Article:
Friedman, S-D, Rathjen, M and Weiermann, A (2013) Slow consistency. Annals of Pure and Applied Logic, 164 (3). 382 - 393 (12). ISSN 0168-0072

https://doi.org/10.1016/j.apal.2012.11.009

Slow Consistency

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Abstract

The fact that "natural" theories, i.e. theories which have something like an "idea" to them, are almost always linearly ordered with regard to logical strength has been called one of the great mysteries of the foundation of mathematics. However, one easily establishes the existence of theories with incomparable logical strengths using self-reference (Rosser-style). As a result, $\mathbf{PA} + \mathrm{Con}(\mathbf{PA})$ is not the least theory whose strength is greater than that of **PA**. But still we can ask: is there a sense in which $\mathbf{PA} + \mathrm{Con}(\mathbf{PA})$ is the least "natural" theory whose strength is greater than that of **PA**? In this paper we exhibit natural theories in strength strictly between **PA** and $\mathbf{PA} + \mathrm{Con}(\mathbf{PA})$ by introducing a notion of slow consistency.

Keywords: Peano arithmetic, consistency strength, interpretation, fast growing function, slow consistency, Orey sentence
2000 MSC: Primary: 03F25, 03F30, Secondary: 03C62, 03F05, 03F15, 03H15.

1. Preliminaries

PA is Peano Arithmetic. $\mathbf{PA}\restriction_k$ denotes the subtheory of **PA** usually denoted by $\mathrm{I}\Sigma_k$. It consists of a finite base theory $\mathbf{P}^-$ (which are the axioms for a commutative discretely ordered semiring) together with a single Π_{k+2} axiom which asserts that induction holds for Σ_k formulae. For functions $F : \mathbb{N} \to \mathbb{N}$ we use exponential notation $F^0(x) = x$ and $F^{k+1}(x) = F(F^k(x))$ to denote repeated compositions of F.

In what follows we require an ordinal representation system for ε_0. Moreover, we assume that these ordinals come equipped with specific fundamental sequences $\lambda[n]$ for each limit ordinal $\lambda \leq \varepsilon_0$. Their definition springs forth from their representation in Cantor normal form (to base ω). For an ordinal α such that $\alpha > 0$, α has a unique representation :

$$\alpha = \omega^{\alpha_1} \cdot n_1 + \cdots + \omega^{\alpha_k} \cdot n_k,$$

where $0 < k, n_1, \ldots, n_k < \omega$, and $\alpha_1, \ldots, \alpha_k$ are ordinals such that $\alpha_1 > \cdots > \alpha_k$.

If the Cantor normal form of $\beta > 0$ is $\omega^{\beta_1} \cdot m_1 + \cdots + \omega^{\beta_l} \cdot m_l$, we write $\alpha \gg \beta$ if $\alpha > \beta$ and $\alpha_k \geq \beta_1$.

Definition 1.1. For α an ordinal and n a natural number, let ω_n^α be defined inductively by $\omega_0^\alpha := \alpha$, and $\omega_{n+1}^\alpha := \omega^{\omega_n^\alpha}$.

We also write ω_n for ω_n^1. In particular, $\omega_0 = 1$ and $\omega_1 = \omega$.

Definition 1.2. For each limit ordinal $\lambda \leq \varepsilon_0$, define a strictly monotone sequence, $\lambda[n]$, of ordinals converging to λ from below. We use the fact, following from the Cantor normal form representation, that if $0 < \alpha < \varepsilon_0$, then there are unique $\beta, \gamma < \varepsilon_0$, and $0 < m < \omega$ such that

$$\alpha = \beta + \omega^\gamma \cdot m$$

and either $\beta = 0$ or β has normal form $\omega^{\beta_1} \cdot m_1 + \cdots + \omega^{\beta_l} \cdot m_l$ with $\beta_l > \gamma$.

The definition of $\lambda[n]$ proceeds by recursion on this representation of λ.

Case 1. $\lambda = \beta + \omega^\gamma \cdot m$ and $\gamma = \delta + 1$.
Put $\lambda[n] = \beta + \omega^\gamma \cdot (m-1) + \omega^\delta \cdot (n+1)$. (Remark: In particular, $\omega[n] = n+1$.)

Case 2. $\lambda = \beta + \omega^\gamma \cdot m$, and $\gamma < \lambda$ is a limit ordinal.
Put $\lambda[n] = \beta + \omega^\gamma \cdot (m-1) + \omega^{\gamma[n]}$.

Case 3. $\lambda = \varepsilon_0$.
Put $\varepsilon_0[0] = \omega$ and $\varepsilon_0[n+1] = \omega^{\varepsilon_0[n]}$. (Remark: Thus $\varepsilon_0[n] = \omega_{n+1}$.)

It will be convenient to have $\alpha[n]$ defined for non-limit α. We set $(\beta+1)[n] = \beta$ and $0[n] = 0$.

Definition 1.3. By "a fast growing " hierarchy we simply mean a transfinitely extended version of the Grzegorczyk hierarchy i.e. a transfinite sequence sequence of number-theoretic functions $F_\alpha : \mathbb{N} \to \mathbb{N}$ defined recursively by iteration at successor levels and diagonalization over fundamental sequences at limit levels. We use the following hierarchy:

$$\begin{aligned} F_0(n) &= n+1 \\ F_{\alpha+1}(n) &= F_\alpha^{n+1}(n) \\ F_\alpha(n) &= F_{\alpha[n]}(n) \quad \text{if } \alpha \text{ is a limit.} \end{aligned}$$

It is closely related to the Hardy hierarchy:

$$\begin{aligned} H_0(n) &= n \\ H_{\alpha+1}(n) &= H_\alpha(n+1) \\ H_\alpha(n) &= H_{\alpha[n]}(n) \quad \text{if } \alpha \text{ is a limit.} \end{aligned}$$

Their relationship is as follows:

$$H_{\omega^\alpha} = F_\alpha \tag{1}$$

for every $\alpha < \varepsilon_0$. If $\alpha = \omega^{\alpha_1} \cdot n_1 + \cdots + \omega^{\alpha_k} \cdot n_k$ is in Cantor normal form and $\beta < \omega^{\alpha_k+1}$, then

$$H_{\alpha+\beta} = H_\alpha \circ H_\beta. \tag{2}$$

Ketonen and Solovay [8] found an interesting combinatorial characterization of the H_α's. Call an interval $[k,n]$ 0-large if $k \leq n$, $\alpha+1$-large if there are $m, m' \in [k,n]$ such that $m \neq m'$ and $[m,n]$ and $[m',n]$ are both α-large; and λ-large (where λ is a limit) if $[k,n]$ is $\lambda[k]$-large.

Theorem 1.4 (Ketonen, Solovay [8])**.** *Let $\alpha < \varepsilon_0$.*

$$\begin{aligned} H_\alpha(n) &= \textit{least } m \textit{ such that } [n,m] \textit{ is } \alpha\textit{-large} \\ F_\alpha(n) &= \textit{least } m \textit{ such that } [n,m] \textit{ is } \omega^\alpha\textit{-large.} \end{aligned}$$

The order of growth of F_{ε_0} is essentially the same as that of the Paris-Harrington function f_{PH}. More details will be provided in section 3.1.

2. Capturing the F_α's in PA

In [8] many facts about the functions F_α, as befits their definition, are proved by transfinite induction on the ordinals $\leq \varepsilon_0$. In [8] there is no attempt to determine whether they are provable in **PA** (let alone in weaker theories). In what follows we will have to assume that some of the properties of the F_α's hold in all models of **PA**. As a consequence, we will revisit some parts of [8], especially section 2, and recast them in such a way that they become provable in **PA**. Statements shown by transfinite induction on the ordinals in [8] will be proved by ordinary induction on the term complexity of ordinal representations, adding extra assumptions.

Definition 2.1. The computation of $F_\alpha(x)$ is closely connected with the step-down relations of [8] and [19]. For $\alpha < \beta \leq \varepsilon_0$ we write $\beta \underset{n}{\longrightarrow} \alpha$ if for some sequence of ordinals $\gamma_0, \ldots, \gamma_r$ we have $\gamma_0 = \beta$, $\gamma_{i+1} = \gamma_i[n]$, for $0 \leq i < r$, and $\gamma_r = \alpha$. If we also want to record the number of steps r, we shall write $\alpha \xrightarrow[n]{r} \beta$.

The definition of the functions F_α for $\alpha \leq \varepsilon_0$ employs transfinite recursion on α. It is therefore not immediately clear how we can speak about these functions in arithmetic. Later on we shall need to refer to a definition of $F_\alpha(x) = y$ in an arbitrary model of **PA**. As it turns out, this can be done via a formula of low complexity.

Lemma 2.2. *There is a Δ_0-formula expressing $F_\alpha(x) = y$ (as a predicate of α, x, y).*

Proof: This is shown in [23, 5.2]. The main idea is that the computation of $F_\alpha(x)$ can be described as a rewrite systems, that is, as a sequence of manipulations of expressions of the form

$$F_{\alpha_1}^{n_1}(F_{\alpha_2}^{n_2}(\ldots(F_{\alpha_k}^{n_k}(n))\ldots)),$$

where $n_1, \ldots, n_k \in \omega - \{0\}$ and $\alpha_1 > \ldots > \alpha_k \geq 0$. $\square$

Let $\mathrm{I}\Delta_0$ be the subsystem of Peano Arithmetic in which induction applies only to formulas with bounded quantifiers (Δ_0-formulas). If we add to $\mathrm{I}\Delta_0$ the axiom $exp = \forall x > 1\, \forall y\, \exists z\, E_0(x, y, z)$, saying that the exponential function is total, then the resulting theory will be denoted by $\mathrm{I}\Delta_0(exp)$. $\mathrm{I}\Delta_0(exp)$ is strong enough to prove all of the results of elementary number theory. For example, Matijasevic's Theorem is provable in it.

Lemma 2.3. *We use $F_\alpha(x)\downarrow$ to denote $\exists y\, F_\alpha(x) = y$. $F_\alpha\downarrow$ stands for $\forall x\, F_\alpha(x)\downarrow$.*

The following are provable in $\mathrm{I}\Delta_0(exp)$:

(i) If $\beta \underset{x}{\longrightarrow} \alpha$ and $F_\beta(x)\downarrow$, then $F_\alpha(x)\downarrow$ and $F_\beta(x) \geq F_\alpha(x)$.

(ii) If $F_\beta(x)\downarrow$ and $x > y$, then $F_\beta(y)\downarrow$ and $F_\beta(x) \geq F_\beta(y)$.

(iii) If $\alpha > \beta$ and $F_\alpha\downarrow$, then $F_\beta\downarrow$.

(iv) If $i > 0$ and $F^i_\alpha(x)\downarrow$ then $x < F^i_\alpha(x)$.

Proof: (i) follows by induction on the length r of the sequence $\gamma_0, \ldots, \gamma_r$ with $\gamma_0 = \beta$, $\gamma_{i+1} = \gamma_i[n]$, for $0 \leq i < r$, and $\gamma_r = \alpha$. In the proof one uses the fact that '$F_\delta(x) = y$' is Δ_0 as a relation with arguments δ, x, y, and also uses [23, Theorem 5.3] (or rather Claim 1 in Appendix A of [22]).

(ii) follows from [23, Proposition 5.4(v)]. (iii) follows from [23, Proposition 5.4(iv)]. (iv) is [23, Proposition 5.4(i)]. □

There is an additional piece of information that is provided by the particular coding and Δ_0 formula denoting $F_\alpha(x) = y$ used in [23, 5.2], namely that there is a fixed polynomial P in one variable such that for all $\alpha \leq \varepsilon_0$, the number of steps it takes to compute $F_\alpha(x)$ is always bounded by $P(F_\alpha(x))$. This has a useful consequence that we are going to exploit in the next lemma.

Lemma 2.4. *The following is provable in* $\mathrm{I}\Delta_0(exp)$*: Let $\alpha \leq \varepsilon_0$. Suppose $F_\alpha(n)\downarrow$. Then $\alpha \xrightarrow[n]{r} 0$ for some $r \leq P(F_\alpha(n))$.*

Proof: We clearly have that the number of steps it takes to compute $F_\alpha(n)$ is a bound for any sequence of ordinals $\gamma_0, \ldots, \gamma_s$ with $\gamma_0 = \alpha$, $\gamma_s > 0$, and $\gamma_{i+1} = \gamma_i[n]$ for $0 \leq i < s$. Hence $s < P(F_\alpha(n))$ and thus $\alpha \xrightarrow[n]{r} 0$ for some $r \leq P(F_\alpha(n))$. □

Convention. For the remainder of this section we will be working in the background theory **PA**, thus all statements are formally provable in **PA**. A cursory glance would reveal that the fragment $\mathrm{I}\Sigma_1$ is certainly capacious enough, and very likely $\mathrm{I}\Delta_0(exp)$ would suffice, too.

Lemma 2.5. *(i) Let $\alpha \xrightarrow[n]{} \beta$, $\alpha \xrightarrow[n]{} \gamma$, $\beta > \gamma$. Then $\beta \xrightarrow[n]{} \gamma$.*

(ii) Let $\alpha \xrightarrow[n]{} \beta$, $\beta \xrightarrow[n]{} \gamma$. Then $\alpha \xrightarrow[n]{} \gamma$.

Proof: This is evident from the definition. □

Definition 2.6. Let α, β be ordinals. Say that α *meshes with* β, if for some ordinals γ, δ, we have $\alpha = \omega^\gamma \cdot \delta$ and $\beta < \omega^{\gamma+1}$.

Note that if α and β have Cantor normal forms $\alpha = \omega^{\alpha_1} \cdot n_1 + \ldots + \omega^{\alpha_k} \cdot n_k$, $\beta = \omega^{\beta_1} \cdot m_1 + \ldots + \omega^{\beta_l} \cdot m_l$, respectively, then the condition that α meshes with β is precisely that $\alpha_k \geq \beta_1$.

Lemma 2.7. *Let $\alpha, \beta < \varepsilon_0$. Let α mesh with $\beta > 0$. Then $(\alpha + \beta)[n] = \alpha + \beta[n]$. Thus if $\beta \underset{n}{\longrightarrow} \gamma$, then $\alpha + \beta \underset{n}{\longrightarrow} \alpha + \gamma$.*

Proof: That α meshes with β implies that the Cantor normal form of $\alpha + \beta$ is basically the concatenation of those for α, β. The first claim thus follows from the way that the definition of $\delta[n]$ focuses on the rightmost term of the Cantor normal form of δ, provided $\delta < \varepsilon_0$. The second claim reduces to the special case when $\gamma = \beta[n]$, using the transitivity of $\underset{n}{\longrightarrow}$. This special claim is evident by the first claim. □

Lemma 2.8. *Let $k < l < \omega$, $\alpha < \varepsilon_0$, and suppose that $\omega^\alpha \cdot l \underset{n}{\longrightarrow} 0$. Then $\omega^\alpha \cdot l \underset{n}{\longrightarrow} \omega^\alpha \cdot k$.*

Proof: This holds by assumption if $k = 0$. So suppose that $n > 0$. Let $\omega^\alpha \cdot k < \delta \leq \omega^\alpha \cdot l$. Then δ can be uniquely written as $\delta = \omega^\alpha \cdot k + \gamma$ for some $\gamma > 0$, and $\omega^\alpha \cdot k$ and γ mesh. Thus it follows from Lemma 2.7 that $\delta[n] = \omega^\alpha \cdot k + \gamma[n]$ and hence $\delta[n] \geq \omega^\alpha \cdot k$. Since $\omega^\alpha \cdot l \underset{n}{\longrightarrow} 0$, we conclude that $\omega^\alpha \cdot l \underset{n}{\longrightarrow} \omega^\alpha \cdot k$. □

Lemma 2.9. *Let $n \geq 1$. Let $\delta < \varepsilon_0$. Suppose $\omega^{\delta+1} \underset{n}{\longrightarrow} 0$. Then $\omega^{\delta+1} \underset{n}{\longrightarrow} \omega^\delta$.*

Proof: $\omega^{\delta+1} \underset{n}{\longrightarrow} \omega^{\delta+1}[n] = \omega^\delta \cdot (n+1)$. Now apply Lemma 2.8 and Lemma 2.5(ii). □

Lemma 2.10. *Let $\alpha_1 < \varepsilon_0$. Let $n \geq 1$. Suppose $\alpha_1 \underset{n}{\longrightarrow} \alpha_2$ and $\omega^{\alpha_1} \underset{n}{\longrightarrow} 0$. Then $\omega^{\alpha_1} \underset{n}{\longrightarrow} \omega^{\alpha_2}$.*

Proof: Let $\alpha_1 \overset{x}{\underset{n}{\longrightarrow}} \alpha_2$. By induction on x we show that $\omega^{\alpha_1} \underset{n}{\longrightarrow} \omega^{\alpha_2}$.

If $x = 0$ this is trivial. Suppose $x > 0$. If α_1 is a successor $\alpha_0 + 1$, then $\alpha_1[n] = \alpha_0 \overset{x-1}{\underset{n}{\longrightarrow}} \alpha_2$ and thus $\alpha^{\alpha_0} \underset{n}{\longrightarrow} \omega^{\alpha_2}$ by the induction hypothesis. Also $\omega^{\alpha_1}[n] = \omega^{\alpha_0} \cdot (n+1)$ and $\omega^{\alpha_0} \cdot (n+1) \underset{n}{\longrightarrow} \omega^{\alpha_0}$ owing to Lemma 2.8. Consequently, $\omega^{\alpha_1} \underset{n}{\longrightarrow} \omega^{\alpha_2}$.

Now let α_1 be a limit. Then $\omega^{\alpha_1}[n] = \omega^{\alpha_1[n]}$. Inductively, as $\alpha_1[n] \xrightarrow[n]{x-1} \alpha_2$, we have that $\omega^{\alpha_1[n]} \underset{n}{\longrightarrow} \omega^{\alpha_2}$. Hence $\omega^{\alpha_1} \underset{n}{\longrightarrow} \omega^{\alpha_2}$. $\square$

Lemma 2.11. *Let $\alpha < \varepsilon_0$. Suppose $\omega^\alpha \xrightarrow[n]{x} 0$. Then $\alpha \xrightarrow[n]{y} 0$ for some $y < x$.*

Proof: We proceed by induction on x. If $\alpha = 0$ then this is obvious. Let $\alpha = \alpha_0 + 1$. Then $\omega^\alpha[n] = \omega^{\alpha_0} \cdot n + \omega^{\alpha_0} \xrightarrow[n]{x-1} 0$. In light of Lemma 2.7 we conclude that $\omega^{\alpha_0} \xrightarrow[n]{u} 0$ for some $u \leq x - 1$. Thus, by the inductive assumption, $\alpha_0 \xrightarrow[n]{v} 0$ for some $v < x - 1$. Therefore $\alpha \xrightarrow[n]{v+1} 0$ with $v + 1 < x$.

Now let α be a limit. Then $\omega^\alpha[n] = \omega^{\alpha[n]} \xrightarrow[n]{x-1} 0$. Inductively we thus have $\alpha[n] \xrightarrow[n]{u} 0$ for some $u < x - 1$, and hence $\alpha \xrightarrow[n]{u+1} 0$ where $u + 1 < x$. $\square$

Proposition 2.12. *Let λ be a limit $\leq \varepsilon_0$. Suppose $i < j < \omega$ and $\lambda[j] \underset{n}{\longrightarrow} 0$. Then $\lambda[j] \underset{n}{\longrightarrow} \lambda[i]$.*

Proof: We proceed by induction on the (term) complexity of λ.

Case 1. $\lambda = \beta + \omega^{\alpha+1} \cdot m$. Then $\lambda[k] = \beta + \omega^{\alpha+1} \cdot (m-1) + \omega^\alpha \cdot (k+1)$. As $\lambda[j] \underset{n}{\longrightarrow} 0$ entails that $\omega^\alpha \cdot (j+1) \underset{n}{\longrightarrow} 0$, it follows from Lemma 2.8 that $\omega^\alpha \cdot (j+1) \underset{n}{\longrightarrow} \omega^\alpha \cdot (i+1)$. But then, by Lemma 2.7,

$$\lambda[j] = \beta + \omega^{\alpha+1} \cdot (m-1) + \omega^\alpha \cdot (j+1) \underset{n}{\longrightarrow} \beta + \omega^{\alpha+1} \cdot (m-1) + \omega^\alpha \cdot (i+1) = \lambda[i].$$

Case 2. $\lambda = \beta + \omega^\gamma \cdot m$, and γ is a limit ordinal. Then $\lambda[k] = \beta + \omega^\gamma \cdot (m-1) + \omega^{\gamma[k]}$. $\lambda[j] \underset{n}{\longrightarrow} 0$ implies that $\omega^{\gamma[j]} \underset{n}{\longrightarrow} 0$, and hence, by Lemma 2.11, $\gamma[j] \underset{n}{\longrightarrow} 0$. Since the term complexity of γ is smaller than that of λ the inductive assumption yields $\gamma[j] \underset{n}{\longrightarrow} \gamma[i]$, and hence $\omega^{\gamma[j]} \underset{n}{\longrightarrow} \omega^{\gamma[i]}$ by Lemma 2.10. As a result, by Lemma 2.7,

$$\lambda[j] = \beta + \omega^\gamma \cdot (m-1) + \omega^{\gamma[j]} \underset{n}{\longrightarrow} \beta + \omega^\gamma \cdot (m-1) + \omega^{\gamma[i]} = \lambda[i].$$

Case 3. $\lambda = \varepsilon_0$. Then $\lambda[j] = \omega_{j+1} = \omega^{\omega_j}$. From the assumption $\lambda[j] \underset{n}{\longrightarrow} 0$, applying Lemma 2.11 iteratively, one deduces that $\omega_k \underset{n}{\longrightarrow} 0$ holds for all $k \leq j + 1$. Obviously, $\omega \underset{n}{\longrightarrow} 1$. Thus, by Lemma 2.10, $\omega_2 = \omega^\omega \underset{n}{\longrightarrow} \omega^1 = \omega = \omega_1$.

Iterating this procedure we have $\omega_{l+1} \underset{n}{\longrightarrow} \omega_l$ for all $l \leq j$. By transitivity of $\underset{n}{\longrightarrow}$ we thus arrive at $\lambda[j] = \omega_{j+1} \underset{n}{\longrightarrow} \omega_{i+1} = \lambda[j]$. $\square$

Lemma 2.13. *Let $n, k < \omega$ and $n > 0$. Suppose $\omega_{k+1} \underset{n}{\longrightarrow} 0$. Then $\omega_{k+1} \underset{n}{\longrightarrow} \omega_k + 1$.*

Proof: From the proof of Proposition 2.12, Case 3, we infer that $\omega_{u+1} \underset{n}{\longrightarrow} 0$ for all $u \leq k$. Now use induction on $u \leq k$ to show that $\omega_{u+1} \underset{n}{\longrightarrow} \omega_u + 1$. If $u = 0$ then $\omega_u = 1$ and $\omega_{u+1} = \omega$, and $\omega \underset{n}{\longrightarrow} 2$ holds since $n \geq 1$. Now suppose $u = v + 1$ and $\omega_{v+1} \underset{n}{\longrightarrow} \omega_v + 1$. Then, as $\omega_{u+1} \underset{n}{\longrightarrow} 0$, we have

$$\omega_{u+1} = \omega^{\omega_{v+1}} \underset{n}{\longrightarrow} \omega^{\omega_v + 1} \tag{3}$$

by applying Lemma 2.10. In particular, $\omega^{\omega_v+1} \underset{n}{\longrightarrow} 0$, and therefore

$$\omega^{\omega_v+1}[n] = \omega^{\omega_v} \cdot (n+1) = \omega_{v+1} \cdot (n+1) \underset{n}{\longrightarrow} \omega_{v+1} + \omega_{v+1} \tag{4}$$

since $n > 0$. Since we also have $\omega_{v+1} \underset{n}{\longrightarrow} \omega_0 = 1$ by Proposition 2.12, (4) implies

$$\omega^{\omega_v+1} \underset{n}{\longrightarrow} \omega_{v+1} + 1. \tag{5}$$

Combining (3) and (5) yields $\omega_{u+1} \underset{n}{\longrightarrow} \omega_u + 1$. $\square$

Corollary 2.14. *Let $k, n < \omega$ and $n > 0$.*

(i) Suppose $\varepsilon_0[k+1] \underset{n}{\longrightarrow} 0$. Then $\varepsilon_0[k+1] \underset{n}{\longrightarrow} \varepsilon_0[k] + 1$.

(ii) Suppose $F_{\varepsilon_0[k+1]}(n)\downarrow$. Then $F_{\varepsilon_0[k+1]}(n) \geq F_{\varepsilon_0[k]}(F_{\varepsilon_0[k]}(n))$.

Proof: As $\varepsilon_0[u] = \omega_{u+1}$, (i) is a consequence of Lemma 2.13.
(ii): By Lemma 2.4, $F_{\varepsilon_0[k+1]}(n)\downarrow$ implies that $\varepsilon_0[k+1] \underset{n}{\longrightarrow} 0$. Thus, using (i), we have $\varepsilon_0[k+1] \underset{n}{\longrightarrow} \varepsilon_0[k] + 1$. Hence, by Lemma 2.3(i),

$$F_{\varepsilon_0[k+1]}(n) \geq F_{\varepsilon_0[k]+1}(n) = F^{n+1}_{\varepsilon_0[k]}(n) \geq F_{\varepsilon_0[k]}(F_{\varepsilon_0[k]}(n)),$$

where the last inequality is a consequence of Lemma 2.3(iv). $\square$

3. Slow consistency

To motivate our notion of slow consistency we recall the concept of interpretability of one theory in another theory. Let S and S' be arbitrary theories. S' is **interpretable in** S or S **interprets** S' (in symbols $S' \trianglelefteq S$) "if roughly speaking, the primitive concepts and the range of the variables of S' are defined in such a way as to turn every theorem of S' into a theorem of S" (quoted from [12] p. 96; for details see [12, section 6]).

To simplify matters, we restrict attention to theories T formulated in the language of **PA** which contain the axioms of **PA** and have a primitive recursive axiomatization, i.e. being an axiom of T is primitive recursively decidable.

For an integer $k \geq 0$, we denote by $T\restriction_k$ the theory consisting of the first k (non-logical) axioms of T. Let $\mathrm{Con}(T)$ be the arithmetized statement that T is consistent.

A theory T is **reflexive** if it proves the consistency of all its finite subtheories, i.e. $T \vdash \mathrm{Con}(T\restriction_k)$ for all $k \in \mathbb{N}$. Note that theories satisfying the conditions spelled out above will always be reflexive.

Another interesting relationship between theories we shall consider is $T_1 \subseteq_{\Pi_1} T_2$, i.e. every Π_1 theorem of T_1 is also a theorem of T_2.

Theorem 3.1. *Let S,T be theories that satisfy the conditions spelled out above. Then:*

$$
\begin{aligned}
S \trianglelefteq T \quad & \textit{if and only if} \quad T \vdash \mathrm{Con}(S\restriction_n) \textit{ holds for all } n \in \mathbb{N} && (6)\\
& \textit{if and only if} \quad S \subseteq_{\Pi_1} T. && (7)
\end{aligned}
$$

Proof: (6) seems to be due to Orey [13]. Another easily accessible proof of (6) can be found in [12, Section 6, Theorem 5]. (7) was first stated in [7] and [11]. A proof can also be found in [12, Section 6, Theorem 6]. □

We know that

$$\mathrm{Con}(\mathbf{PA}) \leftrightarrow \forall x\, \mathrm{Con}(\mathbf{PA}\restriction_x).$$

Given a function $f : \mathbb{N} \to \mathbb{N}$ (say provably total in **PA**) we are thus led to the following consistency statement:

$$\mathrm{Con}_f(\mathbf{PA}) \quad := \quad \forall x\, \mathrm{Con}(\mathbf{PA}\restriction_{f(x)}). \tag{8}$$

It is perhaps worth pointing out that the exact meaning of $\mathrm{Con}_f(\mathbf{PA})$ depends on the representation that we choose for f.

Statements of the form (8) are interesting only if the function f grows extremely slowly, though still has an infinite range but **PA** cannot prove that fact.

Definition 3.2. Define

$$F_{\varepsilon_0}^{-1}(n) \quad = \quad \max(\{k \leq n \mid \exists y \leq n\, F_{\varepsilon_0}(k) = y\} \cup \{0\}).$$

Note that, by Lemma 2.2, the graph of $F_{\varepsilon_0}^{-1}$ has a Δ_0 definition. Thus it follows that $F_{\varepsilon_0}^{-1}$ is a provably recursive function of **PA**.

Let $\mathrm{Con}^*(\mathbf{PA})$ be the statement $\forall x\, \mathrm{Con}(\mathbf{PA} \restriction_{F_{\varepsilon_0}^{-1}(x)})$. Of course, in the definition of $\mathrm{Con}^*(\mathbf{PA})$ we have in mind some standard representation of F_{ε_0} referred to in Lemma 2.2. Note that $\mathrm{Con}^*(\mathbf{PA})$ is equivalent to the statement

$$\forall x\, [F_{\varepsilon_0}(x) \downarrow \rightarrow \mathrm{Con}(\mathbf{PA} \restriction_x)].$$

Proposition 3.3. $\mathbf{PA} \nvdash \mathrm{Con}^*(\mathbf{PA})$.

Proof: Aiming at a contradiction, suppose $\mathbf{PA} \vdash \mathrm{Con}^*(\mathbf{PA})$. Then $\mathbf{PA} \restriction_k \vdash \mathrm{Con}^*(\mathbf{PA})$ for all sufficiently large k. As $\mathbf{PA} \restriction_k \vdash F_{\varepsilon_0}(k) \downarrow$ on account of $F_{\varepsilon_0}(k) \downarrow$ being a true Σ_1 statement, we arrive at $\mathbf{PA} \restriction_k \vdash \mathrm{Con}(\mathbf{PA} \restriction_k)$, contradicting Gödel's second incompleteness theorem. $\square$

Proposition 3.3 holds in more generality.

Corollary 3.4. *If T is a recursive consistent extension of* **PA** *and f is a total recursive function with unbounded range, then*

$$T \nvdash \forall x\, \mathrm{Con}(T \restriction_{f(x)})$$

where $f(x) \downarrow$ is understood to be formalized via some Σ_1 representation of f.

Proof: Basically the same proof as for Proposition 3.3. $\square$

It is quite natural to consider another version of slow consistency where the function $f : \mathbb{N} \to \mathbb{N}$, rather than acting as a bound on the fragments of **PA**, restricts the lengths of proofs. Let $\perp$ be a Gödel number of the canonical inconsistency and let $\mathrm{Proof}_{\mathbf{PA}}(y, z)$ be the primitive recursive predicate expressing the concept that *"y is the Gödel number of a proof in* **PA** *of a formula with Gödel number z"*.

$$\mathrm{Con}^{\ell}_{f}(\mathbf{PA}) \;\; := \;\; \forall x\, \forall y < f(x)\, \neg \mathrm{Proof}_{\mathbf{PA}}(y, \bot) \tag{9}$$

Let $\mathrm{Con}^{\#}(\mathbf{PA})$ be the statement $\mathrm{Con}^{\ell}_{F_{\varepsilon_0}^{-1}}(\mathbf{PA})$.

Note that $\mathrm{Con}^{\#}(\mathbf{PA})$ is equivalent to the following formula:

$$\forall u\, [F_{\varepsilon_0}(u)\!\downarrow\, \rightarrow\; \forall y < u\, \neg \mathrm{Proof}_{\mathbf{PA}}(y, \bot)].$$

As it turns out, by contrast with $\mathrm{Con}^{*}(\mathbf{PA})$, $\mathrm{Con}^{\#}(\mathbf{PA})$ is not very interesting.

Lemma 3.5. $\mathbf{PA} \vdash \mathrm{Con}^{\#}(\mathbf{PA})$.

Proof: First recall that Gentzen showed how to effectively transform an alleged **PA**-proof of an inconsistency (the empty sequent) in his sequent calculus into another proof of the empty sequent such that the latter gets assigned a smaller ordinal than the former. More precisely, there is a reduction procedure $\mathcal{R}$ on proofs P of the empty sequent together with an assignment ord of representations for ordinals $< \varepsilon_0$ to proofs such that $ord(\mathcal{R}(P)) < ord(P)$. Here $<$ denotes the ordering on ordinal representations induced by the ordering of the pertaining ordinals. The functions $\mathcal{R}$ and ord and the relation $<$ are primitive recursive (when viewed as acting on codes for the syntactic objects). With $g(n) = ord(\mathcal{R}^n(P))$, the n-fold iteration of $\mathcal{R}$ applied to P, one has $g(0) > g(1) > g(2) > \ldots > g(n)$ for all n, which is absurd as the ordinals are well-founded.

We will now argue in **PA**. Suppose that $F_{\varepsilon_0}(u)\!\downarrow$. Aiming at a contradiction assume that there is a $p < u$ such that $\mathrm{Proof}_{\mathbf{PA}}(p, \bot)$. We have not said anything about the particular proof predicate $\mathrm{Proof}_{\mathbf{PA}}$ we use, however, whatever proof system is assumed, p will be larger than the Gödel numbers of all formulae occurring in the proof. The proof that p codes, can be primitive recursively transformed into a sequent calculus proof P of the empty sequent in such a way that $ord(P) < \omega_p$ since p is larger than the number of logical symbols occurring in any cut or induction formulae featuring in P (for details see [24, Ch.2]). Inspection of Gentzen's proof, as e.g. presented in [24, 2.12.8], shows there is a primitive recursive function ℓ such that the number of steps it takes to get from $ord(P)$ to 0 by applying the reduction procedure $\mathcal{R}$ is majorized by $\ell(F_{\varepsilon_0}(u))$. As a result we have a contradiction since there is no proof P_0 of the empty sequent with ordinal $ord(P_0) = 0$.

The authors realize that the foregoing proof is merely a sketch. An alternative proof can be obtained by harking back to [1]. The proof will be given in the Appendix. □

The next goal will be to show that $\mathrm{Con}(\mathbf{PA})$ is not derivable in $\mathbf{PA} + \mathrm{Con}^*(\mathbf{PA})$. We need some preparatory definitions.

Definition 3.6. Let E denote the "stack of two's" function, i.e. $E(0) = 0$ and $E(n+1) = 2^{E(n)}$.

Given two elements a and b of a non-standard model $\mathfrak{M}$ of $\mathbf{PA}$, we say that '**b is much larger than a**' if for every standard integer k we have $E^k(a) < b$.

If $\mathfrak{M}$ is a model of $\mathbf{PA}$ and $\mathfrak{I}$ is a substructure of $\mathfrak{M}$ we say that $\mathfrak{I}$ is an **initial segment** of $\mathfrak{M}$, if for all $a \in |\mathfrak{I}|$ and $x \in |\mathfrak{M}|$, $\mathfrak{M} \models x < a$ implies $x \in |\mathfrak{I}|$. We will write $\mathfrak{I} < b$ to mean $b \in |\mathfrak{M}| \setminus |\mathfrak{I}|$. Sometimes we write $a < \mathfrak{I}$ to indicate $a \in |\mathfrak{I}|$.

Theorem 3.7. *Let $\mathfrak{N}$ be a non-standard model of $\mathbf{PA}$ (or $\Delta_0(\exp)$), n be a standard integer, and $e, d \in |\mathfrak{N}|$ be non-standard such that $\mathfrak{N} \models F_{\omega_n^e}(e) = d$. Then there is an initial segment $\mathfrak{I}$ of $\mathfrak{N}$ such $e < \mathfrak{I} < d$ and $\mathfrak{I}$ is a model of Π_{n+1}-induction.*

Proof: This follows e.g. from [23, Theorem 5.25], letting $\alpha = 0$, $c = e$, $a = e$ and $b = d$. The technique used to prove Theorem 5.25 in [23] is a variation of techniques used by Paris in [15]. □

Corollary 3.8. *Let $\mathfrak{N}$ be a non-standard model of $\mathbf{PA}$, $a, e, c \in |\mathfrak{N}|$ be non-standard such that $\mathfrak{N} \models F_{\varepsilon_0}(a) = e$ and $\mathfrak{N} \models F_{\varepsilon_0}(a+1) = c$. Then for every standard n there is an initial segment $\mathfrak{I}$ of $\mathfrak{N}$ such $e < \mathfrak{I} < c$ and $\mathfrak{I}$ is a model of Π_{n+1}-induction.*

Proof: We argue in $\mathfrak{N}$. From $F_{\varepsilon_0}(a+1) = F_{\varepsilon_0[a+1]}(a+1) = c$ we conclude with the help of Corollary 2.14 that

$$c \geq F_{\varepsilon_0[a]}(F_{\varepsilon_0[a]}(a+1)) \geq F_{\varepsilon_0[a]}(F_{\varepsilon_0[a]}(a)) = F_{\varepsilon_0[a]}(e) > e.$$

In view of the previous Theorem we just have to ensure that $F_{\omega_n^e}(e) = d$ for some d with $d \leq c$. From $F_{\varepsilon_0[a]}(e)\downarrow$ we get $\varepsilon_0[a] \underset{e}{\longrightarrow} 0$ by Lemma 2.4.

Proposition 2.12 guarantees that $\varepsilon_0[p] \underset{e}{\rightarrow} e$ holds for all $p \leq a$. In particular, $\varepsilon_0[a-n] \underset{e}{\rightarrow} e$. Applying Lemma 2.10 n-times, we arrive at

$$\varepsilon_0[a] = \omega_n^{\varepsilon_0[a-n]} \underset{e}{\rightarrow} \omega_n^e.$$

In view of Lemma 2.3(i) the latter implies that $F_{\omega_n^e}(e) \downarrow$ and $F_{\varepsilon_0[a]}(e) \geq F_{\omega_n^e}(e)$. $\square$

Definition 3.9. Below we shall need the notion of two models $\mathfrak{M}$ and $\mathfrak{N}$ of **PA** 'agreeing up to e'. For this to hold, the following conditions must be met:

1. e belongs to both models.
2. e has the same predecessors in both $\mathfrak{M}$ and $\mathfrak{N}$.
3. If d_0, d_1, and c are $\leq e$ (in one of the models $\mathfrak{M}$ and $\mathfrak{N}$), then $\mathfrak{M} \models d_0 + d_1 = c$ iff $\mathfrak{N} \models d_0 + d_1 = c$.
4. If d_0, d_1, and c are $\leq e$ (in one of the models $\mathfrak{M}$ and $\mathfrak{N}$), then $\mathfrak{M} \models d_0 \cdot d_1 = c$ iff $\mathfrak{N} \models d_0 \cdot d_1 = c$.

If $\mathfrak{M}$ and $\mathfrak{N}$ agree up to e, $d \leq e$ and $\theta(x)$ is a Δ_0 formula, it follows that $\mathfrak{M} \models \theta(d)$ iff $\mathfrak{N} \models \theta(d)$ (cf. [3, Proposition 1]).

Theorem 3.10. $\mathbf{PA} + \mathrm{Con}^*(\mathbf{PA}) \nvdash \mathrm{Con}(\mathbf{PA})$.

Proof: Let $\mathfrak{M}$ be a countable non-standard model of $\mathbf{PA} + F_{\varepsilon_0}$ *is total*. Let M be the domain of $\mathfrak{M}$ and $a \in M$ be non-standard. Moreover, let $e = F_{\varepsilon_0}^{\mathfrak{M}}(a)$. As a result of the standing assumption, $\mathfrak{M} \models \mathrm{Con}(\mathbf{PA}\restriction_a)$. Owing to a result of Solovay's [21, Theorem 1.1] (or similar results in [9]), there exists a countable model $\mathfrak{N}$ of **PA** such that:

(i) $\mathfrak{M}$ and $\mathfrak{N}$ agree up to e (in the sense of Definition 3.9).

(ii) $\mathfrak{N}$ thinks that $\mathbf{PA}\restriction_a$ is consistent.

(iii) $\mathfrak{N}$ thinks that $\mathbf{PA}\restriction_{a+1}$ is inconsistent. In fact there is a proof of $0 = 1$ from $\mathbf{PA}\restriction_{a+1}$ whose Gödel number is less than 2^{2^e} (as computed in $\mathfrak{N}$).

In actuality, to be able to apply [21, Theorem 1.1] we have to ensure that e is much larger than a, i.e., $E^k(a) < e$ for every standard number k. It is a standard fact (provable in **PA**) that $E(x) \leq F_3(x)$ holds for all sufficiently large x (cf. [8, p. 269]). In particular this holds for all non-standard elements s of $\mathfrak{M}$ and hence

$$E^k(s) \leq F_3^k(s) \leq F_3^s(s) \leq F_4(s) < F_{\varepsilon_0}(s),$$

so that $E^k(a) < e$ holds for all standard k, leading to e being much larger than a.

We will now distinguish two cases.

Case 1: $\mathfrak{N} \models F_{\varepsilon_0}(a+1) \uparrow$. Then also $\mathfrak{N} \models F_{\varepsilon_0}(d) \uparrow$ for all $d > a$ by Lemma 2.3(ii). Hence, in light of (ii), $\mathfrak{N} \models \mathrm{Con}^*(\mathbf{PA})$. As (iii) yields $\mathfrak{N} \models \neg\mathrm{Con}(\mathbf{PA})$, we have

$$\mathfrak{N} \models \mathbf{PA} + \mathrm{Con}^*(\mathbf{PA}) + \neg\mathrm{Con}(\mathbf{PA}). \tag{10}$$

Case 2: $\mathfrak{N} \models F_{\varepsilon_0}(a+1) \downarrow$. We then also have $e = F_{\varepsilon_0}^{\mathfrak{N}}(a)$, for $\mathfrak{M}$ and $\mathfrak{N}$ agree up to e and the formula '$F_{\varepsilon_0}(x) = y$' is Δ_0 by Lemma 2.2. Let $c := F_{\varepsilon_0}^{\mathfrak{N}}(a+1)$. By Corollary 3.8, for every standard n there is an initial segment $\mathfrak{I}$ of $\mathfrak{N}$ such $e < \mathfrak{I} < c$ and $\mathfrak{I}$ is a model of Π_{n+1}-induction. Moreover, it follows from the properties of $\mathfrak{N}$ and the fact that $2^{2^e} < \mathfrak{I}$, that

1. $\mathfrak{I}$ thinks that $\mathbf{PA}\restriction_a$ is consistent.
2. $\mathfrak{I}$ thinks that $\mathbf{PA}\restriction_{a+1}$ is inconsistent.
3. $\mathfrak{I}$ thinks that $F_{\varepsilon_0}(a+1)$ is not defined.

Consequently, $\mathfrak{I} \models \mathrm{Con}^*(\mathbf{PA}) + \neg\mathrm{Con}(\mathbf{PA}) + \Pi_{n+1}$-induction. Since n was arbitrary, this shows that $\mathbf{PA}+\mathrm{Con}^*(\mathbf{PA})+\neg\mathrm{Con}(\mathbf{PA})$ is a consistent theory. □

Proposition 3.3 and Theorem 3.10 can be extended to theories $\mathbf{T} = \mathbf{PA} + \psi$ where ψ is a true Π_1 statement.

Theorem 3.11. *Let* $\mathbf{T} = \mathbf{PA} + \psi$ *where* ψ *is a* Π_1 *statement such that* $\mathbf{T}$ + 'F_{ε_0} is total' *is a consistent theory. Let* $\mathbf{T}\restriction_k$ *to be the theory* $\mathbf{PA}\restriction_k +\psi$ *and* $\mathrm{Con}^*(\mathbf{T}) := \forall x \mathrm{Con}(\mathbf{T}\restriction_{F_{\varepsilon_0}^{-1}(x)})$*. Then the strength of* $\mathbf{T} + \mathrm{Con}^*(T)$ *is strictly between* $\mathbf{T}$ *and* $\mathbf{T} + \mathrm{Con}(\mathbf{T})$*, i.e.*

(i) $\mathbf{T} \nvdash \mathrm{Con}^*(\mathbf{T})$.

(ii) $\mathbf{T} + \mathrm{Con}^*(\mathbf{T}) \nvdash \mathrm{Con}(\mathbf{T})$.

(iii) $\mathbf{T} + \mathrm{Con}(\mathbf{T}) \vdash \mathrm{Con}^*(\mathbf{T})$.

Proof: For (i) the same proof as in Proposition 3.3 works with **PA** replaced by **T**. (iii) is obvious. For (ii) note that Solovay's Theorem also works for **T** so that the proof of Case 1 of Theorem 3.10 can be copied. To deal with Case 2, observe that $\mathfrak{I} \models \psi$ since ψ is Π_1, $\mathfrak{N} \models \psi$ and $\mathfrak{I}$ is an initial segment of $\mathfrak{N}$. □

The methods of Theorem 3.10 can also be used to produce two 'natural' slow growing functions f and g such that the theories $\mathbf{PA} + \mathrm{Con}_f(\mathbf{PA})$ and $\mathbf{PA} + \mathrm{Con}_g(\mathbf{PA})$ are mutually non-interpretable in each other.

Definition 3.12. The even and odd parts of F_{ε_0} are defined as follows:

$$F^{even}_{\varepsilon_0}(2n) = F_{\varepsilon_0}(2n), \qquad F^{even}_{\varepsilon_0}(2n+1) = F_{\varepsilon_0}(2n) + 1\,,$$
$$F^{odd}_{\varepsilon_0}(2n+1) = F_{\varepsilon_0}(2n+1), \qquad F^{odd}_{\varepsilon_0}(2n+2) = F_{\varepsilon_0}(2n+1) + 1, \quad F^{odd}_{\varepsilon_0}(0) = 1,$$

$$\begin{array}{rcl} f(n) & = & \max(\{k \leq n \mid \exists y \leq n\, F^{even}_{\varepsilon_0}(k) = y\} \cup \{0\}) \\ g(n) & = & \max(\{k \leq n \mid \exists y \leq n\, F^{odd}_{\varepsilon_0}(k) = y\} \cup \{0\}). \end{array}$$

By Lemma 2.2, the graphs of f and g are Δ_0 and both functions are provably recursive functions of **PA**.

Remark 3.13. In a much more elaborate form, the method of defining variants of a given computable functions (such as F_{ε_0}) in a piecewise manner has been employed in [10] to obtain results about degree structures of computable functions and in [5] to obtain forcing-like results about provably recursive functions.

Theorem 3.14. *(i)* $\mathbf{PA} + \mathrm{Con}_f(\mathbf{PA}) \nvdash \mathrm{Con}_g(\mathbf{PA})$.

(ii) $\mathbf{PA} + \mathrm{Con}_g(\mathbf{PA}) \nvdash \mathrm{Con}_f(\mathbf{PA})$.

Proof: (i) The proof is a variant of that of Theorem 3.10. Let $\mathfrak{M}$ be a countable non-standard model of $\mathbf{PA} + F_{\varepsilon_0}$ *is total*. Let M be the domain of $\mathfrak{M}$ and $a \in M$ be non-standard such that $\mathfrak{M}$ thinks that a is **odd**. Let $e = F^{\mathfrak{M}}_{\varepsilon_0}(a)$. As before, there exists a countable model $\mathfrak{N}$ of **PA** such that:

(i) $\mathfrak{M}$ and $\mathfrak{N}$ agree up to e.

(ii) $\mathfrak{N}$ thinks that $\mathbf{PA}\restriction_a$ is consistent.

(iii) $\mathfrak{N}$ thinks that $\mathbf{PA}\lceil_{a+1}$ is inconsistent. In fact there is a proof of $0 = 1$ from $\mathbf{PA}\lceil_{a+1}$ whose Gödel number is less than 2^{2^e} (as computed in $\mathfrak{N}$).

Again we distinguish two cases.

Case 1: $\mathfrak{N} \models F_{\varepsilon_0}(a+1)\uparrow$. Then also $\mathfrak{N} \models F_{\varepsilon_0}(d)\uparrow$ for all $d > a$ by Lemma 2.3(ii). Since $\mathfrak{M}$ thinks that $a+1$ is even, so does $\mathfrak{N}$, as both models agree up to e. Thus $\mathfrak{N} \models F^{even}_{\varepsilon_0}(d)\uparrow$ for all $d > a$. As a result, $\mathfrak{N} \models \forall x\, f(x) \leq a$, and hence, $\mathfrak{N} \models \mathrm{Con}_f(\mathbf{PA})$. On the other hand, since $\mathfrak{N} \models F^{odd}_{\varepsilon_0}(a+1) = e+1$ and $\mathfrak{N}$ thinks that $\mathbf{PA}\lceil_{a+1}$ is inconsistent, it follows that $\mathfrak{N} \not\models \mathrm{Con}_g(\mathbf{PA})$.

Case 2: $\mathfrak{N} \models F_{\varepsilon_0}(a+1)\downarrow$. As in the proof of Theorem 3.10, letting $c := F^{\mathfrak{N}}_{\varepsilon_0}(a+1)$, for each n we find an initial segment $\mathfrak{I}$ of $\mathfrak{N}$ such $e < \mathfrak{I} < c$ and $\mathfrak{I}$ is a model of Π_{n+1}-induction. Moreover, it follows from the properties of $\mathfrak{N}$ and the fact that $2^{2^e} < \mathfrak{I}$, that

1. $\mathfrak{I}$ thinks that $\mathbf{PA}\lceil_a$ is consistent.
2. $\mathfrak{I}$ thinks that $\mathbf{PA}\lceil_{a+1}$ is inconsistent.
3. $\mathfrak{I}$ thinks that $F_{\varepsilon_0}(a+1)$ is not defined.

Consequently as $\mathfrak{I}$ thinks that $a+1$ is even, $\mathfrak{I} \models \forall x\, f(x) \leq a$, whence $\mathfrak{I} \models \mathrm{Con}_f(\mathbf{PA})$. On the other hand, since $\mathfrak{I} \models F^{odd}_{\varepsilon_0}(a+1) = e+1$, we also have that $\mathfrak{N} \not\models \mathrm{Con}_g(\mathbf{PA})$. Since n was arbitrary, this shows that $\mathbf{PA} + \mathrm{Con}_f(\mathbf{PA}) + \neg\mathrm{Con}_g(\mathbf{PA})$ is a consistent theory.

(ii). The argument is completely analogous, the only difference being that we start with a non-standard $a \in M$ such that $\mathfrak{M}$ thinks that a is even. $\square$

Corollary 3.15. *Neither is* $\mathbf{PA}+\mathrm{Con}_f(\mathbf{PA})$ *interpretable in* $\mathbf{PA}+\mathrm{Con}_g(\mathbf{PA})$ *nor* $\mathbf{PA}+\mathrm{Con}_g(\mathbf{PA})$ *interpretable in* $\mathbf{PA}+\mathrm{Con}_f(\mathbf{PA})$.

Proof: This follows from Theorem 3.14 and Theorem 3.1. $\square$

3.1. Replacing F_{ε_0} by combinatorial functions

The function F_{ε_0} is defined by reference to ordinal representations. An "ordinal-free" version of slow consistency with similar properties as $\mathrm{Con}^*(\mathbf{PA})$ can be obtained by utilizing the Paris-Harrington function f_{PH} which has roughly the same order of growth as F_{ε_0}.

Definition 3.16. Let X be a finite set of natural numbers and $|X|$ be the number of elements in X. X is **large** if X if X is non-empty, and, letting

s be the least element of X, X has at least s elements. If $d \in \mathbb{N}$ then $[X]^d$ denotes the set of all subsets of X of cardinality d. If $g : [X]^d \to Y$, a subset Z of X is **homogeneous** for g if g is constant on $[Z]^d$. Identify $n \in \mathbb{N}$ with the set $\{0, \ldots, n-1\}$.

Let $a, b, c \in \mathbb{N}$. Then $a \to (\text{large})^b_c$ if for every map $g : [a]^b \to c$, there is a large homogeneous set for g of cardinality greater than b.

Let $\sigma(b, c)$ be the least integer a such that $a \to (\text{large})^b_c$ and $f_{PH}(n) = \sigma(n, n)$.

Theorem 3.17. *(i)* (Harrington, Paris [14]) *The function f_{PH} dominates all* **PA***-provably recursive functions.*

(ii) (Ketonen, Solovay [8]) *For $n \geq 20$:*

$$F_{\varepsilon_0}(n-3) \leq \sigma(n, 8) \leq F_{\varepsilon_0}(n-2)$$
$$f_{PH}(n) \leq F_{\varepsilon_0}(n-1).$$

Below we shall write $T_1 \lhd T_2$ to mean $T_1 \unlhd T_2$ and $T_2 \not\unlhd T_1$.

Theorem 3.18. *Letting $G(n) = \sigma(n+3, 8)$ and $g = G^{-1}$, i.e.*

$$g(n) \quad = \quad \max(\{k \leq n \mid \exists y \leq n \, G(k) = y\} \cup \{0\}),$$

we have

$$\mathbf{PA} \;\lhd\; \mathbf{PA} + \mathrm{Con}_g(\mathbf{PA}) \;\unlhd\; \mathbf{PA} + \mathrm{Con}^*(\mathbf{PA}) \;\lhd\; \mathbf{PA} + \mathrm{Con}(\mathbf{PA}).$$

Proof: The proof of Theorem 3.10 in [8] shows that $F_{\varepsilon_0}(n) \leq G(n)$ holds for $n \geq 5$. Moreover, rumination on the proof reveals that one can prove that if $G(n)$ is defined so is $F_{\varepsilon_0}(n)$ using the means of **PA**. Thus **PA** proves $\forall x\, (G(x) \downarrow \to \;\; F_{\varepsilon_0}(n) \downarrow)$. As a result, $\mathbf{PA} + \mathrm{Con}^*(\mathbf{PA}) \vdash \mathrm{Con}_g(\mathbf{PA})$. The same proof as for Proposition 3.3 shows that $\mathbf{PA} \nvdash \mathrm{Con}_g(\mathbf{PA})$. □

3.2. Some remarks

We add some remarks about related strands of investigation.

3.2.1. Phase transitions

If one defines f_α by

$$f_\alpha(n) \;=\; \max(\{k \leq n \mid \exists y \leq n\, F_\alpha(k) = y\} \cup \{0\})$$

for all $\alpha \leq \varepsilon_0$, then one has $\mathbf{PA}+\mathrm{Con}_{f_\alpha}(\mathbf{PA}) = \mathbf{PA}+\mathrm{Con}(\mathbf{PA})$ for all $\alpha < \varepsilon_0$ whereas $\mathbf{PA}+\mathrm{Con}_{f_{\varepsilon_0}}(\mathbf{PA}) \lhd \mathbf{PA}+\mathrm{Con}(\mathbf{PA})$. This result can be construed as a phase transition. However, one should perhaps bear in mind that this is a phase transition with respect to a particular hierarchy of functions. It is possible to define other hierarchies where the transition occurs at a different ordinal. For instance one could take the inverses of the so-called slow growing hierarchy (see [6, 2, 25, 26]) which catches up with the fast growing hierarchy $\alpha \leq \varepsilon_0$ only at the much bigger Bachmann-Howard ordinal.

3.2.2. Statements weaker than $\mathrm{Con}^*(\mathbf{PA})$

The proof-theoretic literature is awash with fast growing functions. Basically every ordinal analysis of a theory T (see [16, 17, 18]) gives rise to a hierarchy of fast growing functions $(F_\alpha)_{\alpha\leq\tau}$ having the following properties: (i) Every function F_α with $\alpha < \tau$ is provably recursive in T. (ii) Every provably recursive function of T is eventually dominated by some F_α with $\alpha < \tau$. (iv) F_τ is not provably recursive in T and eventually dominates any provably recursive function of T. (v) τ is the proof-theoretic ordinal of T.

Now, if one takes a theory T whose ordinal τ is greater than ε_0 then with the statement $\mathrm{Con}_{F_\tau^{-1}}(\mathbf{PA})$ we conjecture that

$$\mathbf{PA} \;\lhd\; \mathbf{PA}+\mathrm{Con}_{F_\tau^{-1}}(\mathbf{PA}) \;\lhd\; \mathbf{PA}+\mathrm{Con}^*(\mathbf{PA}).$$

Very likely another method for obtaining such intermediate theories will be provided by the inverses of functions coming from miniaturizations of Kruskal's theorem and the graph minor theorem (see [20]).

3.3. A natural Orey sentence

A sentence φ of $\mathbf{PA}$ is called an **Orey sentence** if both $\mathbf{PA}+\varphi \;\trianglelefteq\; \mathbf{PA}$ and $\mathbf{PA}+\neg\varphi \;\trianglelefteq\; \mathbf{PA}$ hold.

Corollary 3.19. *The sentence* $\exists x\,(F_{\varepsilon_0}(x)\!\uparrow\, \wedge \forall y < x\, F_{\varepsilon_0}(y)\!\downarrow\, \wedge\, x \texttt{ is even})$ *is an Orey sentence.*

Proof: Let ψ be the foregoing sentence. In view of Theorem 3.1, it suffices to show that $\mathbf{PA} \vdash \mathrm{Con}(\mathbf{PA}\restriction_k +\psi)$ and $\mathbf{PA} \vdash \mathrm{Con}(\mathbf{PA}\restriction_k +\neg\psi)$ hold for all k. Fix $k > 0$.

First we show that $\mathbf{PA} \vdash \mathrm{Con}(\mathbf{PA}\restriction_k +\psi)$. Note that $\mathbf{PA}$ proves the consistency of $\mathbf{PA}\restriction_k +\forall x\, F_{\omega_{k+1}}(x)\downarrow +\exists x F_{\varepsilon_0}(x)\uparrow$. Arguing in $\mathbf{PA}$ we thus find a non-standard model $\mathfrak{N}$ such that

$$\mathfrak{N} \models \mathbf{PA}\restriction_k +\forall x\, F_{\omega_{k+1}}(x)\downarrow +\exists x F_{\varepsilon_0}(x)\uparrow .$$

In particular there exists a least $a \in |\mathfrak{N}|$ in the sense of $\mathfrak{N}$ such that $\mathfrak{N} \models F_{\varepsilon_0}(a)\uparrow$. If $\mathfrak{N}$ thinks that a is even, then $\mathfrak{N} \models \psi$, which entails that $\mathrm{Con}(\mathbf{PA}\restriction_k +\psi)$. If $\mathfrak{N}$ thinks that a is odd, we define a cut $\mathfrak{I}$ such that $\mathfrak{I} \models \mathbf{PA}\restriction_k$ and $F^{\mathfrak{N}}_{\varepsilon_0}(a-2) < \mathfrak{I} < F^{\mathfrak{N}}_{\varepsilon_0}(a-1)$, applying Theorem 3.7. Then $\mathfrak{I} \models \psi$ which also entails $\mathrm{Con}(\mathbf{PA}\restriction_k +\psi)$.
Next we show that $\mathbf{PA} \vdash \mathrm{Con}(\mathbf{PA}\restriction_k +\neg\psi)$. As $\mathbf{PA}$ proves $\mathrm{Con}(\mathbf{PA}\restriction_k +\forall x\, F_{\omega_{k+1}}(x)\downarrow)$, we can argue in $\mathbf{PA}$ and assume that we have a model $\mathfrak{M} \models \mathbf{PA}\restriction_k +\forall x\, F_{\omega_{k+1}}(x)\downarrow$. If $\mathfrak{M} \models \forall x F_{\varepsilon_0}(x)\downarrow$ then $\mathfrak{M} \models \neg\psi$, and $\mathrm{Con}(\mathbf{PA}\restriction_k +\neg\psi)$ follows. Otherwise there is a least a in the sense of $\mathfrak{M}$ such that $F^{\mathfrak{M}}_{\varepsilon_0}(a)\uparrow$. If $\mathfrak{M}$ thinks that a is odd we have $\mathfrak{M} \models \neg\psi$, too. If $\mathfrak{M}$ thinks that a is even we introduce a cut $F^{\mathfrak{M}}_{\varepsilon_0}(a-2) < \mathfrak{I}' < F^{\mathfrak{M}}_{\varepsilon_0}(a-1)$ such that $\mathfrak{I}' \models \mathbf{PA}\restriction_k$. Since $\mathfrak{I}' \models F_{\varepsilon_0}(a-1)\uparrow$ we have $\mathfrak{I}' \models \neg\psi$, whence $\mathrm{Con}(\mathbf{PA}\restriction_k +\neg\psi)$. $\square$

4. Iterating slow consistency

Recall that we use $T_1 \lhd T_2$ to convey that T_2 interprets T_1 but T_1 does not interpret T_2. The slow consistency operator can be iterated and by Theorem 3.1 and Corollary 3.4 we know that we get a proper hierarchy[1] in the sense of $\lhd$:

$$\begin{aligned}\mathbf{PA} \lhd \mathbf{PA} + \mathrm{Con}^*(\mathbf{PA}) \lhd \mathbf{PA} + \mathrm{Con}^*(\mathbf{PA} + \mathrm{Con}^*(\mathbf{PA}))\\ \lhd \mathbf{PA} + \mathrm{Con}^*(\mathbf{PA} + \mathrm{Con}^*(\mathbf{PA} + \mathrm{Con}^*(\mathbf{PA}))) \lhd \ldots\end{aligned}$$

A natural question arising is where this hierarchy resides with respect to $\mathbf{PA} + \mathrm{Con}(\mathbf{PA})$.

[1]We wish to thank the referee for suggesting to look at this hierarchy.

Theorem 4.1. *Let $\mathbf{T} = \mathbf{PA} + \psi$ where ψ is a Π_1 statement. Let $\mathbf{T}\restriction_k$ to be the theory $\mathbf{PA}\restriction_k + \psi$ and $\mathrm{Con}^*(\mathbf{T}) := \forall x \mathrm{Con}(\mathbf{T}\restriction_{F_{\varepsilon_0}^{-1}(x)})$. Then:*

$$\mathbf{T} + \mathrm{Con}(\mathbf{T}) \vdash \mathrm{Con}(\mathbf{T} + \mathrm{Con}^*(\mathbf{T})).$$

Proof: We will argue in $\mathbf{T} + \mathrm{Con}(\mathbf{T})$. From $\mathrm{Con}(\mathbf{T})$ we infer that there exists a countable non-standard model $\mathfrak{M}$ of $\mathbf{T}$. Let M be the domain of $\mathfrak{M}$. Since $\mathbf{T}$ is reflexive it follows by overspill that there is a non-standard $a \in M$ such that

$$\mathfrak{M} \models \mathrm{Con}(\mathbf{T}\restriction_a). \tag{11}$$

If $\mathfrak{M} \models F_{\varepsilon_0}(a)\uparrow$, then also $\mathfrak{M} \models F_{\varepsilon_0}(d)\uparrow$, for all $d > a$ by Lemma 2.3(ii), and therefore $\mathfrak{M} \models \mathrm{Con}^*(\mathbf{T})$, yielding $\mathrm{Con}(\mathbf{T} + \mathrm{Con}^*(\mathbf{T}))$.

Now assume $\mathfrak{M} \models F_{\varepsilon_0}(a)\downarrow$ for the remainder of the proof. If $\mathfrak{M} \models F_{\varepsilon_0}(a+1)\uparrow$, then $\mathfrak{M}$ will be a model $\mathrm{Con}^*(T)$, too, and hence $\mathrm{Con}(\mathbf{T} + \mathrm{Con}^*(\mathbf{T}))$ holds. So let's assume $\mathfrak{M} \models F_{\varepsilon_0}(a+1)\downarrow$ as well.

Let $e := F_{\varepsilon_0}^{\mathfrak{M}}(a)$ and $c := F_{\varepsilon_0}^{\mathfrak{M}}(a+1)$. By Corollary 3.8, for every standard n there is an initial segment $\mathfrak{I}$ of $\mathfrak{M}$ such $e < \mathfrak{I} < c$ and $\mathfrak{I}$ is a model of Π_{n+1}-induction. Moreover, it follows therefore that:

1. $\mathfrak{I}$ thinks that $\mathbf{T}\restriction_a$ is consistent and that ψ is true, owing to these statements being true in $\mathfrak{M}$ and of Π_1 form.
2. $\mathfrak{I}$ thinks that $F_{\varepsilon_0}(a+1)$ is not defined since it is not defined in $\mathfrak{M}$.

Consequently, $\mathfrak{I} \models \mathrm{Con}^*(\mathbf{T}) + \Pi_{n+1}$-induction. Since n was arbitrary, this shows that $\mathbf{T} + \mathrm{Con}^*(\mathbf{T})$ is a consistent theory.

The only qualms one might have about the preceding proof is whether Corollary 3.8 can be formalized in $\mathbf{PA}$. Corollary 3.8 builds on Theorem 3.7, which is essentially [23, Theorem 5.25]. However, inspection of the proof of the latter result shows that it can be formalized in $\mathbf{PA}$. $\square$

Corollary 4.2. *Letting $\mathbf{T}_0 := \mathbf{PA}$ and $\mathbf{T}_{n+1} := \mathbf{T}_n + \mathrm{Con}^*(\mathbf{T}_n)$, we have*

$$\mathbf{T}_m \lhd \mathbf{PA} + \mathrm{Con}(\mathbf{PA})$$

for all m.

Proof: Using Theorem 4.1 iteratively (induction on n), we have $\mathbf{PA} + \mathrm{Con}(\mathbf{PA}) \vdash \mathrm{Con}(\mathbf{T}_n)$, and hence $\mathbf{T}_n + \mathrm{Con}(\mathbf{T}_n) \subseteq \mathbf{PA} + \mathrm{Con}(\mathbf{PA})$. With Theorem 3.11 we conclude that $\mathbf{T}_m \lhd \mathbf{PA} + \mathrm{Con}(\mathbf{PA})$ holds for all m. $\square$

In the above we could have used the hierarchy $\mathbf{T}'_0 := \mathbf{PA}$ and $\mathbf{T}'_{n+1} := \mathbf{PA} + \mathrm{Con}^*(\mathbf{T}'_n)$. Actually, $\mathbf{T}'_n$ and $\mathbf{T}_n$ are the same theories, i.e., they prove the same theorems.

Remark 4.3. *All extensions of* $\mathbf{PA}$ *considered in this paper are augmentations of* $\mathbf{PA}$ *via true* Π_1 *statements. As a result, all of these theories have the same provably recursive functions. Thus, although the hierarchy of theories* $\mathbf{T}_0 \lhd \mathbf{T}_1 \lhd \mathbf{T}_2 \lhd \ldots$ *is a proper one, the theories share the same "proof-theoretic strength" if the latter notion is identified with a theory's stock of provably recursive functions.*

Remark 4.4. *The hierarchy* $(\mathbf{T}_n)_{n<\omega}$ *could be extended transfinitely. We have not investigated this, but conjecture that all the theories* $\mathbf{T}_\alpha$ *with* $\alpha < \varepsilon_0$ *satisfy* $\mathbf{T}_\alpha \lhd \mathbf{PA} + \mathrm{Con}(\mathbf{PA})$.

Appendix

We will provide an alternative and more detailed proof of Lemma 3.5, namely that $\mathbf{PA} \vdash \mathrm{Con}^{\#}(\mathbf{PA})$.

The reader will be assumed to have access to [1]. That paper uses an infinitary proof system with the ω-rule (of course). But this system is also quite peculiar in that the ordinal assignment adhered to is very rigid and, crucially, it has a so-called accumulation rule. To deal with infinite proofs in **PA**, though, one has to use primitive recursive proof trees instead of arbitrary ones (for details see [4]). The role of the repetition rule (or trivial rule) (cf. [4]) is of central importance to capturing the usual operations on proofs, such as inversion and cut elimination, by primitive recursive functions acting on their codes. In the proof system of [1] the accumulation rule takes over this role. Now assume that everything in [1] has been recast in terms of primitive recursive proof trees. Then the cut elimination for infinitary proofs with finite cut rank (as presented in [4, Theorem 2.19]) can be formalized in **PA**. Working in **PA**, suppose that $F_{\varepsilon_0}(u)\downarrow$. Aiming at a contradiction assume that there is a $p < u$ such that $\mathrm{Proof}_{\mathbf{PA}}(p, \bot)$. As above, the proof that p codes, can be primitive recursively transformed into a proof P of $\bot$ in the sequent calculus of [1] with ordinal ω_p and cut-degree 0 (in the sense of [1, Definition 5]). The plan is to reach a contradiction by constructing an infinite descending sequence of ordinals $(\alpha_i)_{i\in\mathbb{N}}$ such that $\alpha_0 = \omega_p$, $\alpha_{i+1} < \alpha_i$ and $\alpha_{i+1} <_{l_{i+1}} \alpha_i$ for some $l_{i+1} < F_{\omega_p}(2)$. It remains to determine $(\alpha_i)_{i\in\mathbb{N}}$. To this end we construct a branch of the proof-tree P with $\vdash^{\alpha_i} \Delta_i, \Gamma_i$ being the

i-th node of the branch (bottom-up). The sequent Γ_i contains only closed elementary prime formulas and formulas of the form "$n \in N$" whereas Δ_i is of the form $\{n_1 \notin N, \ldots, n_r \notin N\}$ or $\emptyset$. We set $k_{\Delta_i} := \max(\{2\} \cup \{3 \cdot n_1, \ldots, 3 \cdot n_r\})$ in the former and $k_{\Delta_i} := 2$ in the latter case. We say that Γ_i is true in m if Γ_i is true when N is interpreted as the finite set $\{n \mid 3 \cdot n < m\}$. Let $\Gamma_0 = \{0 = 1\}$ and $\Delta_0 = \emptyset$. Clearly, Γ_0 is false in $F_{\alpha_0}(2)$. Now assume $\vdash^{\alpha_i} \Delta_i, \Gamma_i$ has been constructed in such a way that $F_{\alpha_i}(k_{\Delta_i}) \downarrow$ and Γ_i is false in $F_{\alpha_i}(k_{\Delta_i})$ and $F_{\alpha_i}(k_{\Delta_i}) \leq F_{\alpha_0}(2)$. Since Γ_i is false in $F_{\alpha_i}(k_{\Delta_i})$ and $F_{\alpha_i}(k_{\Delta_i}) > k_{\Delta_i}$, it follows that Δ_i, Γ_i is not an axiom. Thus $\vdash^{\alpha_i} \Delta_i, \Gamma_i$ is not an end-node in P and therefore it is the result of an application of an inference rule. As the cut-rank of P is 0, the only possible rules are a cut of rank 0, an N-rule, and Accumulation.

If it is an N-rule, Γ_i contains "$Sn \in N$" for some n and $\vdash^{\beta} \Delta_i, \Gamma'_i, n \in N$ will be a node in P immediately above $\vdash^{\alpha_i} \Delta_i, \Gamma_i$ with $\Gamma'_i \subseteq \Gamma_i$ and $\beta + 1 = \alpha_i$. We let $\alpha_{i+1} = \beta$, $l_{i+1} = 1$, $\Delta_{i+1} = \Delta_i$ and $\Gamma_{i+1} = \Gamma_i, n \in N$. Since Γ_i is false in $F_{\alpha_i}(k_{\Delta_i})$ and $F_{\alpha_{i+1}}(k_{\Delta_i}) + 3 \leq F_{\alpha_i}(k_{\Delta_i})$ it follows that Γ_{i+1} is false in $F_{\alpha_i}(k_{\Delta_{i+1}})$.

If the last rule is Accumulation, $\vdash^{\beta} \Delta_i, \Gamma_i$ will be a node in P immediately above $\vdash^{\alpha_i} \Delta_i, \Gamma_i$ for some $\beta <_{k_{\Delta_i}} \alpha_i$. Then let $\Delta_{i+1} = \Delta_i$, $\Gamma_{i+1} = \Gamma_i$, $\alpha_{i+1} = \beta$, and $l_{i+1} = k_{\Delta_i}$. Since $F_\beta(k_{\Delta_i}) \leq F_{\alpha_i}(k_{\Delta_i})$, Γ_{i+1} is false in $F_{\alpha_{i+1}}(k_{\Delta_{i+1}})$, too. Inductively we also have $F_{\alpha_i}(k_{\Delta_i}) \leq F_{\alpha_0}(2)$, and hence $l_{i+1} < F_{\alpha_0}(2)$.

If the last rule is a cut with a closed elementary prime formula A, the immediate nodes above $\vdash^{\alpha_i} \Delta_i, \Gamma_i$ in P are of the form $\vdash^{\beta} \Delta_i, \Gamma_i, A$ and $\vdash^{\beta} \Delta_i, \Gamma_i, \neg A$, respectively, where $\beta + 1 = \alpha_i$. Let $\Delta_{i+1} = \Delta_i$, $\alpha_{i+1} = \beta$, and $l_{i+1} = 1$. If A is false let $\Gamma_{i+1} = \Gamma_i, A$. If A is true, let $\Gamma_{i+1} = \Gamma_i, \neg A$. Clearly, Γ_{i+1} will be false in $F_{\alpha_{i+1}}(k_{\Delta_{i+1}})$ since this value is smaller than $F_{\alpha_i}(k_{\Delta_i})$.

Finally suppose the last rule is a cut with cut formula "$n \in N$". Then the immediate nodes above $\vdash^{\alpha_i} \Delta_i, \Gamma_i$ in P are of the form $\vdash^{\beta} \Delta_i, n \in N, \Gamma_i$ and $\vdash^{\beta} \Delta_i, n \notin N, \Gamma_i$, respectively, where $\beta + 1 = \alpha_i$. Set $\alpha_{i+1} = \beta$ and and $l_{i+1} = 1$. If $F_\beta(k_{\Delta_i}) \leq 3 \cdot n$, then "$n \in N$" will be false in $F_\beta(k_{\Delta_i})$, and hence, as $F_\beta(k_{\Delta_i}) < F_{\alpha_i}(k_{\Delta_i})$, it follows that $n \in N, \Gamma_i$ will be false in $F_\beta(k_{\Delta_i})$ as well. So in this case let $\Delta_{i+1} = \Delta_i$ and $\Gamma_{i+1} = n \in N, \Gamma_i$.

If on the other hand $3 \cdot n < F_\beta(k_{\Delta_i})$, we compute that

$$F_\beta(k_{\Delta_i, n \notin N}) < F_\beta(F_\beta(k_{\Delta_i})) \leq F_{\alpha_i}(k_{\Delta_i}).$$

Hence Γ_i will be false in $F_\beta(k_{\Delta_i, n \notin N})$, and we put $\Delta_{i+1} = \Delta_i, n \notin N$ and $\Gamma_{i+1} = \Gamma_i$.

This finishes the definition of the $(\alpha_i)_{i\in\mathbb{N}}$. Their construction also guarantees that $F_{\alpha_i}(l_{i+1}) \downarrow$ and $F_{\alpha_{i+1}}(l_{i+1}) \le F_{\alpha_i}(l_{i+1}) \le F_{\omega_p}(2)$. Note also that whenever the inference involving $\vdash^{\alpha_{i+1}} \Delta_{i+1}, \Gamma_{i+1}$ as a premiss and $\vdash^{\alpha_i} \Delta_i, \Gamma_i$ as its conclusion was an application of a rule other than the Accumulation rule, then we have $\alpha_i = \alpha_{i+1} + 1$ and $l_{i+1} = 1$, and hence $F_{\alpha_{i+1}}(l_{i+1}) < F_{\alpha_i}(l_{i+1})$. As a result, there can only be finitely many of those. Hence there exists x_0 such that for $i \ge x_0$ the inference from $\vdash^{\alpha_{i+1}} \Delta_{i+1}, \Gamma_{i+1}$ to $\vdash^{\alpha_i} \Delta_i, \Gamma_i$ is always an instance of Accumulation. Furthermore, this entails that $\Delta_i, \Gamma_i = \Delta_j, \Gamma_j$ and $l_i = l_j$ for all $i, j > x_0$. Hence $\alpha_{i+1} <_k \alpha_i$ for all $i \ge x_0$ where $k = l_{x_0+1}$. However, this is absurd in view of Lemma 2.4 since then the computation of $F_{\alpha_{x_0}}(k)$ (i.e. $F_{\alpha_{x_0}}(l_{x_0+1})$) would never halt. $\square$

Acknowledgements

The research of all authors was supported by Templeton Foundation Grant #13152, the CRM Infinity Project.

The first author also wishes to thank the Austrian Science Fund for its support through research project P22430-N13.

The second author acknowledges support of this research through U.K. EPSRC grant No. EP/G029520/1.

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