explicit use of negation) which the reader can easily check.

One may reasonably ask whether the principles proposed in table 2 are, in fact, *new* principles and are not subsumed by any axioms or group of axioms already considered. It is not hard to see that T-Comp(w) is classically equivalent to T-Comp and on closer inspection $\vee$-Inf is also equivalent to T-Comp provided the truth predicate behaves classically. It is slightly harder to see where $\rightarrow$-Inf fits in. Let $\mathsf{Base}^c_{\mathrm{T}}$ denote $\mathsf{Base}^i_{\mathrm{T}}$ augmented with the axiom of *truth classicism*, the axiom $\forall \ulcorner A \urcorner \, \mathtt{T}(\ulcorner A \vee \neg A \urcorner)$.

The next proposition shows T-Comp is equivalent over $\mathsf{Base}_{\mathrm{T}}$ to each Optional Axiom in table 2, so theorem 3.2 can be extended to involve the additional axioms.

Proposition 3.3

(i). $\mathsf{Base}^i_{\mathrm{T}} \vdash$ T-Comp $\rightarrow$ (T-Comp(w) $\wedge$ ($\vee$-Inf) $\wedge$ ($\rightarrow$-Inf)),

(ii). $\mathsf{Base}^i_{\mathrm{T}} \vdash$ ($\rightarrow$-Inf) $\rightarrow$ T-Comp(w),

(iii). $\mathsf{Base}_{\mathrm{T}} \vdash$ (T-Comp(w) $\vee$ ($\rightarrow$-Inf)) $\rightarrow$ T-Comp,

(iv). $\mathsf{Base}^c_{\mathrm{T}} \vdash$ ($\vee$-Inf) $\rightarrow$ T-Comp.

Proof (i). $(A \vee B) \rightarrow (\neg A \rightarrow B)$ is intuitionistically valid, so by the second axiom of $\mathsf{Base}^i_{\mathrm{T}}$,

$$\mathsf{Base}^i_{\mathrm{T}} \vdash \forall x \forall y \mathtt{T}((x \dot{\vee} y) \dot{\rightarrow} (\dot{\neg} x \dot{\rightarrow} y)).$$

Two applications of T-Imp, yields $\mathsf{Base}^i_{\mathrm{T}} \vdash [\mathtt{T}(x \dot{\vee} y) \wedge \mathtt{T}(\dot{\neg} x)] \rightarrow \mathtt{T}(y)$, whence

$$\mathsf{Base}^i_{\mathrm{T}} \vdash \text{T-Comp} \rightarrow \vee\text{-Inf}.$$

By direct use of the first implication we also see that T-Comp $\rightarrow$ T-Comp(w) is provable in $\mathsf{Base}^i_{\mathrm{T}}$. This leaves only $\rightarrow$-Inf. $(\neg A \vee B) \rightarrow (A \rightarrow B)$ is intuitionistically valid, so

$$\forall \ulcorner A \urcorner \forall \ulcorner B \urcorner [(\mathtt{T}(\ulcorner \neg A \urcorner) \vee \mathtt{T}(\ulcorner B \urcorner)) \rightarrow \mathtt{T}(\ulcorner A \rightarrow B \urcorner)]$$

is a theorem of $\mathsf{Base}^i_{\mathrm{T}}$; but then so is

$$\forall \ulcorner A \urcorner \forall \ulcorner B \urcorner \big[(\mathtt{T}(\ulcorner A \urcorner) \vee \mathtt{T}(\ulcorner \neg A \urcorner)) \rightarrow \\ [(\mathtt{T}(\ulcorner A \urcorner) \rightarrow \mathtt{T}(\ulcorner B \urcorner)) \rightarrow \mathtt{T}(\ulcorner A \rightarrow B \urcorner)]\big].$$

Thus, $\mathsf{Base}^i_{\mathrm{T}} \vdash$ T-Comp $\rightarrow$ ($\rightarrow$-Inf).

(ii). $\mathsf{Base}^i_{\mathrm{T}} \vdash \neg \mathtt{T}(x) \rightarrow (\mathtt{T}(x) \rightarrow \mathtt{T}(\ulcorner \bot \urcorner))$, so

$$\mathsf{Base}^i_{\mathrm{T}} \vdash (\rightarrow\text{-Inf}) \rightarrow \forall \ulcorner A \urcorner [\neg \mathtt{T}(\ulcorner A \urcorner) \rightarrow \mathtt{T}(\ulcorner A \rightarrow \bot \urcorner)],$$

that is, $\mathsf{Base}^i_{\mathrm{T}} \vdash$ $\rightarrow$-Inf $\rightarrow$ T-Comp(w).

(iii). From $\mathtt{T}(\ulcorner A \urcorner) \vee \neg \mathtt{T}(\ulcorner A \urcorner)$ and $\neg \mathtt{T}(\ulcorner A \urcorner) \rightarrow \mathtt{T}(\ulcorner \neg A \urcorner)$ one immediately obtains $\mathtt{T}(\ulcorner A \urcorner) \vee \mathtt{T}(\ulcorner \neg A \urcorner)$; thus $\mathsf{Base}_{\mathrm{T}} \vdash$ T-Comp(w) $\rightarrow$ T-Comp. This with (ii) above, finishes the case.

(iv). Apply $\vee$-Inf to the axiom of truth classicism in $\mathsf{Base}^c_{\mathrm{T}}$. ∎

It is natural to suppose, however, that $\vee$-Inf, T-Comp(w), $\rightarrow$-Inf and T-Comp are not mutually equivalent when the underlying logic and the logic of the truth predicate is non-classical. Let

$$\mathrm{T}_0 = \{\langle 0, \ulcorner B \urcorner\rangle : \mathsf{HA}_{\mathtt{T}} \vdash B\},\ \ \mathrm{T}_1 = \{\langle 1, \ulcorner B \urcorner\rangle : \mathsf{PA}_{\mathtt{T}} \vdash B\},\ \ \mathrm{T}_2 = \{2\} \times \mathbb{N},$$

and let $\leq$ be the standard ordering on $\mathbb{N}$. Define three intuitionistic ω-models as follows.

$$\begin{aligned}\mathfrak{M}_0 =& \langle \{0\}, \leq, \mathrm{T}_0 \rangle,\\ \mathfrak{M}_1 =& \langle \{0,2\}, \leq, \mathrm{T}_0 \cup \mathrm{T}_2 \rangle,\\ \mathfrak{M}_2 =& \langle \{1,2\}, \leq, \mathrm{T}_1 \cup \mathrm{T}_2 \rangle.\end{aligned}$$

$\mathfrak{M}_0$ is a classical model and it is not hard to see that these are all models of $\mathsf{Base}^i_{\mathrm{T}}$. $\mathfrak{M}_0 \models \neg$T-Comp(w) $\wedge$ $\neg$T-Comp and, since $\mathsf{HA}_{\mathtt{T}}$ has the disjunction property $\mathfrak{M}_0 \models \vee$-Inf.

$\mathfrak{M}_1$ also models $\vee$-Inf for the same reason. Since $2 \Vdash_{\mathfrak{M}_1} \mathtt{T}(\bar{n})$ for every $n \in \mathbb{N}$, we see $\mathfrak{M}_1 \models$ T-Comp(w) vacuously. It is clear, though, that $0 \nVdash_{\mathfrak{M}_1} \forall \ulcorner A \urcorner (\mathtt{T}(\ulcorner A \urcorner) \vee \mathtt{T}(\ulcorner \neg A \urcorner))$ so $\mathfrak{M}_1$ is not a model of T-Comp. Similarly $\mathfrak{M}_2 \models$ T-Comp(w) and $\mathfrak{M}_2 \not\models$ T-Comp, but this time $\mathfrak{M}_2 \not\models \vee$-Inf as, for example, if B is the formalised consistency statement for PA, $1 \Vdash \mathtt{T}(\ulcorner B \vee \neg B \urcorner)$, but clearly $1 \nVdash \mathtt{T}(\ulcorner B \urcorner) \vee \mathtt{T}(\ulcorner \neg B \urcorner)$.

This method has successfully furnished us with models that show the implications

$$\begin{aligned}\text{T-Comp(w)} &\rightarrow \text{T-Comp},\\ \vee\text{-Inf} &\rightarrow \text{T-Comp},\\ \text{T-Comp(w)} &\rightarrow \vee\text{-Inf},\\ \vee\text{-Inf} &\rightarrow \text{T-Comp(w)},\end{aligned}$$

are not theorems of $\mathsf{Base}^i_{\mathrm{T}}$. They do not show, however, that the principles T-Comp(w) and $\neg\vee$-Inf, or T-Comp(w) and $\neg$T-Comp are mutually consistent (for example $0 \Vdash_{\mathfrak{M}_1} \neg$T-Comp whereas $2 \Vdash_{\mathfrak{M}_1}$ T-Comp, so $\mathfrak{M}_1 \not\models \neg$T-Comp). Another criticism of the models is that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ satisfy T-Comp(w) vacuously, that is neither satisfies $\neg\mathtt{T}(\bar{n})$ for any $n \in \mathbb{N}$. The next proposition addresses these short-comings, showing there are intuitionistic models which accept one of T-Comp(w), $\vee$-Inf or $\rightarrow$-Inf, while refuting T-Comp.

Proposition 3.4 *There are intuitionistic ω-models of $\mathsf{Base}^i_{\mathrm{T}}$, $\mathfrak{A}_0$ and $\mathfrak{A}_1$, such that*

$$\begin{aligned}\mathfrak{A}_0 \models&\ \ \vee\text{-Inf} \wedge \neg\text{T-Comp(w)} \wedge \neg(\rightarrow\text{-Inf}) \wedge \neg\text{T-Comp},\\ \mathfrak{A}_1 \models& \neg\vee\text{-Inf} \wedge \ \ \text{T-Comp(w)} \wedge \ \ \ \rightarrow\text{-Inf} \ \ \wedge \neg\text{T-Comp}.\end{aligned}$$

Proof $\mathfrak{M}_0$, as defined above, plays the role of $\mathfrak{A}_0$. Since T-Comp(w) is a consequence of $\to$-Inf over $\mathsf{Base}^i_{\mathrm{T}}$, we deduce $\mathfrak{A}_0 \models \neg(\to\text{-Inf})$. A suitable model $\mathfrak{A}_1$ can be obtained from $\mathfrak{M}_1$ by stratifying its construction. Let $2^{<\omega}$ denote the set of finite binary sequences, let $\leq$ be the relation 'initial segment of' on $2^{<\omega}$, $|\sigma|$ denote the length of the sequence σ and suppose $A_0, A_1, \ldots$ is an enumeration of the $\mathcal{L}_{\mathrm{T}}$ formulae. We will define a theory T_σ for every $\sigma \in 2^{<\omega}$ by induction on the length of σ: $\mathrm{T}_{\langle\rangle} := \mathsf{PA}$ where $\langle\rangle$ denotes the empty sequence, and for $i = 0, 1$,

$$\mathrm{T}_{\sigma^\frown i} := \begin{cases} \{B : \mathrm{T}_\sigma + (\neg)^i A_{|\sigma|} \vdash B\}, & \text{if this set is consistent,} \\ \mathrm{T}_\sigma, & \text{otherwise,} \end{cases}$$

where $(\neg)^i A$ abbreviates A, if $i = 0$, and $\neg A$, if $i = 1$. Note that T_σ forms a consistent classical theory for every σ. For this reason whenever $\mathrm{T}_\sigma + B \vdash C$ and $\mathrm{T}_\sigma + \neg B \vdash C$ it must be the case that $\mathtt{T}_\sigma \vdash C$. Thus,

$$C \in \mathrm{T}_{\sigma^\frown 0} \cap \mathrm{T}_{\sigma^\frown 1} \text{ implies } C \in \mathrm{T}_\sigma. \tag{2}$$

Let $\mathfrak{A}_1 = \langle 2^{<\omega}, \leq, \hat{\mathrm{T}}\rangle$, where $\hat{\mathrm{T}} = \{\langle \sigma, \ulcorner B \urcorner\rangle : \sigma \in 2^{<\omega} \wedge B \in \mathrm{T}_\sigma\}$. Since $\mathrm{T}_\sigma \subseteq \mathrm{T}_\tau$ whenever $\sigma \leq \tau$, it follows that $\mathfrak{A}_1$ is an intuitionistic ω-model. In order to show $\mathfrak{A}_1 \models$ T-Comp(w), i.e.

$$\mathfrak{A}_1 \models \forall \ulcorner B \urcorner (\neg \mathtt{T}(\ulcorner B \urcorner) \to \mathtt{T}(\ulcorner \neg B \urcorner)),$$

it suffices to deduce $\sigma \Vdash \neg\mathtt{T}(\ulcorner A_k \urcorner) \to \mathtt{T}(\ulcorner \neg A_k \urcorner)$ for every $\sigma \in 2^{<\omega}$ and $k \in \mathbb{N}$. Fix some $k \in \mathbb{N}$. We show that $A_k \notin \mathrm{T}_\tau$ for every τ extending σ implies $(\neg A_k) \in \mathrm{T}_\sigma$ for every $k \in \mathbb{N}$ and every $\sigma \in 2^{<\omega}$ by induction on the difference $k - |\sigma|$.
Case I. $k - |\sigma| \leq 0$. Then $|\sigma| \geq k$ and A_k has already been seen in the construction of $\mathrm{T}_{\sigma^\frown 0}$. Therefore, assuming $A_k \notin \mathrm{T}_\tau$ for every τ extending σ, A_k must be inconsistent with T_ρ for some initial segment of ρ of σ (if $k = |\sigma|$ take $\rho = \sigma$). In that case A_k is also inconsistent with T_σ, hence $T_\sigma \vdash \neg A_k$ and so $(\neg A_k) \in \mathrm{T}_\sigma$.
Case II. $k - |\sigma| > 0$. Suppose $A_k \notin \mathrm{T}_\tau$ for every τ extending σ. By the induction hypothesis we obtain $(\neg A_k) \in \mathrm{T}_{\sigma^\frown 0} \cap \mathrm{T}_{\sigma^\frown 1}$ and thus $(\neg A_k) \in \mathrm{T}_\sigma$ by eq. (2).

The argument for T-Comp(w) can be generalised to also deduce $\mathfrak{A}_1 \models$ $\to$-Inf: we show by induction on $k + l - |\sigma|$ that if $A_k \in \mathrm{T}_\tau$ implies $A_l \in \mathrm{T}_\tau$ for every τ extending σ, in fact $(A_k \to A_l) \in \mathrm{T}_\sigma$. This suffices to prove

$$\sigma \Vdash (\mathtt{T}(\ulcorner A_k \urcorner) \to \mathtt{T}(\ulcorner A_l \urcorner)) \to \mathtt{T}(\ulcorner A_k \to A_l \urcorner),$$

and hence $\sigma \Vdash \to$-Inf.
Case I. $k + l - |\sigma| \leq 0$. Therefore $|\sigma| \geq k + l$ and the sentences A_k and A_l have already been encountered in the construction of $\mathrm{T}_{\sigma^\frown 0}$. Suppose

$A_k \in \mathrm{T}_\tau$ implies $A_l \in \mathrm{T}_\tau$ whenever τ extends σ. If $A_k \notin \mathrm{T}_{\sigma\frown 0}$ it follows that $(\neg A_k) \in \mathrm{T}_\sigma$ since $k \leq |\sigma|$, and hence $(A_k \to A_l) \in \mathrm{T}_\sigma$. Otherwise, by the assumption, $A_l \in \mathrm{T}_\sigma$; whence also $(A_k \to A_l) \in \mathrm{T}_\sigma$.
Case II. $k + l - |\sigma| > 0$. Suppose $\tau \Vdash \mathtt{T}(\ulcorner A_k \urcorner) \to \mathtt{T}(\ulcorner A_l \urcorner)$ for every τ extending σ. The induction hypothesis entails $(A_k \to A_l) \in \mathrm{T}_{\sigma\frown 0} \cap \mathrm{T}_{\sigma\frown 1}$, whence $(A_k \to A_l) \in \mathrm{T}_\sigma$ by eq. (2).

Furthermore, as T_σ forms a consistent theory for every $\sigma \in 2^{<\omega}$ and at no point will T_σ be complete, $\mathfrak{A}_1 \models \neg$T-Comp. Explicitly, for each $\sigma \in 2^{<\omega}$ let B_σ be the formalised consistency statement for T_σ (which may be defined since T_σ is a finite extension of PA). B_σ is independent of T_σ, hence $\sigma \nVdash \mathtt{T}(\ulcorner B_\sigma \urcorner) \vee \mathtt{T}(\ulcorner \neg B_\sigma \urcorner)$. Thus $\tau \Vdash \neg\forall\ulcorner A \urcorner(\mathtt{T}(\ulcorner A \urcorner) \vee \mathtt{T}(\ulcorner \neg A \urcorner))$ for every $\tau \in 2^{<\omega}$.

Finally, each T_σ is classical, so $\mathfrak{A}_1 \models \mathsf{Base}^c_{\mathrm{T}}$, whence proposition 3.3 implies also $\mathfrak{A}_1 \models \neg\vee$-Inf. In sum,

$$\mathfrak{A}_1 \models \neg\vee\text{-Inf} \wedge \text{T-Comp(w)} \wedge \to\text{-Inf} \wedge \neg\text{T-Comp}. \qquad \blacksquare$$

4 The main theorem

We are now in a position to state the main theorem of this paper, the proof of which constitutes sections 5 to 7.

Theorem 4.1 *The following are the only maximal consistent subsets of the Optional Axioms, over* $\mathsf{Base}^i_{\mathrm{T}}$.

- A^i. *T-In, T-Intro, T-Rep, T-Del, T-Comp, ¬T-Elim, ∀-Inf, ∃-Inf, ∨-Inf, T-Comp(w), →-Inf.*
- B^i. *T-Rep, T-Cons, T-Comp, ∀-Inf, ∃-Inf, T-Comp(w), ∨-Inf, →-Inf.*
- C^i. *T-Del, T-Cons, T-Comp, ∀-Inf, ∃-Inf, T-Comp(w), ∨-Inf, →-Inf.*
- D^i. *T-Intro, T-Elim, T-Cons, T-Comp, ¬T-Intro, ¬T-Elim, ∀-Inf, ∃-Inf, T-Comp(w), ∨-Inf, →-Inf.*
- E^i. *T-Intro, T-Elim, T-Del, T-Cons, ¬T-Intro, ∀-Inf, ∃-Inf, ∨-Inf.*
- F^i. *T-Intro, T-Elim, T-Del, ¬T-Elim, ∀-Inf, ∃-Inf, T-Comp(w), ∨-Inf.*
- G^i. *T-In, T-Intro, T-Elim, T-Rep, ¬T-Elim, ∀-Inf, ∃-Inf, T-Comp(w), ∨-Inf, →-Inf.*
- H^i. *T-Out, T-Rep, T-Elim, ¬T-Intro, T-Del, T-Cons, ∀-Inf, ∃-Inf, ∨-Inf.*
- I^i. *T-Rep, T-Del, T-Elim, ¬T-Elim, ∀-Inf, ∃-Inf, T-Comp(w), ∨-Inf.*

If we ignore the additional axioms presented in table 2 for the time being and examine the effect altering the base theory has on the consistent subsets of the Optional Axioms considered by Friedman and Sheard, one might expect some of the inconsistencies between axioms or rules of inference break down; i.e., the maximal consistent sets grow as we weaken the background logic and even spawn new maximal consistent sets of the Optional Axioms. As it happens, perhaps surprisingly, this does not appear to be the case; most of the inconsistencies are derivable without the use of classical principles. It was observed in [3] that ∃-Inf implies T-Comp over $\mathsf{Base}_{\mathrm{T}}$ and any set consistent with T-Comp is consistent with ∃-Inf. On the other hand with an intuitionistic base theory the dependency dissolves and ∃-Inf becomes consistent with all subsets of the Optional Axioms, just as ∀-Inf is in the classical setting. The only other change is in the consistency of T-In with G. T-In and T-Elim are classically inconsistent, but intuitionistically consistent. We will discuss why this is the case in section 5 where we present a model of G^i.

If we now consider T-Comp(w), ∨-Inf and →-Inf the result appears to offer no significant difference. Like ∃-Inf, ∨-Inf becomes consistent with all subsets of the Optional Axioms. T-Comp(w) and →-Inf on the other hand behave more in line with T-Comp: T-Comp(w) is consistent with *all* subsets of the Optional Axioms excluding those that contain T-Cons and are inconsistent with T-Comp; →-Inf is consistent with those subsets consistent with T-Comp(w) but not containing both T-Del and T-Elim.

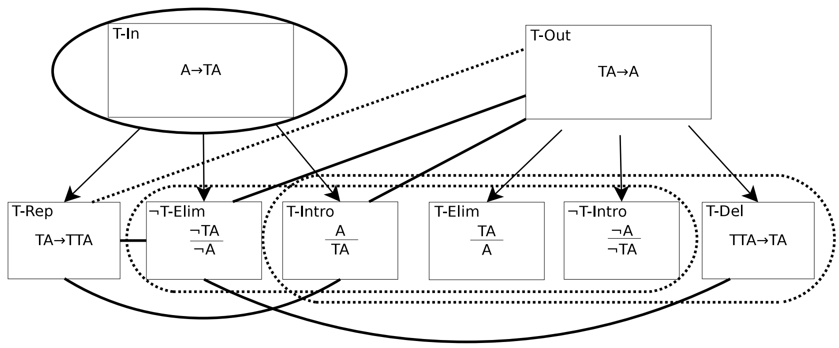

Figure 1: Consistencies and inconsistencies over $\mathsf{Base}^i_{\mathrm{T}}$ + T-Cons.

Figures 1 and 2 show the consistency or inconsistency of subsets of the Optional Axioms over the theories $\mathsf{Base}^i_{\mathrm{T}}$ + T-Cons and $\mathsf{Base}^i_{\mathrm{T}}$ + T-Comp respectively. The arrows represent logical implication; for example closure under T-Intro or inclusion of T-Rep follow from assuming T-In. A thick black line connecting or surrounding principles denotes these are inconsistent over the respective base theory and a dashed line represents the connecting principles are consistent. These two charts are in fact identical to charts 2 and 3 presented in [3] for the classical theories $\mathsf{Base}_{\mathrm{T}}$ + T-Cons

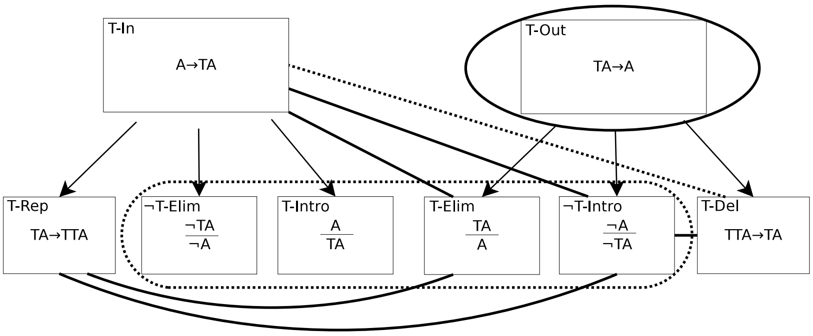

Figure 2: Consistencies and inconsistencies over $\mathsf{Base}^i_{\mathrm{T}}$ + T-Comp.

and $\mathsf{Base}_{\mathrm{T}}$ + T-Comp, and are complete in the sense that the consistency or inconsistency over $\mathsf{Base}^i_{\mathrm{T}}$ + T-Cons (respectively $\mathsf{Base}^i_{\mathrm{T}}$ + T-Comp) of any subset of the Optional Axioms may be determined from these figures; consistency naturally inherits downwards and inconsistency upwards. $\forall$-Inf, $\exists$-Inf, $\vee$-Inf, $\rightarrow$-Inf and T-Comp(w) need not be included in the figures for the reasons descried above. Thus we observe that $\mathsf{Base}_{\mathrm{T}}$ may safely replace $\mathsf{Base}^i_{\mathrm{T}}$ in these charts and they would remain correct and complete (but with different conditions on the treatment of $\exists$-Inf).

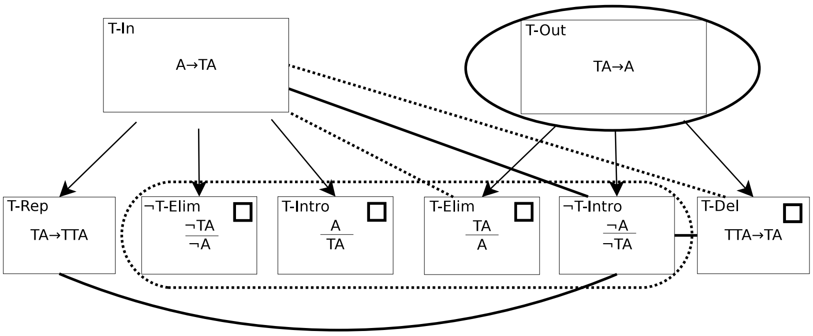

Figure 3: Consistencies and inconsistencies over $\mathsf{Base}^i_{\mathrm{T}}$ + T-Comp(w).

Friedman and Sheard [3] observe a near-perfect symmetry between the two charts with the only exception of the lower-right consistency in fig. 1 being broken by the inconsistency of T-Rep and T-Elim in fig. 2. One may reasonably question whether replacing T-Comp with the weaker T-Comp(w), or perhaps $\rightarrow$-Inf, alters the structure of this figure and if so whether symmetry with figure 1 is gained. This new scenario is presented as figure 3 which is again a complete chart. Compared with fig. 2, the T-Elim, T-Del inconsistency has been replaced by the consistency of $\neg$T-Elim, T-Intro, T-Elim and T-Del, marked by the presence of the four additional boxes in the chart, and the T-Rep, T-Elim inconsistency has been broken by the new consistency line between T-In and T-Elim. Perfect symmetry with fig. 1 is still not obtained, due to the inconsistency lines between T-Out and T-Intro, and between T-Rep and T-Intro in fig. 1. Therefore lack of symmetry between T-Cons and T-Comp(w) over Base_T was not due to the presence of classical logic. Similarly, complete symmetry would not be obtained if the base theory is adjusted to $\mathsf{Base}^c_\mathrm{T}$; since $\mathsf{Base}^c_\mathrm{T}$+T-Elim extends Base_T, T-Rep and T-Elim are inconsistent over $\mathsf{Base}^c_\mathrm{T}$ + T-Comp(w), while T-Del and T-Intro are consistent over $\mathsf{Base}^c_\mathrm{T}$ + T-Cons.

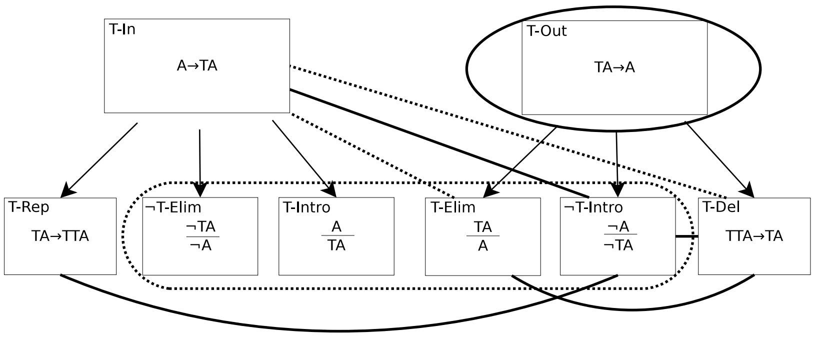

Figure 4: Consistencies and inconsistencies over $\mathsf{Base}^i_\mathrm{T}$ + $\to$-Inf.

One does appear closer to symmetry if one replaces T-Comp(w) by $\to$-Inf, as shown in fig. 4. The T-Elim, T-Del inconsistency present in fig. 2 is regained, corresponding to the inconsistency of T-Rep, T-Intro over $\mathsf{Base}^i_\mathrm{T}$ + T-Cons. However, T-In and T-Elim are still consistent over $\mathsf{Base}^i_\mathrm{T}$ + $\to$-Inf, whereas T-Out and T-Intro are inconsistent over $\mathsf{Base}^i_\mathrm{T}$ + T-Cons (in fact they are inconsistent over $\mathsf{Base}^i_\mathrm{T}$). The duality of truth and falsity might explain the near symmetry of figs. 1 and 4, but the natural principle dual to $\to$-Inf would appear to be the axiom T-Imp, and although there are connections between T-Cons and T-Imp (see, for example, [6, §4.1.2]), the informal meaning behind $\to$-Inf feels a far cry from the intuition behind T-Imp or T-Cons. Perhaps the closeness of figs. 1 and 4 is more to do with some peculiarity inherent in the base theory, rather than a similarity between concepts.

Now consider fig. 5, in which consistencies and inconsistencies over $\mathsf{Base}^i_\mathrm{T}$+ T-Cons + T-Comp(w) are marked. All pairings across consistency lines are contradictory making this a complete chart. It is also identical to the classical case (over Base_T+T-Cons+T-Comp as in [3, Chart 4]) and so we observe that a subset of the Optional Axioms is consistent over $\mathsf{Base}^i_\mathrm{T}$ + T-Cons + T-Comp(w) if and only if it is consistent over Base_T + T-Cons + T-Comp.

5 Consistencies

In this section we will establish the consistency of each of the nine sets of Optional Axioms listed in theorem 4.1. As T-Comp(w), $\vee$-Inf and $\to$-Inf

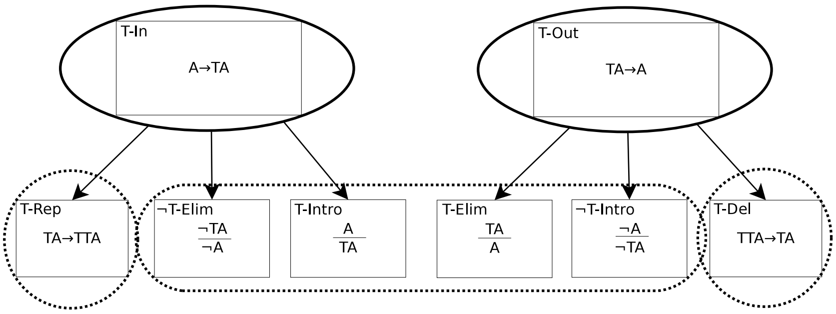

Figure 5: Consistencies and inconsistencies over $\mathsf{Base}^i_{\mathrm{T}}$ + T-Cons + T-Comp(w).

are all consequences of T-Comp, the theories A^i, B^i, C^i and D^i are each a subset of their classical counterpart, and so consistent by theorem 3.2. For completeness, though, we shall present their model constructions as found in [3].

A^i. Let $\mathfrak{A}$ be the classical *everything is true* model $\langle \mathbb{N}, \mathbb{N} \rangle$. Then $\mathfrak{A} \models \mathsf{A}^i$ and as in the classical setting, this is, essentially, the only model of A^i.

B^i, C^i. Let $\mathfrak{A}_0$ be the classical *it is true that everything is true* model $\langle \mathbb{N}, \{\ulcorner B \urcorner : \langle \mathbb{N}, \mathbb{N} \rangle \models B\} \rangle$ and $\mathfrak{A}_1 = \langle \mathbb{N}, \{\ulcorner B \urcorner : \langle \mathbb{N}, \emptyset \rangle \models B\} \rangle$, *it is true that everything is false*. $\mathfrak{A}_0 \models \mathsf{B}^i$ and $\mathfrak{A}_1 \models \mathsf{C}^i$.

D^i. Define a sequence of classical models

$$\begin{aligned} \mathfrak{A}_0 &= \langle \mathbb{N}, \emptyset \rangle, \\ \mathfrak{A}_{n+1} &= \langle \mathbb{N}, \{\ulcorner B \urcorner : \mathfrak{A}_n \models B\} \rangle. \end{aligned}$$

Let $\mathsf{Th}_\infty = \{B : \exists n \forall k > n\, \mathfrak{A}_k \models B\}$. Then Th_∞ is a consistent theory containing D^i and closed under T-Intro, ¬T-Intro, T-Elim and ¬T-Elim.

Each of the remaining theories contain ∨-Inf and ∃-Inf but not T-Comp, so we will necessarily need non-classical interpretations of the truth predicates. Moreover, the presence of T-Elim, coupled with either T-Rep or T-Del means the interpretation of the truth predicate shifts from notions of satisfaction in certain classical ω-models to notions of provability in intuitionistic ω-logic (*cf.* [3, §3]). In order to then establish the consistency of ∃-Inf and ∨-Inf one needs to show these theories of ω-logic have the disjunction and existence property. This can be achieved by replacing the model constructions in [3] by slash constructions.

If the truth predicate is interpreted as provability, the presence of ∀-Inf ensures this is provability in ω-logic. Hence we make substantial use of derivations in intuitionistic ω-logic; writing $\mathsf{S} \vdash_\omega A$ denotes that A is derivable from the axioms and rules of S, which is usually an intuitionistic theory, with the inclusion of the ω-rule in place of generalisation: $\mathsf{S} \vdash_\omega A(\bar{n})$ for every n implies $\mathsf{S} \vdash_\omega \forall x A(x)$. By $\mathsf{S} \vdash A$ we denote ordinary (finitistic) provability in intuitionistic logic. The next proposition is a corollary of Troelstra and van Dalen's proof of the disjunction and existence property for HA [8, chap. 3, thm. 5.10].

Proposition 5.1 $\mathsf{HA}_\mathtt{T}$ *has the disjunction and existence property when formulated in ω-logic.*

5.1 Consistency of E^i

Define a sequence of intuitionistic theories of truth as follows.

$$\begin{aligned} \mathsf{Th}_0 &= \mathsf{Base}^i_\mathtt{T} + \text{T-Cons} + \forall\text{-Inf} + \exists\text{-Inf} + \vee\text{-Inf}, \\ \mathsf{Th}_{n+1} &= \mathsf{Th}_0 + \text{T-Del} + \{\mathtt{T}(\ulcorner A \urcorner) : A \text{ is an } \mathcal{L}_\mathtt{T}\text{-sentence and } \mathsf{Th}_n \vdash A\}. \end{aligned}$$

Provided each Th_n is consistent and $\mathsf{Th}_{n+1} \vdash \mathtt{T}(\ulcorner A \urcorner)$ only if $\mathsf{Th}_n \vdash A$, the theory $\bigcup_n \mathsf{Th}_n$ is a consistent theory, containing E^i. Each Th_n is a finitary theory so, by the presence of $\forall$-Inf in Th_{n+1}, there are sentences such that $\mathsf{Th}_{n+1} \vdash \mathtt{T}(\ulcorner A \urcorner)$, but $\mathsf{Th}_n \nvdash A$. We prove

$$\mathsf{Th}_{n+1} \vdash_\omega \mathtt{T}(\ulcorner A \urcorner) \text{ if and only if } \mathsf{Th}_n \vdash_\omega A.$$

The right-to-left implication holds by definition. In order for the left-to-right direction to hold, the axioms $\exists$-Inf and $\vee$-Inf of Th_{n+1} necessitate that the disjunction and existence property hold for Th_n. The next definition introduces the machinery required to establish this.

Definition 5.2 *Define a slash relation $|_n$ for every n as follows.*

1. $|_n R(s_1, \ldots, s_n)$ *iff* $R(s_1, \ldots, s_n)$ *is true, where R is an n-ary primitive recursive relation.*
2. $|_0 \mathtt{T}(s)$ *iff* $s^\mathbb{N} = \ulcorner A \urcorner$ *for some sentence A with* $\mathsf{HA}_\mathtt{T} \vdash_\omega A$.
3. $|_{n+1} \mathtt{T}(s)$ *iff* $s^\mathbb{N} = \ulcorner A \urcorner$ *for some sentence A with* $\mathsf{Th}_n \vdash_\omega A$.
4. $|_n (A \wedge B)$ *iff* $|_n A$ *and* $|_n B$.
5. $|_n (A \vee B)$ *iff* $|_n A$ *or* $|_n B$.
6. $|_n (A \to B)$ *iff* $|_n A$ *implies* $|_n B$ *and* $\mathsf{Th}_n \vdash_\omega A \to B$.
7. $|_n \forall x F(x)$ *iff* $|_n F(\bar{m})$ *for every m.*
8. $|_n \exists x F(x)$ *iff* $|_n F(\bar{m})$ *holds for some m.*

The slash relation allows the freedom to argue semantically, while still holding on to the important information from the theories of interest. This is made clear by the following propositions.

Proposition 5.3 *Let A be some $\mathcal{L}_{\mathtt{T}}$-sentence. Then*

(i). $\mathsf{HA}_{\mathtt{T}} \vdash_\omega A$ *implies* $\mathsf{Th}_0 \vdash_\omega A \wedge \mathtt{T}(\ulcorner A \urcorner)$,

(ii). $\mathsf{Th}_n \vdash_\omega A$ *implies* $\mathsf{Th}_{n+1} \vdash_\omega A \wedge \mathtt{T}(\ulcorner A \urcorner)$.

Proof (i). Argue by induction on the (transfinite) length of the deduction $\mathsf{HA}_{\mathtt{T}} \vdash_\omega A$. If A is an axiom of $\mathsf{HA}_{\mathtt{T}}$, $\mathsf{Base}^i_{\mathtt{T}} \vdash A$ and $\mathsf{Base}^i_{\mathtt{T}} + \forall$-Inf $\vdash \mathtt{T}(\ulcorner A \urcorner)$ hold immediately. Applications of *modus ponens* in $\mathsf{HA}_{\mathtt{T}}$ are dealt with by T-Imp in Th_0. If $\mathsf{Th}_0 \vdash_\omega \mathtt{T}(\ulcorner B(\bar{m}) \urcorner)$ for every $m \in \mathbb{N}$, the ω-rule entails $\mathsf{Th}_0 \vdash_\omega \forall x\, \mathtt{T}(\ulcorner B(\dot{x}) \urcorner)$, from which $\forall$-Inf implies $\mathsf{Th}_0 \vdash_\omega \mathtt{T}(\ulcorner \forall x\, B(x) \urcorner)$. Therefore if A was derived via the ω-rule we may easily deduce $\mathsf{Th}_0 \vdash_\omega A \wedge \mathtt{T}(\ulcorner A \urcorner)$ from the induction hypothesis.

(ii). Argue by induction on n with a subsidiary induction on the length of the deduction $\mathsf{Th}_n \vdash A$. The main induction hypothesis implies every axiom of Th_{n-1} is an axiom of Th_n and by the above argument (with Th_n in place of $\mathsf{HA}_{\mathtt{T}}$ and Th_{n+1} in place of Th_0) one easily obtains

$$\mathsf{Th}_n \vdash_\omega A \text{ implies } \mathsf{Th}_{n+1} \vdash_\omega A \wedge \mathtt{T}(\ulcorner A \urcorner). \qquad \blacksquare$$

Proposition 5.4 $|_n A$ *holds whenever A is a sentence and* $\mathsf{Base}^i_{\mathtt{T}} \vdash_\omega A$.

Proof By induction on the derivation of $\mathsf{Base}^i_{\mathtt{T}} \vdash_\omega A$. Suppose A is an axiom of $\mathsf{Base}^i_{\mathtt{T}}$. If A is also an axiom of $\mathsf{HA}_{\mathtt{T}}$ but not an instance of the induction schema, $|_n A$ naturally holds. As one can verify $|_n B(\bar{0}) \wedge \forall x(B(x) \to B(x+1))$ implies $|_n B(\bar{m})$ for each m, and so $|_n \forall x\, B(x)$, we also obtain $|_n A$ whenever A is an instance of the induction schema in $\mathsf{HA}_{\mathtt{T}}$. This leaves only the three axioms of truth present in $\mathsf{Base}^i_{\mathtt{T}}$ to consider.

Each theory Th_n is closed under *modus ponens*, thus $|_n \mathtt{T}(\ulcorner A \urcorner)$ and $|_n \mathtt{T}(\ulcorner A \to B \urcorner)$ implies $|_n \mathtt{T}(\ulcorner B \urcorner)$, so $|_n \forall x \forall y (\mathtt{T}(x) \wedge \mathtt{T}(x \dot{\to} y) \to \mathtt{T}(y))$ is easily obtained.

For the second axiom we observe $|_n \mathtt{val}^i(\bar{m})$ holds if and only if m is the code of an intuitionistically valid first-order sentence of $\mathcal{L}_{\mathtt{T}}$, and hence $|_n \mathtt{val}^i(\ulcorner B \urcorner)$ implies $|_n \mathtt{T}(\ulcorner B \urcorner)$. As before, this leads us to conclude $|_n \forall x(\mathtt{val}^i(x) \to \mathtt{T}(u\dot{c}l(x)))$.

Finally, $|_n \mathtt{Ax}_{\mathsf{PRA}}(\bar{m})$ holds if and only if m is the code of a non-logical axiom of PRA; whence we deduce $|_n \forall x(\mathtt{Ax}_{\mathsf{PRA}}(x) \to \mathtt{T}(x))$.

For the induction step A is derived by the ω-rule or *modus ponens*. In either case we may conclude $|_n A$ by the induction hypothesis and the definition of $|_n$. $\blacksquare$

Lemma 5.5 $|_n A$ *implies* $\mathsf{Th}_n \vdash_\omega A$.

Proof By induction on the complexity of A. Suppose A is atomic. If A is arithmetical, $\mathsf{Th}_n \vdash A$, and if $A = \mathtt{T}(\ulcorner B \urcorner)$ either $n = 0$ and $\mathsf{HA}_{\mathtt{T}} \vdash_\omega B$ or $n > 0$ and $\mathsf{Th}_{n-1} \vdash_\omega B$. In either case proposition 5.3 yields $\mathsf{Th}_n \vdash_\omega A$. If A is not atomic the result follows via the induction hypothesis. ■

Theorem 5.6 *The following hold for every n.*

(i). $\mathsf{Th}_n \vdash_\omega A$ *implies* $|_n A$,

(ii). Th_n *is a consistent theory in ω-logic,*

(iii). $\mathsf{Th}_n \vdash_\omega A \vee B$ *implies either* $\mathsf{Th}_n \vdash_\omega A$ *or* $\mathsf{Th}_n \vdash_\omega B$,

(iv). $\mathsf{Th}_n \vdash_\omega \exists x A(x)$ *implies* $\mathsf{Th}_n \vdash_\omega A(t)$ *for some term t,*

(v). $\mathsf{Th}_n \vdash_\omega \mathtt{T}(s)$ *implies there is a sentence A such that* $s^{\mathbb{N}} = \ulcorner A \urcorner$ *and* $\mathsf{Th}_n \vdash_\omega A$.

Proof We prove (i)–(v) simultaneously by *main* induction on n with a *subsidiary* induction on the length of the derivation in Th_n.

We begin with (i) and provide the argument for all n simultaneously. To ease notation it will be convenient to denote $\mathsf{HA}_{\mathtt{T}}$ by Th_{-1}. Suppose $\mathsf{Th}_n \vdash_\omega A$.

Case I. *A is an axiom of* Th_n. This case splits into a number of sub-cases depending on A. Proposition 5.4 deals with the axioms of $\mathsf{Base}^i_{\mathtt{T}}$, and if A is $\mathtt{T}(\ulcorner B \urcorner)$ for some B such that $\mathsf{Th}_{n-1} \vdash B$, $|_n A$ holds by definition.

$\vee$-Inf. The aim is to show $|_n \vee$-Inf. By the main induction hypothesis for (iii) or proposition 5.1 (in the case $n = 0$) we know $|_n \mathtt{T}(\bar{p} \dot{\vee} \bar{q})$ implies $|_n \mathtt{T}(\bar{p}) \vee \mathtt{T}(\bar{q})$ for all p, q. Moreover, $\mathsf{Th}_n \vdash \mathtt{T}(\bar{p} \dot{\vee} \bar{q}) \rightarrow \mathtt{T}(\bar{p}) \vee \mathtt{T}(\bar{q})$ for every p and q, whence

$$|_n \mathtt{T}(\bar{p} \dot{\vee} \bar{q}) \rightarrow \mathtt{T}(\bar{p}) \vee \mathtt{T}(\bar{q}),$$

and thus $|_n \forall x \forall y (\mathtt{T}(x \dot{\vee} y) \rightarrow \mathtt{T}(x) \vee \mathtt{T}(y))$.

$\exists$-Inf. The induction hypothesis for (iv) and proposition 5.1 imply, for a formula $A(x)$ with at most x free, $|_n \mathtt{T}(\ulcorner A(\bar{m}) \urcorner)$ holds for some m whenever $|_n \mathtt{T}(\ulcorner \exists x A(x) \urcorner)$. Thus one obtains

$$|_n \mathtt{T}(\ulcorner \exists x A(x) \urcorner) \rightarrow \exists m \mathtt{T}(\ulcorner A(\dot{m}) \urcorner),$$

and hence $|_n \exists$-Inf.

$\forall$-Inf. By the ω-rule, $|_n \forall x \mathtt{T}(\ulcorner A(\dot{x}) \urcorner)$ implies $|_n \mathtt{T}(\ulcorner \forall x A(x) \urcorner)$. Since $\forall$-Inf is an axiom of Th_n,

$$|_n (\forall x \mathtt{T}(\ulcorner A(\dot{x}) \urcorner) \rightarrow \mathtt{T}(\ulcorner \forall x A(x) \urcorner))$$

holds for every formula A with at most x free, so $|_n \forall$-Inf.

T-Cons. The definition of $|_n$ entails $|_n$T-Cons holds only if $|_n \mathtt{T}(\ulcorner A \urcorner) \wedge \mathtt{T}(\ulcorner \neg A \urcorner)$ fails and $\mathsf{Th}_n \vdash_\omega \mathtt{T}(\ulcorner A \urcorner) \wedge \mathtt{T}(\ulcorner \neg A \urcorner) \to \bot$ holds for every sentence A. The latter obviously holds since Th_n contains T-Cons. To show the former observe that $|_n \mathtt{T}(\ulcorner A \urcorner) \wedge \mathtt{T}(\ulcorner \neg A \urcorner)$ fails if one of $\mathsf{Th}_{n-1} \vdash_\omega A$ or $\mathsf{Th}_{n-1} \vdash_\omega \neg A$ fails, which, of course, must be the case since Th_{n-1} is consistent in ω-logic by the induction hypothesis for (ii).

T-Del. T-Del is an axiom of Th_n only if $n > 0$. Suppose $|_n \mathtt{T}(\ulcorner \mathtt{T}(\bar{k}) \urcorner)$. The definition of $|_n$ implies $\mathsf{Th}_{n-1} \vdash_\omega \mathtt{T}(\bar{k})$. Since $n > 0$, the induction hypothesis for (v) may be applied, yielding a sentence B with $\ulcorner B \urcorner = k$ and $\mathsf{Th}_{n-1} \vdash_\omega B$; whence $|_n \mathtt{T}(\bar{k})$. Thus, $|_n \mathtt{T}(\ulcorner \mathtt{T}(\bar{k}) \urcorner)$ implies $|_n \mathtt{T}(\bar{k})$ and so $|_n$T-Del.

Case II. *A is not an axiom of* Th_n. $\mathsf{Th}_n \vdash_\omega A$ therefore follows by *modus ponens* or the ω-rule. In the case of *modus ponens* the induction hypothesis implies $|_n B$ and $|_n (B \to A)$ for some B. By the definition of the slash relation this means $|_n A$ holds. The other case follows similarly.

Combining (i) and lemma 5.5 one obtains

$$\mathsf{Th}_n \vdash_\omega A \text{ if and only if } |_n A,$$

from which (ii), (iii) and (iv) are immediate consequences. To see (v) suppose $\mathsf{Th}_n \vdash_\omega \mathtt{T}(\ulcorner A \urcorner)$. (i) implies $|_n \mathtt{T}(\ulcorner A \urcorner)$, and so $\mathsf{Th}_{n-1} \vdash_\omega A$. By proposition 5.3 we conclude $\mathsf{Th}_n \vdash_\omega A$. ∎

The next corollary is now an immediate consequence of theorem 5.6.

Corollary 5.7 $\mathsf{Th}_{n+1} \vdash_\omega \mathtt{T}(\ulcorner A \urcorner)$ *if and only if* $\mathsf{Th}_n \vdash_\omega A$.

Let Th_∞ be the (finitary) theory given by $\mathsf{Th}_\infty \vdash A$ if $\mathsf{Th}_n \vdash_\omega A$ for some n. Th_∞ can be axiomatised by $\mathsf{Base}^i_{\mathrm{T}}$, T-Cons, $\forall$-Inf, $\exists$-Inf, $\vee$-Inf, T-Del plus $\{\mathtt{T}(\ulcorner A \urcorner) : \exists n\, \mathsf{Th}_n \vdash A\}$. Corollary 5.7 implies Th_∞ is closed under T-Elim and T-Intro. As we observed earlier, closure under $\neg$T-Intro is a consequence of T-Intro and T-Cons, so Th_∞ contains E^i. Therefore, if $\mathsf{E}^i \vdash A$ there exists an n such that $|_n A$ holds, so E^i is consistent.

5.2 Consistency of F^i

The similarity between F^i and E^i inspires us to define a sequence of intuitionistic theories

$$\begin{aligned} \mathsf{Th}'_0 &= \mathsf{Base}^i_{\mathrm{T}} + \text{T-Comp(w)} + \forall\text{-Inf} + \exists\text{-Inf} + \vee\text{-Inf}, \\ \mathsf{Th}'_{n+1} &= \mathsf{Th}'_0 + \text{T-Del} + \{\mathtt{T}(\ulcorner A \urcorner) : A \text{ is an } \mathcal{L}_{\mathrm{T}}\text{-sentence and } \mathsf{Th}'_n \vdash A\}, \end{aligned}$$

and attempt to prove

$$\mathsf{Th}'_{n+1} \vdash_\omega \mathtt{T}(\ulcorner A \urcorner) \text{ if and only if } \mathsf{Th}'_n \vdash_\omega A$$

for every sentence A. In order to establish this we must first show that Th'_n has the disjunction and the existence property. We can use the same slash relation $|_n$ as before adjusted to refer to Th'_n; explicitly, $|_n$ is defined as in definition 5.2 with clauses 3 and 6 replaced by

3. $|_{n+1}\mathtt{T}(s)$ *iff* $s^{\mathbb{N}} = \ulcorner A \urcorner$ *for some sentence* A *and* $\mathsf{Th}'_n \vdash_\omega A$*;*

6. $|_n(A \to B)$ *iff* $|_n A$ *implies* $|_n B$ *and* $\mathsf{Th}'_n \vdash_\omega A \to B$*.*

Proposition 5.8 *For every* $n \in \mathbb{N}$*,* Th'_n *is a consistent theory. Moreover, the classical* $\mathcal{L}_\mathtt{T}$*-structure* $\langle \mathbb{N}, \mathbb{N} \rangle$ *is a model of* Th'_n*.*

Proof Each Th'_n is a sub-theory of A^i, which is modelled by $\langle \mathbb{N}, \mathbb{N} \rangle$. ∎

The following two lemmata can be proved using the same arguments as the previous section. Again we let $\mathsf{Th}'_{-1} = \mathsf{HA}_\mathtt{T}$.

Lemma 5.9 *For every* $n \geq 0$*,* $\mathsf{Th}'_{n-1} \vdash_\omega A$ *implies* $\mathsf{Th}'_n \vdash_\omega A \wedge \mathtt{T}(\ulcorner A \urcorner)$*.*

Lemma 5.10 $|_n A$ *implies* $\mathsf{Th}'_n \vdash_\omega A$*.*

Theorem 5.11 *The following hold for every* n*,*

(i). $\mathsf{Th}'_n \vdash_\omega A$ *implies* $|_n A$*,*

(ii). Th'_n *has the disjunction and existence property,*

(iii). $\mathsf{Th}'_n \vdash_\omega \mathtt{T}(\ulcorner A \urcorner)$ *implies* $\mathsf{Th}'_n \vdash_\omega A$*.*

Proof In contrast to E^i, here one looks to establish $|_n$T-Comp(w) in place of $|_n$T-Cons.[1] To see that T-Comp(w) is slashed note that, by the definition of $|_n$, $|_n \neg\mathtt{T}(\ulcorner B \urcorner)$ entails $\mathsf{Th}'_n \vdash_\omega \neg\mathtt{T}(\ulcorner B \urcorner)$, but the latter is ruled out by proposition 5.8. Thus $|_n$T-Comp(w) holds vacuously.

(ii) is now a consequence of (i) and lemma 5.10; while (iii) is a consequence of (i) and lemma 5.9. ∎

Let Th'_∞ be the theory extending $\mathsf{Base}^i_\mathtt{T}$ by T-Del, T-Comp(w), ∀-Inf, ∃-Inf, ∨-Inf and $\{\mathtt{T}(\ulcorner A \urcorner) : \exists n\, \mathsf{Th}'_n \vdash A\}$. Th'_∞ is consistent and closed under T-Intro and T-Elim. It is also closed under ¬T-Elim vacuously, since $\langle \mathbb{N}, \mathbb{N} \rangle \models \mathsf{Th}'_\infty$ and so $\mathsf{Th}'_\infty \vdash \neg\mathtt{T}(\ulcorner A \urcorner)$ never occurs. Therefore F^i is a sub-theory of Th'_∞ and so is consistent.

[1] Although $|_n\mathtt{T}(\ulcorner A \urcorner) \wedge \mathtt{T}(\ulcorner \neg A \urcorner)$ fails for every sentence A, one cannot infer $|_n$T-Cons unless, in fact, $\mathsf{Th}'_n \vdash_\omega$ T-Cons. If $\mathsf{Th}'_n \vdash_\omega$ T-Cons, $\bigcup_n \mathsf{Th}'_n$ will be a theory containing T-Cons, ∀-Inf and closed under T-Intro; McGee [7] shows, however, that any such theory is ω-inconsistent, contradicting proposition 5.8.

5.3 Consistency of G^i

We would like to first explore the connection between T-In and T-Elim as this marks a significant change from the classical setting. The classical inconsistency between T-In and T-Elim arises when analysing the consequences of the liar sentence, $B \leftrightarrow \neg\mathtt{T}(\ulcorner B \urcorner)$ (that is, $B \leftrightarrow (\mathtt{T}(\ulcorner B \urcorner) \to \bot)$). T-In implies $B \to \mathtt{T}(\ulcorner B \urcorner)$, so $B \to \bot$ and so $\neg\neg\mathtt{T}(\ulcorner B \urcorner)$. Arguing classically, one may remove the double negation to obtain $\mathtt{T}(\ulcorner B \urcorner)$ and thus derive B by T-Elim, contradicting $\neg B$ from earlier. If one were arguing within intuitionistic logic though, there would be no means to pass from $\neg\neg\mathtt{T}(\ulcorner B \urcorner)$ to $\mathtt{T}(\ulcorner B \urcorner)$, so the contradiction cannot be achieved. However,

$$\mathsf{Base}^i_{\mathtt{T}} \vdash \mathtt{T}(\ulcorner \neg B \urcorner) \to (\mathtt{T}(\ulcorner B \urcorner) \to \mathtt{T}(\ulcorner A \urcorner))$$

for any sentence A of $\mathcal{L}_{\mathtt{T}}$, since $\neg B \to (B \to A)$ is intuitionistically valid, and $\mathsf{Base}^i_{\mathtt{T}} + \text{T-In} \vdash \mathtt{T}(\ulcorner \neg B \urcorner)$. Therefore,

$$\mathsf{Base}^i_{\mathtt{T}} + \text{T-In} \vdash \mathtt{T}(\ulcorner B \urcorner) \to \mathtt{T}(\ulcorner A \urcorner),$$

whence $\mathsf{Base}^i_{\mathtt{T}} + \text{T-In} \vdash \neg\neg\mathtt{T}(\ulcorner B \urcorner) \to \neg\neg\mathtt{T}(\ulcorner A \urcorner)$, and so $\mathsf{Base}^i_{\mathtt{T}} + \text{T-In} \vdash \neg\neg\mathtt{T}(\ulcorner A \urcorner)$, for any sentence A. Thus, for G^i one has the peculiar scenario in which

$$\mathsf{G}^i \vdash \forall \ulcorner A \urcorner \neg\neg\mathtt{T}(\ulcorner A \urcorner), \text{ but} \tag{3}$$

$$\mathsf{G}^i \vdash \mathtt{T}(\ulcorner A \urcorner) \text{ if and only if } \mathsf{G}^i \vdash A. \tag{4}$$

Our first attempt to manage eqs. (3) and (4) will see us mimic the techniques of the preceding sections to obtain a sequence S_i (for $i \in \mathbb{N}$) of theories each containing T-In. Defining a suitable slash relation will provide an elegant means to show each S_i is consistent, has the disjunction and existence property and

$$\mathsf{S}_{i+1} \vdash_\omega \mathtt{T}(\ulcorner A \urcorner) \text{ if and only if } \mathsf{S}_i \vdash_\omega A. \tag{5}$$

Moreover, eq. (5) and the presence of T-In in S_i ensures $\mathsf{S}_{i+1} \subseteq \mathsf{S}_i$, whence we will obtain $\bigcap_n \mathsf{S}_n$, a consistent theory containing T-In, $\vee$-Inf, $\exists$-Inf, $\forall$-Inf, T-Comp(w) and closed under T-Elim, T-Intro and $\neg$T-Elim. Although this method does not incorporate the axiom $\to$-Inf it will provide the motivation for the second approach which does.

Define for each $n \in \mathbb{N}$ an intuitionistic theory S_n by

$$\begin{aligned}
\mathsf{S}_0 = \mathsf{Base}^i_{\mathtt{T}} &+ \text{T-In} + \text{T-Comp(w)} + \vee\text{-Inf} + \exists\text{-Inf} + \forall\text{-Inf} + \\
&+ \{\mathtt{T}(\ulcorner A \urcorner) : A \text{ is an } \mathcal{L}_{\mathtt{T}}\text{-sentence}\},\\
\mathsf{S}_{n+1} = \mathsf{Base}^i_{\mathtt{T}} &+ \text{T-In} + \text{T-Comp(w)} + \vee\text{-Inf} + \exists\text{-Inf} + \forall\text{-Inf} + \\
&+ \{\mathtt{T}(\ulcorner A \urcorner) : A \text{ is an } \mathcal{L}_{\mathtt{T}}\text{-sentence and } \mathsf{S}_n \vdash A\}.
\end{aligned}$$

Let $\widetilde{\mathsf{S}}$ denote the theory whose axioms are given by

$$\text{Ax}(\widetilde{\mathsf{S}}) = \{A : A \text{ is a sentence of } \mathcal{L}_{\mathtt{T}} \text{ and } \forall n\, \mathsf{S}_n \vdash_\omega A\}.$$

The set $\text{Ax}(\widetilde{\mathsf{S}})$ is already deductively closed, that is, if $\widetilde{\mathsf{S}} \vdash A$ and A is an $\mathcal{L}_{\mathtt{T}}$-sentence, $\forall n\, \mathsf{S}_n \vdash_\omega A$ and so $A \in \text{Ax}(\widetilde{\mathsf{S}})$. We begin with the following observations.

Lemma 5.12 *For every $n \in \mathbb{N}$, $\mathsf{S}_n \vdash_\omega A$ implies $\mathsf{S}_{n+1} \vdash_\omega \mathtt{T}(\ulcorner A \urcorner)$.*

Proof We proceed by transfinite induction on the length of the deduction $\mathsf{S}_n \vdash_\omega A$. If no applications of the ω-rule were utilised, $\mathsf{S}_n \vdash A$ and so $\mathtt{T}(\ulcorner A \urcorner)$ is an axiom of S_{n+1}. Otherwise a mixture of the induction hypothesis, ω-rule in S_{n+1} and $\forall$-Inf imply the result. ■

Lemma 5.13 *For each n, $\mathsf{S}_{n+1} \subseteq \mathsf{S}_n$.*

Proof It suffices to show each axiom of S_{n+1} of the form $\mathtt{T}(\ulcorner A \urcorner)$ is derivable in S_n. But if $\mathtt{T}(\ulcorner A \urcorner)$ is such an axiom, $\mathsf{S}_n \vdash A$ by definition and T-In entails $\mathsf{S}_n \vdash \mathtt{T}(\ulcorner A \urcorner)$. ■

Definition 5.14 *For each n define a slash relation $||_n$ as follows.*

1. $||_n R(s_1, \ldots, s_k)$ iff $R(s_1, \ldots, s_k)$ is true, where R is an k-ary primitive recursive relation.
2. $||_0 \mathtt{T}(s)$ iff $s^{\mathbb{N}} = \ulcorner A \urcorner$ for some sentence A.
3. $||_{n+1} \mathtt{T}(s)$ iff $s^{\mathbb{N}} = \ulcorner A \urcorner$ for some sentence A and $\mathsf{S}_n \vdash_\omega A$.
4. $||_n (A \wedge B)$ iff $||_n A$ and $||_n B$.
5. $||_n (A \vee B)$ iff $||_n A$ or $||_n B$.
6. $||_n (A \to B)$ iff $||_n A$ implies $||_n B$ and $\mathsf{S}_n \vdash_\omega A \to B$.
7. $||_n \forall x F(x)$ iff $||_n F(\bar{m})$ for every m.
8. $||_n \exists x F(x)$ iff $||_n F(\bar{m})$ holds for some m.

The significant difference between $||_n$ and $|_n$ as given in definition 5.2 is the behaviour of the base case, $n = 0$. In a similar manner to before we may then deduce the following.

Lemma 5.15 *For every $n \in \mathbb{N}$, $||_n A$ implies $\mathsf{S}_n \vdash_\omega A$.*

Proposition 5.16 *For every $n \in \mathbb{N}$,*

(i). $\mathsf{S}_n \vdash_\omega A$ *implies* $||_n A$,

(ii). S_n *has the disjunction and existence property (in ω-logic).*

Proof By induction on n. (i) consists of a further induction on the length of the derivation.
Case I. $n = 0$. By definition $||_0\mathtt{T}(\ulcorner A \urcorner)$ holds for all sentences A, thus one easily verifies each of $\vee$-Inf, $\exists$-Inf and $\forall$-Inf are slashed. All instances of T-In are also slashed. T-Comp(w) is vacuously slashed, since $||_0\neg\mathtt{T}(\ulcorner A \urcorner)$ fails for every A.
Case II. $n = m+1$. We assess each axiom in turn; those of the form $\mathtt{T}(\ulcorner B \urcorner)$ are slashed by definition.

T-In. Suppose $||_n A$ for some A. Lemma 5.15 implies $\mathsf{S}_n \vdash_\omega A$ and by lemma 5.13 we obtain $\mathsf{S}_m \vdash_\omega A$; whence $||_n\mathtt{T}(\ulcorner A \urcorner)$.

T-Comp(w). $||_n\neg\mathtt{T}(\ulcorner A \urcorner)$ entails $\mathsf{S}_n \vdash_\omega \neg\mathtt{T}(\ulcorner A \urcorner)$ and hence $\mathsf{S}_0 \vdash_\omega \neg\mathtt{T}(\ulcorner A \urcorner)$ by lemma 5.13, which contradicts the consistency of S_0 implied by the induction hypothesis for $n = 0$. Therefore $||_n$T-Comp(w) holds vacuously.

$\vee$-Inf. Suppose $||_n\mathtt{T}(\ulcorner A\vee B \urcorner)$. Then $\mathsf{S}_m \vdash_\omega A\vee B$ and hence, by the induction hypothesis for (ii), either $\mathsf{S}_m \vdash_\omega A$ or $\mathsf{S}_m \vdash_\omega B$, so $||_n\mathtt{T}(\ulcorner A \urcorner) \vee \mathtt{T}(\ulcorner B \urcorner)$.

$\exists$-Inf. Suppose $||_n\mathtt{T}(\ulcorner \exists x\, A(x) \urcorner)$. Then $\mathsf{S}_m \vdash_\omega \exists x A(x)$ and hence, by the induction hypothesis, there is a $k \in \mathbb{N}$ for which $\mathsf{S}_m \vdash_\omega A(\bar{k})$, so $||_n\exists x\, \mathtt{T}(\ulcorner A(\dot{x}) \urcorner)$.

$\forall$-Inf. Suppose $||_n\forall x\, \mathtt{T}(\ulcorner A(\dot{x}) \urcorner)$. Then $||_n\mathtt{T}(\ulcorner A(\bar{k})) \urcorner)$ for every $k \in \mathbb{N}$ and so $\mathsf{S}_m \vdash_\omega A(\bar{k})$ for every k. Thus $\mathsf{S}_m \vdash_\omega \forall x A(x)$ and $||_n\mathtt{T}(\ulcorner \forall x A(x) \urcorner)$.

In the induction step we argue according to the last logical rule applied. All cases are, however, standard and identical to the proof of theorem 5.6.
(ii) is an immediate consequence of lemma 5.15 and (i). ■

Proposition 5.16 and lemma 5.12 imply eq. (5) as desired. We may thus conclude $\widetilde{\mathsf{S}}$, and so G^i without $\rightarrow$-Inf, is a consistent theory.

Theorem 5.17

(i). $\widetilde{\mathsf{S}}$ *is consistent,*

(ii). $\widetilde{\mathsf{S}} \vdash$ *T-In* $+$ *T-Rep* $+ \forall$*-Inf* $+ \exists$*-Inf* $+ \vee$*-Inf* $+$ *T-Comp(w),*

(iii). $\widetilde{\mathsf{S}}$ *is closed under T-Intro, T-Elim and $\neg$T-Elim.*

Proof $||_n\bot$ never holds by clause 1, so (i) is a consequence of proposition 5.16 (i). (ii) holds because all the axioms listed belong to the theory S_n for every n.

(iii). Closure under T-Intro is obvious because of the presence of T-In in $\widetilde{\mathsf{S}}$. If $\widetilde{\mathsf{S}} \vdash \mathtt{T}(\ulcorner A \urcorner)$, the previous proposition implies $||_n \mathtt{T}(\ulcorner A \urcorner)$ holds for every n, thus, by the definition of $||_n$, $\mathsf{S}_n \vdash_\omega A$ for every n, and so $\widetilde{\mathsf{S}} \vdash A$. Therefore $\widetilde{\mathsf{S}}$ is closed under T-Elim. Closure under $\neg$T-Elim is a consequence of T-Comp(w) and T-Elim. ∎

Corollary 5.18 *The theory* G^i *without* $\rightarrow$*-Inf is consistent.*

Proof G^i without $\rightarrow$-Inf is a sub-theory of $\widetilde{\mathsf{S}}$, and hence consistent by theorem 5.17. ∎

Had we attempted to incorporate $\rightarrow$-Inf into the development of $\widetilde{\mathsf{S}}$, we would have required the truth predicate to be interpreted by notions closer to satisfaction and validity than provability. Assuming $\rightarrow$-Inf is an axiom of S_{n+1}, $||_{n+1}(\rightarrow\text{-Inf})$ holds if and only if, for all sentences A, B,

$$\mathsf{S}_{n+1} \vdash_\omega \mathtt{T}(\ulcorner A \urcorner) \rightarrow \mathtt{T}(\ulcorner B \urcorner) \text{ implies } \mathsf{S}_n \vdash_\omega A \rightarrow B.$$

The solution will be to replace the interpretation of truth at each step by validity in a certain Kripke structure $\mathfrak{A}_n$. One naturally requires, amongst other things, the following criteria to be satisfied.

- $\mathfrak{A}_n \models A \vee B$ implies $\mathfrak{A}_n \models A$ or $\mathfrak{A}_n \models B$;
- $\mathfrak{A}_n \models \exists x A(x)$ implies $\mathfrak{A}_n \models A(t)$ for some term t;
- if $\mathfrak{A}_n \models A$ implies $\mathfrak{A}_n \models B$, in fact $\mathfrak{A}_n \models A \rightarrow B$.

Such criteria are often associated with classical models, but as theorem 5.20 below shows, there are non-classical $\mathcal{L}_{\mathrm{T}}$-structures which satisfy them. Let $\preceq$ be the reverse ordering on the natural numbers and define $\mathrm{T}_0 = \{0\} \times \mathbb{N}$ and $\mathfrak{A}_0 = \langle \{0\}, \preceq, \mathrm{T}_0 \rangle$. $\mathfrak{A}_0$ is the '*everything is true*' model used to verify A^i. Assuming $\mathfrak{A}_n$ and T_n are already defined, let

$$\begin{aligned} \mathrm{T}_{n+1} &= \{\langle n+1, \ulcorner A \urcorner \rangle : \mathfrak{A}_n \models A\} \cup \mathrm{T}_n, \\ \mathfrak{A}_{n+1} &= \langle \{k : k \leq n+1\}, \preceq, \mathrm{T}_{n+1} \rangle. \end{aligned}$$

Let $\mathrm{T} = \bigcup_n \mathrm{T}_n$. It should be clear that the set T can safely replace T_n in the definition of $\mathfrak{A}_n$. We claim the following.

a) $\mathfrak{A}_n$ is an intuitionistic Kripke ω-model for every n.

b) The theory $\mathsf{Th}^{\mathsf{G}}_\infty := \{B : \forall n\, \mathfrak{A}_n \models B\}$ is a consistent theory containing G^i.

To deduce a) it is sufficient to show the persistency condition holds for $\mathfrak{A}_n$. However, for every $m \leq n$,

$$m \Vdash_{\mathfrak{A}_n} A \text{ iff } m \Vdash_{\mathfrak{A}_k} A \text{ for every } k \geq m.$$

Thus, $\mathfrak{A}_n \models A$ entails $\mathfrak{A}_m \models A$ for every $m \leq n$, whence

$$\{x : \langle n+1, x\rangle \in \mathrm{T}\} \subseteq \{x : \langle n, x\rangle \in \mathrm{T}\}$$

for every n, as required.

$\mathsf{Th}^{\mathsf{G}}_{\infty}$ is closed under *modus ponens* and contains $\mathsf{Base}^{i}_{\mathtt{T}}$, so $\mathsf{Th}^{\mathsf{G}}_{\infty}$ forms a theory (in fact an infinitary theory since it is also closed under the ω-rule). Moreover, $\mathfrak{A}_0 \models \mathsf{Th}^{\mathsf{G}}_{\infty}$, so $\mathsf{Th}^{\mathsf{G}}_{\infty}$ is consistent. The next proposition and subsequent remarks show each axiom of G^i is contained in $\mathsf{Th}^{\mathsf{G}}_{\infty}$, while theorem 5.20 demonstrates $\mathsf{Th}^{\mathsf{G}}_{\infty}$ is closed under T-Intro, T-Elim and $\neg$T-Elim, whence we conclude $\mathsf{Th}^{\mathsf{G}}_{\infty}$ extends G^i.

We write $m \Vdash A$ to denote $m \Vdash_{\mathfrak{A}_m} A$. By persistency, $n \Vdash A$ implies $m \Vdash_{\mathfrak{A}_n} A$ for every $m \leq n$, so

$$\mathfrak{A}_n \models A \text{ iff } n \Vdash A. \tag{6}$$

Also, for any sentence A of $\mathcal{L}_{\mathtt{T}}$, $n+1 \Vdash \mathtt{T}(\ulcorner A \urcorner)$ if and only if $n \Vdash A$.

Proposition 5.19 *For each $m \in \mathbb{N}$,*

(i). $m \Vdash$ T-In,

(ii). $m \Vdash$ T-Comp(w),

(iii). $m \Vdash$ $\vee$-Inf,

(iv). $m \Vdash$ $\exists$-Inf,

(v). $m \Vdash$ $\forall$-Inf,

(vi). $m \Vdash$ $\rightarrow$-Inf.

Proof By induction on m.
Case I. $m = 0$. $\mathfrak{A}_0 \models \mathtt{T}(\bar{n})$ for every n, so (i)–(vi) hold trivially.
Case II. $m = n+1$. (i). As the induction hypothesis yields

$$k \Vdash A \text{ implies } k \Vdash \mathtt{T}(\ulcorner A \urcorner) \text{ for every } k < m, \tag{7}$$

it suffices to show $m \Vdash A$ implies $m \Vdash \mathtt{T}(\ulcorner A \urcorner)$, so suppose $m \Vdash A$. By persistency $n \Vdash A$, and hence $m \Vdash \mathtt{T}(\ulcorner A \urcorner)$, as desired. Thus $k \Vdash A$ entails $k \Vdash \mathtt{T}(\ulcorner A \urcorner)$ for every $k \leq m$, so $m \Vdash A \rightarrow \mathtt{T}(\ulcorner A \urcorner)$.

(ii). We need to establish $m \Vdash \neg\mathtt{T}(\ulcorner A \urcorner)$ implies $m \Vdash \mathtt{T}(\ulcorner \neg A \urcorner)$ for every sentence A. However, $m \Vdash \neg\mathtt{T}(\ulcorner A \urcorner)$ implies $0 \Vdash \neg\mathtt{T}(\ulcorner A \urcorner)$, contradicting the definition of $\mathfrak{A}_0$. Thus $m \Vdash$ T-Comp(w) vacuously.

(iii). Suppose $m \Vdash \mathtt{T}(\ulcorner A_0 \vee A_1 \urcorner)$. Then $n \Vdash A_0 \vee A_1$ by definition and so either $n \Vdash A_0$ or $n \Vdash A_1$. In either case $m \Vdash \mathtt{T}(\ulcorner A_0 \urcorner) \vee \mathtt{T}(\ulcorner A_1 \urcorner)$, and we may conclude $m \Vdash \vee$-Inf through the induction hypothesis.

(iv). If $m \Vdash \mathtt{T}(\ulcorner \exists x A(x) \urcorner)$ we observe $n \Vdash A(\bar{k})$ for some $k \in \mathbb{N}$, whence $m \Vdash \mathtt{T}(\ulcorner A(\bar{k}) \urcorner)$ and so $m \Vdash \exists x\, \mathtt{T}(\ulcorner A(\dot{x}) \urcorner)$. By the induction hypothesis we obtain $m \Vdash \exists$-Inf.

(v). Since $n \Vdash A(\bar{k})$ for every $k \in \mathbb{N}$ implies $n \Vdash \forall x\, A(x)$, the induction hypothesis entails $m \Vdash \forall$-Inf.

(vi). Suppose $m \Vdash \mathtt{T}(\ulcorner A \urcorner) \to \mathtt{T}(\ulcorner B \urcorner)$. Then $k \Vdash \mathtt{T}(\ulcorner A \urcorner)$ implies $k \Vdash \mathtt{T}(\ulcorner B \urcorner)$ for every $k \leq m$, and so $k \Vdash A$ implies $k \Vdash B$ for every $k \leq n$ by definition. Hence $n \Vdash A \to B$, so $m \Vdash \mathtt{T}(\ulcorner A \to B \urcorner)$ and we may conclude $m \Vdash \to$-Inf. ■

Combining proposition 5.19 with eq. (6) we obtain

$$\mathfrak{A}_n \models \text{T-In} + \text{T-Comp(w)} + \vee\text{-Inf} + \exists\text{-Inf} + \forall\text{-Inf} + \to\text{-Inf}.$$

On the other hand, $\mathfrak{A}_n \models \mathsf{Base}^i_{\mathrm{T}}$, so $\mathfrak{A}_n \models \mathsf{S}_n$.

Theorem 5.20 G^i *is a consistent theory.*

Proof We show $\mathsf{G}^i \vdash A$ implies

$$\mathfrak{A}_n \models A \text{ for every } n \in \mathbb{N}. \tag{8}$$

The preceding remarks verify this for the axioms of G^i and if A was derived via a logical rule, eq. (8) follows from the induction hypothesis. Moreover, applications of T-Intro in G^i are trivialised by T-In. Suppose $\mathsf{G}^i \vdash A$ was a result of T-Elim. Then $\mathsf{G}^i \vdash \mathtt{T}(\ulcorner A \urcorner)$ and, by the induction hypothesis, $\mathfrak{A}_n \models \mathtt{T}(\ulcorner A \urcorner)$ for every $n \in \mathbb{N}$. So $n+1 \Vdash \mathtt{T}(\ulcorner A \urcorner)$, $n \Vdash A$ and hence $\mathfrak{A}_n \models A$ for every $n \in \mathbb{N}$ as required. There is nothing to check for $\neg$T-Elim since if $\mathsf{G}^i \vdash \neg\mathtt{T}(\ulcorner A \urcorner)$, the induction hypothesis yields $\mathfrak{A}_0 \models \neg\mathtt{T}(\ulcorner A \urcorner)$, contradicting the choice of $\mathfrak{A}_0$. ■

5.4 Consistency of H^i

Let $\widehat{\mathsf{Th}}$ be $\mathsf{Base}^i_{\mathrm{T}}$ + T-Intro and define $\mathfrak{M}$ to be the classical model

$$\langle \mathbb{N}, \{\ulcorner B \urcorner : B \text{ is an } \mathcal{L}_{\mathrm{T}}\text{-sentence and } \widehat{\mathsf{Th}} \vdash_\omega B\} \rangle.$$

We claim $\mathfrak{M} \models \mathsf{H}^i$. For this to hold we require $\widehat{\mathsf{Th}}$ to:

a) have the disjunction and existence property (so $\mathfrak{M} \models \vee\text{-Inf} \wedge \exists\text{-Inf}$);

b) be consistent under ω-logic (so $\mathfrak{M} \models$ T-Cons);

c) be closed under T-Intro (so $\mathfrak{M} \models$ T-Rep);

d) be modelled by $\mathfrak{M}$ (so $\mathfrak{M} \models$ T-Out).

c) holds by definition, and b) is a consequence of d). However, $\mathfrak{M} \models \mathsf{Base}^i_{\mathrm{T}}$, so

$$\widehat{\mathsf{Th}} \vdash_\omega A \text{ implies } \mathfrak{M} \models A,$$

whence d) holds. This leaves a), which we also need to hold when $\widehat{\mathsf{Th}}$ is formulated with ω-rule (not simply as a finite theory) so as to also accommodate $\forall$-Inf. We introduce a further slash relation $\widehat{|}$ which is defined as $|_0$ given in definition 5.2 but with clauses 2 and 6 replaced by

2. $\widehat{|}\,\mathtt{T}(s)$ *iff* $s^{\mathbb{N}} = \ulcorner A \urcorner$ *for some sentence* A *and* $\widehat{\mathsf{Th}} \vdash_\omega A$.

6. $\widehat{|}(A \to B)$ *iff* $\widehat{|}\,A$ *implies* $\widehat{|}\,B$ *and* $\widehat{\mathsf{Th}} \vdash_\omega A \to B$.

Lemma 5.21 $\widehat{|}\,A$ *implies* $\widehat{\mathsf{Th}} \vdash_\omega A$.

Proof If A is $\mathtt{T}(s)$, $\widehat{|}\,A$ implies $\widehat{\mathsf{Th}} \vdash_\omega B$ where $\ulcorner B \urcorner = s^{\mathbb{N}}$, whence $\widehat{\mathsf{Th}} \vdash_\omega A$ by T-Intro. The remaining cases are easily verified. ■

Lemma 5.22 $\widehat{\mathsf{Th}} \vdash_\omega A$ *implies* $\widehat{|}\,A$*, and hence* $\widehat{\mathsf{Th}}$ *formulated in* ω*-logic has the disjunction and existence property.*

Proof The first part is shown by induction on the length of the derivation as in theorem 5.6. By the previous lemma this yields $\widehat{|}\,A$ iff $\widehat{\mathsf{Th}} \vdash_\omega A$, from which it is clear $\widehat{\mathsf{Th}}$ has the disjunction and existence property. ■

Combining lemmata 5.21 and 5.22 we obtain the following.

Proposition 5.23 $\mathfrak{M}$ *is a model for the theory extending* $\mathsf{Base}^i_{\mathrm{T}}$ *by T-Out, T-Rep, T-Del,* $\forall$*-Inf,* $\exists$*-Inf,* $\vee$*-Inf and T-Cons.*

Proof We treat each axiom in turn.

T-Rep. Since $\widehat{\mathsf{Th}}$ is closed under T-Intro, $\mathfrak{M} \models$ T-Rep.

$\forall$-Inf. As $\widehat{\mathsf{Th}}$ is formulated in ω-logic, we have

$$\begin{aligned}\mathfrak{M} \models \forall n\, \mathtt{T}(\ulcorner A(\dot{n}) \urcorner) &\Rightarrow \widehat{\mathsf{Th}} \vdash_\omega A(\bar{n}) \text{ for every } n,\\ &\Rightarrow \widehat{\mathsf{Th}} \vdash_\omega \forall x A(x),\\ &\Rightarrow \mathfrak{M} \models \mathtt{T}(\ulcorner \forall x A(x) \urcorner),\end{aligned}$$

and hence $\mathfrak{M} \models \forall$-Inf.

T-Out. $\mathfrak{M} \models \widehat{\mathsf{Th}}$, so

$$\begin{aligned}\mathfrak{M} \models \mathtt{T}(\ulcorner A \urcorner) &\Rightarrow \widehat{\mathsf{Th}} \vdash_\omega A,\\ &\Rightarrow \mathfrak{M} \models A.\end{aligned}$$

T-Del. A consequence of T-Out, above.

T-Cons. Since $\mathfrak{M} \models \widehat{\mathsf{Th}}$, $\widehat{\mathsf{Th}}$ is consistent, and so $\mathfrak{M} \models$ T-Cons.

$\vee$-Inf. Follows from lemma 5.22:

$$\begin{aligned}\mathfrak{M} \models \mathtt{T}(\ulcorner A \vee B \urcorner) &\Rightarrow \widehat{\mathsf{Th}} \vdash_\omega A \vee B,\\ &\Rightarrow \widehat{\mathsf{Th}} \vdash_\omega A \text{ or } \widehat{\mathsf{Th}} \vdash_\omega B,\\ &\Rightarrow \mathfrak{M} \models \mathtt{T}(\ulcorner A \urcorner) \vee \mathtt{T}(\ulcorner B \urcorner),\end{aligned}$$

and hence $\mathfrak{M} \models \vee$-Inf.

$\exists$-Inf. Also follows from lemma 5.22:

$$\begin{aligned}\mathfrak{M} \models \mathtt{T}(\ulcorner \exists x A(x) \urcorner) &\Rightarrow \widehat{\mathsf{Th}} \vdash_\omega \exists x A(x),\\ &\Rightarrow \widehat{\mathsf{Th}} \vdash_\omega A(s), \text{ for some closed term } s,\\ &\Rightarrow \mathfrak{M} \models \mathtt{T}(\ulcorner A(s) \urcorner),\\ &\Rightarrow \mathfrak{M} \models \exists x \mathtt{T}(\ulcorner A(\dot{x}) \urcorner),\end{aligned}$$

and so $\mathfrak{M} \models \exists$-Inf. ■

Theorem 5.24 H^i *is consistent.*

Proof We prove $\mathsf{H}^i \vdash A$ implies $\mathfrak{M} \models A$ by induction on the length of the deduction. In view of points a) to d), only applications of $\neg$T-Intro in H^i need to be considered, so suppose $\mathsf{H}^i \vdash \neg A$ and $\mathfrak{M} \models \neg A$. If $\mathfrak{M} \models \mathtt{T}(\ulcorner A \urcorner)$, $\widehat{\mathsf{Th}} \vdash_\omega A$ and hence $\mathfrak{M} \models A$; thus $\mathfrak{M} \models \mathtt{T}(\ulcorner A \urcorner)$ fails. But $\mathfrak{M}$ is a classical model, so $\mathfrak{M} \models \neg\mathtt{T}(\ulcorner A \urcorner)$. ■

5.5 Consistency of I^i

We will construct a model for I^i based on $\mathfrak{M}$ above. I^i contains T-Comp(w) and $\vee$-Inf, but is inconsistent with T-Comp, thus a model for I^i must necessarily be non-classical as opposed to just having a non-classical interpretation for the truth predicate as was the case with H^i. We will deal with T-Comp(w) in a similar manner to G^i by ensuring no world satisfies a sentence of the form $\neg\mathtt{T}(\ulcorner A \urcorner)$. Before this, however, we consider the sub-theory of I^i without the axiom T-Comp(w). Define $\widetilde{\mathsf{Th}}$ to be the theory extending $\mathsf{Base}^i_{\mathsf{T}}$ by T-Rep, T-Del, $\forall$-Inf, $\vee$-Inf, $\exists$-Inf and the sentence $\mathtt{T}(\ulcorner A \urcorner)$ whenever $\widehat{\mathsf{Th}} \vdash_\omega A$.

Let $\mathfrak{M}$ be the classical structure $\langle \mathbb{N}, \{\ulcorner B \urcorner : \widehat{\mathsf{Th}} \vdash_\omega B\}\rangle$ introduced in the preceding section. Proposition 5.23 demonstrates that $\mathfrak{M}$ is a model of $\widetilde{\mathsf{Th}}$, whence we can deduce $\widetilde{\mathsf{Th}}$ is closed under T-Elim.

Lemma 5.25 *$\widetilde{\mathsf{Th}} \vdash_\omega B$ implies $\mathfrak{M} \models B$, and $\widetilde{\mathsf{Th}} \vdash_\omega \mathtt{T}(\ulcorner A \urcorner)$ implies $\widetilde{\mathsf{Th}} \vdash_\omega A$.*

Proof The first part is an immediate consequence of proposition 5.23, whence

$$\begin{aligned}\widetilde{\mathsf{Th}} \vdash_\omega \mathtt{T}(\ulcorner A \urcorner) &\Rightarrow \mathfrak{M} \models \mathtt{T}(\ulcorner A \urcorner),\\ &\Rightarrow \widehat{\mathsf{Th}} \vdash_\omega A,\\ &\Rightarrow \widetilde{\mathsf{Th}} \vdash_\omega A.\end{aligned}$$

The final implication follows on account of $\widehat{\mathsf{Th}}$ being a sub-theory of $\widetilde{\mathsf{Th}}$. ■

As the classical structure $\langle \mathbb{N}, \mathbb{N} \rangle$ also forms a model of $\widetilde{\mathsf{Th}}$, $\widetilde{\mathsf{Th}}$ is vacuously closed under ¬T-Elim and we may establish the consistency of the sub-theory of I^i without the axiom T-Comp(w).

Corollary 5.26 *The theory I^i formulated without the axiom T-Comp(w), labelled $\widetilde{\mathsf{I}}^i$, is consistent.*

Proof $\widetilde{\mathsf{I}}^i$ is axiomatised by $\mathsf{Base}^i_{\mathrm{T}}$, T-Rep, T-Del, T-Elim, ¬T-Elim, ∀-Inf, ∃-Inf and ∨-Inf. $\widetilde{\mathsf{Th}}$ contains all of these axioms and is closed under T-Elim and ¬T-Elim; thus $\widetilde{\mathsf{I}}^i \vdash B$ implies $\widetilde{\mathsf{Th}} \vdash_\omega B$ for any $\mathcal{L}_{\mathrm{T}}$-sentence B. By lemma 5.25, $\widetilde{\mathsf{Th}}$ is consistent and hence so is $\widetilde{\mathsf{I}}^i$. ■

To conclude that I^i itself is consistent we cannot use lemma 5.25 as $\mathfrak{M} \not\models$ T-Comp(w); for this we must turn to the Kripke model $\widetilde{\mathfrak{A}} = \langle W, \leq, \mathrm{T} \rangle$, defined as the two-world intuitionistic Kripke model given by

$$W = \{0, 1\}, \text{ with } 0 \leq 1,$$
$$\mathrm{T} = \{1\} \times \mathbb{N} \cup \{(0, \ulcorner B \urcorner) : \widehat{\mathsf{Th}} \vdash_\omega B\},$$

where $\widehat{\mathsf{Th}}$ is the theory $\mathsf{Base}^i_{\mathrm{T}}$ + T-Intro used above.

Lemma 5.27 *$\widetilde{\mathsf{Th}} \vdash_\omega A$ implies $\widetilde{\mathfrak{A}} \models A$.*

Proof We need to show $\widetilde{\mathsf{Th}} \vdash_\omega B$ implies $0 \Vdash_{\widetilde{\mathfrak{A}}} B$ and $1 \Vdash_{\widetilde{\mathfrak{A}}} B$ for every sentence B. Since $0 \Vdash B$ if and only if $\langle \mathbb{N}, \mathbb{N} \rangle \models B$, the former is trivial given $\langle \mathbb{N}, \mathbb{N} \rangle \models \widetilde{\mathsf{Th}}$. For the latter, we begin with the axioms of $\widetilde{\mathsf{Th}}$ which we consider in turn. $1 \Vdash B$ holds for every axiom B of $\mathsf{Base}^i_{\mathrm{T}}$ and if $\widehat{\mathsf{Th}} \vdash A$, $1 \Vdash \mathtt{T}(\ulcorner A \urcorner)$ by definition. This leaves the Optional Axioms T-Rep, T-Del, ∀-Inf, ∃-Inf and ∨-Inf to deal with.

T-Rep. $\widehat{\mathsf{Th}}$ is closed under T-Intro, so $1 \Vdash_{\widetilde{\mathfrak{A}}}$ T-Rep.

T-Del.

$$\begin{aligned}1 \Vdash_{\widetilde{\mathfrak{A}}} \mathtt{T}(\ulcorner \mathtt{T}(\ulcorner B \urcorner)\urcorner) &\Rightarrow \widehat{\mathsf{Th}} \vdash_\omega \mathtt{T}(\ulcorner B \urcorner),\\ &\Rightarrow \mathfrak{M} \models \mathtt{T}(\ulcorner B \urcorner),\\ &\Rightarrow \widehat{\mathsf{Th}} \vdash_\omega B,\\ &\Rightarrow 1 \Vdash_{\widetilde{\mathfrak{A}}} \mathtt{T}(\ulcorner B \urcorner),\end{aligned}$$

so $1 \Vdash_{\widetilde{\mathfrak{A}}}$ T-Del.

$\forall$-Inf. Holds since the interpretation of truth at 1 (and also at 0) is closed under ω-logic.

$\vee$-Inf. $1 \Vdash \vee$-Inf because $\widehat{\mathsf{Th}}$ has the disjunction property.

$\exists$-Inf. $1 \Vdash \exists$-Inf since $\widehat{\mathsf{Th}}$ has the existence property.

For the induction step we assume $\widetilde{\mathsf{Th}} \vdash_\omega B$ is derived via *modus ponens*. The induction hypothesis yields $\mathfrak{A} \models A \wedge (A \to B)$ for some sentence A, from which we may readily deduce $\mathfrak{A} \models B$. This is also the case for an application of the ω-rule and any other rule of inference in the derivation.■

Due to the choice of $\widetilde{\mathfrak{A}}$, $0 \Vdash \mathtt{T}(\ulcorner B \urcorner)$ for every sentence B, so $\widetilde{\mathfrak{A}} \models \neg\mathtt{T}(\ulcorner B \urcorner)$ never holds. Thus $\widetilde{\mathfrak{A}} \models$ T-Comp(w) and $\widetilde{\mathfrak{A}}$ is a model of I^i.

Theorem 5.28 $\widetilde{\mathfrak{A}} \models \mathsf{I}^i$*, and so* I^i *is consistent.*

Proof Let $\mathsf{Th}^\#$ denote the theory given by $\widetilde{\mathsf{Th}}$ together with T-Comp(w), that is, $\mathsf{Th}^\#$ denotes the theory

$$\begin{aligned}\mathsf{Base}_{\mathtt{T}} + \text{T-Rep} + \text{T-Del} + \text{T-Comp(w)} + \forall\text{-Inf} +\\ + \exists\text{-Inf} + \vee\text{-Inf} + \{\mathtt{T}(\ulcorner B \urcorner) : \widehat{\mathsf{Th}} \vdash B\}.\end{aligned} \tag{9}$$

We will prove

a) $\mathsf{Th}^\# \vdash_\omega A$ implies $\widetilde{\mathfrak{A}} \models A$; and

b) $\mathsf{Th}^\#$ is closed under T-Elim and $\neg$T-Elim (when formulated in ω-logic), and hence I^i is a sub-theory of $\mathsf{Th}^\#$.

a) is a consequence of the lemma 5.27. For b), closure under $\neg$T-Elim is vacuous, since $\widetilde{\mathfrak{A}} \models \neg\mathtt{T}(s)$ never holds and so $\mathsf{Th}^\# \not\vdash \neg\mathtt{T}(\ulcorner A \urcorner)$ for any sentence A. $\mathsf{Th}^\#$ is closed under T-Elim since $\mathsf{Th}^\# \vdash_\omega \mathtt{T}(\ulcorner A \urcorner)$ implies $\widehat{\mathsf{Th}} \vdash_\omega A$ by the first part and $\widehat{\mathsf{Th}}$ is a sub-theory of $\mathsf{Th}^\#$. ■

6 Inconsistencies

Having shown the consistency of each of the nine theories listed in theorem 4.1, we now turn to the task of showing every subset of the Optional Axioms not contained in one of the theories is inconsistent over $\mathsf{Base}^i_{\mathtt{T}}$. For some subsets the arguments presented in [3] are valid intuitionistically and so no further work is required to deduce their inconsistency. However, many of the derivations do make use of the classical principles inherent in $\mathsf{Base}_{\mathtt{T}}$ and it is not obvious whether or not these can be dispensed with. As we shall see, all but one of the classical inconsistencies has a purely intuitionistic proof. It is important to note that the usual diagonal argument used to construct the liar sentence and its variants may be carried out in purely intuitionistic logic; the argument requires no classical principles. We will abuse notation for the remainder of the section and write $\mathtt{T}(A)$ in place of $\mathtt{T}(\ulcorner A \urcorner)$.

Let B denote the liar sentence; that is, $B \leftrightarrow \neg\mathtt{T}(B)$. Note

$$\mathsf{Base}^i_{\mathtt{T}} \vdash \mathtt{T}(B \leftrightarrow \neg\mathtt{T}(B)),$$

so $\mathsf{Base}^i_{\mathtt{T}} \vdash \mathtt{T}(B) \leftrightarrow \mathtt{T}\neg\mathtt{T}(B)$. However, the sentence $\mathtt{T}(\neg B) \leftrightarrow \mathtt{TT}(B)$, which is a theorem of $\mathsf{Base}_{\mathtt{T}}$, is not derivable in $\mathsf{Base}^i_{\mathtt{T}}$, due to the non-classical nature of the truth predicate.

The subsets of the Optional Axioms for which the argument in [3] suffices to deduce their inconsistency over $\mathsf{Base}^i_{\mathtt{T}}$ are: T-In, T-Out; T-In, ¬T-Intro; T-Out, ¬T-Elim; T-Out, T-Intro; T-Cons, T-Rep, T-Intro. The remaining inconsistencies are presented below with their new proofs according to the order of appearance in [3].

Inconsistencies concerning T-Cons.

T-Cons, T-In:	$B \to \mathtt{T}(B)$ and $B \leftrightarrow \neg\mathtt{T}(B)$, so $\neg B$ and $\neg\neg\mathtt{T}(B)$, but also $\mathtt{T}(\neg B)$, whence $\neg\mathtt{T}(B)$ by T-Cons. $><$
T-Cons, T-Rep, ¬T-Elim:	$\mathtt{T}(B) \to \mathtt{TT}(B)$ and $\mathtt{T}(B) \leftrightarrow \mathtt{T}\neg\mathtt{T}(B)$, so $\neg\mathtt{T}(B)$, and B, but also $\neg B$. $><$
T-Cons, T-Del, ¬T-Elim:	$\mathtt{TT}(B) \to \mathtt{T}(B)$ and $\mathtt{T}(B) \leftrightarrow \mathtt{T}\neg\mathtt{T}(B)$, so $\mathtt{TT}(B) \to \mathtt{T}\neg\mathtt{T}(B)$, $\neg\mathtt{TT}(B)$, $\neg\mathtt{T}(B)$, and B but also $\neg B$. $><$

Inconsistencies concerning T-Comp(w) or T-Comp.

T-Comp(w), T-Out:	$\mathtt{T}(B) \to B$ and $\mathtt{T}(B) \to \neg B$, so $\neg\mathtt{T}(B)$, $\mathtt{T}(\neg B)$, $\neg B$, but also B. $><$
T-Comp(w), T-Rep, $\neg$T-Intro:	$\neg$T-Intro yields T-Cons, so $\neg(\mathtt{TT}(B) \wedge \mathtt{T}\neg\mathtt{T}(B))$. Also $\mathtt{T}(B) \to \mathtt{TT}(B)$ and $\mathtt{T}(B) \leftrightarrow \mathtt{T}\neg\mathtt{T}(B)$, so $\neg\mathtt{T}(B)$, $\neg\mathtt{TT}(B)$, $\mathtt{T}\neg\mathtt{T}(B)$ and so $\mathtt{T}(B)$. $><$
T-Comp(w), T-Del, $\neg$T-Intro:	$\neg\mathtt{T}(B) \to \neg\mathtt{TT}(B)$, $\neg\mathtt{TT}(B) \to \mathtt{T}\neg\mathtt{T}(B)$ and $\mathtt{T}\neg\mathtt{T}(B) \leftrightarrow \mathtt{T}(B)$, so $\neg\mathtt{T}(B) \to \mathtt{T}(B)$, $\neg\neg\mathtt{T}(B)$, $\neg B$, whence also $\neg\mathtt{T}(B)$. $><$
T-Comp, T-Rep, T-Elim:	$\mathtt{T}\neg\mathtt{T}(B) \leftrightarrow \mathtt{T}(B)$, $\mathtt{T}(B) \to \mathtt{TT}(B)$ and $\mathtt{TT}(B) \vee \mathtt{T}\neg\mathtt{T}(B)$, so $\mathtt{TT}(B)$, $\mathtt{T}(B)$, B and $\neg\mathtt{T}(B)$. $><$
T-Comp, T-Del, T-Elim:	$\mathtt{TT}(B) \vee \mathtt{T}\neg\mathtt{T}(B)$, so $\mathtt{T}(B)$, $\neg B$ but also B. $><$

Inconsistencies concerning T-Cons and T-Comp(w).

T-Cons, T-Comp(w), T-Rep, T-Elim:	T-Rep and T-Cons yield $\neg\mathtt{T}(B)$, so $\mathtt{T}(\neg B)$, $\neg B$, but also B. $><$
T-Cons, T-Comp(w), T-Del, T-Elim:	T-Comp(w) and T-Del imply $\neg\neg\mathtt{T}(B)$, so $\neg B$, but also $\mathtt{T}(\neg\neg B)$, whence $\neg\neg B$. $><$
T-Cons, T-Comp(w), T-Rep, T-Del:	T-Rep and T-Cons yield $\neg\mathtt{T}(B)$, while T-Del and T-Comp(w) implies $\neg\neg\mathtt{T}(B)$. $><$
T-Cons, T-Comp(w), T-Del, T-Intro:	$\neg\neg\mathtt{T}(B)$, so $\neg B$, $\mathtt{T}(\neg B)$, $\neg\mathtt{T}(B)$, and B. $><$

T-Comp implies T-Comp(w), so all the inconsistencies listed above involving T-Comp(w) also hold for T-Comp. The above list covers almost all cases and leaves only two subsets to consider: the triple of principles T-Del, T-Elim and $\to$-Inf; and the quadruple T-Intro, T-Rep, T-Del and T-Elim. The first set is inconsistent due to the special behaviour of $\to$-Imp.

Lemma 6.1 *All subsets of the Optional Axioms containing the principles T-Del, $\to$-Inf and T-Elim are inconsistent over* $\mathsf{Base}^i_{\mathtt{T}}$.

Proof T-Del and $\to$-Inf imply $\mathtt{T}(\mathtt{T}(A) \to A)$ for every sentence A, whence T-Elim yields T-Out. On the other hand, $\to$-Inf implies T-Comp(w) which is inconsistent with T-Out as shown above. ∎

To deal with the second subset, Friedman and Sheard use of a form of Löb's Theorem. Their proof, however, makes use of classical principles which are not obviously redundant. The next lemma provides an intuitionistic, and also simpler, proof of the theorem under the same assumptions.

Lemma 6.2 *(Schematic Löb's Theorem) If* S *is a theory extending* $\mathsf{Base}^i_{\mathtt{T}}$*, T-Rep and T-Intro, and* A *is a sentence of* $\mathcal{L}_{\mathtt{T}}$*,*

$$\mathsf{S} \vdash \mathtt{T}(\mathtt{T}(A) \to A) \to \mathtt{T}(A).$$

Proof By the diagonal lemma pick a sentence F such that

$$\mathsf{S} \vdash F \leftrightarrow \mathtt{T}(F \to A).$$

An application of T-Intro yields

$$\mathsf{S} \vdash \mathtt{T}(F \leftrightarrow \mathtt{T}(F \to A)). \tag{10}$$

We now argue informally within S. Assume

$$\mathtt{T}(F \to A).$$

T-Rep implies $\mathtt{T}(\mathtt{T}(F \to A))$, so $\mathtt{T}(F)$ by eq. (10), and so $\mathtt{T}(A)$. Thus,

$$\mathsf{S} \vdash \mathtt{T}(F \to A) \to \mathtt{T}(A), \tag{11}$$

whence a further application of T-Intro entails

$$\mathsf{S} \vdash \mathtt{T}(\mathtt{T}(F \to A) \to \mathtt{T}(A)). \tag{12}$$

Now assume, within S, $\mathtt{T}(\mathtt{T}(A) \to A)$. Then we deduce

$$\begin{aligned} &\mathtt{T}(\mathtt{T}(F \to A) \to A), \text{ by eq. (12)}, \\ &\mathtt{T}(F \to A), \text{ by eq. (10)}, \\ &\mathtt{T}(A), \text{ by eq. (11)}, \end{aligned}$$

that is, $\mathsf{S} \vdash \mathtt{T}(\mathtt{T}(A) \to A) \to \mathtt{T}(A)$. ■

It is worth remarking that had we assumed the axiom T-Rep was given in its quantified form, the above proof may be generalised to deduce, under the same assumptions,

$$\mathsf{S} \vdash \forall \ulcorner A \urcorner (\mathtt{T}(\mathtt{T}(A) \to A) \to \mathtt{T}(A)).$$

In place of the diagonal lemma one makes use of its parametrised form which allows the construction of a formula $F(x)$ such that

$$\mathsf{S} \vdash \forall \ulcorner A \urcorner (F(\ulcorner A \urcorner) \leftrightarrow \mathtt{T}(\ulcorner F(\ulcorner A \urcorner) \to A \urcorner).$$

As the sentence A does not occur outside the scope of the truth predicate in the proof of lemma 6.2, the remainder of the proof may proceed as before. Notice that the parametrised form of T-Intro follows from the non-parametrised form due to the fact that $\mathsf{Base}^i_{\mathtt{T}} \vdash \mathtt{T}(\ulcorner \forall x\, A(x) \urcorner) \to \mathtt{T}(\ulcorner A(\dot{x}) \urcorner)$, and thus this form is available for use in the proof.

The remaining inconsistency is now easily verified.

Lemma 6.3 *All subsets of the Optional Axioms containing the four axioms T-Intro, T-Elim, T-Rep and T-Del are inconsistent over* $\mathsf{Base}^i_{\mathtt{T}}$.

Proof If we assume T-Del we have $\mathtt{TT}(A) \to \mathtt{T}(A)$ for every sentence A. Assuming, further, T-Intro yields $\mathtt{T}(\mathtt{TT}(A) \to \mathtt{T}(A))$ for every A. The previous lemma thus shows the triple T-Del, T-Intro and T-Rep implies $\mathtt{T}(A)$ for every $\mathcal{L}_{\mathtt{T}}$-sentence A and so with the presence of T-Elim one obtains a contradiction. ∎

7 Completing the proof of the main theorem

We can now complete the proof of theorem 4.1. Section 5 shows that each of the nine theories are consistent and section 6 provides the necessary results to see they are maximally so. All that remains is to show these are the only maximal consistent subsets of the Optional Axioms. Let OA^c denote the set of classical Optional Axioms, excluding $\exists$-Inf, and let OA^i be the set of axioms {$\exists$-Inf, $\vee$-Inf, T-Comp(w), $\to$-Inf}. Suppose, in search of a contradiction, that R is a consistent subset of the Optional Axioms (over $\mathsf{Base}^i_{\mathtt{T}}$) which is not a subset of any of the nine theories listed in theorem 4.1. R can be viewed as $\mathrm{S}_1 \cup \mathrm{S}_2$ where $\mathrm{S}_1 \subseteq \mathrm{OA}^c$ and $\mathrm{S}_2 \subseteq \mathrm{OA}^i$. By the combined work of [3, §4] and section 6 we see that S_1 must be a subset of (at least) one of the nine theories A^i to I^i and the only situation where one may obtain a consistent subset of the Optional Axioms which is not included in the list is if S_2 contains $\to$-Inf and S_1 is a subset of one of F^i or I^i, or S_2 contains T-Comp(w) and S_1 is a subset of either E^i or H^i. We thus have two cases to consider based on S_2, each with a further two sub-cases dealing with the choice of S_1.

Case Ia. S_2 *contains* $\to$*-Inf and* $\mathrm{S}_1 \subseteq \mathsf{F}^i$. Since $\to$-Inf logically implies T-Comp(w) we may assume, without loss of generality, that S_2 also contains T-Comp(w). Lemma 6.2 entails S_1 does not contain one of T-Elim or T-Del. Without T-Elim, R is a subset of A^i; and without T-Del, R is contained in D^i, contradicting the assumption.

Case Ib. S_2 *contains* $\to$*-Inf and* $\mathrm{S}_1 \subseteq \mathsf{I}^i$. Again one of T-Del and T-Elim is not contained in S_1 and hence R is contained in either A^i or G^i.

Case IIa. S_2 *contains T-Comp(w) and* $\mathrm{S}_1 \subseteq \mathsf{E}^i$. Unless R is a subset of D^i, S_1 must contain T-Del. Likewise, to avoid F^i (and thus case Ia above), S_1 must contain T-Cons. But then, each of T-Elim, T-Intro and $\neg$T-Intro is inconsistent with R, and R is a subset of C^i.

Case IIb. S_2 *contains T-Comp(w) and* $\mathrm{S}_1 \subseteq \mathsf{H}^i$. T-Out is inconsistent with T-Comp(w), so S_1 does not contain T-Out. We may assume S_1 contains T-Rep as otherwise S_1 is a subset of E^i and by the previous case, we are done. To avoid I^i (and hence case Ib above), S_1 must contain T-Cons or $\neg$T-Intro; either way R is consistent with T-Cons. So R is consistent with T-Rep, T-Comp(w) and T-Cons and to avoid inconsistency we see that S_1

may not contain T-Del, ¬T-Intro or T-Elim. Thus R is a subset of T-Rep, T-Comp(w), T-Cons, ∀-Inf, T-Comp, ∃-Inf, ∨-Inf, →-Inf, and hence is contained in B^i.

8 Conclusion

With intuitionistic logic we obtain more freedom to assert additional natural principles about truth. The principles ∨-Inf and T-Comp(w), for example, are independent over $\mathsf{Base}_{\mathrm{T}}^i$ but equivalent over the fully classical $\mathsf{Base}_{\mathrm{T}}$. Although we still obtain exactly nine maximal consistent sets of the Optional Axioms, more would appear if we allow mixed scenarios, e.g. if the underlying logic of the base theory is classical but the logic of the truth predicate is intuitionistic. For instance, the theory $\widetilde{\mathsf{I}}^i$ of corollary 5.26 may be formulated in classical logic while still maintaining an intuitionistic truth predicate since the models used in the proof of its consistency are all classical.[2] Let us denote this new theory by I^c. Due to the presence of classical logic, T-Comp(w) is inconsistent with I^c and so I^c is not a subset of any of the intuitionistic theories A^i–I^i. I^c, however, maintains the axioms ∨-Inf and ∃-Inf, so nor is it contained in any of Friedman and Sheard's classical theories. Thus I^c is a new maximal consistent theory. Likewise theorem 5.24 shows H^i can also be consistently formulated in classical logic, again with an intuitionistic truth predicate; let us denote this theory by H^c. Note, however, H^c does not represent an additional maximal consistent theory as H^c extends H^i. Furthermore, formulating any theory containing T-Intro in classical logic results in a classical truth predicate and hence in a theory extending $\mathsf{Base}_{\mathrm{T}}$.[3] Thus, adding the law of excluded middle to the collection of Optional Axioms and allowing the user to insist upon a theory based on classical logic yields as possible maximal consistent theories A–G, E^i–G^i, I^i, H^c, and I^c, a total of thirteen theories.

Theorem 8.1 *Let* OA^+ *denote the set containing all fifteen Optional Axioms and the law of excluded middle (the axiom schema '$A \vee \neg A$'). Every consistent subset of* OA^+ *over* $\mathsf{Base}_{\mathrm{T}}^i$ *is contained in one of the following theories.*

- A, B, C, D, E, F, G *(Classical theories with a classical truth predicate);*
- H^c *or* I^c *(Classical theories with an intuitionistic truth predicate);*

[2] In particular, the theory $\widetilde{\mathsf{Th}}$, upon which the consistency of $\widetilde{\mathsf{I}}^i$ relies, can also be formulated in classical logic; lemmata 5.25 and 5.25 still hold.

[3] This is not entirely true as the combination of classical logic and T-Intro would yield, in general, only $\mathtt{T}(\ulcorner A \vee \neg A \urcorner)$ for every sentence A whereas $\mathsf{Base}_{\mathrm{T}} \vdash \forall \ulcorner A \urcorner \mathtt{T}(\ulcorner A \vee \neg A \urcorner)$. This discrepancy, however, has no affect on the problems of consistency we are addressing here.

– E^i, F^i, G^i, I^i *(Intuitionistic theories with an intuitionistic truth predicate).*

Moreover, each theory in the above list contains a subset of OA^+ *which is not contained in any other theory.*

Proof That each of these theories is consistent has already been established. This leaves two remaining tasks: show every consistent subset of OA^+ is contained in one of the thirteen theories; and associate to each theory a unique maximal consistent subset of OA^+.

For the former it suffices to show every consistent subset of the Optional Axioms not extended by any of the proposed classical theories is inconsistent with the law of excluded middle. To that aim, suppose S is a consistent subset of the Optional Axioms but is not a subset of any of the nine classical theories listed above and suppose, in search of a contradiction, that S is consistent with the law of excluded middle. In particular S must be a subset of one of E^i, F^i, G^i or I^i (any other intuitionistic theory is a sub-theory of one of the classical theories in the list), and be inconsistent with the axiom T-Comp. Consider the following facts.

a) The pair T-In, T-Elim is inconsistent with classical logic;

b) T-Comp(w) entails T-Comp over classical logic;

c) $\rightarrow$-Inf implies T-Comp(w) over $\mathsf{Base}^i_{\mathtt{T}}$;

d) $\vee$-Inf and $\mathtt{T}(\ulcorner A \vee \neg A \urcorner)$ logically implies $\mathtt{T}(\ulcorner A \urcorner) \vee \mathtt{T}(\ulcorner \neg A \urcorner)$;

e) $\exists$-Inf and $\mathtt{T}(\ulcorner A \vee \neg A \urcorner)$ imply $\mathtt{T}(\ulcorner A \urcorner) \vee \mathtt{T}(\ulcorner \neg A \urcorner)$ over $\mathsf{Base}^i_{\mathtt{T}}$.

The first has already been remarked, b) and c) were proved in proposition 3.3 and d) is immediate. e) is a result of the argument that $\exists$-Inf implies T-Comp over $\mathsf{Base}_{\mathtt{T}}$.

Combining these facts we see that S cannot contain both T-In and T-Elim, nor can S contain either T-Comp(w) or $\rightarrow$-Inf, as part of the assumptions entail T-Comp is inconsistent with S. This means S must now be a subset of one of E^i, F^i or G^i and, moreover, S must contain at least one of the axioms $\exists$-Inf, $\vee$-Inf. But, if S does not contain T-Intro it is also a subset of either H^c, I^c or A, and if S contains T-Intro, either d) or e) entails that S is consistent with T-Comp, yielding a contradiction.

We now move to the second task, namely associating with each theory in the list a subset of OA^+ which is unique to that theory. For the theories A–D, E^i, F^i, G^i, H^c and I^i simply pick the corresponding maximal consistent set given by theorem 4.1. For the remainder, E, F, G and I^c, pick the set of Optional Axioms proscribed by theorem 3.2 and add the law of excluded middle. ■

Another candidate for inclusion in the list of Optional Axioms (and arguably a more natural choice than the excluded middle) is the principle $\forall \ulcorner A \urcorner \mathtt{T}(\ulcorner A \vee \neg A \urcorner)$ stating that the truth predicate contains classical logic. At first sight, it might appear that this axiom would enable one to construct new maximal consistent theories based on intuitionistic logic while maintaining a classical truth predicate. The next theorem, however, demonstrates one does not obtain any theories not already encountered.

Theorem 8.2 *Let* T-Class *denote the axiom* $\forall \ulcorner A \urcorner \mathtt{T}(\ulcorner A \vee \neg A \urcorner)$*, and let* S *be some subset of the Optional Axioms.* $\mathsf{Base}^i_{\mathrm{T}} + \mathrm{S} + \text{T-Class}$ *is consistent if and only if* $\mathsf{Base}_{\mathrm{T}} + \mathrm{S}$ *is consistent.*

Proof $\mathsf{Base}^i_{\mathrm{T}} + \text{T-Class}$ is a sub-theory of $\mathsf{Base}_{\mathrm{T}}$ so the right-to-left implication holds trivially. To show the converse suppose, in search of a contradiction, $\mathsf{Base}^i_{\mathrm{T}} + \mathrm{S} + \text{T-Class}$ is consistent, but $\mathsf{Base}_{\mathrm{T}} + \mathrm{S}$ is inconsistent. S must therefore be a subset of one of the nine intuitionistic theories A^i to I^i but not a subset of any of the classical Friedman-Sheard theories A to I. Thus S contains one of T-Comp(w), $\vee$-Inf, $\rightarrow$-Inf or $\exists$-Inf, and $\mathsf{Base}^i_{\mathrm{T}} + \mathrm{S} + \text{T-Comp}$ is inconsistent. Furthermore, the proof of the preceding theorem (specifically points d) and e) on the previous page) shows S cannot contain $\vee$-Inf or $\exists$-Inf. Therefore, S is a subset of F^i, G^i or I^i, and we may assume S contains T-Comp(w) (as $\rightarrow$-Inf implies T-Comp(w) over $\mathsf{Base}^i_{\mathrm{T}}$). S cannot contain T-Elim, as otherwise $\mathsf{Base}^i_{\mathrm{T}} + \mathrm{S} + \text{T-Class}$ would extend $\mathsf{Base}_{\mathrm{T}}$; but then S is a subset of A, contradicting the assumptions. ∎

Although the addition of T-Class does not create any extra theories of truth, it does allow one to differentiate between the classical theories E–I and the intuitionistic theories E^i–I^i. In particular, the triple $\{\text{T-Out}, \vee\text{-Inf}, \text{T-Class}\}$ is inconsistent, but the two sets $\{\text{T-Out}, \vee\text{-Inf}\}$ and $\{\text{T-Out}, \text{T-Class}\}$ are consistent over $\mathsf{Base}^i_{\mathrm{T}}$; they correspond to the theories H^c and H respectively. Using only subsets of the original fifteen Optional Axioms and the law of excluded middle, one is unable to differentiate between the two cases.

Theorem 8.3 *Allowing the axiom T-Class as an additional Optional Axiom one obtains exactly fourteen maximal consistent theories, whereas allowing both T-Class and the law of excluded middle provides exactly fifteen maximal consistent theories.*

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